STABILIZATION OF WALLS FOR NANO-WIRES OF FINITE LENGTH

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Abstract. In this paper we study a one dimensional model of ferromagnetic nano-wires of finite length. First we justify the model by Γ-convergence arguments. Furthermore we prove the existence of wall profiles. These walls being unstable, we stabilize them by the mean of an applied magnetic field.

1. Introduction

This paper is concerned with stabilization of wall configurations in a mono-dimensional model of finite length nanowire. This kind of object can be found in nano electronic devices. The three dimensional model is the following (see [2,11,20]). We denote by $m : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}^3$ the magnetic moment, defined on the ferromagnetic domain Ω. We assume that the material is saturated, so that m satisfies the constraint:

$$\text{for almost every } (t, X) \in \mathbb{R}^+ \times \Omega, \ |m(t, X)| = 1.$$

The magnetic moment links the magnetic induction B with the magnetic field H by the relation $B = H + \bar{m}$, where $\bar{m}$ is the extension of m by zero outside Ω. The behavior of m is governed by the Landau-Lifschitz equation:

$$
\begin{cases}
\dfrac{\partial m}{\partial t} = -m \wedge H_e - m \wedge (m \wedge H_e), \\[2mm]
H_e = A^2 \Delta m + H_d(m) + H_a, \\[2mm]
\dfrac{\partial m}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where

- A^2 is the exchange constant, depending on the material;

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- the demagnetizing field $H_d(m)$ is characterized by

$$\text{curl } H_d(m) = 0 \text{ and div } (H_d(m) + \bar{m}) = 0 \text{ in } \mathbb{R}^3. \tag{1.2}$$

We remark that $-H_d(m)$ is the orthogonal projection of $\bar{m}$ (for the L^2 inner product) onto the curl free vector fields, so that

$$\|H_d(m)\|_{L^2(\mathbb{R}^3)} \leq \|m\|_{L^2(\Omega)};$$

- H_a is an applied magnetic field;
- $\mathbf{n}$ is the outward unit normal on $\partial\Omega$.

The effective field is derived from the micromagnetism energy: $H_e = -\partial_m \mathcal{E}_{\text{mic}}$ with

$$\mathcal{E}_{\text{mic}}(m) = \frac{A^2}{2} \int_\Omega |\nabla m|^2 \, \mathrm{d}X + \frac{1}{2} \int_{\mathbb{R}^3} |H_d(m)|^2 \, \mathrm{d}X - \int_\Omega H_a \cdot m \, \mathrm{d}X.$$

Existence of weak solutions for (1.1) is established in [5,14,19]. Existence of strong solutions is proved in [6,7]. Numerical simulations are performed in [15]. For thin domains, equivalent 2-d models are justified in [1,4,10,16]. For nanowires, 1-d models are discussed in [8,9,13,17].

In this paper we deal with ferromagnetic nanowires. We assume that the wire is a cylinder of length $2L$ and radius η. Taking the limit by Γ-convergence arguments when η tends to zero, we obtain an asymptotic one dimensional model (see Sect. 2).

After renormalization, the one dimensional wire is assimilated to the segment $]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}[e_1$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. The magnetic moment m is then defined on $\mathbb{R}_t^+ \times]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}[$. The equivalent demagnetizing field (after renormalization) is given by

$$h_d(m) = -m_2 e_2 - m_3 e_3,$$

i.e. $-h_d$ is the orthogonal projection onto the plane orthogonal to the wire. In addition we assume that we apply a magnetic field in the direction of the wire axis, and we denote by h_a its renormalized intensity. Therefore we deal with the following system:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \wedge H_e - m \wedge (m \wedge H_e), \\[2mm] H_e = \dfrac{\partial^2 m}{\partial x^2} - m_2 e_2 - m_3 e_3 + h_a e_1, \\[2mm] \dfrac{\partial m}{\partial x}\left(t, -\dfrac{L}{A\sqrt{2}}\right) = \dfrac{\partial m}{\partial x}\left(t, \dfrac{L}{A\sqrt{2}}\right) = 0, \\[2mm] \text{with } m : \mathbb{R}_t^+ \times]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}[\longrightarrow \mathbb{R}^3, \ |m| = 1, \end{cases} \tag{1.3}$$

associated to the energy

$$\mathcal{E}(m) = \int_{-L/A\sqrt{2}}^{L/A\sqrt{2}} \left(\frac{1}{2}|\partial_x m|^2 + \frac{1}{2}(|m_2|^2 + |m_3|^2) - h_a m_1 \right) \mathrm{d}x.$$

In [8,9], we studied a similar model of infinite nanowire. After renormalization, the model was the following: the wire is assimilated to the real axis $\mathbb{R}e_1$, and the one dimensional Landau-Lifschitz equation is in this case:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \wedge H_e - m \wedge (m \wedge H_e), \\[2mm] H_e = \dfrac{\partial^2 m}{\partial x^2} - m_2 e_2 - m_3 e_3 + h_a e_1, \\[2mm] \text{with } m : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow \mathbb{R}^3, \ |m| = 1. \end{cases} \tag{1.4}$$

This system is invariant by translation in the x variable and by rotation around the wire axis, that is, if m is a solution of (1.4), then for σ and τ in $\mathbb{R}$, $(t,x) \mapsto \rho_\tau(m(t,x-\sigma))$ is another solution of (1.4), where ρ_τ is the rotation of angle τ around the e_1 axis:

$$\rho_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix}.$$

We dealt with wall configurations for this model, that is solutions separating a left hand side domain where the magnetization is closed to $-e_1$ to a right hand side domain where the magnetization is closed to $+e_1$. Such solutions are described by rotations and translations of the canonical profile $x \mapsto (\operatorname{th} x, 1/\operatorname{ch} x, 0)$. In [8,9], we proved the asymptotic stability and the controllability for these configurations.

Here, in the case of a finite wire, the situation is quite different.

Definition 1.1. We call canonical wall profile a static solution for (1.3) of the form $M_0 = (\sin\theta_0, \cos\theta_0, 0)$ such that $\theta_0 : [-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}] \longrightarrow [-\pi/2, \pi/2]$ is a non decreasing map satisfying $\theta_0(-\frac{L}{A\sqrt{2}}) < 0 < \theta_0(\frac{L}{A\sqrt{2}})$.

In the following theorem, we claim that the wall profiles exist if and only if the wire is long enough compared to the exchange length:

Theorem 1.2. *There exists a canonical wall profile if and only if* $\frac{L}{A\sqrt{2}} > \pi/2$. *This profile is unique and is centered in the middle of the wire, that is* $\theta_0(0) = 0$. *In addition,* θ_0 *is odd.*

Remark 1.1. In the infinite wire case, the corresponding canonical profiles are obtained taking $\theta_0(x) = \operatorname{Arcsin} \operatorname{th} x$, and all its translation in the x variable (because of the invariance of (1.4) by translation). In the finite wire case, we only have invariance by rotation, and we loose the invariance by translation, so we have only one canonical profile.

Concerning the stability, we obtained in [8] the stability of the wall profile. In that case, the invariance by rotation-translation induces that 0 is a double eigenvalue of the linearized version of (1.4) around the wall profile. In the finite wire case, the lack of invariance by translation induces that the linearized version of (1.3) around the canonical profile has one negative eigenvalue, and therefore we can prove that the wall profile given by Theorem 1.2 is unstable.

Theorem 1.3. *Assume that* $\frac{L}{A\sqrt{2}} > \pi/2$. *Let* θ_0 *given by Theorem* 1.2. *The static solution* $M_0 = (\sin\theta_0, \cos\theta_0, 0)$ *in linearly unstable for the Landau-Lifschitz equation* (1.3) *with* $h_a = 0$.

Remark 1.2. This phenomenon was expected. Let us consider a small translation of the centered wall. Without energetic cost, the Landau-Lifschitz equation induces then a displacement of the wall and pushes it outside the wire. Then the magnetic moment tends to $+\pi/2$ or $-\pi/2$ (*i.e.* the minimizers of the ferromagnetism energy $\mathcal{E}$). In the case of an infinite wire, obviously, this translation cannot make the wall disappear, and we have stability.

We prove now a stabilization result. We control the system by an applied field parallel to the wire axis.
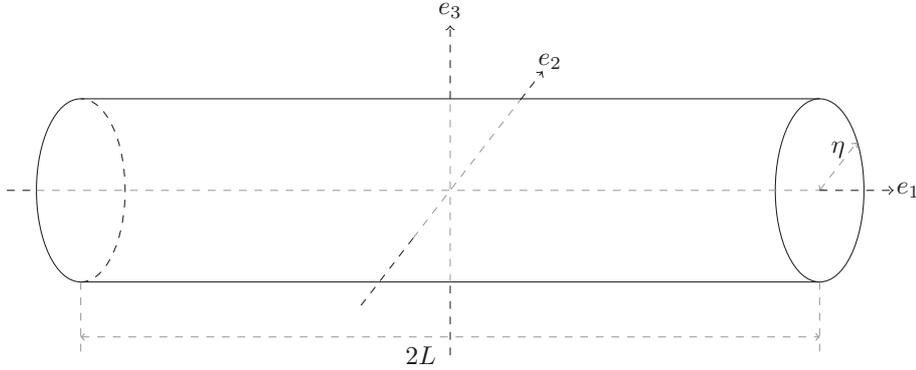
FIGURE 1. Domain Ω_η.

Theorem 1.4. *Let L and A as in Theorem 1.2, and let $M_0 = (\sin\theta_0, \cos\theta_0, 0)$ be the canonical profile given by this theorem. We consider the following control:*

$$h(m(t,.)) = \left[-\frac{A}{\sqrt{2}L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} m_1(t,s)\mathrm{d}s \right].$$

Then M_0 is stable for the Landau Lifschitz equation controlled with the applied field $h_a = h(m)$.

Remark 1.3. The control given here is quite natural: for example when the wall is translated to the right hand side, then the average of the profile first component is negative and our applied field he_1 (with $h > 0$) pushes the wall to the left hand side.

The paper is organized as follows. In Section 2 we justify the one dimensional model by Γ-convergence arguments. In Section 3 we prove the existence of a canonical wall profile. We address the unstability of this profile in Section 4 by linearization of the Landau-Lifschitz equation. The last section is devoted to the stabilization of the wall by a convenient applied magnetic field.

2. MODELIZATION

In this section, we address the justification of the one dimensional model by Γ convergence arguments.

At the beginning we deal with the three-dimensional static case. The finite 3d wire is the cylinder $\Omega_\eta =]-L, L[\times B_2(0, \eta)$, where $B_d(X, r)$ is the ball of radius r and center X in $\mathbb{R}^d$. We assume that we apply on the wire a magnetic field in the direction e_1: $H_a = h_a e_1$, where $h_a \in \mathbb{R}$. See Figure 1.

The micromagnetism energy on Ω_η is given by

$$\mathcal{E}_\eta(v) = \frac{A^2}{2} \int_{\Omega_\eta} |\nabla v|^2 \ \mathrm{d}X + \frac{1}{2} \int_{\mathbb{R}^3} |H_d(v)|^2 \ \mathrm{d}X - h_a \int_{\Omega_\eta} v \cdot e_1 \ \mathrm{d}X \text{ for } v \in H^1(\Omega_\eta; S^2).$$

The static configurations satisfy the minimization problem:

$$\begin{cases} \text{find } u \text{ in } H^1(\Omega_\eta; S^2) \text{ such that} \\ \mathcal{E}_\eta(u) = \min_{v \in H^1(\Omega_\eta; S^2)} \mathcal{E}_\eta(v). \end{cases} \tag{2.1}$$

In order to work with a fixed domain, we introduce the following rescaling: for $v \in H^1(\Omega_\eta; S^2)$, we define $\underline{v} \in H^1(\Omega_1; S^2)$ by

$$\underline{v}(x, y, z) = v(x, \eta y, \eta z).$$

In addition, for $\underline{v} \in H^1(\Omega_1; S^2)$, we denote by $\underline{H_d^\eta}(\underline{v})$ the rescaled demagnetizing field:

$$\underline{H_d^\eta}(\underline{v})(x, y, z) = H_d(v)(x, \eta y, \eta z),$$

where $v \in H^1(\Omega_\eta; S^2)$ is deduced from $\underline{v}$ by $v(x, y, z) = \underline{v}(x, y/\eta, z/\eta)$.

We deal with the rescaled energy defined for $\underline{v} \in H^1(\Omega_1; S^2)$ by $\underline{\mathcal{E}_\eta}(\underline{v}) = \frac{1}{\eta^2}\mathcal{E}_\eta(v)$:

$$\mathcal{E}_\eta(v) = \frac{A^2}{2}\int_{\Omega_1}\left(|\partial_x\underline{v}|^2 + \frac{1}{\eta^2}|\partial_y\underline{v}|^2 + \frac{1}{\eta^2}|\partial_z\underline{v}|^2\right) \, \mathrm{d}X + \frac{1}{2}\int_{\mathbb{R}^3}|\underline{H_d^\eta}(\underline{v})|^2 \, \mathrm{d}X - h_a\int_{\Omega_1}\underline{v}\cdot e_1 \, \mathrm{d}X$$

and the problem (2.1) is rescaled in the following equivalent problem:

$$\begin{cases} \text{find } \underline{u} \text{ in } H^1(\Omega_1; S^2) \text{ such that} \\[2mm] \underline{\mathcal{E}_\eta}(\underline{u}) = \min_{\underline{v}\in H^1(\Omega_1;S^2)}\underline{\mathcal{E}_\eta}(\underline{v}). \end{cases} \tag{2.2}$$

Proposition 2.1. *For all $(\underline{v}_\eta)_{\eta\in\mathbb{R}_*^+}$ in $H^1(\Omega_1; S^2)$, sequence of minimizers of (2.2), $(\underline{\mathcal{E}_\eta}(\underline{v}_\eta))_{\eta\in\mathbb{R}_*^+}$ is a bounded sequence of $\mathbb{R}$.*

Proof. Let $(\underline{v}_\eta)_{\eta\in\mathbb{R}_*^+}$ be a sequence of minimizers of (2.2) in $H^1(\Omega_1; S^2)$. We denote by v_η the rescaled of $\underline{v}_\eta$: $v_\eta(x, y, z) = \underline{v}_\eta(x, y/\eta, z/\eta)$.

In order to exhibit an upper bound for $\underline{\mathcal{E}_\eta}(\underline{v}_\eta)$, we write:

$$\forall\eta \in \mathbb{R}_*^+, \ \mathcal{E}_\eta(v_\eta) \le \mathcal{E}_\eta(e_1),$$

then

$$\mathcal{E}_\eta(v_\eta) \le -2h_a\eta^2\pi L + \tfrac{1}{2}\int_{\mathbb{R}^3}|H_d(e_1)|^2 \, \mathrm{d}X$$

$$\le 2\eta^2 L\pi(1 - h_a).$$

Then, the lower bound is obtained by canceling the positive contributions of the energy and maximizing the external contribution:

$$-\eta^2 2h_a\pi L \le \mathcal{E}_\eta(v_\eta).$$

So, we can conclude

$$-2h_a\pi L \le \underline{\mathcal{E}_\eta}(\underline{v}_\eta) \le 2L\pi(1 - h_a). \qquad \square$$

We introduce the limit energy: we define $\underline{H}^1(\Omega_1; S^2)$ by

$$\underline{H}^1(\Omega_1; S^2) = \{u \in H^1(\Omega_1; S^2), \ \partial_y u = \partial_z u = 0\}.$$

For $\underline{v} \in \underline{H}^1(\Omega_1; S^2)$, we set:

$$\underline{\mathcal{E}}(\underline{v}) = \frac{A^2}{2}\int_{\Omega_1}|\partial_x\underline{v}|^2 \, \mathrm{d}X + \frac{1}{4}\int_{\Omega_1}\left(|\underline{v}_2|^2 + |\underline{v}_3|^2\right) \, \mathrm{d}X - h_a\int_{\Omega_1}\underline{v}\cdot e_1 \, \mathrm{d}X$$

$$= \frac{A^2\pi}{2}\int_{]-L,L[}|\partial_x\underline{v}|^2\mathrm{d}x + \frac{\pi}{4}\int_{]-L,L[}\left(|\underline{v}_2|^2 + |\underline{v}_3|^2\right) \mathrm{d}x - \pi h_a\int_{]-L,L[}\underline{v}\cdot e_1\mathrm{d}x.$$

The limit minimization problem is given by:

$$
\begin{cases}
\text{find } u \text{ in } \underline{H}^1(\Omega_1, S^2) \text{ such that} \\[2mm]
\underline{\mathcal{E}}(u) = \min_{v \in \underline{H}^1(\Omega_1, S^2)} \underline{\mathcal{E}}(v).
\end{cases}
\tag{2.3}
$$

The main result of this section is the following:

Theorem 2.1. $\underline{\mathcal{E}}_\eta$ *Γ-converges to* $\underline{\mathcal{E}}$ *in sense of* $H^1(\Omega_1; \mathbb{R}^3)$ *weak, it is to say:*

(i) *(lower semi continuity)* *for all sequence* $(\underline{v}_\eta)_{\eta \in \mathbb{R}_*^+}$ *of* $H^1(\Omega_1; S^2)$ *such that* $\underline{v}_\eta \rightharpoonup \underline{v}_0$ *in* $H^1(\Omega_1; \mathbb{R}^3)$ *weakly when* $\eta \to 0$, *and* $(\underline{\mathcal{E}}_\eta(\underline{v}_\eta))_{\eta \in \mathbb{R}_*^+}$ *is bounded, then the limit* $\underline{v}_0$ *is an element of* $\underline{H}^1(\Omega_1; S^2)$ *such that:*

$$
\liminf_{\eta \to 0} \underline{\mathcal{E}}_\eta(\underline{v}_\eta) \geq \underline{\mathcal{E}}(\underline{v}_0);
$$

(ii) *(reconstruction)* *for all* $\underline{u}_0 \in \underline{H}^1(\Omega_1; S^2)$, *there exists* $(\underline{v}_\eta)_{\eta \in \mathbb{R}_*^+}$, *sequence of* $H^1(\Omega_1; S^2)$, *such that:*

$$
\lim_{\eta \to 0} \underline{v}_\eta = \underline{u}_0,
$$

and

$$
\limsup_{\eta \to 0} \underline{\mathcal{E}}_\eta(\underline{v}_\eta) \leq \underline{\mathcal{E}}(\underline{u}_0).
$$

Proof. **(i) Lower semi-continuity.** Let $(\underline{v}_\eta)_\eta$ be a sequence in $H^1(\Omega_1; \mathcal{S}^2)$ such that $\underline{v}_\eta$ tends to $\underline{v}$ in $H^1(\Omega_1)$ weak and such that there exists c such that for all η

$$
\underline{\mathcal{E}}_\eta(\underline{v}_\eta) \leq c.
\tag{2.4}
$$

Extracting a subsequence, we can assume that $\underline{v}_\eta$ tends to $\underline{v}$ almost everywhere, so $|\underline{v}| = 1$ *a.e.*

Using (2.4), we remark that since $\left| \int_{\Omega_1} \underline{v}_\eta e_1 \right| \leq \operatorname{meas}(\Omega_1)$,

$$
\|\partial_y \underline{v}_\eta\|^2_{L^2(\Omega_1)} + \|\partial_z \underline{v}_\eta\|^2_{L^2(\Omega_1)} \leq C\eta^2,
$$

so $\partial_y \underline{v} = \partial_z \underline{v} = 0$, that is

$$
\underline{v} \in \underline{H}^1(\Omega_1; S^2).
$$

From straightforward arguments,

$$
\begin{aligned}
&\frac{A^2}{2} \int_{\Omega_1} |\partial_x \underline{v}|^2 \, dX - h_a \int_{\Omega_1} \underline{v} \cdot e_1 \, dX \\
&\leq \liminf \left(\frac{A^2}{2} \int_{\Omega_1} \left(|\partial_x \underline{v}_\eta|^2 + \frac{1}{\eta^2} |\partial_y \underline{v}_\eta|^2 + \frac{1}{\eta^2} |\partial_z \underline{v}_\eta|^2 \right) dX - h_a \int_{\Omega_1} \underline{v}_\eta \cdot e_1 \, dX \right).
\end{aligned}
$$

Concerning the demagnetizing field, the sequence $(\underline{H}_d^\eta(\underline{v}_\eta))_\eta$ is bounded in $L^2(\mathbb{R}^3)$. Indeed, we denote by v_η the rescaled of $\underline{v}_\eta$: $v_\eta(x, y, z) = \underline{v}_\eta(x, y/\eta, z/\eta)$. We have

$$
\begin{aligned}
\|\underline{H}_d^\eta(\underline{v}_\eta)\|_{L^2(\mathbb{R}^3)} &= \frac{1}{\eta}\|H_d(v_\eta)\|_{L^2(\mathbb{R}^3)} \text{ (by change of variable in the integral)} \\[2mm]
&\leq \frac{1}{\eta}\|v_\eta\|_{L^2(\mathbb{R}^3)} \text{ (since } -H_d \text{ is an orthogonal projection in } L^2) \\[2mm]
&\leq \frac{1}{\eta}(2L\pi\eta^2)^{\frac{1}{2}} \text{ (since } |v_\eta| = 1) \\[2mm]
&\leq \sqrt{2\pi\, L}.
\end{aligned}
$$

So, up to the extraction of a subsequence, we can assume that $\underline{H}_d^\eta(\underline{v}_\eta) \rightharpoonup \underline{H}$ for the weak topology in $L^2(\mathbb{R}^3)$.

We denote the coordinates of $\underline{H}_d^\eta(\underline{v}_\eta)$ and those of $\underline{H}$ by:

$$
\underline{H}_d^\eta(\underline{v}_\eta) = \begin{pmatrix} h_\eta^1 \\ h_\eta^2 \\ h_\eta^3 \end{pmatrix} \text{ and } \underline{H} = \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix}.
$$

Writing (1.2) in the rescaled coordinates, we obtain first that

$$
\eta\partial_x(h_\eta^1 + \overline{v_\eta^1}) + \partial_y(h_\eta^2 + \overline{v_\eta^2}) + \partial_z(h_\eta^3 + \overline{v_\eta^3}) = 0,
$$

so taking the weak limit, we have:

$$
\partial_y(h^2 + \overline{v^2}) + \partial_z(h^3 + \overline{v^3}) = 0. \tag{2.5}
$$

In addition, from the curl free condition, we obtain that

$$
\begin{cases}
\partial_y h_\eta^3 - \partial_z h_\eta^2 = 0 \\[1mm]
\eta\partial_x h_\eta^2 - \partial_y h_\eta^1 = 0 \\[1mm]
\eta\partial_x h_\eta^3 - \partial_z h_\eta^1 = 0,
\end{cases}
$$

so, taking the weak limit we get

$$
\partial_y h^1 = \partial_z h^1 = 0 \text{ so } h^1 = 0, \text{ since } h^1 \in L^2(\mathbb{R}^3),
$$

and

$$
\partial_y h^3 - \partial_z h^2 = 0.
$$

So $\underline{H}$ is deduced from $\underline{v}$ by the relations:

$$
\begin{cases}
\partial_y(h^2 + \overline{v^2}) + \partial_z(h^3 + \overline{v^3}) = 0, \\[1mm]
\partial_y h^3 - \partial_z h^2 = 0, \\[1mm]
h^1 = 0,
\end{cases}
$$

that is for a fixed x, $\underline{H}(x, \cdot)$ is defined from $\underline{v}(x, \cdot)$ by the 2-d demagnetizing field. In the following lemma we calculate the 2-d demagnetizing field induced by a constantly magnetized 2-d disk.

Lemma 2.2. *Let $\xi = (\xi_2, \xi_3) \in \mathbb{R}^2$ be a fixed vector. Let $V = (V^2, V^3) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ such that*

$$\begin{cases} \operatorname{curl}_{2D} V := \partial_y V^3 - \partial_z V^2 = 0, \\ \operatorname{div}_{2D}(V + \overline{\xi}) := \partial_y(V^2 + \overline{\xi_2}) + \partial_z(V^3 + \overline{\xi_3}) = 0, \end{cases}$$

where $\overline{\xi}(y, z) = \xi \chi_{B(0,1)}(y, z)$ (we denote by $\chi_{B(0,1)}$ the characteristic function of $B(0, 1)$). Then

$$V(y, z) = \begin{cases} -\dfrac{1}{2}\xi & \text{if } (y, z) \in B_2(0, 1), \\[2ex] \dfrac{1}{2}\dfrac{1}{(y^2 + z^2)^2} \begin{pmatrix} \xi_2(y^2 - z^2) + 2\xi_3 yz \\[1ex] -\xi_3(y^2 - z^2) - 2\xi_2 yz \end{pmatrix} & \text{if } (y, z) \notin B_2(0, 1), \end{cases} \tag{2.6}$$

and

$$\int_{\mathbb{R}^2} |V|^2 \, \mathrm{d}y\mathrm{d}z = \frac{\pi}{2}|\xi|^2.$$

Proof. We have (see [3] or [14]), for $X = (y, z) \in \mathbb{R}^2$:

$$V(X) = \frac{1}{2\pi}\int_{Y \in \partial B_2(0,1)} \frac{X - Y}{|X - Y|^2}\xi \cdot \nu(Y)\mathrm{d}\sigma(Y),$$

where ν is the unit normal on $\partial B_2(0, 1)$.

Using complex notations, we write $V(X) = V^2(X) + iV^3(X)$, $X = y + iz$, $\xi = \xi_2 + i\xi_3$ and we have:

$$V(X) = \frac{1}{2i\pi}\int_{z \in \mathcal{C}(0,1)} \frac{\xi + w^2\overline{\xi}}{2\overline{X}w\left(w - \frac{1}{\overline{X}}\right)}\mathrm{d}w.$$

If $|X| < 1$, the only pole is 0, and by the residue formula, $V(X) = -\frac{1}{2}\xi$. If $|X| > 1$, there are two poles, 0 and $1/\overline{X}$, and we obtain $V(X) = \frac{1}{2}\frac{\overline{\xi}}{\overline{X}^2}$ which is equivalent to (2.6).

In addition, by a direct calculation, we obtain that:

$$\int_{\mathbb{R}^2} |V(y, z)|^2 \, \mathrm{d}y\mathrm{d}z = \frac{\pi}{2}|\xi|^2. \qquad \square$$

We apply the previous lemma to calculate the weak limit $\underline{H}$:

$$\underline{H}(x, y, z) = \begin{cases} -\dfrac{1}{2}\begin{pmatrix} 0 \\ \underline{v}^2(x) \\ \underline{v}^3(x) \end{pmatrix} & \text{if } (y, z) \in B_2(0, 1), \\[3ex] \dfrac{1}{2}\dfrac{1}{(y^2 + z^2)^2}\begin{pmatrix} 0 \\ \underline{v}^2(x)(y^2 - z^2) - 2\underline{v}^3(x)yz \\ -\underline{v}^3(y^2 - z^2) - 2\underline{v}^2 yz \end{pmatrix} & \text{if } (y, z) \notin B_2(0, 1), \end{cases}$$

and by a direct calculation,

$$\int_{\mathbb{R}^3} |\underline{H}(x, y, z)|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\pi}{2}\int_{\mathbb{R}}(|\underline{v}^2(x)|^2 + |\underline{v}^3(x)|^2)\mathrm{d}x.$$

We remark now that since $-H_d$ is an orthogonal projection for the L^2 inner product,

$$\int_{\mathbb{R}^3} |\underline{H_d^\eta}(\underline{v}_\eta)|^2 \; \mathrm{d}X = -\int_{\Omega_1} \underline{H_d^\eta}(\underline{v}_\eta) \cdot \underline{v}_\eta \; \mathrm{d}X.$$

Thus we can take the limit in this integral, and we obtain that:

$$\int_{\mathbb{R}^3} |\underline{H_d^\eta}(\underline{v}_\eta)|^2 \; \mathrm{d}X \longrightarrow -\int_{\Omega_1} \underline{H} \cdot \underline{v} \; \mathrm{d}X = \int_{\mathbb{R}^3} |\underline{H}|^2 \; \mathrm{d}X.$$

So, in fact, $\underline{H_d^\eta}(\underline{v}_\eta)$ tends to $\underline{H}$ strongly in $L^2(\mathbb{R}^3)$, and we obtain that

$$\frac{\pi}{4} \int_{\mathbb{R}} (|\underline{v}^2(x)|^2 + |\underline{v}^3(x)|^2) \mathrm{d}x = \liminf \frac{1}{2} \int_{\mathbb{R}^3} |\underline{H_d^\eta}(\underline{v}_\eta)|^2 \; \mathrm{d}X.$$

Therefore,

$$\underline{\mathcal{E}}(\underline{v}) \le \liminf \underline{\mathcal{E}}_\eta(\underline{v}_\eta).$$

(ii) Reconstruction. Let $\underline{u_0} \in \underline{H}^1(\Omega_1; S^2)$. For $\eta > 0$, we define $\underline{u}_\eta = \underline{u_0}$. Then,

$$\underline{\mathcal{E}}_\eta(\underline{u}_\eta) = \frac{A^2}{2} \int_{\Omega_1} |\partial_x \underline{u_0}|^2 \; \mathrm{d}X + \frac{1}{2} \int_{\mathbb{R}^3} |\underline{H_d^\eta}(\underline{u_0})|^2 \; \mathrm{d}X - h_a \int_{\Omega_1} \underline{u_0} \cdot e_1 \; \mathrm{d}X.$$

As in the previous step, we have:

$$\frac{1}{2} \int_{\mathbb{R}^3} |\underline{H_d^\eta}(\underline{u_0})|^2 \; \mathrm{d}X \longrightarrow \frac{\pi}{4} \int_{\Omega_1} (|\underline{u_0^2}|^2 + |\underline{u_0^3}|^2) \; \mathrm{d}X,$$

that is

$$\underline{\mathcal{E}}_\eta(\underline{u}_\eta) \longrightarrow \underline{\mathcal{E}}(\underline{u_0}). \qquad \square$$

This theorem gives the behavior of minimizers for the one dimensional limit case: the limit energy, for $u \in H^1(]-L, L[; S^2)$ is then $\mathcal{E}(u) = \underline{\mathcal{E}}(u \cdot \chi_{B^2(0,1)})$:

$$\mathcal{E}(u) = \frac{\pi A^2}{2} \int_{]-L,L[} \left| \frac{\mathrm{d}u}{\mathrm{d}x} \right|^2 \; \mathrm{d}x + \frac{\pi}{4} \int_{]-L,L[} (|u_2|^2 + |u_3|^2) \; \mathrm{d}x - h_a \pi \int_{]-L,L[} v \cdot e_1 \; \mathrm{d}x,$$

then, in order to find out the effective field, we write (see [11]):

$$H_e = -\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}u},$$

it is to say

$$H_e = \pi A^2 \frac{\partial^2 u}{\partial x^2} - \frac{\pi}{2}(u_2 \; e_2 + u_3 \; e_3) + \pi h_a \; e_1.$$

The boundary conditions for the minimizers are:

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = -L \text{ and } x = L,$$

since we minimize the energy without Dirichlet boundary conditions.

Then, the limit dynamic system is obtain using the Landau Lifschitz combined with the new effective field H_e.

We perform the following rescaling in time $t' = \frac{\pi}{2}t$ and in space $x' = x/A\sqrt{2}$, we denote by $h'_a = 2h_a$ the rescaled applied field. The resulting rescaled system is then given by (1.3) as we expected.

3. Existence of canonical wall profiles

In this section we are interested in characterizing equilibrium states of the magnetization in a finite nano-wire when $h = 0$, it is to say when there is no external magnetic field. The constant stable solutions are those equal to $-e_1$ or e_1, since the demagnetizing field behaves like a $\mathbb{R}e_1$ easy axes anisotropic field. We are interested in profiles describing one wall connecting a left hand side $-e_1$ domain to a right hand side e_1 domain. So, *a priori*, we look for solutions which can be written as:

$$\begin{pmatrix} \sin\theta_0 \\ \cos\theta_0\cos\tau \\ \cos\theta_0\sin\tau \end{pmatrix}$$

where θ_0 is a non decreasing map from $\left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[$ into $\mathbb{R}$ satisfying $\theta_0\left(-\frac{L}{A\sqrt{2}}\right) < 0 < \theta_0\left(\frac{L}{A\sqrt{2}}\right)$. If we write the energy of such a profile, we obtain:

$$\mathcal{E}(\theta_0, \tau) = \frac{1}{2}\int_{-\frac{L}{A\sqrt{2}}}^{-\frac{L}{A\sqrt{2}}} \left(|\theta_0'|^2 + |\tau'|^2\cos^2\theta_0\right) + \frac{1}{2}\int_{-\frac{L}{A\sqrt{2}}}^{-\frac{L}{A\sqrt{2}}} |\cos^2\theta_0|^2.$$

So in order to decrease the energy, we obtain minimizing walls assuming that $\tau' = 0$, that is for constant τ. Without loss of generality (by rotational invariance), we assume that $\tau = 0$. Therefore, we look for wall profiles which can be written as follows:

$$M_0(x) = \begin{pmatrix} \sin\theta_0 \\ \cos\theta_0 \\ 0 \end{pmatrix}, \ \forall x \in \left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[,$$

where θ_0 is a non decreasing map from $\left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[$ into $\mathbb{R}$ satisfying $\theta_0\left(-\frac{L}{A\sqrt{2}}\right) < 0 < \theta_0\left(\frac{L}{A\sqrt{2}}\right)$, and such that M_0 is a stationary solution to (1.3). In fact, we want M_0 to verify:

$$-M_0 \wedge H_e - M_0 \wedge (M_0 \wedge H_e) = 0, \ \forall x \in \left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[,$$

with

$$H_e = \frac{\partial^2 M_0}{\partial x^2} - \cos\theta_0 \ e_2,$$

then, one has the following relation

$$-\theta_0'' - \sin\theta_0 \ \cos\theta_0 = 0, \ \forall x \in \left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[, \tag{3.1}$$

with on the boundary

$$\theta_0'\left(-\frac{L}{A\sqrt{2}}\right) = \theta_0'\left(\frac{L}{A\sqrt{2}}\right) = 0. \tag{3.2}$$

Setting $\gamma_0 = \theta_0(\frac{L}{A\sqrt{2}})$ (γ_0 is supposed to be non negative, that is we look for a non decreasing solution θ_0), we have, integrating the equation (3.1) and using (3.2):

$$(\theta_0')^2 + \sin^2 \theta_0 = \sin^2 \gamma_0. \tag{3.3}$$

The length of the nano-wire has to be such that the function θ_0 goes from $-\gamma_0$ to γ_0. From formula (3.3), we deduce the length:

$$\ell(\gamma_0) = \int_{-\gamma_0}^{\gamma_0} \frac{\mathrm{d}\theta}{\sqrt{\sin^2 \gamma_0 - \sin^2 \theta}}.$$

Using the length expression computed above, we prove now Theorem 1.2.

Proof. In order to ensure the existence of an equilibrium state as defined above, we must verify that there exists $\gamma_0 \in]0, \pi/2[$ such that $\ell(\gamma_0) = 2\frac{L}{A\sqrt{2}}$ (the case $\gamma_0 = \pi/2$ corresponds to the constant solution $M_2 = -e_1$, which does not describe a wall). First of all, one has

$$\ell(\gamma_0) = 2\int_0^1 \frac{\gamma_0}{\sqrt{\sin^2 \gamma_0 - \sin^2 u\gamma_0}}\mathrm{d}u,$$

we then see that

$$\lim_{\gamma_0 \to \frac{\pi}{2}} \ell(\gamma_0) = +\infty.$$

Let us prove that ℓ is a non decreasing map. We denote by

$$F(\gamma, u) = \frac{\gamma}{(\sin^2 \gamma - \sin^2 u\gamma)^{\frac{1}{2}}}$$

so that

$$\ell(\gamma) = 2\int_0^1 F(\gamma, u)\mathrm{d}u.$$

For $u \in [0, 1[$, we have

$$\frac{\partial F}{\partial \gamma} = \frac{1}{(\sin^2 \gamma - \sin^2 u\gamma)^{\frac{3}{2}}}(g(\gamma) - g(u\gamma))$$

with $g(\tau) = \sin\tau(\sin\tau - \tau\cos\tau)$.

We remark that for $\tau \in [0, \pi/2]$, $\tau \mapsto \sin\tau$ is increasing, $\tau \mapsto -\tau\cos\tau$ is increasing, and $(\sin\tau - \tau\cos\tau)$ remains non negative, so g is an increasing map on $[0, \pi/2]$. Hence for $\gamma \in [0, \pi/2]$ and for $u \in [0, 1[$, $g(\gamma) - g(u\gamma)$ is non negative, therefore for $u \in [0, 1[$, $\gamma \mapsto F(\gamma, u)$ is non decreasing, that is ℓ is strictly increasing on $]0, \frac{\pi}{2}]$.

Then, the comparison could be done at the limit $\gamma_0 = 0$:

$$\begin{aligned}
\lim_{\gamma_0 \to 0} \ell(\gamma_0) &= \lim_{\gamma_0 \to 0} \int_{-1}^1 \frac{\gamma_0}{\sin\gamma_0}\frac{\mathrm{d}u}{\sqrt{1 - \frac{\sin^2(u\ \gamma_0)}{\sin^2\gamma_0}}} \\
&= \int_{-1}^1 \frac{\mathrm{d}u}{\sqrt{1 - u^2}} \\
&= \pi,
\end{aligned}$$

then, we require that $2\frac{L}{A\sqrt{2}} > \pi$. □

Remark 3.1. In the case of nanowires, in order to describe a wall, the magnetization must point out the domain so we are in presence of a Bloch type wall (see [18]).

4. LINEAR UNSTABILITY OF CANONICAL WALL PROFILES WITHOUT APPLIED FIELD

Let us consider L, A and $\theta_0 > 0$ given by Theorem 1.2. We denote by M_0 the canonical wall profile

$$M_0(x) = \begin{pmatrix} \sin\theta_0(x) \\ \cos\theta_0(x) \\ 0 \end{pmatrix}.$$

This profile is a static solution to (1.3) with $h_a = 0$. We aim to calculate the linearized equation of (1.3) around M_0. In order to calculate this linearization taking into account that we are only interested in the perturbations of M_0 satisfying the constraint $|m| = 1$, we describe the small perturbations of M_0 as follows:

$$m(t, x) = r_1 \, M_1 + r_2 \, M_2 + \sqrt{1 - r_1^2 - r_2^2} \, M_0,$$

where the mobile frame $(M_0(x), M_1(x), M_2)$ is given by

$$M_0(x) = \begin{pmatrix} \sin\theta_0(x) \\ \cos\theta_0(x) \\ 0 \end{pmatrix}, \ M_1(x) = \begin{pmatrix} -\cos\theta_0(x) \\ \sin\theta_0(x) \\ 0 \end{pmatrix}, \ M_2(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.1}$$

for $x \in \left]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right[$.

The new unknown is now $r = (r_1, r_2)$ taking its values into $\mathbb{R}^2$ and the profile M_0 corresponds to $r = 0$. We rewrite the Landau-Lifschitz system (1.3) in the unknown r and we linearize this new equation around $r = 0$. In the new unknowns, the effective field writes:

$$H_e = g_0 M_0 + (2\theta_0' \partial_x r_1 + r_1 \theta_0'' - \cos\theta_0 \sin\theta_0 \, r_1) M_0$$
$$+ (\partial_{xx} r_1 + \cos^2\gamma_0 r_1) M_1 + \partial_{xx} r_2 M_2 + \text{non linear terms}$$

where

$$g_0 = \sin^2\theta_0 - (\theta_0')^2. \tag{4.2}$$

After tedious calculations, we obtain the following linearized equation:

$$\partial_t r = J \begin{pmatrix} (\mathcal{L} - \cos^2\gamma_0) r_1 \\ \mathcal{L} r_2 \end{pmatrix},$$

with

- $J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$;
- $\gamma_0 = \theta_0(L/A\sqrt{2})$;
- $\mathcal{L} r_i = -\partial_{xx} r_i + g_0 r_i$, where g_0 is given by (4.2).

The linear unstability of the wall structure computed in the previous section is given by the study of the operator $\mathcal{L}$.

Proposition 4.1. $\mathcal{L}$ is a linear, positive operator. Its first eigenvalue, 0, is associated to the eigenfunction $\cos\theta_0$ and its second eigenvalue, 1, is associated to the eigenfunction $\sin\theta_0$.

Proof. We set: $f = \theta_0' \, \tan\theta_0$, then

$$\mathcal{L} = \ell^* \ell, \text{ where } \ell = \partial_x + f,$$

then, we can conclude that $\mathcal{L}$ is a positive operator and that $\cos\theta_0$ is in the kernel of $\mathcal{L}$. Thus 0 is the first eigenvalue of $\mathcal{L}$ and $\text{Ker } \mathcal{L} = \mathbb{R}\cos\theta_0$.

Furthermore, we have:

$$\mathcal{L}(\sin\theta_0) = \sin\theta_0,$$

it is to say that 1 is an eigenvalue of $\mathcal{L}$ associated to the eigenfunction $\sin\theta_0$. In addition we remark that $\sin\theta_0$ vanishes once in the domain, so by Sturm-Liouville theorem, 1 is the second eigenvalue of $\mathcal{L}$. □

We can now prove Theorem 1.3.

Proof. We first remark that by definition of the canonical profile, then $\gamma_0 < \pi/2$. Indeed $\gamma_0 = \pi/2$ should imply that θ_0 is constant equal to $\pi/2$ and does not satisfy the condition $\theta_0(-L/\sqrt{2}A) < 0$. This case corresponds to the case $M_0 \equiv e_1$, which is a stable (but constant) configuration.

From the previous proposition, since $-\cos^2\gamma_0 < 0$, we conclude that $\mathcal{L} - \cos^2\gamma_0$ has one strictly negative eigenvalue, then, zero is unstable for the linearized of (1.3) around M_0. □

5. STABILIZATION OF WALLS

Now, we discuss the stabilization of M_0 by the command H_a (the applied field). We recall that we introduced the following command:

$$h_a = h_a(m) = \frac{-A}{\sqrt{2}L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} m_1(t,s) \, \mathrm{d}s.$$

We want to prove that the profile M_0 is a stable stationary solution for the following system:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \wedge H_e - m \wedge (m \wedge H_e), \\[2mm] \dfrac{\partial m}{\partial x}\left(-\dfrac{L}{A\sqrt{2}}\right) = \dfrac{\partial m}{\partial x}\left(\dfrac{L}{A\sqrt{2}}\right) = 0, \\[2mm] H_e = \dfrac{\partial^2 m}{\partial x^2} - m_2 \, e_2 - m_3 \, e_3 + h_a(m) \, e_1. \end{cases} \tag{5.1}$$

Proof. To start with, let us introduce the wall canonical profile $M_0 = (\sin\theta_0, \cos\theta_0, 0)$ given in Theorem 1.2. We recall that

$$\theta_0'' + \sin\theta_0 \cos\theta_0 = 0,$$

$$\theta'\left(-\tfrac{L}{A\sqrt{2}}\right) = \theta'\left(\tfrac{L}{A\sqrt{2}}\right) = 0.$$

Furthermore, on $\left[-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right]$,

$$\cos^2\theta_0 - (\theta_0')^2 = \cos^2\gamma_0,$$

where $\gamma_0 = \theta_0(\frac{L}{A\sqrt{2}})$. In addition, we remark that θ_0 is an odd map.

Since $h_a(M_0) = 0$ (as θ_0 is odd), we remark that M_0 is a stationary solution of (5.1).

5.1. First step: moving frame

As in the previous section, in the spirit of [8], we will describe the problem in the moving frame

$$(M_0(x), M_1(x), M_2),$$

given by (4.1).

We write the solutions to (5.1) as:

$$m(t, x) = M_0(x) + r_1(t, x)M_1(x) + r_2(t, x)M_2 + \nu(r(t, x))M_0(x)$$

where $\nu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$. In this moving frame, we get

$$H_e = (g_0 + a_0)M_0 + (a_1 + \tilde{a}_1)M_1 + a_2 M_2$$

with

$$g_0 = \sin^2 \theta_0 - (\theta_0')^2$$

$$a_0 = 2\theta_0' \partial_x r_1 + r_1 \theta_0'' - \cos \theta_0 \sin \theta_0\ r_1 + \partial_{xx} \nu - \nu(\theta_0')^2 + \nu \sin^2 \theta_0 + h(r) \sin \theta_0,$$

$$a_1 = \partial_{xx} r_1 + \cos^2 \gamma_0 r_1 - S(r_1) \cos \theta_0,$$

$$\tilde{a}_1 = -2\partial_x \nu \theta_0' - \varphi(r) \cos \theta_0,$$

$$a_2 = \partial_{xx} r_2,$$

where

$$S(r_1) = \frac{A}{\sqrt{2}L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} r_1 \cos \theta_0\ \mathrm{d}s, \quad \varphi(r) = -\frac{A}{\sqrt{2}L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} \nu(r) \sin \theta_0\ \mathrm{d}s,$$

and $h(r) = S(r_1) + \varphi(r)$.

Using these coordinates in the Landau-Lifschitz equation (3.2) and projecting on M_1 and M_2 yield:

$$\partial_t r = \Lambda r + F(x, r, \partial_x r, \partial_{xx} r), \tag{5.2}$$

where

$$\Lambda r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{L}}_1(r_1) \\ \mathcal{L}(r_2) \end{pmatrix}$$

with

$$\mathcal{L} = -\partial_{xx} + g_0,$$

$$\tilde{\mathcal{L}}_1 = \mathcal{L} - \cos^2 \gamma_0 + \cos \theta_0 S,$$

and where the non linear part F is given by

$$F(x, r, \partial_x r, \partial_{xx} r) = F_1(r)\partial_{xx} r + F_2(r)(\partial_x r, \partial_x r) + F_3(x, r)\partial_x r + F_4(r) + F_5(r),$$

with

- $F_1 \in \mathcal{C}^\infty(\mathbb{R}^2; \mathcal{M}_2(\mathbb{R}))$:

$$
F_1(r) = \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) & \nu(r) - r_1 r_2 \\ r_1 r_2 - \nu(r) & (r_1)^2 + (\nu(r))^2 + 2\nu(r) \end{pmatrix}
$$
$$
+ \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix} \nu'(r);
$$

- $F_2 \in \mathcal{C}^\infty(\mathbb{R}^2; \mathcal{L}_2(\mathbb{R}^2; \mathbb{R}^2))$:

$$
F_2(r)(\partial_x r, \partial_x r) = \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix} \nu''(r)(\partial_x r, \partial_x r);
$$

- $F_3 \in \mathcal{C}^\infty\left(\left[-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right] \times \mathbb{R}^2; \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)\right)$:

$$
F_3(x, r)\partial_x r = 2\theta_0'(x) \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix} \partial_x r_1
$$
$$
- 2\theta_0'(x) \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) + 1 \\ r_1 r_2 - \nu(r) - 1 \end{pmatrix} \nu'(r)(\partial_x r);
$$

- $F_4(r) \in \mathcal{C}^\infty\left(\left[-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}\right] \times \mathbb{R}^2; \mathbb{R}^2\right)$:

$$
F_4(r) = (\cos^2\gamma_0\, r_1) \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) \\ r_1 r_2 - \nu(r) \end{pmatrix} - g_0\nu(r) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}
$$
$$
+ ((\theta_0'' - \cos\theta_0\sin\theta_0)r_1 + g_0\nu(r)) \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix};
$$

- F_5 is given by

$$
F_5(r) = -h(r)\cos\theta_0 \begin{pmatrix} (r_2)^2 + (\nu(r))^2 + 2\nu(r) \\ r_1 r_2 - \nu(r) \end{pmatrix} - \varphi(r)\cos\theta_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}
$$
$$
+ h(r)\sin\theta_0 \begin{pmatrix} -r_1 - r_2 - \nu(r)r_1 \\ r_1 - r_2 - \nu(r)r_2 \end{pmatrix}.
$$

Remark 5.1. The command h makes the linear part of (5.2) positive. Indeed, on one hand, we know that $\mathcal{L} \geq 0$ with $\text{Ker}\,\mathcal{L} = \mathbb{R}\cos\theta_0$. On the other hand, $\tilde{\mathcal{L}}_1 = \mathcal{L} + \cos\theta_0 S - \cos^2\gamma_0$. On $\mathbb{R}\cos\theta_0$, $\tilde{\mathcal{L}}_1\cos\theta_0 = \alpha_0\cos\theta_0$

with

$$\alpha_0 = \frac{A}{\sqrt{2}L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} \cos^2\theta_0(x)\mathrm{d}x - \cos^2\gamma_0 > 0 \tag{5.3}$$

since for $x \in]-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}}[$, $\cos^2\theta_0(x) > \cos^2\gamma_0$, and on $(\mathbb{R}\cos\theta_0)^\perp$,

$$\tilde{\mathcal{L}}_1|_{(\cos\theta_0)^\perp} = \mathcal{L} - \cos^2\gamma_0 \geq 1 - \cos^2\gamma_0$$

since $\mathcal{L} \geq 1$ on $(\mathbb{R}\cos\theta_0)^\perp$ (see Prop. 4.1).

5.2. Second step

5.2.1. *New unknown*

The Landau Lifschitz equation (1.3) is invariant by rotation around the wire axis, so we can build a one parameter family of static solutions for (5.2).

For $\tau \in \mathbb{R}$ let us introduce the rotation around the x-axis given by

$$\rho_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix}.$$

We denote $M_\tau(x) = \rho_\tau M_0(x)$, and R_τ its projection on the moving frame:

$$R_\tau(x) = \begin{pmatrix} M_\tau(x).M_1(x) \\ M_\tau(x).M_2(x) \end{pmatrix} = \begin{pmatrix} \cos\theta_0(x)\sin\theta_0(x)(\cos\tau - 1) \\ \cos\theta_0(x)\sin\tau \end{pmatrix}.$$

Since M_τ is solution to (5.1), R_τ is a static solution of (5.2), that is

$$\Lambda R_\theta + F(x, R_\theta, \partial_x R_\theta, \partial_{xx} R_\theta) = 0. \tag{5.4}$$

Now in order to avoid the problems due to the zero eigenvalue of Λ, we describe r in the new coordinates (τ, σ, W) defined by

$$r(t, x) = R_{\tau(t)}(x) + \sigma(t)\begin{pmatrix} \cos\theta_0(x) \\ 0 \end{pmatrix} + W(t, x), \tag{5.5}$$

where $(\tau, \sigma) \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^2)$ and $W \in \mathcal{C}^1(\mathbb{R}^+; H^2)$ such that both coordinates of W are in $(\mathbb{R}\cos\theta_0)^\perp$.

Indeed, as in [12], we can prove that for a given $r \in H^2(-\frac{L}{A\sqrt{2}}, \frac{L}{A\sqrt{2}})$ in a neighbourhood of 0, there exists a unique $(\tau, \sigma, W) \in \mathbb{R} \times \mathbb{R} \times \mathcal{W}$ such that

$$\rho = R_\theta + \sigma\begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} + W,$$

where $\mathcal{W}$ is the set of the $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in H^2$ satisfying the homogeneous Neumann Boundary condition $\frac{\partial W}{\partial x}\left(-\frac{L}{A\sqrt{2}}\right) = \frac{\partial W}{\partial x}\left(\frac{L}{A\sqrt{2}}\right) = 0$ and the orthogonality condition:

$$\int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} W_1(x)\cos\theta_0(x)\mathrm{d}x = \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} W_2(x)\cos\theta_0(x)\mathrm{d}x = 0.$$

Remark 5.2. On $(\mathbb{R}\cos\theta_0)^\perp$, the operators $\mathcal{L}$ and $\mathcal{L} - \cos^2\theta_0$ are non negative, so we introduce the following norms on $\mathcal{W}$, respectively equivalent to the H^2 and the H^3 norms:

$$\|W\|_{\mathcal{H}^2} = \left(\|\mathcal{L}^{\frac{1}{2}}(\mathcal{L} - \cos^2\gamma_0)^{\frac{1}{2}} W_1\|^2_{L^2} + \|\mathcal{L}W_2\|^2_{L^2} \right)^{\frac{1}{2}},$$

$$\|W\|_{\mathcal{H}^3} = \left(\|\mathcal{L}^{\frac{1}{2}}(\mathcal{L} - \cos^2\gamma_0) W_1\|^2_{L^2} + \|\mathcal{L}^{\frac{3}{2}} W_2\|^2_{L^2} \right)^{\frac{1}{2}}.$$

5.2.2. *Equation* (5.2) *with these unknowns*

Plugging the decomposition (5.5) in (5.2) and using (5.4) yield the following equivalent form for the Landau-Lifschitz equation in the coordinates (τ, σ, W), valid for small perturbations of M_0:

$$\frac{d\tau}{dt} R'_\tau + \frac{d\sigma}{dt} \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} + \partial_t W = \sigma \begin{pmatrix} -\alpha_0 \cos\theta_0 \\ \alpha_0 \cos\theta_0 \end{pmatrix} + \Lambda W + G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W). \tag{5.6}$$

Let us describe the different terms in this equation:
Concerning the time derivatives, we have

$$\partial_t r = \frac{d\tau}{dt} R'_\tau(x) + \frac{d\sigma}{dt} \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} + \partial_t W,$$

where

$$R'_\tau(x) = \begin{pmatrix} -\sin\tau\cos\theta_0\sin\theta_0 \\ \cos\tau\cos\theta_0 \end{pmatrix}.$$

Concerning the linear part, we have

$$\Lambda r = \Lambda R_\tau + \begin{pmatrix} -\alpha_0 \cos\theta_0 \\ \alpha_0 \cos\theta_0 \end{pmatrix} \sigma + \Lambda(W)$$

(see Rem. 4.1 for the definition of α_0).
Concerning the non linear part, we have

$$F(x, r, \partial_x r, \partial_{xx} r) = F(x, R_\tau, \partial_x R_\tau, \partial_{xx} R_\tau) + G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W).$$

The last term G is obtained from F with the Taylor formula around R_τ:

$$\begin{aligned}
G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W) = \ & F_1(r)(\partial_{xx} w) + \tilde{F}_1(r)(w)(\partial_{xx} R_\tau) \\
& + 2F_2(r)(\partial_x w, \partial_x R_\tau) + F_2(r)(\partial_x w, \partial_x w) \\
& + \tilde{F}_2(r)(w)(\partial_x R_\tau, \partial_x R_\tau) \\
& + F_3(x, r)(\partial_x w) + \tilde{F}_3(x, r)(w)(\partial_x R_\tau) + \tilde{F}_4(r)(w) \\
& + F_5(r),
\end{aligned}$$

where
- $w = W + \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} \sigma;$

- for $i = 1, \ldots, 4$, $\tilde{F}_i(r) \in \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)$ is given by

$$\tilde{F}_i(r) = \int_0^1 F_i'(R_\tau + sw)\mathrm{d}s.$$

From straightforward calculations, we see that:

$$F_1(r) = \mathcal{O}(|r|^2), \ \tilde{F}_1(r) = \mathcal{O}(|r|^2)$$

$$F_2(r) = \mathcal{O}(|r|), \ \tilde{F}_2(r) = \mathcal{O}(1)$$

$$F_3(x, r) = \mathcal{O}(|r|), \ \tilde{F}_3(x, r) = \mathcal{O}(1)$$

$$\tilde{F}_4(r)\mathcal{O}(|r|).$$

- Concerning F_5, on one hand we remark that

$$S(r_1) = S(W_1) + \sigma S(\cos \theta_0)$$

(since $S(R_\tau) = 0$).
 On the other hand, $\varphi(R_\tau) = 0$ and

$$\varphi(r) = -\frac{\varepsilon}{2L} \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} \tilde{\nu}(r)(w) \sin \theta_0,$$

where

$$\tilde{\nu}(r) = \int_0^1 \nu'(R_\tau + sw)\mathrm{d}s = \mathcal{O}(|r|).$$

Therefore with all these estimates, if $\|r\|_{H^2}$ is sufficiently small, we have

$$\|G\|_{L^2} \le K\|r\|_{L^\infty}[|\sigma| + \|W\|_{\mathcal{H}^2}],$$

$$\|\partial_x G\|_{L^2} \le K\|r\|_{L^\infty}[|\sigma| + \|W\|_{\mathcal{H}^3}]$$

(5.7)

(see Rem. 5.2).

5.2.3. *Separation of the unknowns*

In order to separate the unknowns τ, σ and W, we first take the inner product of (5.6) with $\begin{pmatrix} \cos \theta_0 \\ 0 \end{pmatrix}$ and with $\begin{pmatrix} 0 \\ \cos \theta_0 \end{pmatrix}$. We remark that both $\partial_t W$ and ΛW are orthogonal to these vectors, so that we obtain:

$$\rho_0 \sigma' = -\alpha_0 \rho_0 \sigma + \overline{G}_1,$$

$$g_\tau \tau' = \alpha_0 \rho_0 \sigma + \overline{G}_2,$$

where

$$\rho_0 = \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} \cos^2 \theta_0,$$

$$\overline{G}_1 = \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W) \cdot \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} \, dx,$$

$$g_\tau = \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} R'_\tau(x) \cdot \begin{pmatrix} 0 \\ \cos\theta_0 \end{pmatrix} \, dx,$$

$$\overline{G}_2 = \int_{-\frac{L}{A\sqrt{2}}}^{\frac{L}{A\sqrt{2}}} G(x, \tau, \sigma, W, \partial_x W, \partial_{xx} W) \cdot \begin{pmatrix} 0 \\ \cos\theta_0 \end{pmatrix} \, dx.$$

By subtraction, we have:

$$\partial_t W = \Lambda W + \widetilde{G}$$

with

$$\widetilde{G} = G - \frac{\overline{G}_2}{g_\tau} R'_\tau - \frac{\overline{G}_1}{\rho_0} \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} + \alpha_0 \sigma \left[\begin{pmatrix} 0 \\ \cos\theta_0 \end{pmatrix} - \frac{\rho_0}{g_\tau} R'_\tau \right].$$

5.2.4. *Equivalent formulation*

We are then led to study the following equation

$$\tau' = \alpha_0 \frac{\rho_0}{g_\tau} \sigma + \frac{1}{g_\tau} \overline{G}_2, \tag{5.8}$$

together with the system coupling:

$$\sigma' = -\alpha_0 \sigma + \frac{1}{\rho_0} \overline{G}_1, \tag{5.9}$$

with

$$\partial_t W = \begin{pmatrix} -(\mathcal{L} - \cos^2\gamma_0)W_1 - \mathcal{L}W_2 \\ (\mathcal{L} - \cos^2\gamma_0)W_1 - \mathcal{L}W_2 \end{pmatrix} + \widetilde{G}. \tag{5.10}$$

From (5.7), with Remark 5.2 we have

$$\left| \frac{1}{\rho_0} \overline{G}_1 \right| (t) \le K \|r\|_{L^\infty} \left[|\sigma(t)| + \|W(t)\|_{\mathcal{H}^3} \right]. \tag{5.11}$$

In addition, since $g_\tau = \rho_0 + \mathcal{O}(\tau^2)$, since $R'_\tau = \begin{pmatrix} 0 \\ \cos\theta_0 \end{pmatrix} + \mathcal{O}(\tau)$, we get:

$$\|\tilde{G}\|_{H^1} \le K \|r\|_{L^\infty} \left[|\sigma(t)| + \|W(t)\|_{\mathcal{H}^3} \right]. \tag{5.12}$$

5.3. **Last step: Variational estimates**

Taking the inner product of (5.10) with $\begin{pmatrix} \mathcal{L}(\mathcal{L} - \cos^2\gamma_0)W_1 \\ \mathcal{L}^2 W_2 \end{pmatrix}$, we obtain using (5.12) that

$$\frac{d}{dt} \left(\|W\|_{\mathcal{H}^2}^2 \right) + \|W\|_{\mathcal{H}^3}^2 \le K \|r\|_{L^\infty} \left[|\sigma(t)| + \|W(t)\|_{\mathcal{H}^3} \right].$$

Remark 5.3. In order to avoid the boundary terms when we integrate by part, we perform the previous estimates on the Galerkin approximation of (5.10) built on the eigenvectors of $\mathcal{L}$ with the homogeneous Neumann boundary condition.

Multiplying (5.9) by σ, (5.11) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^2 + \alpha_0\sigma^2 \leq K\|r\|_{L^\infty}\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^3}^2\right].$$

Summing up the previous estimates, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^2}^2\right] + \alpha_0\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^3}^2\right](1 - K\|r\|_{L^\infty}) \leq 0.$$

So there exists $\delta > 0$ such that while $\|r\|_{L^\infty} \leq \frac{1}{2K}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^2}^2\right] + \delta\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^2}^2\right] \leq 0,$$

that is

$$\left[|\sigma(t)|^2 + \|W(t)\|_{\mathcal{H}^2}^2\right] \leq \left[|\sigma_0|^2 + \|W_0\|_{\mathcal{H}^2}^2\right]\mathrm{e}^{-\delta t}.$$

Now, with equation (5.8), we have

$$|\tau'| \leq K|\sigma| + K\left[|\sigma(t)| + \|W(t)\|_{\mathcal{H}^2}\right]$$

so while $\|r\|_{L^\infty} \leq \frac{1}{2K}$,

$$|\tau| \leq |\tau_0| + K\left[|\sigma_0| + \|W_0\|_{\mathcal{H}^2}\right]\mathrm{e}^{-\delta t/2}.$$

Therefore, if τ_0, σ_0 and $\|W_0\|_{H^2}$ are small enough, we remain in the domain $\{\|r\|_{L^\infty} \leq \frac{1}{2K}\}$ and all the previous estimates remain valid for all times. This concludes the proof of Theorem 1.4. $\square$

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