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ON EXACT CONTROLLABILITY FOR THE NAVIER-STOKES EQUATIONS

O.YU. IMANUVILOV

ABSTRACT. We study the local exact controllability problem for the Navier-Stokes equations that describe an incompressible fluid flow in a bounded domain Ω with control distributed in a subdomain $\omega \subset \Omega$. The result that we obtain in this paper is as follows. Suppose that $\hat{v}(x)$ is a given steady-state solution of the Navier-Stokes equations. Let $v_0(x)$ be a given initial condition and $\|\hat{v}(\cdot) - v_0\| < \varepsilon$ where ε is small enough. Then there exists a locally distributed control u, with supp $u \subset (0,T) \times \omega$ such that the solution $v(t,x)$ of the Navier-Stokes equations:

$$\partial_t v - \Delta v + (v, \nabla)v = \nabla p + u + f, \ \mathbf{div}\, v = 0, \ v|_{\partial\Omega} = 0, \ v|_{t=0} = v_0$$

coincides with $\hat{v}(x)$ at instant T, $v(T,x) \equiv \hat{v}(x)$.

1. INTRODUCTION

This paper is concerned with the local exact controllability of the Navier-Stokes equations, defined on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2,3)$ with boundary $\partial\Omega \in C^\infty$. More precisely, the problem under study is as follows. Let us consider the nonstationary Navier-Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v, \nabla)\, v + \nabla p = f(x) + \chi_\omega u \ \text{ in } \Omega, \quad \mathbf{div}\, v = 0, \tag{1.1}$$

with initial and boundary conditions

$$v|_\Sigma = 0, \quad v\,|_{t=0} = v_0(x), \tag{1.2}$$

where $v(t,x) = (v_1(t,x), ..., v_n(t,x))$ is the fluid velocity, p the pressure, $f(x) = (f_1(x), .., f_n(x))$ a density of external forces, $u(t,x)$ a control distributed in an arbitrary fixed subdomain ω of the domain Ω and χ_ω is the characteristic function of the set ω:

$$\chi_\omega(x) = \begin{cases} 1, & \text{for } x \in \omega \\ 0, & \text{for } x \in \Omega \setminus \omega. \end{cases}$$

Let $(\hat{v}(x), \hat{p}(x))$ be a steady-state solution of the Navier-Stokes equations

$$-\Delta\hat{v} + (\hat{v}, \nabla)\hat{v} + \nabla\hat{p} = f(x) \text{ in } \Omega, \quad \mathbf{div}\,\hat{v} = 0, \quad \hat{v}|_{\partial\Omega} = 0 \tag{1.3}$$

Korea Institute for Advanced Study, 207-43 Chungryangri-dong Dongdaemoon-ku Seoul 130-012, Korea. E-mail: oleg@kias.kaist.ac.kr

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close enough to the initial condition

$$\|v_0 - \hat{v}\|_{V^1(\Omega)} \le \varepsilon, \tag{1.4}$$

where the parameter ε is sufficiently small. We want to find a control u such that for given $T > 0$, the following equality holds:

$$v(T, x) = \hat{v}(x). \tag{1.5}$$

We assume

CONDITION 1.1. *The boundary $\partial\Omega = \cup_{i=1}^K \Gamma_i \in C^\infty$, ($\Gamma_i \cap \Gamma_j = \{\emptyset\}$ for all $i \neq j$) where Γ_i is a $n-1$-dimensional connected manifold of class C^∞. For each Γ_i, there exists a neighborhood $\mathfrak{A}_i \subset \mathbb{R}^n$ and a diffeomorphism $\eta_i \in C^\infty(\mathfrak{A}_i, \mathbb{R}^n)$ such that $\eta_i(\Gamma_i) = S_1^n$.**

The main result of this paper is the following Theorem.

THEOREM 1.2. *Let $v_0 \in V^1(\Omega)$ and pair $(\hat{v}, \hat{p}) \in (V^1(\Omega) \bigcap (W^1_\infty(\Omega))^n) \times W^1_2(\Omega)$ is a given steady state solution of the Navier-Stokes equations (1.3) such that* $\mathbf{supp}\, \hat{v} \subset\subset \Omega$. *Then for sufficiently small ε there exists a solution $(v, p, u) \in V^{1,2}(Q) \times L^2(0, T; W^1_2(\Omega)) \times (L^2(Q_\omega))^n$ of problem (1.1), (1.2), (1.4), (1.5).*

To explain this result, let us assume that $\hat{v}|_{\partial\Omega} = 0$ and $\hat{v}$ is an unstable singular point of the dynamical system generated by equation (1.1) in the phase space of solenoidal vector fields with adherence conditions on $\partial\Omega$. Let v_0 be an initial condition in a neighborhood of the function $\hat{v}$. This work shows that one can construct a locally distributed control such that the trajectory goes out of point v_0 and reaches $\hat{v}$ in finite time. In other words, by means of the locally distributed control, one can suppress the generation of turbulence. This result clarifies the question of the connection between turbulence and controllability (see J.-L. Lions [26]).

The result we obtain in Theorem 1.2 is local. On the other hand, for the linearized Navier-Stokes system, we can prove global zero-controllability, see Theorem 4.3.

One important special case is the following controllability problem for the Stokes system:

$$\partial_t v(t, x) - \Delta v(t, x) = \nabla p + f(t, x) + \chi_\omega u, \text{ in } \Omega, \quad \mathbf{div}\, v = 0, \tag{1.6}$$

$$v|_\Sigma = 0, \quad v|_{t=0} = v_0(x), \quad v|_{t=T} \equiv 0. \tag{1.7}$$

We have

THEOREM 1.3. *Let $v_0 \in V^1(\Omega), f \in L^2(0, T; V^0(\Omega))$ and there exists $\varepsilon > 0$ such that $\int_Q |f|^2 e^{\frac{1}{(T-t)^{2+\varepsilon}}} dx dt < \infty$. Then there exists a solution $(v, p, u) \in V^{1,2}(Q) \times L^2(0, T; W^1_2(\Omega)) \times (L^2(Q_\omega))^n$ to problem (1.6),(1.7).*

This paper is organized as follows. To prove Theorem 1.2 we use a variant of the implicit function theorem. The only nontrivial condition to be checked is to show that the derivative of the corresponding mapping at some

* $S^n_r = \partial B_r, B_r = \{x \in \mathbb{R}^n; |x| < r\}$

point is an epimorphism. In our case, this problem is equivalent to the zero controllability of the linearization of the Navier-Stokes equations at point $\hat{v}$. (see problem (4.1)-(4.3).) Sections 2-4 are devoted to this problem. One of the usual ways to solve the controllability problem for evolution equations is to reduce it to an observability problem for the adjoint equation. Thus, in section 2 we introduce a linear operator (see equation (2.1)) which after the change $t \to -t$ is formally adjoint to the derivative of the Navier-Stokes equations at point $\hat{v}(x)$. The observability problem for this operator is solved in three steps. First in Theorem 2.11, we get an appropriate estimate for the pressure p. Then in Theorem 3.1, we obtain a Carleman estimate for the velocity y of the fluid via a weighted L^2-norm of the density of external forces f and the pressure p. Moreover, for the pressure, by Theorem 2.11, one can choose a weighted L^2 norm over $(0,T)\times\omega$. And finally in Theorem 3.6, we prove an estimate (not of Carleman type) for the velocity where p and an initial condition are absent from the right-hand side. In section 4, this observability estimate is converted into a controllability result in Theorem 4.3. In section 5, all conditions for the implicit function theorem are checked.

We close this section by mentioning some of the previous results regarding our problems. The solvability of (1.1), (1.2),(1.5) was first proved in A.V. Fursikov, O.Yu. Imanuvilov [12] in the case when (1.1) is Burgers' equation. For a control distributed in a domain ω such that $\partial\Omega \subset \overline{\omega}$, this problem was studied in the case of the Navier-Stokes equations and $\hat{v} \equiv 0$ in A.V. Fursikov, O.Yu. Imanuvilov [13] in dimension $n = 2$ and in A.V. Fursikov [10] when $n = 3$. The case of the Navier-Stokes equations and $\hat{v} \neq 0$ has been studied in A.V. Fursikov, O.Yu. Imanuvilov [14], [17], O.Yu. Imanuvilov [21] and for the Boussinesq system in [15] (see also [16]). On the other hand, in pioneering works [3]-[5], J.-M. Coron proved the global approximate controllability for the 2-D Euler equations and the 2-D Navier-Stokes equations with slip boundary conditions. In [6], combining results on global approximate and local exact controllability results, J.-M. Coron and A.V. Fursikov obtained the global exact controllability for the Navier-Stokes system on a 2-D manifold without boundary.

In [7], C. Fabre obtained an approximate controllability for "cut off" Navier-Stokes equations.

2. Estimate for the pressure

Let us consider the system

$$\frac{\partial y}{\partial t} - \Delta y + B^*(y,\hat{v}) + B^*(\hat{v},y) = \nabla p + f \text{ in } Q, \tag{2.1}$$

$$\mathbf{div}\, y = 0, \quad y|_{\partial\Omega} = 0, \quad y(0,x) = y_0(x), \tag{2.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \subset C^\infty, Q = (0,T)\times\Omega$ and the operators $B^*(\hat{v},\cdot)$, $B^*(\cdot,\hat{v})$ are defined by the formulas:

$$B^*(y,\hat{v}) = ((y, \frac{\partial \hat{v}}{\partial x_1}),\ldots,(y,\frac{\partial \hat{v}}{\partial x_n})),\ B^*(\hat{v},y) = -(\hat{v},\nabla)y. \tag{2.3}$$

Denote $Q_\omega = (0,T)\times\omega, \Sigma = (0,T)\times\partial\Omega$. Let ν be the outward unit normal to $\partial\Omega$. In this paper we use the following functional spaces. Recall that $W_p^k(\Omega)$, $k \geq 0$, $1 \leq p < \infty$ is the Sobolev space of functions with finite norm

$$\|u\|_{W_p^k(\Omega)} = (\sum_{|\alpha|\leq k}\int_\Omega \left|\partial^{|\alpha|}u(x)\,/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}\right|^p dx)^{1/p},$$

where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)\,,\ |\alpha| = \alpha_1+\cdots+\alpha_n,$

$$W^{1,2}(Q) = \{w(t,x)|w\in L^2(0,T;W_2^2(\Omega)),\ \frac{\partial w}{\partial t}\in L^2(0,T;L^2(\Omega))\},$$

$$V^1(\Omega) = \{v(x) = (v_1,\ldots,v_n)\ \in\ (W_2^1(\Omega))^n; \mathbf{div}\, v = 0,\ v|_{\partial\Omega} = 0\},$$

$$V^0(\Omega) = \{v(x) = (v_1,\ldots,v_n)\ \in\ (L^2(\Omega))^n; \mathbf{div}\, v = 0,\ (v,\nu)|_{\partial\Omega} = 0\},$$

$$V^{-1}(\Omega) = (V^1(\Omega))^*,$$

$$V^{1,2}(Q) = \{v(t,x)\ \in\ (W^{1,2}(Q))^n; \mathbf{div}\, v = 0,\ v|_{\partial\Omega} = 0\},$$

$$L^2(Q,\rho) = \{v(t,x); \int_Q \rho v^2 dxdt < \infty\}.$$

We have

PROPOSITION 2.1. *([27]) Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega\in C^2$. Then*

$$(L^2(\Omega))^n = V^0(\Omega)\oplus(V^0(\Omega))^\perp,$$

where

$$(V^0(\Omega))^\perp = \{v(x) = (v_1,\ldots,v_n)\ \in\ (L^2(\Omega))^n; v = \nabla p, p\in W_2^1(\Omega)\}.$$

Here and below we assume that the pair $(\hat v,\hat p)$ satisfies (1.3) and

$$(\hat v,\hat p)\in (V^1(\Omega)\bigcap(W^1_\infty(\Omega))^n)\times W_2^1(\Omega)\quad \mathrm{supp}\hat v\subset\subset\Omega.$$

Let $\omega\subset\subset\Omega$ be an arbitrary fixed subdomain and η_i be the mapping from Condition 1.1.

Without loss of generality we can assume that $\eta_i(\mathfrak{A}_i\cap\Omega)\subset B_1$. (Otherwise we can make the change $x\to x/|x|^2$.)

Set $\mathfrak{U}_i = \eta_i^{-1}(\{x\in\mathbb{R}^n|1-\varepsilon<|x|<1\})$, where $\varepsilon\in(0,1)$. For all sufficiently small ε, the set $\mathfrak{U}_i$ is correctly defined and

$$\overline{\mathfrak{U}_i}\cap\overline{\mathfrak{U}_j} = \{\emptyset\}\quad\text{for all } i\neq j,\quad \partial\mathfrak{U}_i = \Gamma_i\cup\gamma_i, \tag{2.4}$$

where $\gamma_i = \eta_i^{-1}(S^n_{1-\varepsilon})$ and

$$(\overline{\omega}\cup\mathrm{supp}\,\hat v)\cap\overline{\mathfrak{U}_i} = \{\emptyset\}\ \forall i = 1,\ldots,K. \tag{2.5}$$

Let $G\subset\mathbb{R}^n$ be a domain which satisfies the following condition:

Condition 2.2. *The domain G is diffeomorphic to the cylinder $\Gamma \times [0, T_0]$, where $T_0 > 0$ is a number and $\Gamma \subset \mathbb{R}^n$ is a closed $(n-1)$-dimensional manifold of class C^∞.*

This condition implies immediately that $\partial G \in C^\infty$ and $\partial G = \sigma_0 \cup \sigma_1$, where σ_i is a $(n-1)$-dimensional connected manifold of class C^∞.

Let $w(x)$ be a harmonic function in G:

$$\Delta w = 0 \text{ in } G, \tag{2.6}$$

such that

$$\frac{\partial w}{\partial \nu}|_{\sigma_1} = 0, \tag{2.7}$$

$$\int_{\sigma_0} |w|^2 d\sigma \le (2M)^2, \ \int_{\sigma_1} |w|^2 d\sigma \le (2\varepsilon)^2. \tag{2.8}$$

Let $\theta(x) \in C^\infty(\overline{G})$ be a function satisfying the conditions

$$0 < C_1 \le |\nabla \theta(x)| \ \forall x \in G; \ \theta|_{\sigma_1} = 0, \ \theta|_{\sigma_0} = 1. \tag{2.9}$$

Then the set

$$\kappa_t = \{x \in G; \theta(x) = t\}, \ t \in [0,1] \tag{2.10}$$

is a smooth manifold diffeomorphic to σ_0 and σ_1. We have:

Theorem 2.3.*([28]) There exist a constant $C_2 > 0$ and a function $\theta(x) \in C^\infty(\overline{G})$, satisfying condition (2.9), such that for any function $w \in W_2^{\frac{1}{2}}(G)$ for which (2.6)-(2.8) are satisfied*

$$\int_{\kappa_t} |w|^2 d\sigma \le C_2 \|w\|^{2(1-t)}_{L^2(\sigma_1)} \|w\|^{2t}_{L^2(\sigma_0)} \le C_2 (2\varepsilon)^{2(1-t)} (2M)^{2t}, \tag{2.11}$$

where κ_t is the manifold (2.10).

Obviously the domain $\mathfrak{U}_i$ satisfies Condition 2.2 for all $i \in \{1, \dots, K\}$. Denote by $\theta_i(x)$ the function from Theorem 2.3 which corresponds to the domain $\mathfrak{U}_i$. Similar to (2.10) we set

$$\kappa_t(i) = \{x \in \mathfrak{U}_i; \theta_i(x) = t\}. \tag{2.12}$$

For all $r \in (0,1)$ we introduce the auxiliary domains

$$\mathfrak{D}_i(r) = \{x_i \in \mathfrak{U}_i | r < \theta_i(x) < 1\}, \quad \mathfrak{D}(r) = \cup_{i=1}^K \mathfrak{D}_i(r). \tag{2.13}$$

Let $w_0 \Subset w$ be an arbitrary subdomain. We have:

Lemma 2.4. *There exists a function $\psi(x) \in C^\infty(\overline{\Omega})$ such that*

$$\psi(x) = 1 - \theta_i(x) \ \forall x \in \mathfrak{U}_i, \ \psi(x) > 0 \textit{ in } \Omega, \ |\nabla \psi(x)| > 0 \ \forall x \in \Omega \setminus \omega_0. \tag{2.14}$$

Proof. First we construct an auxiliary function $\beta(x) \in C^\infty(\overline{\Omega})$ such that

$$\beta(x) = 1 - \theta_i(x) \ \forall x \in \mathfrak{U}_i, \ \beta(x) > 0 \text{ in } \Omega. \tag{2.15}$$

To do this, we consider the sequence of domains $\tilde{\mathfrak{U}}_i \subset \mathbb{R}^n$, $i = \{1, \dots, K\}$ with the following properties

$$\partial\tilde{\mathfrak{U}}_i = \Gamma_i \cup \tilde{\gamma}_i \in C^\infty, \quad \mathfrak{U}_i \subset \tilde{\mathfrak{U}}_i, \gamma_i \subset\subset \tilde{\mathfrak{U}}_i,\ \overline{\tilde{\mathfrak{U}}_i} \cap \overline{\tilde{\mathfrak{U}}_j} = \{\emptyset\} \text{ for } i \neq j,$$

where $\tilde{\gamma}_i$ is a connected $(n-1)$-dimensional C^∞ manifold. (For example we can choose $\tilde{\mathfrak{U}}_i$ as $\tilde{\mathfrak{U}}_i = \eta_i^{-1}(\{x \in \mathbb{R}^n | 1 - \tilde{\varepsilon} < |x| < 1\})$ for some $\tilde{\varepsilon}$.)

Since by assumption γ_i is a C^∞ surface, one can extend the function $\theta_i(x)$ to a smooth function of $C^\infty(\tilde{\mathfrak{U}}_i)$ such that $\theta_i \equiv 0$ in some neighborhood of $\tilde{\gamma}_i$.

Set $\tilde{\beta}(x)|_{\tilde{\mathfrak{U}}_i} \equiv 1 - \theta_i(x)$ and $\tilde{\beta} \equiv 0$ in $\Omega \setminus \cup_{i=1}^K \tilde{\mathfrak{U}}_i$.

Let $\mu(x) \in C_0^\infty(\Omega \setminus \cup_{i=1}^K \mathfrak{U}_i)$ be a nonnegative function such that

$$\mu(x) > 0 \quad \forall x \in \overline{\{x \in \Omega \setminus \cup_{i=1}^K \mathfrak{U}_i | \tilde{\beta}(x) = 0\}}.$$

Then the function $\beta(x) = \tilde{\beta} + s\mu$ satisfies (2.15) for all $s > 0$ sufficiently large.

Now let us show that the function $\beta(x)$ which satisfies (2.15) can be chosen as a Morse function. Since by (2.9), (2.15)

$$|\nabla\beta(x)| > 0 \quad \forall x \in \overline{\cup_{i=1}^K \mathfrak{U}_i}$$

there exists a sequence of domains $\tilde{\tilde{\mathfrak{U}}}_i \subset \mathbb{R}^n$ such that

$$\partial\tilde{\tilde{\mathfrak{U}}}_i = \Gamma_i \cup \tilde{\tilde{\gamma}}_i \in C^\infty, \quad \mathfrak{U}_i \subset \tilde{\tilde{\mathfrak{U}}}_i \subset \tilde{\mathfrak{U}}_i, \gamma_i \subset\subset \tilde{\tilde{\mathfrak{U}}}_i,\ \tilde{\tilde{\mathfrak{U}}}_i \cap \tilde{\tilde{\mathfrak{U}}}_j = \{\emptyset\} \text{ for } i \neq j,$$

where $\tilde{\tilde{\gamma}}_i$ is a connected $(n-1)$-dimensional C^∞ manifold, and

$$|\nabla\beta(x)| > 0 \quad \forall x \in \overline{\cup_{i=1}^K \tilde{\tilde{\mathfrak{U}}}_i}.$$

Let $\rho \in C_0^\infty(\Omega \setminus \cup_{i=1}^K \mathfrak{U}_i)$ such that

$$\rho(x) = 1 \ \forall x \in \Omega \setminus \cup_{i=1}^K \tilde{\tilde{\mathfrak{U}}}_i, \quad \rho(x) = 0 \ \forall x \in \cup_{i=1}^K \mathfrak{U}_i.$$

For every $\varepsilon > 0$ there exists a Morse function β_ε such that $\|\beta - \beta_\varepsilon\|_{C^2(\overline{\Omega})} \leq \varepsilon$.

Set $\psi_\varepsilon(x) = (1 - \rho(x))\beta(x) + \rho(x)\beta_\varepsilon(x)$. Obviously

$$\psi_\varepsilon(x) = 1 - \theta_i(x) \ \forall\, x \in \mathfrak{U}_i,$$

and for all sufficiently small $\varepsilon > 0$

$$\psi_\varepsilon(x) > 0 \text{ in } \Omega.$$

Let us show that for all small ε, ψ_ε is a Morse function. Actually, in $\cup_{i=1}^K \mathfrak{U}_i$ function ψ_ε has no critical points. In $\Omega \setminus \cup_{i=1}^K \tilde{\tilde{\mathfrak{U}}}_i$, ψ_ε coincides with the Morse function β_ε. Short calculations give the inequality

$$|\nabla\psi_\varepsilon(x)| = |\nabla\beta(x) - \nabla\rho(x)(\beta - \beta_\varepsilon)(x) - \rho(x)(\nabla\beta - \nabla\beta_\varepsilon)(x)| \geq C - 2\|\rho\|_{C^1(\overline{\Omega})}\|\beta - \beta_\varepsilon\|_{C^2(\overline{\Omega})} \geq C - 2\|\rho\|_{C^1(\overline{\Omega})}\varepsilon,$$

where $C > 0$, $x \in \overline{\cup_{i=1}^{k}\tilde{\tilde{\mathfrak{U}}}_i}$.

Since for all sufficiently small ε, the right-hand side of this inequality is positive, the function ψ_ε has no critical points in $\overline{\cup_{i=1}^{K}\tilde{\tilde{\mathfrak{U}}}_i}$.

Denote by $\mathcal{M}$ the set of critical points of the function ψ_ε. Exactly in the same way as it was done in [2], [20], one can construct a diffeomorphism $r : \Omega \to \Omega$ such that

$$r(x) = x \quad \forall x \in \cup_{i=1}^{K}\mathfrak{U}_i, \quad r^{-1}(\mathcal{M}) \subset \omega_0.$$

Thus, the function $\psi(x) = \psi_\varepsilon(r(x))$ satisfies all the conditions of our lemma. □

We set

$$\varphi(t,x) = e^{\lambda\psi(x)}/(t(T-t))^2, \tag{2.16}$$

$$\begin{aligned} &\alpha(t,x) = (e^{\lambda\psi} - e^{\lambda^2\|\psi\|_{C(\overline{\Omega})}})/(t(T-t))^2, \\ &\overline{\alpha}(t) = \alpha(t,x_0), \quad \overline{\varphi}(t) = \varphi(t,x_0), \end{aligned} \tag{2.17}$$

where $\lambda > 1$, function ψ from Lemma 2.4 and $x_0 \in \partial\Omega$. By (2.9), (2.14) the functions $\overline{\alpha}, \overline{\varphi}$ are independent of the selection of $x_0 \in \partial\Omega$. Note that (2.9), (2.14) imply the obvious inequality

$$0 > \alpha(t,x) \geq \overline{\alpha}(t); \quad \varphi(t,x) \geq \overline{\varphi}(t) \quad \forall\,(t,x) \in Q.$$

Let us introduce a function $\ell(t) \in C^\infty[0,T]$ by the formula

$$\ell(t) > 0, \quad t \in (0,T); \quad \ell(t) = \begin{cases} 1, & t \in (0, \frac{1}{4}T) \\ t, & t \in (\frac{3}{4}T, T) \end{cases}. \tag{2.18}$$

In this section, our aim is to get an estimate for the function p using the trace of p on $\partial\Omega$ and the restriction of p on $[0,T] \times \omega_0$. To prove this estimate, we need to recall some previous results on Carleman inequalities for the Laplace operator.

Let us consider the analog of problem (2.6)-(2.8) in the domain $\mathfrak{U}_i$:

$$\Delta w = 0 \text{ in } \mathfrak{U}_i, \tag{2.19}$$

$$\frac{\partial w}{\partial \nu}|_{\gamma_i} = 0, \tag{2.20}$$

Set

$$A = A(\lambda) = \max_{\tau \in [\frac{1}{4},1]} \frac{e^{\lambda(1-\tau)} - 1}{1-\tau}. \tag{2.21}$$

By (2.9), (2.14) there exists $\lambda_0 > 1$ such that

$$\begin{aligned} &1 + A(\lambda)\psi(x) \geq e^{\lambda\psi(x)} \quad \forall \lambda > \lambda_0,\ x \in \mathfrak{O}(\frac{1}{4}), \\ &e^\lambda = (e^{\lambda\psi(x)})|_{\gamma_i} > 1 + A(\lambda)\psi(x) > e^{\lambda\psi(x)} \\ &\quad \forall \lambda > \lambda_0, x \in \overline{\mathfrak{O}(\frac{1}{2}) \setminus \mathfrak{O}(\frac{3}{4})}. \end{aligned} \tag{2.22}$$

We have:

LEMMA 2.5. *Let the function $w \in W_2^{\frac{1}{2}}(\mathfrak{U}_i)$ be a solution of problem (2.19), (2.20). Then there exist $\hat{\delta} \in (0,1)$ and $\hat{\lambda} > 1$ such that for $\lambda > \hat{\lambda}$*

$$\frac{s}{((T-t)t)^2}\int_{\mathfrak{D}_i(\frac{1}{4})} |w|^2 e^{2s\frac{1+A\psi}{((T-t)t)^2}}dx \le C_4(\int_{\partial\Omega}|w|^2e^{2s\varphi}d\sigma + \int_{\gamma_i}|w|^2e^{2s\hat{\delta}\varphi}d\sigma), \tag{2.23}$$

where the domain $\mathfrak{D}_i(r)$ is defined in (2.13), $i \in \{1,\dots,K\}$.

Proof. Set

$$(2M)^2 = \int_{\Gamma_i}|w|^2d\sigma,\ (2\varepsilon)^2 = \int_{\gamma_i}|w|^2d\sigma.$$

Let $\lambda > 1$ and $\kappa_\tau(i)$ be the manifold defined by (2.12). Note that, by (2.14)

$$A = \max_{\tau\in[\frac{1}{4},1]}\frac{e^{\lambda(1-\tau)}-1}{1-\tau} = \max_{x\in\mathfrak{D}_i(\frac{1}{4})}\frac{e^{\lambda\psi(x)}-1}{\psi(x)} \le \lambda e^{\frac{3}{4}\lambda} = \lambda(e^{\lambda\psi}|_{\gamma_i})^{\frac{3}{4}}. \tag{2.24}$$

Using inequality (2.11), we obtain

$$\begin{aligned}\int_{\mathfrak{D}_i(\frac{1}{4})}|w|^2e^{2sA\psi}dx &\le C_6\int_{\frac{1}{4}}^1\int_{\kappa_\tau}|w|^2e^{2sA(1-\tau)}d\sigma d\tau\\ &\le C_7\int_{\frac{1}{4}}^1\|e^{sA}w\|^{2(1-\tau)}_{L^2(\gamma_i)}\|w\|^{2\tau}_{L^2(\Gamma_i)}d\tau \le C_7\int_0^1\|e^{sA}w\|^{2(1-\tau)}_{L^2(\gamma_i)}\|w\|^{2\tau}_{L^2(\Gamma_i)}d\tau\\ &\le C_8\int_0^1(\varepsilon e^{sA})^{2(1-\tau)}M^{2\tau}d\tau = I. \end{aligned}\tag{2.25}$$

Short calculations give the equality

$$I = \frac{M^2}{2}\left[\left(\frac{\varepsilon e^{sA}}{M}\right)^2 - 1\right]/ln\left(\frac{\varepsilon e^{sA}}{M}\right). \tag{2.26}$$

Obviously, by (2.24) there exist $\hat{\delta} \in (0,1)$ and $\hat{\lambda} > 1$ such that for all $\lambda > \hat{\lambda} > 1$

$$A + 1 \le \lambda e^{\frac{3}{4}\lambda} + 1 \le \hat{\delta}\min_{x\in\gamma_i} e^{\lambda\psi(x)} - 1 = \hat{\delta}e^{\lambda} - 1. \tag{2.27}$$

Let us consider two cases.

A) Let

$$\frac{\varepsilon e^{s(A+1)}}{M} \le 1. \tag{2.28}$$

Thus $ln(\frac{\varepsilon e^{sA}}{M}) \le 0$ and

$$-ln(\frac{\varepsilon e^{sA}}{M}) = s - ln(\frac{\varepsilon e^{s(A+1)}}{M}) \ge s. \tag{2.29}$$

Inequalities (2.26), (2.28) and (2.29) imply

$$I \leq \frac{M^2}{2s}.$$

By this inequality and (2.25), keeping in mind that $e^{\lambda\psi}|_{\partial\Omega} = 1$, we obtain

$$s\int_{\mathfrak{O}_i(\frac{1}{4})} |w|^2 e^{2s(1+A\psi)}dx \leq C_{10}\int_{\Gamma_i} |w|^2 e^{2se^{\lambda\psi}}d\sigma. \tag{2.30}$$

B) Let

$$\frac{\varepsilon e^{s(A+1)}}{M} \geq 1.$$

Then by (2.25), (2.27)

$$I \leq C_{11}\int_0^1 \varepsilon^2 e^{2sA}e^{2s\tau}d\tau \leq C_{11}\varepsilon^2 e^{2s(\lambda e^{\frac{3}{4}\lambda}+1)} \leq C_{12}\int_{\gamma_i} w^2 e^{2s(\hat{\delta}e^{\lambda\psi}-1)}d\sigma,$$

where $\lambda > \hat{\lambda} > 1$. From this inequality, increasing the parameter $\hat{\delta} \in (0,1)$ if necessary, we obtain

$$s\int_{\mathfrak{O}_i(\frac{1}{4})} |w|^2 e^{2s(1+A\psi)}dx \leq C_{13}\int_{\gamma_i} |w|^2 e^{2s\hat{\delta}e^{\lambda\psi}}d\sigma. \tag{2.31}$$

Inequalities (2.30), (2.31) after the change $s \to s/(t(T-t))^2$ imply (2.23). □

We have:

LEMMA 2.6. *Let $p \in W_2^{\frac{1}{2}}(\mathfrak{U}_i)$ be a harmonic function in $\mathfrak{U}_i$. Then there exists $\hat{\lambda} > 1$ such that for $\lambda > \hat{\lambda}$ there exists $s_0(\lambda) > 0$ such that*

$$\begin{aligned} &\frac{s}{((T-t)t)^2}\int_{\mathfrak{O}(\frac{1}{4})} |p|^2 e^{2s\frac{(1+A\psi)}{((T-t)t)^2}}dx \\ &\leq C\Big(\int_{\partial\Omega} |p|^2 e^{2s\varphi}d\sigma + \int_{\gamma_i}(|\nabla p|^2+p^2)e^{2s\delta\varphi}d\sigma\Big) \quad \forall s \geq s_0(\lambda), \end{aligned} \tag{2.32}$$

for some $\delta \in (0,1)$.

Proof. Let $\hat{\delta}$ be defined in Lemma 2.5 and $\hat{\lambda}$ be the maximum of the corresponding parameter from Lemma 2.5 and λ_0 from (2.22). We are looking for a function p of the form: $p = z_1 + z_2$,

$$\Delta z_1 = 0 \text{ in } \mathfrak{U}_i,\ z_1|_{\Gamma_i} = 0, \frac{\partial z_1}{\partial\vec{n}(i)}|_{\gamma_i} = \frac{\partial p}{\partial\vec{n}(i)}|_{\gamma_i} \tag{2.33}$$

and

$$\Delta z_2 = 0 \text{ in } \mathfrak{U}_i,\ z_2|_{\Gamma_i} = p, \frac{\partial z_2}{\partial\vec{n}(i)}|_{\gamma_i} = 0, \tag{2.34}$$

where $\vec{n}(i)$ is a unit outward normal to $\mathfrak{U}_i$. Note that

$$\varphi|_{\gamma_i} = \frac{e^\lambda}{((T-t)t)^2} \quad \forall\, i = \{1,\dots,K\}.$$

Also by (2.22) there exists $\delta \in (\hat{\delta}, 1), \delta > \frac{3}{4}$ such that

$$\frac{1 + A\max_{x\in\mathfrak{O}_i(\frac{1}{4})}\psi(x)}{((T-t)t)^2} < \delta \frac{e^\lambda}{((T-t)t)^2} = \delta\varphi|_{\gamma_i}.$$

From this inequality, (2.14) and (2.33) we obtain

$$\begin{aligned} &\int_{\gamma_i} |z_1|^2 e^{2s\delta\varphi} d\sigma + \frac{s}{((T-t)t)^2}\int_{\mathfrak{O}_i(\frac{1}{4})} |z_1|^2 e^{2s\frac{(1+A\psi)}{((T-t)t)^2}} dx \\ &\le C_{13}\int_{\gamma_i}\left(\frac{\partial p}{\partial \vec{n}(i)}\right)^2 e^{2s\delta\varphi} dx. \end{aligned} \tag{2.35}$$

Then, by Lemma 2.5, the function z_2 satisfies the estimate

$$\begin{aligned} &\frac{s}{((T-t)t)^2}\int_{\mathfrak{O}_i(\frac{1}{4})} |z_2|^2 e^{2s\frac{1+A\psi}{((T-t)t)^2}} dx \\ &\qquad \le C_{13}(\int_{\partial\Omega} |p|^2 e^{2s\varphi} d\sigma + \int_{\gamma_i} |z_2|^2 e^{2s\delta\varphi} d\sigma) \\ &\qquad \le C_{14}(\int_{\partial\Omega} |p|^2 e^{2s\varphi} d\sigma + \int_{\gamma_i} (|z_1|^2 + p^2) e^{2s\delta\varphi} d\sigma). \end{aligned} \tag{2.36}$$

By (2.35), (2.36) we have

$$\begin{aligned} &\frac{s}{((T-t)t)^2}\int_{\mathfrak{O}_i(\frac{1}{4})} |z_2|^2 e^{2s\frac{1+A\psi}{((T-t)t)^2}} dx \\ &\le C_{15}(\int_{\partial\Omega} |p|^2 e^{2s\varphi} d\sigma + \int_{\gamma_i} |\nabla p|^2 e^{2s\delta\varphi} d\sigma) \\ &\le C_{15}(\int_{\delta\Omega} |p|^2 e^{2s\varphi} d\sigma + \int_{\gamma_i} (|\nabla p|^2 + p^2) e^{2s\delta\varphi} d\sigma). \end{aligned} \tag{2.37}$$

Inequalities (2.35), (2.37) imply (2.32). □

Let us consider the Dirichlet boundary value problem for the Laplace operator

$$\Delta z(x) = f(x) \qquad x \in \Omega, \qquad z|_{\partial\Omega} = 0. \tag{2.38}$$

We have:

Lemma 2.7. *Let $f(x) \in L^2(\Omega)$. There exists $\hat{\lambda} > 1$ such that for $\lambda > \hat{\lambda}$, there exists $s_0(\lambda) > 0$ such that for any $s > s_0$ the solution $z(x) \in W_2^2(\Omega)$ of problem (2.38) satisfies the estimate*

$$\begin{aligned} &\int_\Omega \left(\frac{1}{s\varphi}\sum_{i,j=1}^n \left|\frac{\partial^2 z}{\partial x_i \partial x_j}\right|^2 + s\lambda^2\varphi|\nabla z|^2 + s^3\lambda^4\varphi^3 z^2\right) e^{2s\alpha}\, dx \\ &\le C_{16}(\int_\Omega f^2 e^{2s\alpha}\, dx + \int_{\omega_0} s^3\lambda^4\varphi^3 z^2 e^{2s\alpha}\, dx), \end{aligned} \tag{2.39}$$

where $C_1 > 0$ does not depend on s, t.

The Carleman inequality (2.39) can be proved in the same way as the corresponding Carleman estimate for a parabolic equation in [20], [17]. Note that for the case $\partial\Omega \subset \overline{\omega}_0$ this estimate was proved in [19].

Now to continue estimating the pressure p, we have to use equations (2.1), (2.2). Applying the operator **div** to both parts of equation (2.1) we obtain

$$\Delta p = \mathbf{div}(B^*(\hat{v}, y) + B^*(y, \hat{v})) \text{ in } \Omega \tag{2.40}$$

for a.e. $t \in [0, T]$.

The following lemma gives an estimate of the L^2-norm of the pressure p.

LEMMA 2.8. *Let $f \in L^2(0, T; V^1(\Omega))$ and $p(t, \cdot) \in L^2(\Omega)$ satisfies (2.40). There exists $\hat{\lambda} > 1$ such that for all $\lambda > \hat{\lambda}$, there exists $s_0(\lambda)$ such that*

$$\begin{aligned}\int_\Omega s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\, dx \le C_{10}(\lambda) \Bigg(\int_{\omega_0} s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\, dx \\ + \int_{\partial\Omega} s\lambda\varphi p^2 e^{2s\alpha}\, d\sigma + \int_\Omega \left(\frac{|\nabla y|^2}{s\lambda^2\varphi} + s\lambda^2\varphi|y|^2 \right) e^{2s\alpha} dx \Bigg) \quad \forall\, s > s_0(\lambda),\end{aligned} \tag{2.41}$$

where the constant C_{10} is independent of s and t.

Proof. Let $\hat{\lambda}$ be a maximum of the corresponding parameters from Lemma 2.6 and Lemma 2.7 and λ_0 from (2.22). Note that $\operatorname{supp} \mathbf{div}(B^*(\hat{v}, y) + B^*(y, \hat{v})) \subset \operatorname{supp} \hat{v}$. By (2.5)

$$\operatorname{supp} \hat{v} \cap \overline{\mathfrak{U}_i} = \{\emptyset\} \text{ for all } i \in \{1, \dots, K\}.$$

So there exists a neighborhood of γ_i, a domain G_i such that p is the harmonic function in G_i and $\cup_{i=1}^K G_i \subset \Omega \setminus \cup_{i=1}^K \mathfrak{O}_i(\frac{1}{8}) = \{\emptyset\}$. Note also that

$$\varphi|_{\gamma_i} = \frac{e^\lambda}{((T-t)t)^2} \quad \forall\, i = \{1, \dots, K\}.$$

Thus, by the properties of interior regularity of solutions of elliptic equations, there exists a constant C such that

$$\sum_{i=1}^K \int_{\gamma_i} |\nabla p|^2 e^{2s\delta\varphi} d\sigma \le C \int_{\cup_{i=1}^K G_i} |p|^2 e^{2s\varphi} dx, \tag{2.42}$$

where $\delta \in (0, 1)$ is defined in Lemma 2.6.

By (2.13), (2.14)

$$\lim_{\varepsilon \to +0} \min_{x \in \mathfrak{O}(\frac{3}{4}-\varepsilon)\setminus\mathfrak{O}(\frac{3}{4})} \psi(x) = \lim_{\varepsilon \to +0} \max_{x \in \mathfrak{O}(\frac{3}{4}-\varepsilon)\setminus\mathfrak{O}(\frac{3}{4})} \psi(x) = \frac{1}{4}.$$

Thus, by (2.22), there exists $\varepsilon_0 \in (0, \frac{1}{100})$ such that

$$1 + A \inf_{x \in \mathfrak{O}(\frac{3}{4}-\varepsilon_0)\setminus\mathfrak{O}(\frac{3}{4})} \psi(x) > \sup_{x \in \mathfrak{O}(\frac{3}{4}-\varepsilon_0)\setminus\mathfrak{O}(\frac{3}{4})} e^{\lambda\psi}. \tag{2.43}$$

Again, using the property of interior regularity of harmonic functions, by (2.43) we have

$$\begin{aligned}&\int_{\mathfrak{O}(\frac{3}{4}(1-\varepsilon_0))\setminus\mathfrak{O}(\frac{3}{4}-\frac{3}{8}\varepsilon_0)}(|\nabla p|^2+p^2)e^{2s\varphi}dx\\&\le C\int_{\mathfrak{O}(\frac{3}{4}-\varepsilon_0)\setminus\mathfrak{O}(\frac{3}{4})}p^2e^{2s\frac{1+A\psi}{((T-t)t)^2}}dx\quad\forall s>1.\end{aligned}\tag{2.44}$$

Let $\rho(x)$ be a function such that

$$\rho(x)=1\ \forall x\in\Omega\setminus\mathfrak{O}(\frac{3}{4}(1-\varepsilon_0));\ \rho\ge 0\text{ in }\Omega,\quad \rho|_{\mathfrak{O}(\frac{3}{4}-\frac{3}{8}\varepsilon_0)}\equiv 0.$$

Then by (2.40) the function $z=\rho p$ satisfies the equation

$$\Delta z=2(\nabla\rho,\nabla p)+p\Delta\rho+\rho\,\mathbf{div}(B^*(\hat v,y)+B^*(y,\hat v))\text{ in }\Omega,\quad z|_{\partial\Omega}=0.\tag{2.45}$$

Note that

$$\operatorname{supp}(2(\nabla\rho,\nabla p)+p\Delta\rho)\subset\mathfrak{O}(\frac{3}{4}(1-\varepsilon_0))\setminus\mathfrak{O}(\frac{3}{4}-\frac{3}{8}\varepsilon_0).$$

Short calculations give the inequality:

$$\begin{aligned}&|\mathbf{div}(B^*(\hat v,y)+B^*(y,\hat v))(t,x)|\\&\le C(|\nabla\hat v(x)|+|\hat v(x)|)(|\nabla v(t,x)|+|v(t,x)|)\quad\text{in }Q.\end{aligned}\tag{2.46}$$

By (2.44), (2.46), (2.39) we obtain from (2.45)

$$\begin{aligned}s^2\int_\Omega\varphi^2\rho^2p^2e^{2s\varphi}dx&\le s^2\int_{\Omega\setminus\mathfrak{O}(\frac{3}{4}(1-\varepsilon_0))}\varphi^2p^2e^{2s\varphi}dx\\&\le C(\lambda)(\int_\Omega\frac{1}{s\varphi}(|\nabla v|^2+|v|^2)e^{2s\varphi}dx\\&+\int_{\omega_0}s^2\varphi^2p^2e^{2s\varphi}dx+\frac{1}{s\varphi}\int_{\mathfrak{O}(\frac{3}{4}-\varepsilon_0)\setminus\mathfrak{O}(\frac{3}{4})}p^2e^{2s\frac{1+A\psi}{((T-t)t)^2}}dx)\quad\forall s>s_0(\lambda).\end{aligned}\tag{2.47}$$

On the other hand, by (2.32), (2.41) we have

$$\begin{aligned}&s^2\int_{\mathfrak{O}(\frac{1}{4})}\varphi^2p^2(e^{2s\varphi}+e^{2s\frac{1+A\psi}{((T-t)t)^2}})dx\\&\le C(\int_{\partial\Omega}s\varphi p^2e^{2s\varphi}d\sigma+\int_{\cup_{i=1}^K G_i}s\varphi|p|^2e^{2s\varphi}dx).\end{aligned}\tag{2.48}$$

Since $\cup_{i=1}^K G_i\subset\Omega\setminus\mathfrak{O}(\frac{3}{4}(1-\varepsilon_0))$ and $\mathfrak{O}(\frac{3}{4}-\varepsilon_0)\setminus\mathfrak{O}(\frac{3}{4})\subset\mathfrak{O}(\frac{1}{4})$ inequalities (2.47), (2.48) imply (2.41). □

Now in right-hand side of (2.41), we have to estimate the integral on Σ containing pressure p.

LEMMA 2.9. *Let $p \in L^2(Q)$ satisfies (2.1), (2.2), and $\lambda > \hat{\lambda}$, $s > s_0(\lambda)$ where $\hat{\lambda}, s_0(\lambda)$ are defined in Lemma 2.8. Then the following estimate holds*

$$\begin{aligned}&\int_\Sigma s\lambda\varphi p^2 e^{2s\alpha}\,d\Sigma\\ &\le C_{13}\left(\int_{Q_{\omega_0}} s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}dxdt + \int_Q s^2\lambda^2\varphi^3|y|^2 e^{2s\alpha}dxdt\right.\\ &\quad\left.+\int_\Sigma s\lambda\varphi\left|\frac{\partial y}{\partial\nu}\right|^2 e^{2s\alpha}\,d\Sigma + \int_Q s\lambda\varphi|f|^2 e^{2s\alpha}\,dxdt\right),\end{aligned} \tag{2.49}$$

where the constant C_{13} is independent of s.

Proof. Let us introduce a function $\gamma(t,x)$ by the formula

$$\Delta\gamma(t,\cdot) = 0 \text{ in } \Omega, \quad \left.\frac{\partial\gamma}{\partial\nu}\right|_{\partial\Omega} = p - a_0\int_{\partial\Omega} p d\sigma \text{ for a.e.} t\in(0,T), \tag{2.50}$$

where $a_0 = \left(\int_{\partial\Omega} 1\,d\sigma\right)^{-1}$. Note that solutions of (2.50) satisfy the estimate

$$\|\gamma\|_{W_2^{\frac{3}{2}}(\Omega)} \le C_{14}\|p\|_{L^2(\partial\Omega)}. \tag{2.51}$$

Taking the scalar product of (2.1) with $\nabla\gamma$ in $(L^2(\Omega))^n$ and integrating by parts we have

$$\begin{aligned}\int_{\partial\Omega} p^2 d\sigma = &-\int_{\partial\Omega}(\frac{\partial y}{\partial\nu},\nabla\gamma)d\sigma + a_0(\int_{\partial\Omega} pd\sigma)^2\\ &+\int_\Omega (B^*(\hat{v},y)+B^*(y,\hat{v})-f,\nabla\gamma)dx.\end{aligned} \tag{2.52}$$

Multiplying (2.52) by $s\lambda\overline{\varphi}e^{2s\overline{\alpha}(t)}$ and integrating on the segment $(0,T)$ we obtain

$$\begin{aligned}&\int_\Sigma s\lambda\varphi p^2 e^{2s\overline{\alpha}(t)}\,d\Sigma = -\int_\Sigma s\lambda\varphi(\frac{\partial y}{\partial\nu},\nabla\gamma)e^{2s\overline{\alpha}(t)}\,d\Sigma\\ &+s\lambda\int_0^T\overline{\varphi}a_0(\int_{\partial\Omega} p\,d\sigma)^2 e^{2s\overline{\alpha}(t)}dt\\ &+s\lambda\int_Q\overline{\varphi}(B^*(\hat{v},y)+B^*(y,\hat{v})-f,\nabla\gamma)e^{2s\overline{\alpha}(t)}dxdt\\ &\le C_{15}\left(\int_\Sigma s\lambda\varphi|(\nabla y,\nabla\gamma)|e^{2s\overline{\alpha}(t)}\,d\Sigma\right.\\ &+s\lambda\int_0^T\overline{\varphi}(\int_{\partial\Omega} p\,d\sigma)^2 e^{2s\overline{\alpha}(t)}dt\\ &\left.+s\lambda|\int_Q\overline{\varphi}(B^*(\hat{v},y)+B^*(y,\hat{v})-f,\nabla\gamma)e^{2s\overline{\alpha}(t)}dxdt|\right).\end{aligned}$$

After estimating of functions $B^*(\hat{v},y), B^*(y,\hat{v})$, we deduce from (2.51) and this inequality

$$\int_\Sigma s\lambda\varphi p^2 e^{2s\overline{\alpha}(t)}\,d\Sigma$$
$$\leq C_{16}\left(\int_\Sigma s\lambda\varphi\left|\frac{\partial y}{\partial\nu}\right|^2 e^{2s\overline{\alpha}(t)}\,d\Sigma + s\lambda\int_0^T\overline{\varphi}(\int_{\partial\Omega}p\,d\sigma)^2 e^{2s\overline{\alpha}(t)}dt\right. \tag{2.53}$$
$$\left.+s\lambda\int_Q\overline{\varphi}(|\nabla y|^2+|y|^2+|f|^2)e^{2s\overline{\alpha}(t)}\,dxdt\right)+\frac{s\lambda}{2}\int_\Sigma\varphi p^2e^{2s\overline{\alpha}(t)}d\Sigma.$$

Let us introduce the function $\gamma_1(t,x)$ by formula

$$\Delta\gamma_1(t,\cdot)=a\chi_{\omega_0},\quad \left.\frac{\partial\gamma_1}{\partial\nu}\right|_{\partial\Omega}=1,$$

where $a=\int_{\partial\Omega}1d\sigma/\int_{\omega_0}1dx$. Multiplying (2.1) bt $\nabla\gamma_1$ scalarly in $(L^2(\Omega))^n$ and integrating by parts for a.e. $t\in(0,T)$ we have

$$\int_{\partial\Omega}p\,d\sigma=-\int_{\partial\Omega}(\frac{\partial y}{\partial\nu},\nabla\gamma_1)d\sigma$$
$$+\int_{\omega_0}apdx+\int_\Omega(B^*(\hat{v},y)+B^*(y,\hat{v})-f,\nabla\gamma_1)dx.$$

Thus we obtain

$$|\int_{\partial\Omega}p\,d\sigma|\leq C_{17}\left((\int_{\partial\Omega}\left|\frac{\partial y}{\partial\nu}\right|^2d\sigma)^{\frac{1}{2}}+|\int_{\omega_0}pdx|+\int_\Omega(|\nabla y|+|y|+|f|)dx\right). \tag{2.54}$$

Taking the scalar product of equation (2.1) with $s\lambda\overline{\varphi}ye^{2s\overline{\alpha}(t)}$ in $(L^2(\Omega))^n$ and using Gronwall's inequality, we have

$$\int_Q s\lambda\overline{\varphi}|\nabla y|^2e^{2s\overline{\alpha}(t)}\,dxdt$$
$$\leq C_{18}(\int_Q s^2\lambda^2\overline{\varphi}^3|y|^2e^{2s\overline{\alpha}(t)}\,dxdt+\int_Q s\lambda\overline{\varphi}|f|^2e^{2s\overline{\alpha}(t)}dxdt) \tag{2.55}$$
$$\leq C_{18}(\int_Q s^2\lambda^2\varphi^3|y|^2e^{2s\alpha}\,dxdt+\int_Q s\lambda\varphi|f|^2e^{2s\alpha}dxdt).$$

By (2.53) - (2.55) we obtain (2.49). □

The following lemma plays a decisive role in the estimation of the pressure.

Lemma 2.10. *Let $y\in L^2(0,T;V^0(\Omega))$ and $p\in L^2(Q)$ satisfy (2.1), (2.2), $f\in(L^2(Q))^n$, $\lambda>\hat{\lambda}$, $s>s_0(\lambda)$ where $\hat{\lambda},s_0(\lambda)$ are defined in Lemma 2.8. Then the following estimate holds*

$$\int_\Sigma\frac{1}{((T-t)t)^2}\left|\frac{\partial y}{\partial\nu}\right|^2e^{2s\alpha}\,d\Sigma$$
$$\leq C_{19}\left(\int_Q\frac{p^2}{((T-t)t)^2}e^{2s\alpha}dxdt\right. \tag{2.56}$$
$$\left.+\int_Q\frac{s|y|^2}{((T-t)t)^6}e^{2s\alpha}dxdt+\int_Q\frac{|f|^2}{((T-t)t)^2}e^{2s\alpha}dxdt\right),$$

where the constant C_{19} is independent of s.

Proof. Set $u(t,x) = ye^{s\overline{\alpha}(t)}/((T-t)t), q(t,x) = pe^{s\overline{\alpha}(t)}/((T-t)t), m = (f - B^*(y,\hat{v}) - B^*(\hat{v},y))e^{s\overline{\alpha}(t)}/((T-t)t)$. Then the pair (u,q) satisfies the equations

$$\frac{\partial u}{\partial t} - \Delta u + l(t)u = \nabla q + m \text{ in } Q, \tag{2.57}$$

$$\mathbf{div}\, u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0,x) = u(T,x) = 0, \tag{2.58}$$

where $l(t) = -s\frac{\partial\overline{\alpha}(t)}{\partial t} + \frac{1}{t} - \frac{1}{(T-t)}$. Obviously the estimate holds

$$|l(t)| \le C_{20}(\lambda)\frac{s}{((T-t)t)^3}, \quad \left|\frac{dl(t)}{dt}\right| \le C_{21}(\lambda)\frac{s}{((T-t)t)^4}.$$

Taking the scalar product of equation (2.57) with $\frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u$ in $(L^2(Q))^n$ and integrating by parts with respect to variables x and t we have:

$$\begin{aligned}
&\int_Q (m, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u)\, dxdt \\
&= \int_Q \{|\frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u|^2 - (\Delta u, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u) \\
&\quad + (l(t)u - (\nabla\psi,\nabla)u, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u) - (\nabla q, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u)\}\, dxdt \\
&= \int_Q \{|\frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u|^2 + (\nabla u, \nabla((\nabla\psi,\nabla)u)) + q\mathbf{div}((\nabla\psi,\nabla)u) \\
&\quad + (l(t)u - (\nabla\psi,\nabla)u, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u)\}\, dxdt \\
&\quad - \int_\Sigma (\frac{\partial u}{\partial \nu}, (\nabla\psi,\nabla)u) d\Sigma - \int_\Sigma q(\nu, (\nabla\psi,\nabla)u) d\Sigma.
\end{aligned} \tag{2.59}$$

Since $u|_{\partial\Omega} = 0$, we can rewrite equation $\mathbf{div}\, u = 0$ on the boundary as follows: $(\nu, \frac{\partial u}{\partial\nu}) = 0$. Thus

$$\int_\Sigma q(\nu, (\nabla\psi,\nabla)u)\, d\Sigma = -\int_\Sigma q|\nabla\psi|(\nu, \frac{\partial u}{\partial\nu})\, d\Sigma = 0. \tag{2.60}$$

Also, by (2.9) and Lemma 2.4 $\nu_i(x) = -\nabla\psi(x)/|\nabla\psi(x)|$, so

$$-\int_\Sigma (\frac{\partial u}{\partial\nu}, (\nabla\psi,\nabla)u) d\Sigma = \int_\Sigma |\nabla\psi| \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma. \tag{2.61}$$

Taking into account (2.60), (2.61), we can deduce from (2.59)

$$\int_Q (m, \frac{\partial u}{\partial t} + (\nabla\psi, \nabla)u)\, dxdt$$
$$= \int_Q \{ \left| \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u \right|^2 + q\sum_{k,i=1}^n \frac{\partial^2\psi}{\partial x_k \partial x_i}\frac{\partial u_k}{\partial x_i}$$
$$+ \sum_{k=1}^n \frac{1}{2}\frac{\partial\psi}{\partial x_k}\frac{\partial |\nabla u|^2}{\partial x_k} + \sum_{i,j,k=1}^n \frac{\partial^2\psi}{\partial x_j\partial x_i}\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i} - \frac{dl}{dt}|u|^2 - \frac{l(t)}{2}\Delta\psi |u|^2$$
$$- ((\nabla\psi,\nabla)u, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u) \}\, dxdt + \int_\Sigma |\nabla\psi| \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma$$
$$= \int_Q \{ \left| \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u \right|^2$$
$$+ q\sum_{k,i=1}^n \frac{\partial^2\psi}{\partial x_k\partial x_i}\frac{\partial u_k}{\partial x_i} - \frac{\Delta\psi}{2}|\nabla u|^2 + \sum_{i,j,k=1}^n \frac{\partial^2\psi}{\partial x_j\partial x_i}\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i} - \frac{dl}{dt}|u|^2$$
$$- \frac{l(t)}{2}\Delta\psi|u|^2 - ((\nabla\psi,\nabla)u, \frac{\partial u}{\partial t} + (\nabla\psi,\nabla)u)\}dxdt + \frac{1}{2}\int_\Sigma |\nabla\psi| \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma.$$

From this equality, taking into account that, by (2.14), $\min_{x\in\partial\Omega}|\nabla\psi(x)| > 0$ we obtain the estimate:

$$\int_\Sigma \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma \le C_{22}\left(\int_Q q^2 dxdt + \int_Q \left(|\nabla u|^2 + \left(\left|\frac{dl}{dt}\right| + 1\right)|u|^2 + m^2\right) dxdt\right). \tag{2.62}$$

Multiplying equation (2.57) by u scalarly in $(L^2(\Omega))^n$, by Gronwall's inequality, we have

$$\int_Q |\nabla u|^2\, dxdt \le C_{23}(\int_Q |l(t)||u|^2\, dxdt + \int_Q \frac{|f|^2}{((T-t)t)^2} e^{2s\overline{\alpha}} dxdt). \tag{2.63}$$

By (2.62), (2.63) and Lemma 2.4 we deduce

$$\int_\Sigma \left|\frac{\partial u}{\partial\nu}\right|^2 d\Sigma$$
$$\le C_{24}\left(\int_Q q^2 dxdt + \int_Q \left(\left|\frac{dl}{dt}\right| + |l(t)| + 1\right)|u|^2\, dxdt \right. \tag{2.64}$$
$$\left. + \int_Q \frac{|f|^2}{((T-t)t)^2} e^{2s\alpha} dxdt\right).$$

After returning in (2.64) to the variables y, p, f we get (2.56). □

Now we can prove main theorem of this section:

THEOREM 2.11. *Let $y \in L^2(0,T;V^1(\Omega)), p \in L^2(Q)$ satisfies (2.1), (2.2), $f \in L^2(0,T;V^1(\Omega))$, $\lambda > \hat{\lambda}$, $s > s_0(\lambda)$ where $\hat{\lambda}, s_0(\lambda)$ are defined in Lemma*

2.8. Then the following estimate holds

$$
\begin{aligned}
&\int_Q s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\,dxdt \\
&\le C_{25}\left(\int_{Q_{\omega_0}} s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}dxdt + \int_Q s^2\lambda^3\varphi^3|y|^2 e^{2s\alpha}\,dxdt\right. \\
&\quad \left. + \int_Q \frac{|\nabla y|^2}{s\lambda^2\varphi} e^{2s\alpha}\,dxdt + \int_Q s\lambda\varphi|f|^2 e^{2s\alpha}\,dxdt\right),
\end{aligned}
\tag{2.65}
$$

where the constant C_{24} is independent of s.

Proof. Inequalities (2.41), (2.49) imply that

$$
\begin{aligned}
&\int_Q s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\,dxdt \\
&\le C_{10}(\lambda)\left(\int_{Q_{\omega_0}} s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\,dxdt + \int_\Sigma s\lambda\varphi\left|\frac{\partial y}{\partial\nu}\right|^2 e^{2s\alpha}\,d\Sigma\right. \\
&\quad \left. + \int_Q \left(\frac{|\nabla y|^2}{s\lambda^2\varphi} + s^2\lambda^2\varphi^3|y|^2\right) e^{2s\alpha}dxdt + \int_Q s\lambda\varphi|f|^2 e^{2s\alpha}dxdt\right).
\end{aligned}
$$

Note that

$$\varphi(t,x)|_{\partial\Omega} = ((T-t)t)^{-2}.$$

Then, estimating the normal derivative of the function y by (2.56), we have

$$
\begin{aligned}
&\int_Q s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\,dxdt \\
&\le C_{10}(\lambda)\left(\int_{Q_{\omega_0}} s^2\lambda^2\varphi^2 p^2 e^{2s\alpha}\,dxdt + \int_Q \left(\frac{|\nabla y|^2}{s\lambda^2\varphi} + s^2\lambda^2\varphi^3|y|^2\right) e^{2s\alpha}dxdt\right. \\
&\quad \left. + \int_Q s\lambda\varphi|f|^2 e^{2s\alpha}dxdt + \int_Q s\varphi p^2 e^{2s\alpha}\,dxdt\right).
\end{aligned}
\tag{2.66}
$$

Hence, increasing the parameter s if necessary in (2.66), we obtain (2.65). □

REMARK 2.12. In system (2.1), (2.2), the pressure p is defined up to an arbitrary constant. In the next section we will fix it by setting

$$p(t,x_0) = 0 \quad \forall\, t\in(0,T)$$

for some $x_0 \in \omega_0$.

3. Carleman estimate for the Stokes system

In this section, our aim is to solve the observability problem for system (2.1), (2.2). In other words, we would like to obtain an *a priori* estimate for solutions of (2.1), (2.2) via function f and restriction of y on Q_ω.

Let ω_1 be an arbitrary subdomain of Ω such that

$$\omega_0 \Subset \omega_1 \Subset \omega.$$

We start from the following theorem.

THEOREM 3.1. *Let the pair (y,p) satisfy (2.1), (2.2), $f \in (L^2(Q))^n$. Then there exists a $\hat{\lambda} > 1$ such that for any $\lambda > \hat{\lambda}$ one can find $s_0(\lambda)$ such that the following inequality holds*

$$\begin{aligned}\int_Q \left(\frac{1}{s\varphi}\left(\left|\frac{\partial y}{\partial t}\right|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right|^2\right) + s\lambda^2\varphi|\nabla y|^2 + s^3\lambda^4\varphi^3|y|^2\right) e^{s\alpha} dxdt \\ \le C_1 \left(\int_Q |f|^2 e^{s\alpha}\, dxdt + \int_{Q_{\omega_1}} s^3\lambda^4\varphi^3|y|^2\, e^{s\alpha}\, dxdt \right. \\ \left. + \int_Q s^2\lambda^2\varphi^2 p^2 e^{s\alpha}\, dxdt\right) \quad \forall s \ge s_0, \quad (3.1)\end{aligned}$$

where the constant C_1 is independent of s.

Proof. Let us denote $w(t,x) = y(t,x)e^{s\alpha}$, $q(t,x) = p(t,x)e^{s\alpha}$.
By (2.14), (2.17) we have

$$w(T,\cdot) = w(0,\cdot) = 0 \quad \text{in} \quad \Omega; \quad w|_{\partial\Omega} = 0. \quad (3.2)$$

We define the operator P by formula

$$Pw = e^{s\alpha}(\partial_t - \Delta)e^{-s\alpha}w. \quad (3.3)$$

The operator P can be written explicitly as follows:

$$\begin{aligned}Pw = \frac{\partial w}{\partial t} - \Delta w + 2s\lambda\varphi(\nabla\psi,\nabla)w + s\lambda^2\varphi|\nabla\psi|^2 w \\ - s^2\lambda^2\varphi^2|\nabla\psi|^2 w + s\lambda\Delta\psi\varphi w - s\alpha_t w.\end{aligned}$$

Let us introduce the operators L_1, L_2 as follows:

$$L_1(w,q) = -\Delta w - s^2\lambda^2\varphi^2|\nabla\psi|^2 w - \nabla q + s\lambda\varphi\nabla\psi q, \quad (3.4)$$

$$L_2 w = \frac{\partial w}{\partial t} + 2s\lambda\varphi(\nabla\psi,\nabla)w + 2s\lambda^2\varphi|\nabla\psi|^2 w. \quad (3.5)$$

It follows from (2.1), (2.2), (3.4), (3.5) that

$$L_1(w,q) + L_2 w = f_s \text{ in } Q, \quad (3.6)$$

$$\mathbf{div}\, w = s\lambda\varphi(\nabla\psi, w), \quad (3.7)$$

where

$$f_s = (f - B^*(y,\hat{v}) - B^*(\hat{v},y))e^{s\alpha} + s\alpha_t w + s\lambda^2\varphi|\nabla\psi|^2 w - s\lambda\varphi\Delta\psi w.$$

Taking the L_2-norm of both sides of (3.6), we obtain

$$||f_s||^2_{(L^2(Q))^n} = ||L_1(w,q)||^2_{(L^2(Q))^n} + ||L_2 w||^2_{(L^2(Q))^n} + 2(L_1(w,q), L_2 w)_{(L^2(Q))^n}. \tag{3.8}$$

By (3.4) and (3.5) we have the following equality:

$$(L_1(w,q), L_2 w)_{(L^2(Q))^n} = (L_1(w,q), 2s\lambda\varphi(\nabla\psi,\nabla)w)_{(L^2(Q))^n} + (L_1(w,q), \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2 w)_{(L^2(Q))^n} = A_0 + A_1. \tag{3.9}$$

Integrating by parts in the first term of right-hand side of (3.9) we obtain

$$\begin{aligned} A_0 &= (-\Delta w - s^2\lambda^2\varphi^2|\nabla\psi|^2 w - \nabla q + s\lambda\varphi\nabla\psi q, 2s\lambda\varphi(\nabla\psi,\nabla)w)_{(L^2(Q))^n} \\ &= \int_Q \Bigg\{ 2s\lambda^2\varphi|(\nabla\psi,\nabla)w|^2 + 2s\lambda\varphi\sum_{l=1}^n\left(\nabla w_l, \sum_{i=1}^n \frac{\partial w_l}{\partial x_i}\nabla\frac{\partial\psi}{\partial x_i}\right) \\ &\quad + 2s\lambda\varphi\sum_{l=1}^n\left(\nabla w_l, \sum_{i=1}^n\frac{\partial\psi}{\partial x_i}\frac{\partial\nabla w_l}{\partial x_i}\right) - s^3\lambda^3\varphi^3|\nabla\psi|^2(\nabla\psi,\nabla|w|^2) \\ &\quad + 2s^2\lambda^2\varphi^2 q(\nabla\psi,(\nabla\psi,\nabla)w) + 2s\lambda q\,\mathbf{div}(\varphi(\nabla\psi,\nabla)w)\Bigg\} dxdt \\ &\quad - \int_\Sigma 2s\lambda\varphi(\frac{\partial w}{\partial\nu},(\nabla\psi,\nabla)w)d\Sigma - \int_\Sigma 2s\lambda\varphi q(\nu,(\nabla\psi,\nabla)w)d\Sigma. \end{aligned} \tag{3.10}$$

Now let us transform some terms in (3.10).

$$\begin{aligned} &2s\lambda q\,\mathbf{div}(\varphi(\nabla\psi,\nabla)w) \\ &= 2s\lambda^2\varphi q(\nabla\psi,(\nabla\psi,\nabla)w) + 2s\lambda\varphi q\,\mathbf{div}((\nabla\psi,\nabla)w) \\ &= 2s\lambda^2\varphi q(\nabla\psi,(\nabla\psi,\nabla)w) \\ &\quad + 2s\lambda\varphi q\sum_{i,j=1}^n\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial w_i}{\partial x_j} + 2s\lambda\varphi q(\nabla\psi,\nabla\mathbf{div}w) \\ &= 2s\lambda^2\varphi q(\nabla\psi,(\nabla\psi,\nabla)w) \\ &\quad + 2s\lambda\varphi q\sum_{i,j=1}^n\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial w_i}{\partial x_j} + 2s\lambda\varphi q(\nabla\psi,\nabla(s\lambda\varphi(\nabla\psi,w))) \\ &= 2s\lambda^2\varphi q(\nabla\psi,(\nabla\psi,\nabla)w) \\ &\quad + 2s\lambda\varphi q\sum_{i,j=1}^n\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial w_i}{\partial x_j} + 2s^2\lambda^3\varphi^2 q|\nabla\psi|^2(\nabla\psi,w) \\ &\quad + 2s^2\lambda^2\varphi^2 q\sum_{i,j=1}^n\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}w_j + 2s^2\lambda^2\varphi^2 q(\nabla\psi,(\nabla\psi,\nabla)w). \end{aligned} \tag{3.11}$$

Similar to (2.61)

$$-\int_{\Sigma} 2s\lambda\varphi(\frac{\partial w}{\partial \nu}, (\nabla\psi, \nabla)w)d\Sigma = \int_{\Sigma} 2s\lambda\varphi|\nabla\psi| \left|\frac{\partial w}{\partial \nu}\right|^2 d\Sigma. \tag{3.12}$$

By virtue of (3.2), (3.7) we have

$$(\nu, \frac{\partial w}{\partial \nu})\big|_{\partial\Omega} = 0.$$

Bearing this equality in mind, we deduce that the last term in (3.10) equals zero:

$$\int_{\Sigma} 2s\lambda\varphi q(\nu, (\nabla\psi, \nabla)w)\, d\Sigma = -\int_{\Sigma} 2s\lambda\varphi q|\nabla\psi|(\nu, \frac{\partial w}{\partial \nu})\, d\Sigma = 0. \tag{3.13}$$

Integrating by parts in equation (3.10) and taking into account equations (3.11) - (3.13) we get

$$\begin{aligned} A_0 = \int_Q \Bigg\{ & 2s\lambda^2\varphi|(\nabla\psi, \nabla)w|^2 + 2s\lambda\varphi\sum_{l=1}^{n}\left(\nabla w_l, \sum_{i=1}^{n}\frac{\partial w_l}{\partial x_i}\nabla\frac{\partial \psi}{\partial x_i}\right) \\ & - s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 - s\lambda\varphi\Delta\psi|\nabla w|^2 + 3s^3\lambda^4\varphi^3|\nabla\psi|^4 w^2 \\ & + s^3\lambda^3\varphi^3\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\nabla\psi|^2\frac{\partial \psi}{\partial x_i}\right)|w|^2 + 4s^2\lambda^2\varphi^2 q(\nabla\psi, (\nabla\psi, \nabla)w) \\ & + 2s\lambda^2\varphi q(\nabla\psi, (\nabla\psi, \nabla)w) \\ & + 2s^2\lambda^2\varphi^2 q\sum_{i,j=1}^{n}\frac{\partial \psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}w_j + 2s^2\lambda^3\varphi^2 q|\nabla\psi|^2(\nabla\psi, w) \\ & + 2s\lambda\varphi q\sum_{i,j=1}^{n}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial w_i}{\partial x_j}\Bigg\} dxdt + \int_{\Sigma} s\lambda\varphi|\nabla\psi|\left|\frac{\partial w}{\partial \nu}\right|^2 d\Sigma. \end{aligned} \tag{3.14}$$

Let us now transform the second term in the right hand side of (3.9) keeping (3.7) in mind

$$
\begin{aligned}
A_1 &= (L_1(w,q), \frac{\partial w}{\partial t} + 2s\lambda^2\varphi|\nabla\psi|^2 w)_{(L^2(Q))^n} \\
&= \int_Q \left(\frac{1}{2}\frac{\partial}{\partial t}|\nabla w|^2 - \frac{1}{2}s^2\lambda^2\varphi^2|\nabla\psi|^2\frac{\partial |w|^2}{\partial t} \right) dxdt \\
&+ \int_Q \left(q\,\mathbf{div}\frac{\partial w}{\partial t} + s\lambda\varphi q(\nabla\psi, \frac{\partial w}{\partial t}) \right) dxdt \\
&+ \int_Q \left\{ 2s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 - 2s^3\lambda^4\varphi^3|\nabla\psi|^4|w|^2 \right. \\
&\left. + 2s\lambda^2\sum_{k=1}^n \left(\frac{\partial w}{\partial x_k}, w\frac{\partial}{\partial x_k}(\varphi|\nabla\psi|^2) \right) \right\} dxdt \\
&+ \int_Q (2s\lambda^2 q\,\mathbf{div}(\varphi|\nabla\psi|^2 w) + q2s^2\lambda^3\varphi^2|\nabla\psi|^2(\nabla\psi, w))\,dxdt \\
&= \int_Q \left\{ \frac{1}{2}s^2\lambda^2|\nabla\psi|^2\frac{\partial\varphi^2}{\partial t}|w|^2 + qs\lambda\varphi_t(\nabla\psi, w) \right. \\
&+ 2qs\lambda\varphi(\nabla\psi, \frac{\partial w}{\partial t}) + 2s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 - 2s^3\lambda^4\varphi^3|\nabla\psi|^4|w|^2 \\
&\left. - s\lambda^2|w|^2\Delta(\varphi|\nabla\psi|^2) \right\} dxdt \\
&+ \int_Q \left\{ 2s\lambda^2 q\sum_{i=1}^n \frac{\partial}{\partial x_i}(\varphi|\nabla\psi|^2)w_i + 4s^2\lambda^3\varphi^2 q|\nabla\psi|^2(\nabla\psi, w) \right\} dxdt.
\end{aligned}
\tag{3.15}
$$

Hence, by virtue of (3.8), (3.9), (3.14) and (3.15), we finally obtain

$$
\begin{aligned}
&\|f_s\|^2_{(L^2(Q))^n} \\
&= \|L_1(w,q)\|^2_{(L^2(Q))^n} + \|L_2 w\|^2_{(L^2(Q))^n} + 2\int_Q (2s\lambda^2\varphi|(\nabla\psi, \nabla)w|^2 \\
&+ s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 + s^3\lambda^4\varphi^3|\nabla\psi|^4|w|^2 \\
&+ 2s\lambda\varphi\sum_{l=1}^n \left(\nabla w_l, \sum_{i=1}^n \frac{\partial w_l}{\partial x_i}\nabla\frac{\partial\psi}{\partial x_i} \right) \\
&- s\lambda\varphi\Delta\psi|\nabla w|^2 + s^3\lambda^3\varphi^3\sum_{i=1}^n \frac{\partial}{\partial x_i}(|\nabla\psi|^2\frac{\partial\psi}{\partial x_i})|w|^2)\,dxdt \\
&+ 2\int_\Sigma s\lambda\varphi|\nabla\psi| \left| \frac{\partial w}{\partial\nu} \right|^2 d\Sigma + 2X_1 + 2X_2,
\end{aligned}
\tag{3.16}
$$

where

$$
X_1 = \int_Q s^2\lambda^2\varphi\varphi_t|\nabla\psi|^2|w|^2\,dxdt, \tag{3.17}
$$

and

$$X_2 = \int_Q \Big\{ q2s\lambda\varphi(\nabla\psi, L_2 w) + qs\lambda\varphi_t(\nabla\psi, w) + 2s^2\lambda^3\varphi^2 q|\nabla\psi|^2(\nabla\psi, w) + 2s\lambda^2 q\sum_{i=1}^n \frac{\partial}{\partial x_i}(\varphi|\nabla\psi|^2)w_i + 2s^2\lambda^2\varphi^2 q\sum_{i,j=1}^n \frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}w_j - s\lambda^2|w|^2\Delta(\varphi|\nabla\psi|^2) + 2s\lambda^2\varphi q(\nabla\psi,(\nabla\psi,\nabla)w) + 2s\lambda\varphi q\sum_{i=1}^n \frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial w_i}{\partial x_j}\Big\}\, dxdt. \quad (3.18)$$

W can easily obtain the following estimate:

$$|X_2| \le \frac{1}{4}||L_2 w||^2_{(L^2(Q))^n} + C_2\int_Q (s^2\lambda^2\varphi^2 q^2 + \lambda^4|\nabla w|^2 + s^2\lambda^4\varphi^2|w|^2)\, dxdt. \quad (3.19)$$

To estimate X_1, we observe that by definition (2.16) of the function φ,

$$|\varphi_t(t,x)| \le C_3|\varphi(t,x)|^{\frac{3}{2}} \quad \forall (t,x)\in Q,$$

where C_3 is some independent constant. Therefore

$$|X_1| \le C_4\int_Q s^2\lambda^2\varphi^3|\nabla\psi|^2|w|^2\, dxdt. \quad (3.20)$$

We recall that by Lemma 2.4

$$|\nabla\psi(x)| > \beta > 0 \quad \forall x\in\overline{\Omega\setminus\omega_0}.$$

Thus, there exists $\hat\lambda > 1$ independent on w such that for any $\lambda > \hat\lambda$ and $s > 1$

$$\int_{[0,T]\times(\Omega\setminus\omega_0)} \Big(2s\lambda^2\varphi|(\nabla\psi,\nabla w)|^2 + \frac{1}{4}s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 + \frac{1}{4}s^3\lambda^4\varphi^3|\nabla\psi|^4|w|^2 + 2s\lambda\varphi\sum_{l=1}^n\Big(\nabla w_l, \sum_{i=1}^n\frac{\partial w_l}{\partial x_i}\nabla\frac{\partial\psi}{\partial x_i}\Big) - s\lambda\varphi\Delta\psi|\nabla w|^2 + s^3\lambda^3\varphi^3\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(|\nabla\psi|^2\frac{\partial\psi}{\partial x_i}\Big)|w|^2\Big)\, dxdt \ge 0. \quad (3.21)$$

Combining (3.21) and (3.16), we deduce

$$||L_1(w,q)||^2_{(L^2(Q))^n} + ||L_2 w||^2_{(L^2(Q))^n} + \int_Q (s\lambda^2\varphi\beta^2|\nabla w|^2 + s^3\lambda^4\varphi^3\beta^4|w|^2)dxdt + 2\int_\Sigma s\lambda|\nabla\psi|\varphi\left|\frac{\partial w}{\partial\nu}\right|^2 d\Sigma \le 2|X_1| + 2|X_2| + \|f_s\|^2_{(L^2(Q))^n} + C_5\int_{Q_{\omega_0}}(s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3|w|^2)dxdt, \quad (3.22)$$

for all $\lambda > \hat{\lambda}, s > 1$. Let now $\lambda \geq \hat{\lambda}$ be fixed. By (3.19), (3.20), (3.22) there exists $s_1(\lambda)$ such that

$$
\begin{aligned}
&||L_1(w,q)||^2_{(L^2(Q))^n} + \frac{1}{2}||L_2 w||^2_{(L^2(Q))^n} + \frac{1}{2}\int_Q (s\lambda^2\varphi\beta^2|\nabla w|^2 \\
&+ s^3\lambda^4\varphi^3\beta^4|w|^2)\,dxdt + 2\int_\Sigma s\lambda|\nabla\psi|\varphi\left|\frac{\partial w}{\partial\nu}\right|^2\,d\Sigma \\
&\leq \|f_s\|^2_{(L^2(Q))^n} \\
&+ C_6(\int_{Q_{\omega_0}} (s\lambda^2\varphi|\nabla w|^2 + s^3\lambda^4\varphi^3|w|^2)dxdt + \int_Q s^2\lambda^2\varphi^2 q^2\,dxdt)
\end{aligned}
\tag{3.23}
$$

for all $s \geq s_1(\lambda)$.

Let $\rho \in C_0^\infty(\omega_1), \rho = 1\ \forall x \in \omega_0$. Taking the scalar product of (3.6) with $s\lambda^2\varphi\rho w$ in $(L^2(Q))^n$ and integrating by parts we have

$$
\begin{aligned}
\int_{Q_{\omega_0}} s\lambda^2\varphi|\nabla w|^2\,dxdt \leq &C_7\bigg(\int_{Q_{\omega_1}} s^3\lambda^4\varphi^3|w|^2\,dxdt \\
&+ \|f_s\|^2_{(L^2(Q))^n} + \int_Q s^2\lambda^2\varphi^2 q^2\,dxdt\bigg).
\end{aligned}
\tag{3.24}
$$

The inequalities (3.23), (3.24) imply the estimate

$$
\begin{aligned}
&||L_1(w,q)||^2_{(L^2(Q))^n} + \frac{1}{2}||L_2 w||^2_{(L^2(Q))^n} + \frac{1}{2}\int_Q (s\lambda^2\varphi\beta^2|\nabla w|^2 \\
&+ s^3\lambda^4\varphi^3\beta^4|w|^2)\,dxdt + 2\int_\Sigma s\lambda|\nabla\psi|\varphi\left|\frac{\partial w}{\partial\nu}\right|^2\,d\Sigma \\
&\leq \|f_s\|^2_{(L^2(Q))^n} \\
&+ C_8(\int_{Q_{\omega_1}} s^3\lambda^4\varphi^3|w|^2 dxdt + \int_Q s^2\lambda^2\varphi^2 q^2\,dxdt)
\end{aligned}
\tag{3.25}
$$

for all $s \geq s_2(\lambda)$.

We observe that

$$
|\alpha_t(t,x)| \leq C_9(\lambda)|\varphi(t,x)|^{\frac{3}{2}} \quad \forall\,(t,x) \in Q.
$$

Then the definition of the function f_s implies

$$
\begin{aligned}
\|f_s\|^2_{L^2(Q)} \leq &C_{10}(\lambda)(\int_Q (|f|^2 + |\nabla y|^2 + |y|^2)e^{2s\alpha}dxdt \\
&+ \int_Q (s^2\lambda^4\varphi^2|\nabla\psi|^4|w|^2 + s^2\varphi^3|w|^2)dxdt).
\end{aligned}
\tag{3.26}
$$

Then, estimating the term with f_s in (3.25) by (3.26), we obtain (3.1). □

Combining the statements of Theorem 2.11 and Theorem 3.1 we obtain.

Corollary 3.2. *Let the pair* $(y,p) \in V^{1,2}(Q) \times L^2(0,T;W_2^1(\Omega))$ *satisfy (2.1), (2.2),* $f \in L^2(0,T;V^1(\Omega))$. *Then there exists a* $\hat{\lambda} > 1$ *such that for any* $\lambda > \hat{\lambda}$ *there exists* $s_0(\lambda)$ *such that the following inequality holds*

$$\int_Q \left(\frac{1}{s\varphi} \left(\left| \frac{\partial y}{\partial t} \right|^2 + \sum_{i,j=1}^n \left| \frac{\partial^2 y}{\partial x_i \partial x_j} \right|^2 \right) + s\lambda^2\varphi|\nabla y|^2 + s^3\lambda^4\varphi^3|y|^2 \right) e^{s\alpha} dxdt$$
$$\leq C_{10} \left(\int_{Q_{\omega_1}} (s^2\lambda^2\varphi^2 p^2 + s^3\lambda^4\varphi^3|y|^2) e^{s\alpha}\, dxdt \right.$$
$$\left. + \int_Q s\lambda\varphi|f|^2 e^{s\alpha} dxdt \right) \quad \forall s \geq s_0(\lambda), \tag{3.27}$$

where the constant C_{10} *is independent of* s.

Proof. Since all assumptions of Theorem 3.1 are fulfilled, there exists $\hat{\lambda}_1$ such that for every $\lambda > \hat{\lambda}_1$, there exists $s_0(\lambda)$ such that for $s > s_0(\lambda), \lambda > \hat{\lambda}_1$ inequality (3.1) holds. For λ sufficiently large, we can estimate the last term in (3.1) by the right part of inequality (2.65). Then, increasing magnitude of the parameter $\hat{\lambda}_1$ and $s_0(\lambda)$ if necessary, we get (3.27). □

Remark 3.3. In particular, estimate (3.27) implies the ε-controllability for the system formally adjoint to (2.1), (2.2), as well as the unique continuation property for the Navier-Stokes equations. Such results were obtained for the Stokes system in [18] and for the Navier-Stokes equations in [8] under weak regularity assumptions of the function $\hat{v}$.

Let us consider the system of partial differential equations which is obtained from (2.1), (2.2) by the change of variables $t \to -t$:

$$L^*z = -\frac{\partial z}{\partial t} - \Delta z + B^*(z,\hat{v}) + B^*(\hat{v},z) = \nabla q + f \text{ in } Q, \tag{3.28}$$

$$\mathbf{div}\, z = 0, \quad z|_{\partial\Omega} = 0, \quad z(T,\cdot) = z_0, \tag{3.29}$$

where the operators $B^*(\hat{v},\cdot), B^*(\cdot,\hat{v})$ are defined in (2.3). Short calculations shows that L^* is formally adjoint to the operator which is the linearization of the Navier-Stokes equations at point $\hat{v}$.

Using the energy method, we can prove

Lemma 3.4. *Let* $z_0 \in V^0(\Omega)$. *Then the solutions of problem (3.28), (3.29) satisfy the estimate*

$$\left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,T/4;V^0(\Omega))} + \|z\|_{L^2(0,T/4;V^2(\Omega))}$$
$$\leq C_{11}(\|z\|_{L^2(0,T/2;V^1(\Omega))} + \|f\|_{L^2(0,T/2;V^0(\Omega))}). \tag{3.30}$$

Let us consider the boundary value problem for the stationary Stokes system

$$\Delta v = \nabla q + g \text{ in } \Omega, \ \mathbf{div}\, v = 0, \ v|_{\partial\Omega} = 0. \tag{3.31}$$

The following lemma is proved in [27].

LEMMA 3.5. *For any $g \in V^{-1}(\Omega)$ there exists a unique solution $v \in V^1(\Omega)$ of problem (3.31) and this solution satisfies the estimate*

$$\|v\|_{V^1(\Omega)} \le C_{11}\|g\|_{V^{-1}(\Omega)}. \tag{3.32}$$

Let us introduce the function κ by formula

$$\kappa(t,x) = (e^{\tilde\lambda\psi} - e^{\tilde\lambda^2\|\psi\|_{C(\overline{\Omega})}})/(\ell(t)(T-t))^2, \qquad \hat\kappa(t) = \min_{x\in\Omega}\kappa(t,x), \qquad \tilde\kappa(t) = \max_{x\in\Omega}\kappa(t,x), \tag{3.33}$$

where the function $\ell(t)$ is defined in (2.18). The parameter $\tilde\lambda$ is such that

$$\tilde\lambda > \hat\lambda,$$

where $\hat\lambda$ is defined in Corollary 3.2 and

$$\max_{x\in\Omega}\kappa(t,x) < \frac{9}{10}\min_{x\in\Omega}\kappa(t,x) \quad \forall t \in [0,T]. \tag{3.34}$$

Note that

$$\kappa(t,x) = \alpha_{\tilde\lambda}(t,x) \quad \forall\,(t,x) \in (\frac{3}{4}T, T)\times\Omega.$$

We have:

THEOREM 3.6. *Let the pair $(z,q) \in V^{1,2}(Q)\times L^2(0,T;W^1_2(\Omega))$ satisfy (3.28), (3.29), $f \in L^2(0,T;V^1(\Omega))$. Then there exists $\hat s > 1$ such that the following inequality holds*

$$\begin{aligned}\|z(0,\cdot)\|^2_{V^0(\Omega)} &+ \int_Q (T-t)^2|z|^2e^{\hat s\kappa}\,dxdt \\ &\le C_{12}(\int_{Q_\omega}|z|^2e^{\frac{9}{10}\hat s\hat\kappa}\,dxdt + \int_Q |f|^2e^{\frac{9}{10}\hat s\hat\kappa}\,dxdt).\end{aligned} \tag{3.35}$$

Proof. Let us introduce the functions $r, g, \tilde f$ by formulas

$$r(t,x) = \int_0^t z(\tau,x)\,d\tau, \quad g(t,x) = \int_0^t q(\tau,x)\,d\tau, \quad \tilde f(t,x) = \int_0^t f(\tau,x)\,d\tau.$$

Short calculations show that the pair (r,g) satisfies the equations

$$L^*r = \nabla g - z(0,\cdot) + \tilde f \text{ in } Q, \tag{3.36}$$

$$\mathbf{div}\,r = 0, \quad r|_{\partial\Omega} = 0. \tag{3.37}$$

Let us show that the function g satisfies the estimate

$$\begin{aligned}\|g(t,\cdot)\|_{L^2(\omega_1)} \le &C_{13}(\|z(0,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n} \\ &+ \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|\tilde f(t,\cdot)\|_{(L^2(\omega))^n}),\end{aligned} \tag{3.38}$$

where C_{13} is independent of t. Using the definition of the function r we can rewrite equation (3.36) as follows

$$-\Delta r = \nabla g - z(0,\cdot) + z(t,\cdot) - B^*(r,\hat{v}) - B^*(\hat{v},r) + \tilde{f} \quad \text{in } \Omega. \tag{3.39}$$

Note that the function g in (3.39) is defined up to an arbitrary constant. To fix it, we set

$$p(t,x_0) = 0 \ \forall t \in [0,T]$$

for some $x_0 \in \omega_0$. This equality implies

$$g(t,x_0) = 0 \quad \forall t \in [0,T]. \tag{3.40}$$

We are looking for functions r, g in the form $r = r_1 + r_2, g = g_1 + g_2$, where

$$\begin{aligned} &-\Delta r_1 = \nabla g_1 - z(0,\cdot) + z(t,\cdot) - B^*(r,\hat{v}) - B^*(\hat{v},r) + \tilde{f} \text{ in } \omega, \\ &\mathbf{div}\, r_1 = 0,\ r_1|_{\partial\omega} = 0,\ g_1(t,x_0) = 0\ t \in (0,T). \end{aligned} \tag{3.41}$$

By Lemma 3.5, the unique solution of problem (3.41) exists and satisfies the estimate

$$\begin{aligned} \|r_1\|_{V^1(\omega)} + \|g_1\|_{L^2(\omega)} \le & C_{14}(\|z(0,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n} \\ & + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n}). \end{aligned} \tag{3.42}$$

By virtue of (3.39), (3.37), (3.41) the functions r_2, q_2 should satisfy the equations

$$\Delta r_2 = \nabla g_2 \text{ in } \omega,\ \mathbf{div}\, r_2 = 0. \tag{3.43}$$

Applying the Laplace operator Δ to this equation, we have $\Delta^2 r_2 = 0$. Thus, by (3.42) and well-known estimates for interior regularity of solutions of elliptic equations (see [23]), we have:

$$\begin{aligned} &\|r_2(t)\|_{(C^2(\overline{\omega}_1))^n} \le C\|r(t) - r_1(t)\|_{(L^2(\omega))^n} \\ &\le C_{15}(\|z(0,\cdot)\|_{(L^2(\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n} + \|\tilde{f}\|_{(L^2(\omega))^n}). \end{aligned} \tag{3.44}$$

By (3.44) equality (3.43) implies the estimate

$$\begin{aligned} \|\nabla g_2(t,\cdot)\|_{(C(\overline{\omega}_1))^n} \le & C_{16}(\|z(0,\cdot)\|_{(L^2(\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} \\ & + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n}). \end{aligned} \tag{3.45}$$

By (3.40), (3.41)

$$g_2(t,x_0) = 0 \ \forall\, t \in [0,T].$$

Thus inequality (3.45) yields

$$\begin{aligned} \|g_2(t,\cdot)\|_{L^2(\omega_1)} \le & C_{17}(\|z(0,\cdot)\|_{(L^2(\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} \\ & + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n}). \end{aligned} \tag{3.46}$$

This inequality and (3.42) imply (3.38).

Applying the Carleman inequality (3.27) to equations (3.36) and (3.37), we have:

$$\int_Q \left(\frac{1}{s\varphi} \left(|z|^2 + \sum_{i,j=1}^{n} \left| \frac{\partial^2 r}{\partial x_i \partial x_j} \right|^2 \right) + s\varphi\tilde{\lambda}^2 |\nabla r|^2 + s^3\varphi^3\tilde{\lambda}^4 |r|^2 \right) e^{s\alpha}\, dxdt$$
$$\leq C_{18}(\tilde{\lambda}) \left(\int_{Q_{\omega_1}} s^2\tilde{\lambda}^2\varphi^2 g^2 e^{s\alpha}\, dxdt + \int_{Q_{\omega_1}} s^3\tilde{\lambda}^4\varphi^3 |r|^2 e^{s\alpha}\, dxdt \right.$$
$$\left. + \int_Q s\varphi\tilde{\lambda}(|\tilde{f}|^2 + |z(0,x)|^2) e^{s\alpha}\, dxdt \right), \tag{3.47}$$

where $s \geq s_0(\tilde{\lambda})$.

The parameter $s_0(\tilde{\lambda})$ is defined in Corollary 3.2. Set $\hat{s} = s_0(\tilde{\lambda})$. Using the a *priori* estimate (3.30) for system (3.28), (3.29) in the right-hand side of inequality (3.47), we can replace the function α by κ, the function φ by $(T-t)^{-2}$ and the constant C by $C(s)$:

$$\int_Q (T-t)^2 |z|^2 e^{s\kappa}\, dxdt + \|z(0,\cdot)\|^2_{V^1(\Omega)}$$
$$\leq C_{19}(s)(\int_{Q_{\omega_1}} \frac{g^2}{(T-t)^4} e^{s\kappa}\, dxdt$$
$$+ \int_{Q_{\omega_1}} \frac{|r|^2}{(T-t)^6} e^{s\kappa} dxdt + \int_Q \frac{1}{(T-t)^2} |\tilde{f}|^2 e^{s\kappa}\, dxdt + \|z(0,\cdot)\|^2_{V^0(\Omega)}), \tag{3.48}$$

where $s \geq \hat{s}$. Using estimate (3.38), we can rewrite (3.48) as follows:

$$\begin{aligned} &\int_Q (T-t)^2 |z|^2 e^{\hat{s}\kappa}\, dxdt + \|z(0,\cdot)\|^2_{V^1(\Omega)} \\ &\leq C_{20}(\int_{Q_{\omega_1}} (T-t)^{-6} |r|^2 e^{\hat{s}\hat{\kappa}}\, dxdt \\ &+ \int_{Q_\omega} \frac{(|\tilde{f}|^2 + |z|^2 + |r|^2)}{(T-t)^4} e^{\hat{s}\hat{\kappa}}\, dxdt \\ &+ \int_Q \frac{1}{(T-t)^2} |\tilde{f}|^2 e^{\hat{s}\kappa}\, dxdt + \|z(0,\cdot)\|^2_{V^0(\Omega)}). \end{aligned} \tag{3.49}$$

Note that by (3.34)

$$\int_{Q_\omega} \frac{(|\tilde{f}|^2 + |z|^2 + |r|^2)}{(T-t)^6} e^{\hat{s}\hat{\kappa}}\, dxdt + \int_Q \frac{|\tilde{f}|^2}{(T-t)^2} e^{\hat{s}\kappa}\, dxdt$$
$$\leq C_{21}(\int_Q |f|^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}}\, dxdt + \int_{Q_\omega} |z|^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}} dxdt),$$

where C_{21} is an independent constant. By this inequality, we deduce from (3.49)

$$\begin{aligned} &\int_Q |z|^2 (T-t)^2 e^{\hat{s}\kappa}\, dxdt + \|z(0,\cdot)\|^2_{V^1(\Omega)} \\ &\leq C_{22}(\int_{Q_\omega} |z|^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}}\, dxdt + \int_Q |f|^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}}\, dxdt + \|z(0,\cdot)\|^2_{V^0(\Omega)}). \end{aligned} \tag{3.50}$$

Let us finish the proof by contradiction. If the estimate (3.35) is not true, then by (3.50) there exists a sequence (z_k, q_k, f_k) such that

$$L^* z_k = \nabla q_k + f_k \text{ in } Q, \ \mathbf{div}\, z_k = 0, \ z_k|_{\partial\Omega} = 0, \ \|z_k(0,\cdot)\|_{(L^2(\Omega))^n} = 1, \tag{3.51}$$

$$f_k \to 0 \text{ in } (L^2(Q, e^{\frac{9}{10}\hat{s}\hat{\kappa}}))^n, \ \int_{Q_\omega} |z_k|^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}}\, dxdt \to 0 \text{ as } k \to \infty,$$

$$z_k(0,\cdot) \to z(0,\cdot) \text{ in } L^2(\Omega), \ z_k \to z \text{ weakly in } V^{1,2}((0, T-\epsilon)\times\Omega) \tag{3.52}$$

for all $\varepsilon \in (0,T)$. Passing to the limit in (3.51), taking into account (3.52) we obtain

$$L^* z = \nabla q \text{ in } Q, \quad \mathbf{div}\, z = 0, \quad z|_{\partial\Omega} = 0, \quad z|_{Q_\omega} \equiv 0, \tag{3.53}$$

$$\|z(0,\cdot)\|_{(L^2(\Omega))^n} = 1. \tag{3.54}$$

By (3.27), (3.53)

$$z \equiv 0,$$

but this is impossible by virtue of (3.54). The proof of the lemma is complete. □

4. Solvability of the linear controllability problem

Let us consider the problem of exact controllability of the linearized Navier-Stokes equations:

$$Ly = \frac{\partial y}{\partial t} - \Delta y + B(y, \hat{v}) + B(\hat{v}, y) = \nabla p + f + \chi_\omega u \text{ in } Q, \tag{4.1}$$

$$\mathbf{div}\, y = 0, \quad y|_{\partial\Omega} = 0, \quad y(0,x) = y_0(x), \tag{4.2}$$

$$y(T,x) = 0, \tag{4.3}$$

where the functions y_0, f are given and u is a control from the space $(L^2(Q_\omega))^n$. Before studying the solvability of problem (4.1)-(4.3), let us recall some existence theorems for boundary value problem (4.1), (4.2), assuming that u is a fixed function.

Lemma 4.1. *Let $\hat{v} \in V^1(\Omega) \cap (W^1_\infty(\Omega))^n$. Then for any $f \in L^2(0,T;V^0(\Omega))$, $u \in (L^2(Q))^n$, and $y_0 \in V^1(\Omega)$ there exists a solution $(y,p) \in V^{1,2}(Q) \times L^2(Q)$ of problem (4.1), (4.2). Moreover this solution is unique in the space $C(0,T;V^0(\Omega)) \times L^2(0,T;V^{-2}(\Omega))$ and satisfies the estimate*

$$\|y\|_{V^{1,2}(Q)} \le C_0(\|f\|_{L^2(0,T;V^0(\Omega))} + \|u\|_{(L^2(Q_\omega))^n} + \|y_0\|_{V^1(\Omega)}). \tag{4.4}$$

Set

$$\eta(t,x) = -\hat{s}\kappa(t,x), \tag{4.5}$$

where the function κ is defined in (3.33), (3.34) and the parameter $\hat{s}$ from Theorem 3.6. Since the function $\kappa(t,x)$ is negative, $\eta(t,x)$ is positive. Moreover $\lim_{t\to T-0} \eta(t,x) = +\infty$.

We use below the following weight function:

$$\theta(t,x) = (1-\chi_\omega)\frac{e^\eta}{(T-t)^2} + \chi_\omega. \tag{4.6}$$

To formulate our results, we need to introduce some nonstandard functional spaces

$$F(Q,\theta) = \{f \in (L_2(Q))^n;\ \exists\ f_1 \in (L_2(Q,\theta))^n,\ \exists\ f_2 \in L_2(0,T;W_2^1(\Omega)) \text{ such that } f = f_1 + \nabla f_2\}.$$

The norm in $F(Q,\theta)$ is defined by the relation

$$\|f\|_{F(Q,\theta)} = \inf_{\substack{f_1,\nabla f_2\\ f=f_1+\nabla f_2}} (\|f_1\|^2_{(L_2(Q,\theta))^n} + \|\nabla f_2\|^2_{(L_2(Q))^n})^{1/2}.$$

We are looking for solutions of the controllability problem in the following space:

$$Y(Q) = \{y \in V^{1,2}(Q); Ly \in F(Q,\theta), e^{-\frac{2}{5}\hat{s}\hat{\kappa}} y \in V^{1,2}(Q)\}$$

with the norm

$$||y||^2_{Y(Q)} = ||Ly||^2_{F(Q,\theta)} + \|e^{-\frac{2}{5}\hat{s}\hat{\kappa}} y\|^2_{V^{1,2}(Q)}.$$

Remark 4.2. The space of solutions $Y(Q)$ depends, at least formally, on the function $\hat{v}$. To construct a suitable Banach space of solutions of (4.1)-(4.3) independent of $\hat{v}$, we have to prove sharper estimates on the rate of convergence of $y(t,\cdot)$ to zero near $t=T$ than those we obtain below. This is probably possible.

We have:

Theorem 4.3. *Let $f \in F(Q,\theta)$, $y_0 \in V^1(\Omega)$. Then there exists a solution of problem (4.1)-(4.3) $(y,p,u) \in Y(Q) \times L^2(0,T;W_2^1(\Omega)) \times (L^2(Q_\omega))^n$ which satisfies the estimate*

$$||(y,p,u)||_{Y(Q)\times L^2(0,T;W_2^1(\Omega))\times(L^2(Q_\omega))^n} \le C_1(||y_0||_{V^1(\Omega)} + ||f||_{F(Q,\theta)}). \tag{4.7}$$

Proof. We first assume that $f \in L^2(Q,\theta)$ and $f|_\omega \equiv 0$. Let us consider the extremal problem

$$\mathcal{J}_k(y,u) = \frac{1}{2}\int_Q \rho_k |y|^2\, dxdt + \frac{1}{2}\int_Q m_k |u|^2 dxdt \to \inf, \tag{4.8}$$

$$\begin{aligned} &Ly = u + \nabla p + f \quad \text{in} \quad Q,\ \mathbf{div}\, y = 0,\ y\big|_\Sigma = 0,\\ &y(0,x) = y_0(x), \quad y(T,x) = 0, \end{aligned} \tag{4.9}$$

where

$$\rho_k(t) = e^{\frac{-9\hat{s}\hat{\kappa}(t)(T-t)^2}{10(T-t+1/k)^2}}, \qquad m_k(t,x) = \begin{cases} e^{\frac{-9}{10}\frac{\hat{s}\hat{\kappa}(T-t)^2}{(T-t+1/k)^2}}, x \in \overline{\omega},\\ k, x \in \Omega \setminus \overline{\omega}. \end{cases}$$

Obviously the functions ρ_k, m_k are bounded in Q for every $k > 0$.

By Proposition 2.1 and Lemma 4.1, there exists an admissible element to the problem (4.8), (4.9). So it is easy to prove (see [24], [25]) that the problem (4.8)-(4.9) has a unique solution, which we denote by $(\hat{y}_k, \hat{u}_k) \in V^{1,2}(Q) \times (L^2(Q))^n$.

Thus, applying the Lagrange principle to problem (4.8) - (4.9) (see [1], [9]), we obtain

$$L\hat{y}_k = f + \nabla p_k + \hat{u}_k \text{ in } Q, \mathbf{div}\, \hat{y}_k = 0,\ \hat{y}_k\big|_\Sigma = 0,\ \hat{y}_k(T,\cdot) \equiv 0,\ \hat{y}_k(0,\cdot) = y_0, \tag{4.10}$$

$$L^* z_k = \nabla q_k + \rho_k \hat{y}_k \text{ in } Q, \quad z_k\big|_\Sigma = 0,\ \mathbf{div}\, z_k = 0,\ z_k = -m_k \hat{u}_k \text{ in } Q, \tag{4.11}$$

where the operator L^* defined in (3.28) is formally conjugate to the operator L.

Since the function ρ_k depends only on the variable t, by Lemma 4.1 $\rho_k \hat{y}_k \in L^2(0,T;V^1(\Omega))$. So we can apply estimate (3.35) to equation (4.11):

$$\begin{aligned} &\int_Q (T-t)^2 e^{\hat{s}\kappa} |z_k|^2 dxdt + \|z_k(0,\cdot)\|^2_{(L^2(\Omega))^n} \\ &\le C_2 \left(\int_Q \rho_k^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}} |\hat{y}_k|^2 dxdt + \int_{Q_\omega} e^{\frac{9}{10}\hat{s}\hat{\kappa}} |z_k|^2 dxdt \right). \end{aligned} \tag{4.12}$$

We observe that $|\rho_k(t) e^{\frac{9}{10}\hat{s}\hat{\kappa}(t)}| \le 1$ for all $(t,x) \in Q$ and $|m_k(t,x) e^{\frac{9}{10}\hat{s}\hat{\kappa}(t)}| \le 1$ for all $(t,x) \in Q_\omega$. Actually,

$$\begin{aligned} &-\frac{9\hat{s}\hat{\kappa}(t)(T-t)^2}{10(T-t+1/k)^2} + \frac{9}{10}\hat{s}\hat{\kappa}(t) \\ &= -\frac{9}{10}\frac{\hat{s}}{\ell(t)} \max_{x\in\Omega}(e^{\tilde{\lambda}^2\|\psi\|_{C(\overline{\Omega})}} - e^{\tilde{\lambda}\psi(x)}) \left(\frac{1}{(T-t)^2} - \frac{1}{(T-t+1/k)^2} \right) < 0, \end{aligned}$$

where the function $\ell(t)$ is defined in (2.18). Keeping in mind these inequalities and substituting z_k by the right-hand side of (4.11_4) in the last integral of equality (4.12) we have

$$\begin{aligned} &\int_\Omega |z_k(0,x)|^2 dx + \int_Q |z_k|^2 (T-t)^2 e^{\hat{s}\kappa} dxdt \\ &\le C_3 (\int_Q \rho_k |\hat{y}_k|^2 dxdt + \int_{Q_\omega} m_k |\hat{u}_k|^2 dxdt). \end{aligned} \tag{4.13}$$

Taking the scalar product of (4.11_1) with $\hat{y}_k$ in $(L^2(Q))^n$ and integrating by parts with respect to t and x, bearing in mind (4.10), after simplifications we have

$$\begin{aligned} 0 &= (L^* z_k - \nabla q_k - \rho_k \hat{y}_k, \hat{y}_k)_{(L^2(Q))^n} \\ &= -\int_Q \rho_k |\hat{y}_k|^2 dxdt + (z_k, L\hat{y}_k)_{(L^2(Q))^n} + (z_k(0,\cdot), \hat{y}_k(0,\cdot))_{(L^2(\Omega))^n} \\ &= -\int_Q \rho_k |\hat{y}_k|^2 dxdt - \int_Q m_k |\hat{u}_k|^2 dxdt + \int_Q (f, z_k) dxdt + (z_k(0,\cdot), y_0)_{(L^2(\Omega))^n}. \end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{J}_k(\hat{y}_k, \hat{u}_k) &= \frac{1}{2}\int\limits_Q \left(\rho_k|\hat{y}_k|^2 + m_k|\hat{u}_k|^2\right) dxdt \\ &= \frac{1}{2}\left(\int\limits_Q (f, z_k)dxdt + (z_k(0,\cdot), y_0)_{(L^2(\Omega))^n}\right).\end{aligned} \tag{4.14}$$

Note that

$$\left|\int_Q (f, z_k)dxdt\right| \leq C_4\|f\|_{(L^2(Q,\theta))^n}\|(T-t)e^{\hat{s}\kappa/2}z_k\|_{(L^2(Q))^n}. \tag{4.15}$$

By (4.13), (4.14), (4.15), we obtain

$$\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \leq C_5(||f||_{F(Q,\theta)} + ||y_0||_{(L^2(\Omega))^n})\sqrt{\mathcal{J}_k(\hat{y}_k, \hat{u}_k)}.$$

Hence,

$$\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \leq C_5^2(||f||^2_{F(Q,\theta)} + ||y_0||^2_{(L^2(\Omega))^n}). \tag{4.16}$$

By virtue of (4.16),(4.4), we have a subsequence $\{(\hat{y}_k, \hat{u}_k)\}_{k=1}^{\infty}$ such that

$$\begin{aligned}&(\hat{y}_k, \hat{u}_k) \to (y, u) \quad \text{weakly in} \quad V^{1,2}(Q) \times (L^2(Q))^n, \\ &\hat{u}_k \to 0 \quad \text{in} \quad (L^2((0,T) \times (\Omega \setminus \omega)))^n, \\ &\int_{Q_\omega} m_k|\hat{u}_k|^2dxdt + \int_Q \rho_k|\hat{y}_k|^2dxdt \leq C_6.\end{aligned} \tag{4.17}$$

By (4.13), (4.17) it also follows from (4.11) that

$$\|m_k\hat{u}_k\|_{V^{1,2}((0,T-\varepsilon)\times\Omega)} \leq C_7(\varepsilon)$$

for all $\varepsilon \in (0,T)$. Hence, without loss of generality, we can assume

$$\hat{u}_k(t,x) \to u(t,x) \quad \text{almost everywhere in } Q_\omega. \tag{4.18}$$

Using (4.17), we pass to the limit in (4.10) to obtain that the pair (y, u) is a solution of problem (4.1)-(4.3). The relations (4.16), (4.17) and Lemma 4.1 imply the estimate

$$\begin{aligned}&||(y,p,u)||_{V^{1,2}(Q)\times L^2(0,T;W_2^1(\Omega))\times(L^2(Q_\omega))^n} \\ &\leq C_8(||y_0||_{V^1(\Omega)} + ||f||_{F(Q,\theta)}).\end{aligned} \tag{4.19}$$

Furthermore, by (4.17), (4.18) and Fatou's theorem (see [22 p. 307]), we have:

$$\|(y,u)\|_{(L^2(Q,e^{-\frac{9}{10}\hat{s}\hat{\kappa}}))^n\times(L^2(Q_\omega,e^{-\frac{9}{10}\hat{s}\hat{\kappa}}))^n} \leq C_6. \tag{4.20}$$

Now, to prove (4.7), we need only to estimate the norm of the function $e^{-\frac{2}{5}\hat{s}\hat{\kappa}}y$ in the space $V^{1,2}(Q)$. Let us make the change in (4.1). Set $\tilde{y} =$

$ye^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{p} = pe^{-\frac{2}{5}\hat{s}\hat{\kappa}}, g = (f - \frac{2}{5}\hat{s}\frac{\partial\hat{\kappa}}{\partial t}y)e^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{u} = ue^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{y}_0 = y_0 e^{-\frac{2}{5}\hat{s}\hat{\kappa}(0)}$. Note that by (4.19), (4.20)

$$\|g\|_{(L^2(Q))^n} + \|\tilde{u}\|_{(L^2(Q))^n} \leq C_7(\|y_0\|_{V^0(\Omega)} + \|f\|_{F(Q,\theta)}). \tag{4.21}$$

We observe that the pair $(\tilde{y}, \tilde{p})$ satisfies the equation

$$L\tilde{y} = \nabla\tilde{p} + g + \chi_\omega \tilde{u} \text{ in } Q, \tag{4.20}$$

$$\mathbf{div}\, \tilde{y} = 0, \quad \tilde{y}|_{\partial\Omega} = 0,\ \tilde{y}(0, x) = \tilde{y}_0(x). \tag{4.22}$$

Thus (4.19)-(4.21) and *a priori* estimate (4.4) imply (4.7).

Now let $f \in F(Q, \theta)$ be an arbitrary function. Then there exist functions $f_1 \in (L^2(Q, \theta))^n$ and $f_2 \in L^2(0, T; W_2^1(\Omega))$ such that $f = f_1 + \nabla f_2$. Above we proved that there exists a solution (y, p, u) of the exact controllability problem (4.1)-(4.3) with initial datum $(y_0, (1 - \chi_\omega)f_1)$ which satisfies estimate (4.7). Evidently $(y, p + f_2, u + \chi_\omega f_1)$ is a solution of this problem for the initial datum (y_0, f). □

Note that the statement of Theorem 1.3 follows from Theorem 4.3.

5. Proof of the main theorem

Our next aim is to reduce the proof of Theorem 1.2 to the case of a linear controllability problem. We are looking for a solution of problem (1.1)-(1.3) of the form

$$v(t, x) = \hat{v}(x) + y(t, x). \tag{5.1}$$

The substitution of (5.1) into equations (1.1)-(1.3) and subtraction from them of the same equation for $\hat{v}$ yields

$$\mathcal{N}(y, q, u) = \partial_t y(t, x) - \Delta y + B(\hat{v}, y) + B(y, \hat{v}) + B(y, y) - \nabla q - \chi_\omega u = 0 \text{ in } \Omega, \tag{5.2}$$

$$\mathbf{div}\, y = 0, \tag{5.3}$$

$$y(0, x) = v_0(x) - \hat{v}(x), \tag{5.4}$$

$$y(T, x) = 0. \tag{5.5}$$

We will solve the problem (5.2)-(5.5) with the help of the following variant of the implicit function theorem (see [1]).

Theorem 5.1. (on a right inverse operator). *Suppose that X, Z are Banach spaces and*

$$\mathcal{A} : X \to Z$$

is a continuously differentiable map. We assume that for $x_0 \in X$, $z_0 \in Z$ the equality

$$\mathcal{A}(x_0) = z_0 \tag{5.6}$$

holds and the derivative $\mathcal{A}'(x_0) : X \to Z$ of the map $\mathcal{A}$ at x_0 is an epimorphism. Then there exists $\varepsilon > 0$ such that for any $z \in Z$ which satisfies the condition

$$\|z - z_0\|_Z < \varepsilon$$

there exists a solution $x \in X$ of the equation

$$\mathcal{A}(x) = z.$$

In our case the space

$$X = Y(Q) \times L^2(0,T;W_2^1(\Omega)) \times (L^2(Q_\omega))^n \tag{4.7}$$

and

$$Z = F(Q,\theta) \times V^1(\Omega). \tag{5.8}$$

The operator $\mathcal{A}$ defined by formula

$$\mathcal{A}(y,q,u) = (\mathcal{N}(y,q,u), y(0,\cdot)). \tag{5.9}$$

We have :

LEMMA 5.2. *Let $\hat{v} \in V^1(\Omega)\bigcap(W^1_\infty(\Omega))^n$, then $\mathcal{A} \in C^1(X,Y)$.*

Proof. It follows directly from the definitions of the spaces X, Z that the operator

$$(y,q,u) \to (\partial_t y(t,x) - \Delta y + B(\hat{v},y) + B(y,\hat{v}) - \nabla q - \chi_\omega u, y(0,\cdot)) : X \to Z$$

is continuous and by virtue of linearity continuously differentiable. The operator B is bilinear. So, to prove this theorem, it is sufficient to establish the continuity of the bilinear operator

$$B : Y(Q) \times Y(Q) \to (L^2(Q,\theta))^n,$$

where the function θ is defined in (4.6). Note that by (2.4), (3.33), (4.5)

$$\eta(t,x) \le -\hat{s}\hat{\kappa}(t,x) \quad \forall\,(t,x) \in Q. \tag{5.10}$$

Then (5.10) and simple transformations give the estimate

$$\begin{aligned}
&\|B(y_1,y_2)\|^2_{(L^2(Q,\theta))^n}\\
&\le C_1 \sum_{i,j=1}^{2} \Big(\int_{Q\setminus Q_\omega} \frac{|y_i|^2|\nabla y_j|^2 e^\eta}{(T-t)^2}\,dxdt + \int_{Q_\omega} |y_i|^2|\nabla y_j|^2\,dxdt\Big)\\
&\le C_2 \sum_{i,j=1}^{2} \int_Q \frac{|y_i|^2|\nabla y_j|^2}{(T-t)^2} e^{-\hat{s}\hat{\kappa}}\,dxdt\\
&\le C_3 \|e^{-\frac{4}{5}\hat{s}\hat{\kappa}} y_1\|^2_{V^{1,2}(Q)} \|e^{-\frac{4}{5}\hat{s}\hat{\kappa}} y_2\|^2_{V^{1,2}(Q)}.
\end{aligned}$$

This inequality proves the theorem. □

Proof of Theorem 1.2. First, we apply the inverse operator theorem to problem (5.2)-(5.5). Let $\mathcal{A}$ be defined by formulas (5.9), (5.2), and the spaces X, Z defined in (5.7), (5.8). Set $x_0 = (0,0,0)$, $z_0 = (0,0)$. Then equation (5.6) obviously holds. By Lemma 5.2, $\mathcal{A} \in C^1(X,Z)$ and by Theorem 4.3, $Im\mathcal{A}'(0) = Z$. So all necessary conditions needed to apply the theorem on the inverse operator are fulfilled. Therefore there exists $\varepsilon > 0$ such that for any initial data (5.4) satisfying the inequality

$$\|y_0\| \leq \varepsilon$$

problem (5.2)-(5.5) has a solution $(y, p, u) \in X$. Then the triple $(y + \hat{v}, p + \hat{p}, u)$ is a solution of problem (1.1)-(1.3), (1.5). □

REMARK 5.3. If we assume that $\hat{v} \in (C^\infty(Q))^n, supp\, \hat{v} \subset\subset [0,T] \times \Omega$ is solution of the nonstationary Navier-Stokes system :

$$\partial_t \hat{v}(t,x) - \Delta \hat{v}(t,x) + (\hat{v}, \nabla)\, \hat{v} + \nabla \hat{p} = f(t,x) \text{ in } \Omega, \quad \mathbf{div}\, \hat{v} = 0$$

and interchange inequality (1.4) on

$$\|\hat{v}(0,\cdot) - v_0\|_{V^1(\Omega)} \leq \epsilon$$

the statement of Theorem 1.2 holds true. Some small changes have to be done in the proof of Theorem 3.6.

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