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A GENERAL COMPARISON PRINCIPLE FOR HAMILTON JACOBI BELLMAN EQUATIONS ON STRATIFIED DOMAINS

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Abstract. This manuscript aims to study finite horizon, first order Hamilton Jacobi Bellman equations on stratified domains. This problem is related to optimal control problems with discontinuous dynamics. We use nonsmooth analysis techniques to derive a strong comparison principle as in the classical theory and deduce that the value function is the unique viscosity solution. Furthermore, we prove some stability results of the Hamilton Jacobi Bellman equation. Finally, we establish a general convergence result for monotone numerical schemes in the stratified case.

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1. Introduction

In this paper, we study the well-posedness of a system of Hamilton Jacobi Bellman (HJB) equations defined on a stratification of $\mathbb{R}^N$. A stratification of $\mathbb{R}^N$ is a finite collection of disjoint open sets of $\mathbb{R}^N$ denoted $(\mathcal{M}_i)_{i=1,\ldots,n}$ such that

$$\mathbb{R}^N = \bigcup_{i=1}^{n} \overline{\mathcal{M}}_i, \quad \text{and} \quad \mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \text{ whenever } i \neq j.$$

The union of the open sets $\cup_{i=1}^{n} \mathcal{M}_i$ is called the regular part of the stratification. The singular part of the stratification is the union of all the interfaces between the open sets $(\mathcal{M})_{i=1,\ldots,n}$. It is the set

$$\Lambda := \mathbb{R}^N \setminus \bigcup_{i=1}^{n} \mathcal{M}_i.$$

We consider a system of finite horizon HJB equations

$$\begin{cases} -\partial_t u(t,x) + H_{F_i}(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i \\ u(T,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $T > 0$ is the final time, $\psi : \mathbb{R}^N \to \mathbb{R}$ is the final cost, assumed to be Lipschitz continuous, and $H_{F_i} : \mathcal{M}_i \times \mathbb{R}^N \to \mathbb{R}$ are Bellman Hamiltonians defined the following way:

$$H_{F_i}(x, p) = \sup_{q \in F_i(x)} \{-\langle p, q \rangle\},$$

with $F_i : \mathcal{M}_i \rightsquigarrow \mathbb{R}^N$ are set-valued maps, called the dynamics, that satisfy some standing hypotheses (see Sect. 2). In the case when there exists a Lipschitz continuous set-valued map F such that the restriction $F|_{\mathcal{M}_i} = F_i$, equation (1.1) admits a unique solution, in the viscosity sense, called the value function defined as follows (see [3, 18])

$$u(t, x) := \inf\{\psi(y(T)) \mid \dot{y}(s) \in F(y(s)) \text{ a. e. } s \in (t, T) \text{ and } y(t) = x\}. \tag{1.2}$$

In viscosity theory, the uniqueness of the solution comes from the so-called *comparison principle*. This principle asserts that if an upper semicontinuous sub-solution u is inferior to a lower semicontinuous super-solution v at time T, then $u \leq v$ at all time $t \in (0, T]$.

The viscosity notion has been extended by Ishii to the case of discontinuous Hamiltonians, see [27]. In the particular case of the stratified system (1.1), Ishii's extension provides a condition on the singular set Λ in the following form

$$\begin{cases} -\partial_t u(t, x) + \max_{i=1,\dots n}\{H_{F_i}(x, \partial_x u(t, x))\} \geq 0, & (t, x) \in (0, T) \times \Lambda \\ -\partial_t u(t, x) + \min_{i=1,\dots n}\{H_{F_i}(x, \partial_x u(t, x))\} \leq 0, & (t, x) \in (0, T) \times \Lambda. \end{cases} \tag{1.3}$$

However, using Ishii's extension to the singular set does not guarantee uniqueness of the viscosity solution in general.

Bressan and Hong [11] were the first to study a stratified system similar to (1.1). In particular, they introduced a regularized upper semicontinuous convex valued dynamics F on $\mathbb{R}^N$ such that the restriction of F on $\mathcal{M}_i$ coincides with F_i. They showed that when the value function of the control problem (1.2) is continuous, then it is the unique solution that satisfies some HJB inequalities on the interface. They also showed that a comparison result holds under the assumption that every subsolution is continuous.

Later, Barles *et al.* [5, 6] considered the case where the stratification is constituted by two disjoint open sets, the interface being a hypersurface. In this setting, the authors provided a very thorough analysis of the viscosity solution in Ishii sense. In particular, they proved that the value function of (1.2) with the regularized upper semicontinuous convex valued dynamics F is a viscosity solution of (1.1) in Ishii sense. They also described the minimal supersolution and maximal subsolution in Ishii sense. A sufficient condition for uniqueness is proved by Barles-Chasseigne in Corollary 10.32.2 of [8].

In [7, 8], Barles and Chasseigne considered a more general stratified setting than the one considered in the present paper, and introduced a different condition on the interface. Furthermore, different conditions under which the comparison principle is satisfied have been investigated in [8] (a comparison between our results and those derived in [7] will be given in Section 2.6).

We also mention some works on Hamilton Jacobi equations on stratified networks that share the same kind of difficulties as our layout: Imbert and Monneau [25], Imbert *et al.* [26], Achdou, Camilli, Cutrì and Tchou [1], Camilli and Marchi [12]. Besides, Lions and Souganidis [29] investigated Hamilton Jacobi equations on networks and considered a different class of Hamiltonians that are not necessarily convex, but only continuous and coercive.

HJB equations are also related to a geometric notion known as flow invariance in the theory of differential inclusions ([13], chap. 12). More precisely, nonsmooth analysis tools provide an interpretation of the sub-solution property of the value function as the strong invariance of the hypograph of the value function and the super-solution property as the weak invariance of its epigraph. The classical case without stratification (*i.e.*, when there exists $F : \mathbb{R}^N \to \mathbb{R}^N$ Lipschitz continuous such that $F_i = F|_{\mathcal{M}_i}$), has been treated thoroughly in the literature [3, 13]. In the case of stratified domains and discontinuous dynamics at the interfaces, Barnard and Wolenski [10] investigated the characterization of the weak and strong invariance principles with a new Hamiltonian that they called *the essential Hamiltonian*. However their statement of strong invariance was inaccurate. Despite their valid intuition regarding the choice of the Hamiltonian, the choice of the "test functions" (in analogy with the viscosity theory) did not take into account the singular geometry of the problem which turns out to be crucial for comparison type results. Let us also mention that a different control problem with a stratified set of state-constraints has been studied in [23, 24]. In these papers the dynamics is Lipschitz continuous everywhere. Since the problem is with state constraints, the value function might not be continuous. Then, the stratified structure of the set of constraints is exploited in [23, 24] to establish some HJB equations on the boundary and to analyze a characterization of the value function with a generalized notion of solution, called *bilateral viscosity solution*.

In this present work, we aim to prove the well-posedness of the HJB equation (1.1) on stratified domains. We will first define a Hamiltonian H_Λ at the interface Λ and consider the HJB equation:

$$\begin{cases} -\partial_t u(t,x) + H_{F_i}(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i, \\ -\partial_t u(t,x) + H_\Lambda(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \Lambda, \\ u(T,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

The Hamiltonian H_Λ is defined on the interface Λ by using the notion of *essential Hamiltonian* obtained along the same lines of [10]. The essential Hamiltonian comes from the optimal control interpretation of the system (1.1). It is defined from the set-valued map that represents the "essential velocities" of the system, meaning the velocities that are actually taken by the trajectories of the control problem (1.2) with the classical regularized (upper semicontinuous convex valued map F).

A similar definition for the Hamiltonian H_Λ has been considered in [20, 31, 32]. However, these papers analyzed a comparison result only under the assumption that the sub-solution is continuous at the interface Λ.

In the present work, we will revisit the definition of viscosity solutions and give a new one that encodes the nature of the singular geometry of the problem. This new definition will allow us to extend the strong comparison type results known when the Hamiltonian is Lipschitz continuous to the present setting. More precisely, we prove the following result: Let u and v be respectively upper semicontinuous and lower semicontinuous functions on $(0,T] \times \mathbb{R}^N$. If u is a sub-solution of (1.4), and if v is a super-solution of (1.4) and $u(T,.) \le v(T,.)$, then $u \le v$. The proof of this result relies on new results in nonsmooth analysis. In particular, we will establish new weak and strong invariance principles in the stratified setting. We would like to emphasize that the extension of the invariance principles is also a contribution of this paper.

The strong comparison principle that we prove in the present paper will have two major consequences. First, it will allow to obtain some stability results in the stratified setting and in the presence of perturbations on the dynamics. We prove that if there exist sequences $\left(F_i^j\right)_j$ of set-valued maps such that $F_i^j \longrightarrow F_i$ with respect to the Hausdorff distance, and a sequence $(v^j : \mathbb{R}^N \to \mathbb{R})_j$ of lower semicontinuous (respectively upper semicontinuous) functions such that $v^j \to v$ locally uniformly in $\mathbb{R}^N$ and suppose for all j, v^j is a super-solution (respectively sub-solution) of

$$\begin{cases} -\partial_t u(t,x) + H_{F_i^j}(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i, \\ -\partial_t u(t,x) + H_\Lambda^j(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \Lambda, \\ u(T,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^N. \end{cases}$$

then v is a super-solution (respectively sub-solution) of (1.4).

Finally, we will extend the classical result due to Barles and Souganidis [9] for the convergence of monotone numerical schemes to the stratified setting. In the context of one dimensional networks, convergence results of finite differences numerical schemes have been established by Guerand and Koumaiha [21] and by Morfe [30].

However, to the best of our knowledge, the case of stratified systems has not been yet studied in the literature. Here, we consider a numerical scheme in the following form

$$\begin{cases} S_i^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h,x_h)}) = 0 \quad \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \bigcap ((0,T) \times \mathcal{M}_i), \\ S_\Lambda^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h,x_h)}) = 0 \quad \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \bigcap ((0,T) \times \Lambda), \\ u^h(T, x_h) = \psi(x_h), \quad \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \cap \{t_h = T\}, \end{cases}$$

where $\Pi^{\Delta t}$ is a time grid, $\mathcal{G}^{\Delta x}$ is a spatial grid, $h = (\Delta t, \Delta x)$ is the step of the grid and $[u^h]_{(t_h,x_h)}$ are all the values of of u^h on $\mathcal{G}^{\Delta x}$ at other points than (t_h, x_h) on the grid. We show that under the usual hypotheses of monotonicity, stability and consistency, the numerical scheme converges locally uniformly to the viscosity solution of (1.4).

The paper is organized as follows: in Section 2, we define the notations and conventions used throughout the paper. We also define the geometry of the problem, the dynamics of the HJB equation and we state the main results. Furthermore, in the same section, we provide a comparison between our results and the results of [7] through several examples. Section 3 is devoted to the invariance principles, a nonsmooth analysis point of view of the HJB equation. In Section 4, we first define the optimal control problem associated to the HJB equation, we introduce the value function and we prove that the super-optimality and sub-optimality properties of the value function are equivalent to it being a viscosity super-solution and sub-solution respectively. Then we prove the strong comparison result. Section 5 is devoted to the proofs of the stability results. Finally, we prove in Section 6 a general convergence result for monotone numerical schemes.

2. Main results

2.1. Notations

Throughout the manuscript, we denote by $\mathbb{R}^N$ the Euclidean space where the stratification is defined, $\mathbb{B}$ the unit ball of center 0 of $\mathbb{R}^N$ and $\mathbb{B}(x,r) = x + r\mathbb{B}$. For any set $S \subset \mathbb{R}^N$, we denote $\overline{S}$, ∂S its closure and topological boundary. We denote by $\mathrm{co}(S)$ the convex hull of S and by $\mathscr{L}$, the Lebesgue measure on $\mathbb{R}$.

The distance function associated to S is $d_S(x) = \inf\{|x-y| \ : \ y \in S\}$ and the set of solutions where the infimum is attained is called the projection of x on S and denoted by $proj_S(x)$ (note that it might be empty).

The Bouligand tangent cone of S at x, denoted $\mathcal{T}_S(x)$ is defined the following way:

$$\mathcal{T}_S(x) = \left\{ v \in \mathbb{R}^N \ : \ \liminf_{t \to 0^+} \frac{d_S(x+tv)}{t} = 0 \right\}.$$

If A and B are two sets of $\mathbb{R}^N$, we define a distance between them by $d(A,B) = \inf\{|a-b| : \ (a,b) \in A \times B\}$, with the convention $d(\emptyset,\emptyset) = 0$ and $d(\emptyset, B) = +\infty$ if $B \neq \emptyset$. For K_1 and K_2 two compact sets of $\mathbb{R}^N$, the Hausdorff distance is given by

$$d_{\mathcal{H}}(K_1, K_2) = max\left\{ \sup_{x \in K_2} d_{K_1}(x) \ , \ \sup_{x \in K_1} d_{K_2}(x) \right\},$$

with the convention $d_{\mathcal{H}}(\emptyset,\emptyset) = 0$ and $d_{\mathcal{H}}(\emptyset, S) = +\infty$ if $S \neq \emptyset$.

For a given function $f : \mathbb{R}^N \to \mathbb{R}$, $epi(f)$ and $hyp(f)$ denote respectively its epigraph and hypograph:

$$epi(f) = \left\{(x,r) \in \mathbb{R}^N \times \mathbb{R} \ : \ f(x) \leq r\right\}, \quad hyp(f) = \left\{(x,r) \in \mathbb{R}^N \times \mathbb{R} \ : \ f(x) \geq r\right\}.$$

If Γ is a set-valued map, then $dom(\Gamma)$ is the set of points x such that $\Gamma(x) \neq \emptyset$.

Let $\mathcal{M}$ be a C^2 embedded submanifold in $\mathbb{R}^N$ and let $\Gamma : \mathcal{M} \rightsquigarrow \mathbb{R}^N$ be a set-valued map. For $T > 0$, we define the differential inclusion associated to Γ, with the initial condition $(t, x) \in (0, T) \times \mathbb{R}^N$, by

$$(DI)_\Gamma(t,x) = \begin{cases} \dot{y}(s) \in \Gamma(y(s)) & a.e.\ s \in [t,T] \\ y(t) = x. \end{cases}$$

Finally the abbreviations 'u.s.c.', 'l.s.c', 'HJB' and 'w.r.t' respectively stand for: 'upper semicontinuous', 'lower semicontinuous', 'Hamilton Jacobi Bellman' and 'with respect to'.

2.2. Stratification

Let $N, n \geq 1$ be two integers. Let $\mathcal{M}_i$, $i = 1, \dots, n$ be pairwise disjoint, connected open sets of $\mathbb{R}^N$. We suppose that $\mathbb{R}^N = \cup_{i=1}^n \overline{\mathcal{M}}_i$ and we denote by $\Lambda := \mathbb{R}^N \setminus \cup_{i=1}^n \mathcal{M}_i$ the interfaces or the singular set. Furthermore, we suppose that Λ is equal to a union of l, pairwise disjoint, C^2 embedded submanifolds $\mathcal{M}_{n+1}, \dots, \mathcal{M}_{n+l}$ of lower dimension than N, so that we have

$$\mathbb{R}^N = \bigcup_{i=1}^n \overline{\mathcal{M}}_i = \left(\bigcup_{i=1}^n \mathcal{M}_i \right) \bigcup \Lambda = \bigcup_{i=1}^{n+l} \mathcal{M}_i.$$

Finally, we suppose that each $\overline{\mathcal{M}}_i$, $i = 1, \dots, n+l$, is proximally smooth and relatively wedged. All these assumptions on the stratification are summarized as following:

$$(H_1) \begin{cases} (i) & \text{Each } \mathcal{M}_i \text{ is a } C^2 \text{ embedded submanifold,} \\ (ii) & dim(\mathcal{M}_1) = ... = dim(\mathcal{M}_n) = N, \quad \text{and} \quad dim(\mathcal{M}_{n+1}), ..., dim(\mathcal{M}_{n+l}) < N, \\ (iii) & \mathbb{R}^N = \cup_{i=1}^n \overline{\mathcal{M}}_i = \bigcup_{i=1}^{n+l} \mathcal{M}_i, \\ (iv) & \forall i,j = 1, \dots, n+l, \quad \mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \text{ if } i \neq j, \\ (v) & \text{if } \mathcal{M}_i \cap \overline{\mathcal{M}}_j \neq \emptyset, \text{ then } \mathcal{M}_i \subset \overline{\mathcal{M}}_j \text{ or } \mathcal{M}_j \subset \overline{\mathcal{M}}_i, \\ (vi) & \text{each } \overline{\mathcal{M}}_i \text{ is proximally smooth and relatively wedged.} \end{cases}$$

We call $\bigcup_{i=1}^n \mathcal{M}_i$ the regular part of the stratification and $\Lambda := \bigcup_{i=1}^l \mathcal{M}_{n+i}$ the singular part or the interfaces.

Comments on the Hypothesis (H_1)

Hypotheses $(H_1)(i)$ to $(H_1)(v)$ are standard for a stratification of $\mathbb{R}^N$. As for $(H_1)(vi)$, a closed set $X \subset \mathbb{R}^N$ is said to be proximally smooth if there exists $r > 0$ such that the projection map $proj_X(.)$ is a singleton on the tube $\{x \in X, d_X(x) < r\}$ [16]. The class of proximally smooth sets includes convex subsets of $\mathbb{R}^N$ and C^2 compact submanifolds of $\mathbb{R}^N$. Relative wedgeness hypothesis was introduced in [10] for C^2 submanifolds of $\mathbb{R}^N$ such that their closure is proximally smooth. Roughly speaking, relative wedgeness of $\overline{\mathcal{M}}_i$, with $i \in \{1, \dots, n+l\}$, means that the dimension of the Bouligand tangent cone at every point of $\overline{\mathcal{M}}_i$ is equal to the dimension of the manifold $\mathcal{M}_i$ [10]. The precise definition of this property is presented in Appendix A.

Example 2.1. Figure 1 shows an example of the stratified setting, where $N = 1$, $n = 2$, $l = 1$.

$$\mathcal{M}_1 = (0, +\infty)e_1, \quad \mathcal{M}_2 = (0, +\infty)e_2, \quad \mathcal{M}_3 = \{0\}.$$

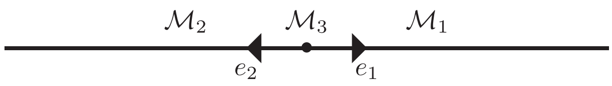

FIGURE 1. Example of a stratification of $\mathbb{R}$.

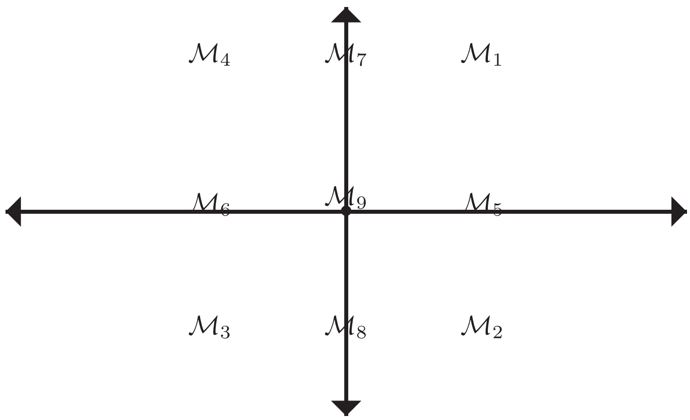

FIGURE 2. Example of a stratification of $\mathbb{R}^2$.

Example 2.2. Figure 2 shows an example of a stratified setting in $\mathbb{R}^2$ with $n = 4$ where the open sets are given by:

$$\mathcal{M}_1 = \mathbb{R}^*_+ \times \mathbb{R}^*_+, \quad \mathcal{M}_2 = \mathbb{R}^*_+ \times \mathbb{R}^*_-, \quad \mathcal{M}_3 = \mathbb{R}^*_- \times \mathbb{R}^*_-, \quad \mathcal{M}_4 = \mathbb{R}^*_- \times \mathbb{R}^*_+.$$

The interface in Figure 2 is constituted by the following submonifolds of dimension 1 or 0:

$$\mathcal{M}_5 = \mathbb{R}^*_+ \times \{0\}, \quad \mathcal{M}_6 = \mathbb{R}^*_- \times \{0\}, \quad \mathcal{M}_7 = \{0\} \times \mathbb{R}^*_+, \quad \mathcal{M}_8 = \{0\} \times \mathbb{R}^*_-, \quad \mathcal{M}_9 = \{0\}.$$

We set for any $x \in \mathbb{R}^N$, the index set-valued map

$$I(x) := \{\, i \in \{1, \dots, n + l\} \ : \ x \in \overline{\mathcal{M}_i}\, \}.$$

Remark 2.3. It is clear from the definition of the stratification that for $x \in \mathbb{R}^N$ fixed, and $y \in \mathbb{R}^N$ close enough to x, we have $I(y) \subseteq I(x)$.

2.3. Setting of the problem

We begin by defining the dynamics for the Hamiltonians presented in the introduction. On each $\mathcal{M}_i$ with $i = 1, \dots, n$, we are given a set-valued map $F_i : \mathcal{M}_i \rightsquigarrow \mathbb{R}^N$ that satisfies the standard hypotheses

$$(SH) \begin{cases} (i) & x \rightsquigarrow F_i(x) \text{ has non empty convex and compact images,} \\ (ii) & \exists\, \lambda > 0 \text{ such that } \max\{\, |p|,\ p \in F_i(x)\} \leq \lambda(1+|x|), \\ (iii) & F_i \text{ is Lipschitz continuous on bounded sets of } \mathcal{M}_i \text{ w.r.t the Hausdorff metric,} \\ & \textit{i.e.} \text{ for each } R > 0, \text{ there are constants } K_{1,R}, \dots, K_{n,R} > 0 \text{ such that} \\ & d_{\mathcal{H}}(F_i(x), F_i(y)) \leq K_{i,R}|x - y| \quad \text{if } x, y \in \mathbb{B}(0, R) \cap \mathcal{M}_i,\ i \in \{1, \dots, n\}. \end{cases}$$

We are interested in studying the well-posedness of the following HJB equation.

$$\begin{cases} -\partial_t u(t,x) + \sup_{\nu \in F_i(x)} \{ -\langle \nu, \partial_x u(t,x) \rangle \} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i, \ i = 1, \dots, n, \\ u(T,x) = \psi(x), \end{cases} \tag{2.1}$$

where $T > 0$ is the final time and $\psi : \mathbb{R}^N \to \mathbb{R}$ is the final cost required to satisfy the following assumption.

$$(H\psi) : \quad \psi \text{ is locally Lipschitz continuous.}$$

The study of HJB equations is done using a weak notion of solutions, called viscosity solutions. This setting requires the HJB equation to be defined at every point. Hence, we need to find suitable interfaces conditions in order to guarantee the well-posedness of the system. To do so, we aim to define the appropriate dynamics to consider at the interfaces.

Notice first that since the dynamics F_i, $i = 1, \dots, n$ verify hypothesis (SH), then they can be extended to $\overline{\mathcal{M}}_i$ while verifying the same hypothesis (SH). *We denote this extension by F_i as well.* In order to define the dynamics on the whole space, a classical idea is to consider the upper semicontinuous and convex valued regularization of $(F_i)_{i=1,\dots,n}$, denoted $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ and defined by

$$F(x) := \bigcap_{\varepsilon > 0} \overline{co} \bigcup_{y} \{ \cup_{i \in \{1 \dots n\}} F_i(y) \ : |x - y| \leq \varepsilon \}.$$

It is straightforward to check that F has a linear growth. However, it might not be Lipschitz in general. By the nature of our problem, the regularization is equal to

$$F(x) = co \{ F_i(x) \ : \ i \in \{1 \dots n\} \}.$$

For $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, we define the Hamiltonian associated to F by

$$H_F(x,p) = \sup_{q \in F(x)} \{ -\langle p, q \rangle \}.$$

Since F is only upper semicontinuous, the Hamiltonian $H_F(.,p)$ is also only upper semicontinuous. If $H_F(.,p)$ were to be Lipschitz continuous, we would have defined our HJB equation using the Hamiltonian associated to F and the well-posedness of the HJB system would have followed from the classical theory, see [13, 19]. This is generally not the case in a stratified domain.

Nevertheless, the next step is to use F to define suitable dynamics at the interfaces. We define the dynamics $F_{n+i} : \mathcal{M}_{n+i} \rightsquigarrow \mathbb{R}^N$, for $i = 1, \dots, l$ on each interface $\mathcal{M}_{n+i}$ by

$$F_{n+i}(x) = F(x) \cap \mathcal{T}_{\mathcal{M}_{n+i}}(x),$$

where $\mathcal{T}_{\mathcal{M}_{n+i}}(x)$ is the Bouligand tangent cone which coincides with the classical tangent space of $\mathcal{M}_{n+i}$ at x since it is a C^2 manifold. Furthermore, we suppose that all the interface dynamics are Lipschitz continuous on bounded sets as well:

$$(H_D) \quad \text{for } i = 1, \dots, l, \ F_{n+i}(.) \text{ is Lipschitz continuous on bounded sets of } \mathcal{M}_{n+i}.$$

We point out that since we have the conventions $d_{\mathcal{H}}(\emptyset, S) = +\infty$ if $S \neq \emptyset$ and $d_{\mathcal{H}}(\emptyset, \emptyset) = 0$, it follows that (H_D) implies that F_{n+i} is either identically the empty set or nonempty on the whole domain $\mathcal{M}_{n+i}$. Under assumption (H_D), each $F_{n+i} : \mathcal{M}_{n+i} \rightsquigarrow \mathbb{R}^N$, $(i = 1, \dots, l)$ satisfies (SH). Thus each F_{n+i} can be extended to $\overline{\mathcal{M}}_{n+i}$ while verifying the same hypothesis (SH). *We denote this extension by F_{n+i} as well.*

A sufficient condition for (H_D) to be satisfied is full controllability near Λ. We mean by full controllability the following assumption:

$$(CH)\ \exists\, r > 0 \ : \quad \text{for all } \ i = 1, \ldots, n, \ \text{and } x \in \Lambda \cap \overline{\mathcal{M}}_i \ : \ \mathbb{B}(0, r) \subseteq F_i(x).$$

Proposition 2.4. *[32], Lemma 2.2. Assume (H_1) and (CH). Then, (H_D) holds.*

For $x \in \overline{\mathcal{M}}_i$, $i = 1, \ldots, n + l$, and $p \in \mathbb{R}^N$, we define the Hamiltonian

$$H_{F_i}(x, p) := \sup_{q \in F_i(x)} \{-\langle p, q\rangle\}.$$

At this point, we are tempted to define the HJB equation on the stratified domain using the dynamics $F_i(.)$ (defined above), the following way:

$$\begin{cases} -\partial_t u(t, x) + H_{F_i}(x, \partial_x u(t, x)) = 0 \quad \text{for } (t, x) \in (0, T) \times \mathcal{M}_i, \ i = 1, \ldots, n, \\ -\partial_t u(t, x) + \max_{i \in I(x)} \{H_{F_i}(x, \partial_x u(t, x))\} = 0 \quad \text{for } (t, x) \in (0, T) \times \Lambda, \\ u(T, x) = \psi(x). \end{cases} \tag{2.2}$$

However, it turns out that the set of dynamics in equation (2.2) is too large. These dynamics may contain velocities that are not *useful* for the evolution of the solution at the interface. This claim is analyzed in the next subsection.

2.4. The essential dynamics

We define the notion of *essential dynamics* on each domain, introduced in [10] for stratified Euclidean spaces. For $i = 1, ...n + l$, we define the essential dynamics on each $\overline{\mathcal{M}}_i$ by

$$F_i^\sharp(x) := F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x), \quad \text{for all } x \in \overline{\mathcal{M}}_i,$$

where $\mathcal{T}_{\overline{\mathcal{M}}_i}(x)$ is the *Bouligand tangent cone* of $\overline{\mathcal{M}}_i$ at x. Notice that if $x \in \mathcal{M}_i$, we have $F_i^\sharp(x) = F_i(x)$. The associated Hamiltonian is defined as

$$H_{F_i^\sharp}(x, p) = \sup_{q \in F_i^\sharp(x)} \{-\langle p, q\rangle\}.$$

The essential dynamics $F_i^\sharp$ on each domain represent the *inward pointing* velocities of F_i on $\overline{\mathcal{M}}_i$. We suppose that each $F_i^\sharp$ is l.s.c.

$$(H_{ESS}) \quad \text{for all } i = 1, \ldots, n + l, \quad F_i^\sharp \text{ is } \textit{l.s.c.}$$

Hypothesis (H_{ESS}) holds for many cases. In particular, if we assume the controllability assumption (CH) to hold for the dynamics, then (H_{ESS}) holds for all stratifications presented in Examples 2.1 and 2.2. A discussion about sufficient conditions to ensure (H_{ESS}) is given in Appendix B.

The essential dynamics on $\mathbb{R}^N$ is defined as the union of the essential dynamics on each domain.

$$\forall x \in \mathbb{R}^N, \quad F^\sharp(x) = \bigcup_{i=1}^{n+l} \{\, F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) \ : \ x \in \overline{\mathcal{M}}_i \,\}.$$

Its associated Hamiltonian is also defined as usual. For $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, we have

$$H_{F^\sharp}(x,p) = \sup_{q \in F^\sharp(x)} \{-\langle p, q\rangle\}.$$

Example 2.5. We consider the stratification of $\mathbb{R}$ defined in Example 2.1. Let $c_i \geq 0$ with $i = 1,2$ be real positive constants. We define the following dynamics on each branch

$$F_i(x) = [-c_i, c_i] \quad , i = 1, 2.$$

The resulting HJB system is the *Eikonal equation* on the stratification 2.1. The dynamics at the interface $\mathcal{M}_3$ and the essential dynamics are respectively equal to

$$F_3(.) \equiv \{0\} \quad , \quad F^\sharp(x) = \begin{cases} [-c_i, c_i] & x \in \mathcal{M}_i \;\; i = 1, 2, \\ [-c_2, c_1] & x = 0. \end{cases}$$

Note that this simple example, in dimension 1, can be seen as a network problem. Here the essential Hamiltonian defined at the junction is the same one that is used in [26]. Let $T > 0$ be a given time horizon, and consider the following HJB associated to the dynamics $F_i^\sharp$

$$\begin{cases} -\partial_t u(t,x) + \max_{i \in I(x)} \{H_{F_i^\sharp}(x, \partial_x u(t,x))\} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(T,x) = \psi(x), \end{cases} \tag{2.3}$$

where $\psi : \mathbb{R}^N \to \mathbb{R}$ is the final cost and satisfies $(H\psi)$.
Notice that in the HJB equation (2.3), if x belongs to the regular part of the stratification ($i.e.$ $x \in \bigcup_{i=1}^n \mathcal{M}_i$), then the HJB equation (2.3) is the same as the HJB equation (2.1). So equation (2.3) has the following form

$$\begin{cases} -\partial_t u(t,x) + H_{F_i}(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i, \;\; i = 1, \cdots, n, \\ -\partial_t u(t,x) + \max_{i \in I(x)} \{H_{F_i^\sharp}(x, \partial_x u(t,x))\} = 0 \quad \text{for } (t,x) \in (0,T) \times \Lambda, \;\; i = n+1, \cdots, n+l, \\ u(T,x) = \psi(x). \end{cases} \tag{2.4}$$

Given the singular nature of the stratification, one cannot use the classical notion of viscosity solutions. We are going to define a new one that will turn out to be appropriate for obtaining a strong comparison result.

Definition 2.6. (Viscosity super-solution). Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a l.s.c function. We say that u is a **super-solution** of (2.3) at $(t,x) \in (0,T) \times \mathbb{R}^N$ if and only if there exists $i \in I(x)$ such that for all $\phi \in C^1((0,T) \times \mathbb{R}^N)$, $u - \phi$ attains a local minimum in $(0,T) \times \overline{\mathcal{M}}_i$ at (t,x), we have

$$-\partial_t \phi + H_{F_i^\sharp}(x, \partial_x \phi) \geq 0.$$

Definition 2.7. (Viscosity sub-solution). Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a u.s.c function. We say that u is a **sub-solution** of (2.3) at $(t,x) \in (0,T) \times \mathbb{R}^N$ if and only if for all $i \in I(x)$, for all $\phi \in C^1((0,T) \times \mathbb{R}^N)$, $u - \phi$ attains a local maximum in $(0,T) \times \overline{\mathcal{M}}_i$ at (t,x), we have

$$-\partial_t \phi + H_{F_i^\sharp}(x, \partial_x \phi) \leq 0.$$

Definition 2.8. (Viscosity solution) u is a viscosity solution of (2.3) if and only if it is both a super-solution and a sub-solution and satisfies the final condition $u(T,.) = \psi(.)$.

The above definitions of viscosity super- and sub-solutions can be rewritten using the viscosity sub-gradient and super-gradient (also known as the semijets ([17], P. 10) or Dini sub/super gradient [13], Def. 11.18).

Definition 2.9. (Viscosity sub/super gradient)

- Let $u : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c function. The viscosity sub-gradient (or subjet) at a point $x \in dom(u)$ is defined the following way,

$$D^-u(x) := \left\{ p \in \mathbb{R}^N \ : \ \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}.$$

- Similarly, for an u.s.c function $u : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$, the viscosity super-gradient (or superjet) at a point $x \in dom(u)$ is defined the following way,

$$D^+u(x) := -D^-(-u)(x) = \left\{ p \in \mathbb{R}^N \ : \ \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

Remark 2.10.

- Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a l.s.c function. We say that u is a **super-solution** of (2.3) at $(t,x) \in (0,T) \times \mathbb{R}^N$ if and only if there exists $i \in I(x)$, such that

$$-\theta + H_{F_i^\sharp}(x, \xi_i) \geq 0 \quad \forall (\theta, \xi_i) \in D^-u_i(t,x),$$

  with $u_i \equiv u$ on $(0,T] \times \overline{\mathcal{M}}_i$ and $u_i \equiv +\infty$ elsewhere.
- Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be an u.s.c function. u is a **sub-solution** of (2.3) at $(t,x) \in (0,T) \times \mathbb{R}^N$ if and only if for all $i \in I(x)$, we have

$$-\theta + H_{F_i^\sharp}(x, \xi) \leq 0 \qquad \forall (\theta, \xi) \in D^+u_i(t,x),$$

  with $u_i \equiv u$ on $(0,T] \times \overline{\mathcal{M}}_i$ and $u_i \equiv -\infty$ elsewhere.

Indeed, if for example $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a l.s.c function, we set: $u_i \equiv u$ on $(0,T] \times \overline{\mathcal{M}}_i$ and $u_i \equiv +\infty$ elsewhere, for some $i = 1, \ldots, n+l$. Then, for $(t,x) \in (0,T) \times \overline{\mathcal{M}}_i$, we have

$$(\theta, \xi_i) \in D^-u_i(t,x) \iff \exists\, \phi^{(i)} \in C^1((0,T) \times \mathbb{R}^N), \text{ such that } u_i - \phi^{(i)} \text{ attains a local minimum at } (t,x).$$

Since $u_i - \phi^{(i)} \equiv +\infty$ whenever $x \notin \overline{\mathcal{M}}_i$, we get that $\phi^{(i)}$ satisfies the requirements of Definition 2.6. Conversely, if there exists such a function ϕ in the sense of Definition 2.6, then $u_i - \phi$ attains a local maximum in $\mathbb{R}^N$ at (t,x). The exact same reasoning holds for sub-solutions.

Next, we state the main results. In particular, we will show that equation (2.3) has a unique viscosity solution (following Def. 2.8).

2.5. Statement of the main results

Theorem 2.11. *Assume* (H_1), (SH), $(H\psi)$, (H_{ESS}) *and* (CH). *Then the HJB equation* (2.3) *has a unique continuous solution in the sense of Definition 2.8.*

Theorem 2.12. *(Strong comparison principle). Assume* (H_1), (SH), (H_D) *and* (H_{ESS}). *Let* $u_1, u_2 : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ *be respectively a l.s.c super-solution and an u.s.c sub-solution in the sense of Definition 2.8 with* $u_2(T,.) \le u_1(T,.)$. *Then*

$$u_2(t,x) \le u_1(t,x) \quad \forall (t,x) \in (0,T] \times \mathbb{R}^N.$$

It is worth-noticing that, unlike the previous literature on the subject [20, 31, 32] or [5–7], the strong comparison principle stated in the above theorem does not require the sub-solution to be continuous nor to have any particular behavior on the interface. The proof of this result will clearly show the importance of the use of essential dynamics with the notion of viscosity as it is defined in Definitions 2.6–2.7 (and more precisely the choice of the test functions in those definitions). Furthermore, the unique viscosity solution in Theorem 2.11 is the *value function* associated to the Mayer optimal control problem with the dynamics $F(.)$. A study of the value function and the associated optimal control problem is presented in Section 4. Hypothesis (CH) in Theorem 2.11 is only used to give sufficient conditions for the value function to be continuous. Therefore, Theorem 2.11 holds if one assumes that the value function is continuous instead of assuming (CH).

The proofs of Theorems 2.11, 2.12 are given in Section 4. The proofs will rely on invariance theorems stated in Section 3 and proven in Appendix C. Furthermore, we will establish stability results of the super-solution and sub-solution in presence of perturbations of the Hamiltonian in Section 5. Section 6 is devoted to stating and proving a general convergence result of monotone numerical schemes. The numerical scheme has the following form in each $\overline{\mathcal{M}}_i$, $i = 1, \ldots, n+l$,

$$S_i^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h,x_h)}\} = 0 \quad \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x}),$$

where $\Pi^{\Delta t}$ is time grid, $\mathcal{G}_i^{\Delta x}$ is a spatial grid of $\overline{\mathcal{M}}_i$ and $h = (\Delta t, \Delta x)$ is the step of the grid. We show that under the usual hypotheses of monotonicity, stability and consistency, the numerical scheme converges. This result generalizes the classical convergence theorem of monotone numerical schemes in the classical case, due to Barles and Souganidis [9].

2.6. Comparison with existing literature

Recently, control problems and Hamilton Jacobi equations on stratified structures have been investigated in several works. A similar setting to the one considered in this article can be found in [5, 6, 20, 31, 32]. The techniques used in [20, 31, 32] are also all based on invariance principles and on the use of the essential Hamiltonian to describe the behavior of the value function. Here, we investigate further the essential dynamics and its corresponding Hamiltonian. In particular, we show that the invariance principles (weak and strong) can be fully characterized by using the essential Hamiltonian (for both principles). This result is new, it generalizes to the stratified case the invariance principles known in the literature for a Lipschitz dynamics. As consequence of the invariance principles, particularly the strong invariance principle, we obtain a strong comparison principle for equation (2.3) by assuming further that the essential dynamics $F_i^\sharp(.)$ are l.s.c in their domains. The comparison principle states that for any u_1 u.s.c sub-solution of (2.3) and for any u_2 l.s.c super-solution of (2.3), we have $u_1 \le u_2$ in $\mathbb{R}^N \times (0,T]$. Unlike the results established in [5, 6, 20, 31, 32], the comparison principle does not require any additional controllability assumption nor the continuity of the sub-solution around the interface.

The setting of control problems considered in [7, 11] is very close to ours. However, in those papers, the HJB equation considered on the singular set Λ is different from the one we use in (2.3). Indeed, in [7, 11], the

Hamiltonian on each stratum is built by using only *local* information with the dynamics defined on the stratum without taking into account the behavior of the dynamics at the relative boundary of each stratum. Therefore, the Hamiltonian on the interface does not take into account the information coming from neighboring strata. As consequence, the comparison principle in [11] requires an additional controllability condition and the continuity of sub-solutions. The work of [7] is more general, it requires a weaker controllability assumption and gives a comparison between u.s.c solutions and l.s.c solutions if the sub-solution satisfies Ishii's condition or a weak continuity requirement.

To compare our results with those presented in [7], we will analyze different examples.

Example 2.13. Notice that the setting in [7] concerns optimal control problems, with final and running costs, on a general stratifications of $\mathbb{R}^N$. For a simple comparison, we will restrict ourselves to a framework without a distributed cost. We also suppose that $N = 2$ and the stratification is composed simply of two half-spaces of $\mathbb{R}^2$ separated by a line ($n = 2$ and $l = 1$)

$$\mathbb{R}^2 = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3,$$

with

$$\mathcal{M}_1 := \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0\}, \quad \mathcal{M}_2 := \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0\}, \quad \mathcal{M}_3 := \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 = 0\}.$$

This stratification satisfies (H_1) and it is a *regular* stratification in the sense of [7] as well. Let $F(.)$ be an u.s.c dynamics defined on $\mathbb{R}^N$, with

$$F_i(x) = F(x) \cap \mathcal{T}_{\mathcal{M}_i}(x) \quad \text{for } x \in \mathcal{M}_i, \text{ and for } i = 1, 2, 3. \tag{2.5}$$

We define the *value function* the following way

$$\vartheta(x, t) := \inf\{\psi(y(T)),\ \dot{y}(s) \in F(y(t)) \text{ for a.e. } s \in [t, T],\ y(t) = x\}.$$

We will discuss the properties of the value function in Section 4. In [7], the cost function ψ and the dynamics $F_i(.)$ satisfy

$$(H_{D,[7]}) \begin{cases} (i) \text{ The dynamics } F \text{ is uniformly bounded on } \mathbb{R}^N. \\ (ii) \text{ For } i = 1, 2, 3, \quad F_i \text{ is Lipschitz continuous on } \mathcal{M}_i. \\ (iii) \text{ The cost function } \psi \text{ is bounded and uniformly continuous on } \mathbb{R}^N. \end{cases}$$

Besides, a normal controllability assumption is introduced in [7]. In the simple setting of this example, this normal controllability is the following

$(H_{N,[7]})$ For $i \in \{1, 2\}$ and every $x \in \mathcal{M}_i$, and $r > 0$, there exists $C > 0$ and $\delta > 0$ such that

$$H_{F_i}(y, p) \geq \delta|p_2| - C(1 + |p_1|) \quad \forall y \in B(x, r) \cap \mathcal{M}_3,\ p = (p_1, p_2) \in \mathcal{T}_{\mathcal{M}_3}(x) \times \mathcal{T}^{\perp}_{\mathcal{M}_3}(x).$$

The set $\mathcal{T}^{\perp}_{\mathcal{M}_3}(x)$ is the orthogonal complement of the tangent space $\mathcal{T}_{\mathcal{M}_3}(x)$ in $\mathbb{R}^N$. According to [7], under assumptions $(H_{D,[7]})$ and $(H_{N,[7]})$, the value function ϑ is bounded and continuous. Moreover, ϑ is a super-solution of the equation

$$-\partial_t v(x, t) + H_F(x, \partial_x v(x, t)) \geq 0, \tag{2.6a}$$

and ϑ is a sub-solution to the system of equations

$$-\partial_t v(x,t) + H_{F_i}(x, \partial_x v(x,t)) \leq 0 \quad \forall x \in \mathcal{M}_i, \text{ and for } i = 1,2,3. \tag{2.6b}$$

Furthermore, let v_2 be l.s.c super-solution of (2.6a), and let v_1 be a u.s.c sub-solution of (2.6b) satisfying one of the following conditions:

- (i) v_1 is continuous on $\mathcal{M}_3$,
- (ii) v_1 satisfies Ishii's condition, *i.e.*, u is sub-solution to

$$-\partial_t v(x,t) + H_*(x, \partial_x v(x,t)) \leq 0, \quad \text{for } x \in \mathcal{M}_3,$$

where H_* is the l.s.c envelope of H:

$$H_*(x,p) := \liminf_{(y,q)\to(x,p)} H_F(y,q).$$

Then, under assumptions $(H_{D,[7]})$ and $(H_{N,[7]})$, by Theorem 4.1 of [7] we have $v_1 \leq v_2$ on $\mathbb{R}^N \times [0,T]$. The result is even more precise and provides a *local* strong comparison result.

Now, let us see how our work differs from [7]. Our assumptions (SH), (H_D) and (H_{ESS}) require the dynamics $F_i(.)$ to be Lipschitz continuous on bounded sets of $\mathcal{M}_i$ with a linear growth. No boundedness is required. Furthermore, the essential dynamics $F_i^\sharp(.)$ are l.s.c. on $\overline{\mathcal{M}}_i$ under assumption (H_{ESS}), the result of Theorem 2.12 provides a comparison between sub-solutions and super-solutions of the following HJB system

$$\begin{cases} -\partial_t u(t,x) + H_{F_i}(x, \partial_x u(t,x)) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i, \ i = 1,2, \\ -\partial_t u(t,x) + \max\limits_{i=1,2,3} \{ H_{F_i^\sharp}(x, \partial_x u(t,x)) \} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_3, \\ u(T,x) = \psi(x). \end{cases} \tag{2.7}$$

This result does not require a controllability assumption and it states that for every u.s.c sub-solution u_1 of (2.7) and for every l.s.c super-solution of (2.7), we have $u_1 \leq u_2$ on $(0,T] \times \mathbb{R}^N$. Note that in the system (2.7), we use the same Hamiltonian $\max_i H_{F_i^\sharp}$ on $\mathcal{M}_3$ to describe the sub and super optimality. This Hamiltonian includes all the dynamics of trajectories starting from a position in $\mathcal{M}_3$. Therefore, the Hamiltonian $\max_i H_{F_i^\sharp}$ *contains more information* than Hamiltonian H_{F_3}. Theorem 2.12 gives a comparison principle for (2.7) without requiring additional information on sub-solutions.

The controllability assumption (CH), which is stronger than $(H_{N,[7]})$ is used only to prove that the value function is continuous (see Sect. 4). The assumption (CH) is not necessary and the continuity of the value function can be obtained in some cases without assumption $(H_{N,[7]})$ or (CH).

Example 2.14. Consider the same stratification as in Example 2.13. The dynamics $F_1(.)$ and $F_2(.)$ defined on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, with

$$F_1(x) := \mathbb{B}(0,1) \text{ on } \mathcal{M}_1, \quad F_2(x) := \mathbb{B}(0,2) \ \text{ on } \mathcal{M}_2.$$

We define $F(.)$ as the regularization of the dynamics $F_1(.)$ and $F_2(.)$. It is given by

$$F(x) = \begin{cases} F_1(x) = \mathbb{B}(0,1) & \text{if } x \in \mathcal{M}_1, \\ F_2(x) = \mathbb{B}(0,2) & \text{if } x \in \mathcal{M}_2, \\ \mathbb{B}(0,2) & \text{if } x \in \mathcal{M}_3. \end{cases}$$

The essential dynamics are given by

$$F_1^\sharp(.) = F_1(.) \text{ on } \mathcal{M}_1 \quad \text{and} \quad F_1^\sharp(.) = [-1,0]\times[-1,1] \text{ on } \mathcal{M}_3 = \overline{\mathcal{M}}_1 \setminus \mathcal{M}_1,$$

$$F_2^\sharp(.) = F_2(.) \text{ on } \mathcal{M}_2 \quad \text{and} \quad F_2^\sharp = [0,2]\times[-2,2] \text{ on } \mathcal{M}_3 = \overline{\mathcal{M}}_2 \setminus \mathcal{M}_2,$$

$$F_3^\sharp(.) = \{0\}\times[-2,2] \text{ on } \overline{\mathcal{M}}_3 = \mathcal{M}_3.$$

The dynamics of this example satisfy assumptions $(SH), (H_D)$, (H_{ESS}) and (CH). Furthermore, it satisfies hypotheses $(H_{D,[7]})$ and $(H_{N,[7]})$ from [7]. Hence, our results give a comparison principle for any u.s.c sub-solution u_1 and any l.s.c.super-solution u_2, in the sense of Theorem 2.12, of the following equation

$$\begin{cases} -\partial_t v(t,x)+|\partial_x v(t,x)| = 0 \quad \text{on } (0,T)\times\mathcal{M}_1, \\ -\partial_t v(t,x) + 2|\partial_x v(t,x)| = 0 \quad \text{on } (0,T)\times\mathcal{M}_2, \\ -\partial_t v(t,x) + \max\Big(\max_{\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]}\langle \partial_x v(t,x), \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}\rangle, \quad 2\max_{\theta\in[\frac{\pi}{2},\frac{3\pi}{2}]}\langle \partial_x v(t,x), \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}\rangle\Big) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0,T)\times\mathcal{M}_3, \\ v(T,x) = \psi(x), \quad x\in\mathbb{R}^2. \end{cases}$$

Moreover, by Theorem 2.11, the above equation admits a unique continuous viscosity solution. The viscosity solution is the value function ϑ associated to the optimal control problem defined using the dynamics $F(.)$ and the final cost ψ (see Sect. 4). By Theorem 4.1 of [7], ϑ is also the unique continuous function that satisfies $v(T,x) = \psi(x)$ and is both a viscosity super-solution of

$$\begin{cases} -\partial_t v(t,x)+|\partial_x v(t,x)| \geq 0 & \text{on } (0,T)\times\mathcal{M}_1, \\ -\partial_t v(t,x) + 2|\partial_x v(t,x)| \geq 0 & \text{on } (0,T)\times\mathcal{M}_2\cup\mathcal{M}_3. \end{cases} \tag{2.8}$$

and a viscosity sub-solution of

$$\begin{cases} -\partial_t v(t,x)+|\partial_x v(t,x)| \leq 0 & \text{on } (0,T)\times\mathcal{M}_1, \\ -\partial_t v(t,x) + 2|\partial_x v(t,x)| \leq 0 & \text{on } (0,T)\times\mathcal{M}_2, \\ -\partial_t v(t,x) + \max\left(\partial_x v(t,x)\cdot\begin{pmatrix}0\\2\end{pmatrix}, \partial_x v(t,x)\cdot\begin{pmatrix}0\\-2\end{pmatrix}\right) \leq 0 & \text{on } (0,T)\times\mathcal{M}_3, \end{cases} \tag{2.9}$$

The comparison theorem in [7] allows also to compare any l.s.c super-solution of (2.8) and any u.s.c sub-solution of (2.9).

Example 2.15. This example is inspired from Section 10.4 of [8]. Consider the same stratification as in the previous example and introduce a new dynamics defined by

$$F_1(x) = \left\{\begin{pmatrix}a\\a\end{pmatrix},\ a\in[-1,1]\right\} \text{ for } x\in\mathcal{M}_1, \quad F_2(x) = \left\{\begin{pmatrix}a\\-a\end{pmatrix},\ a\in[-1,1]\right\} \text{ for } x\in\mathcal{M}_2.$$

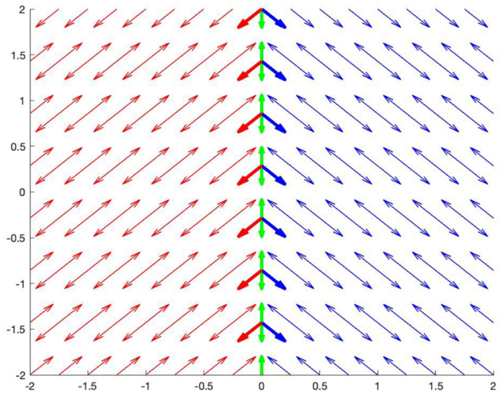

FIGURE 3. Dynamics of Example 2.15.

For $x \in \mathcal{M}_3$, the convexified dynamics is $F(x) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, |a| \leq 1,\ |b| \leq 1 \right\}$, and the tangent dynamics on $\mathcal{M}_3$ is given by $F_3(x) = \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix}, a \in [-1, 1] \right\}$.

For this setting, the essential dynamics is given by

$$\begin{aligned} F_1^\sharp(x) &= F_1(x) \ \text{ on } \mathcal{M}_1, \quad F_1^\sharp(x) = \left\{ \begin{pmatrix} -a \\ -a \end{pmatrix},\ a \in [0,1] \right\} \quad \text{on } \mathcal{M}_3, \\ F_2^\sharp(x) &= F_2(x) \ \text{ on } \mathcal{M}_2, \quad F_2^\sharp(x) = \left\{ \begin{pmatrix} a \\ -a \end{pmatrix},\ a \in [0,1] \right\} \quad \text{on } \mathcal{M}_3, \\ \text{and} \ \ F_3^\sharp(x) &= F_3(x) \ \text{ on } \mathcal{M}_3. \end{aligned}$$

Figure 3 shows the essential dynamics where $F_1^\sharp$ is represented in red, $F_2^\sharp$ in blue and $F_3^\sharp$ in green. Theorem 2.12 states a comparison principle for the following HJB equation

$$\begin{cases} -\partial_t u(t,x) + |\partial_{x_1} u(t,x) + \partial_{x_2} u(t,x)| = 0 \qquad \text{for } (t,x) \in (0,T) \times \mathcal{M}_1, \\ -\partial_t u(t,x) + |\partial_{x_1} u(t,x) - \partial_{x_2} u(t,x| = 0 \qquad \text{for } (t,x) \in (0,T) \times \mathcal{M}_2, \\ -\partial_t u(t,x) + \max(|\partial_{x_1} u(t,x)| + \partial_{x_2} u(t,x), |\partial_{x_2} u(t,x)|) = 0 \qquad \text{for } (t,x) \in (0,T) \times \mathcal{M}_3, \\ u(T,x) = |x_1| + x_2, \end{cases} \tag{2.10}$$

where $\psi(x) = |x_1| + x_2$ is the final cost. We can also prove that this equation has a unique continuous viscosity solution that happens to be the value function associated with the following optimal control problem:

$$u(t,x) = \inf \ \{|y_1(T)| + y_2(T) \ : \ \dot{y}(s) \in F(y(t)) \text{ for a.e. } s \in [t,T],\ y(t) = x\}.$$

Let us notice that in this setting, the viscosity solution in Ishii's sense is not unique, as shown in Section 10.4 of [8].

Example 2.16. In this example, we consider a different stratification of $\mathbb{R}^2$ given by

$$\mathcal{M}_1 := \{x = (x_1, x_2) \in \mathbb{R}^2, x_1 < 0\}, \ \ \mathcal{M}_2 := \{x = (x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0\},$$

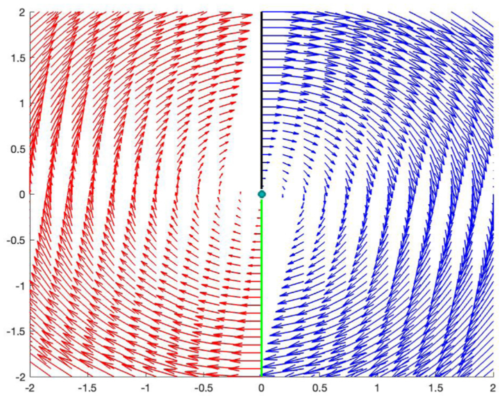

FIGURE 4. The dynamics of Example 2.16: In red the dynamics $F_1(.)$ and in blue the dynamics $F_2(.)$.

$$\mathcal{M}_3 := \{0\}\times]-\infty,0[,\ \ \mathcal{M}_4 := \{0\}\times]0,+\infty[,\ \text{ and } \mathcal{M}_5 := \{0\}.$$

The stratification satisfies (H_1). Consider the dynamics $F_1(.)$ and $F_2(.)$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively defined as follows:

$$F_1(x) := c_1 \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \text{ on } \mathcal{M}_1, \quad F_2(x) := c_2 \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \text{ on } \mathcal{M}_2.$$

The convexified dynamics $F(.)$ is given by

$$F(x) = \begin{cases} F_1(x), & x \in \mathcal{M}_1, \\ F_2(x), & x \in \mathcal{M}_2, \\ [\min(c_1,c_2), \max(c_1,c_2)] x_2 e_{x_1} & \text{elsewhere.} \end{cases}$$

Assume that $c_1, c_2 > 0$. Then we have

$$F_3(.) = \emptyset \text{ on } \overline{\mathcal{M}}_3, \quad F_4(.) = \emptyset \text{ on } \overline{\mathcal{M}}_4, \quad F_5(.) = \{0\} \text{ on } \overline{\mathcal{M}}_5.$$

Figure 4 shows the essential dynamics (F_1 is represented in blue while F_2 is represented in red). Notice that in this example the normal controllability is not satisfied and the results of [7] do not apply (see some remarks on problems without controllability in Section 12.2 of [8].

The essential dynamics are defined by

$$F_1^\sharp(x) = \begin{cases} F_1(x), & \text{on } \mathcal{M}_1, \\ F_1(x), & \text{on } \mathcal{M}_3, \\ \emptyset, & \text{on } \mathcal{M}_4, \\ \{0\}, & \text{on } \mathcal{M}_5. \end{cases} \quad F_2^\sharp(x) = \begin{cases} F_2(x) & \text{on } \mathcal{M}_2, \\ \emptyset, & \text{on } \mathcal{M}_3, \\ F_2(x), & \text{on } \mathcal{M}_4, \\ \{0\}, & \text{on } \mathcal{M}_5. \end{cases}$$

$$F_3^\sharp(x) = \begin{cases} \emptyset, & \text{on } \mathcal{M}_3, \\ \{0\} & \text{on } \mathcal{M}_5. \end{cases} \quad F_4^\sharp(x) = \begin{cases} \emptyset, & \text{on } \mathcal{M}_4, \\ \{0\} & \text{on } \mathcal{M}_5. \end{cases} \quad F_5^\sharp(0) = \{0\}.$$

We consider the following HJB system

$$\begin{cases} -\partial_t u(t,x) + \sup_{\nu \in F_i(x)}\{-\langle \partial_x u(t,x), \nu\rangle\} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_i,\ i = 1,2, \\ -\partial_t u(t,x) + \sup_{\nu \in F_1(x)}\{-\langle \partial_x u(t,x), \nu\rangle\} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_3, \\ -\partial_t u(t,x) + \sup_{\nu \in F_2(x)}\{-\langle \partial_x u(t,x), \nu\rangle\} = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{M}_4, \\ -\partial_t u(t,0) = 0 \quad \text{for } t \in (0,T), \\ u(T,x) = \psi(x) \quad \text{for } x \in \mathbb{R}^N. \end{cases}$$

The dynamics of this example satisfy the assumptions (SH), (H_D) and (H_{ESS}) but not the controllability assumptions $(H_{N,[7]})$. However, whenever we choose ψ Lipschitz continuous and bounded, we can prove that there exists a unique continuous solution to the above HJB system and it is the value function associated to the optimal control problem defined with $F(.)$ (see Sect. 4). Moreover, Theorem 2.12 provides a strong comparison principle for the above HJB system.

3. Invariance principles

In this section, we present weak and strong invariance principles. These principles are known in the classical case, when the dynamics $F(.)$ is Lipschitz continuous, see Chapter 11 of [13]. Here, we give an extension of the weak and strong principles in the stratifies case.

First, we recall some tools from nonsmooth analysis.

Definition 3.1. (Proximal sub-gradient and super-gradient).

• Let $u : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c function. We say that ζ is proximal sub-gradient at a point $x \in dom(u)$ for some $\sigma = \sigma(x,\zeta)$ and some neighborhood $V = V(x,\zeta)$ of x if we have

$$u(y) - u(x) + \sigma|y-x|^2 \geq \langle \zeta, y - x\rangle, \ \ \forall\, y \in V.$$

The collection of such ζ form the proximal sub-gradient. It is denoted $\partial_p u(x)$.

• Similarly, Let $u : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be an u.s.c function. We say that ζ is proximal super-gradient at a point $x \in dom(u)$ for some $\sigma = \sigma(x,\zeta)$ and some neighborhood $V = V(x,\zeta)$ of x if we have

$$u(y) - u(x) + \sigma|y-x|^2 \leq \langle \zeta, y - x\rangle, \ \ \forall\, y \in V.$$

The collection of such ζ forms the proximal super-gradient. It is denoted $\partial^p u(x)$. We also have the property $\partial^p u(x) = -\partial_p(-u)(x)$.

Definition 3.2. (Proximal normal cone).
Let $S \subseteq \mathbb{R}^N$ be a closed set and $x \in S$. A vector ζ is a proximal normal to the closed set S at the point x if there exists $\sigma > 0$ such that $\langle \zeta, y - x\rangle \leq \frac{|\zeta|}{2\sigma}|y-x|^2 \ \ \forall y \in S$. The set of all proximal normal vectors at x is denoted by $N_S^p(x)$.

Definition 3.3. (Weak invariance).
Let $\Gamma : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be a set-valued map. Let $S \subseteq \mathbb{R}^N$ be a closed set. We say that (S, Γ) is weakly invariant provided that for $x \in S$, $t \in [0,T]$ there exists $y(.)$ a solution of $(DI)_\Gamma(t,x)$ such that $y(\tau) \in S$ for all τ in $[t,T]$.

Definition 3.4. (Strong invariance).
Let $\Gamma : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be a set-valued map. Let $S \subseteq \mathbb{R}^N$ be a closed set. We say that (S, Γ) is strongly invariant provided that for $x \in S$, $t \in [0,T]$ and every $y(.)$ a solution of $(DI)_\Gamma(t,x)$ we have $y(\tau) \in S$ for all τ in $[t,T]$.

Theorem 3.5. *Assume (H_1), (SH) and (H_D). Let S be a closed set of $\mathbb{R}^N$. We denote by $S_i := \overline{\mathcal{M}}_i \cap S$. The following assertions are equivalent:*

- *(i)* (S, F) *is weakly invariant,*
- *(ii)* $\forall x \in S,\ \exists\, i \in I(x)\ :\ \forall \eta_i \in N^p_{S_i}(x), \quad H_{F_i}(x, \eta_i) \geq 0,$
- *(iii)* $\forall x \in S,\ \exists\, i \in I(x)\ :\ \forall \eta_i \in N^p_{S_i}(x), \quad H_{F_i^\sharp}(x, \eta_i) \geq 0.$

Theorem 3.6. *Assume* (H_1), (SH) *and* (H_D). *Let* S *be a closed set of* $\mathbb{R}^N$. *We denote by* $S_i := \overline{\mathcal{M}}_i \cap S$. *The following assertions are equivalent:*

- *(i)* (S, F) *is strongly invariant,*
- *(ii)* $\forall x \in S, \quad \forall i \in I(x), \quad \forall \eta_i \in N^p_{S_i}(x), \quad H_{F_i^\sharp}(x, -\eta_i) \leq 0.$

The complete proof of Theorems 3.5 and 3.6 are given in Appendix C.

For stratified systems, the first attempt to prove the invariance results was in [10], using the essential Hamiltonian strategy. In particular, for the strong invariance principle, under assumptions (H_1), (SH) and (H_D), ([10], Thm. 5.1) states that (S, F) is strongly invariant for some closed set $S \subseteq \mathbb{R}^N$ if and only if

$$\forall x \in S,\ \forall \xi \in N^p_S(x), \quad H_{F^\sharp}(x, -\xi) \leq 0. \tag{3.1}$$

Although the intuition of using the essential Hamiltonian was very interesting, the proximal normal cone $N^p_S(x)$ used in (3.1) is the same as the one in the classical case, which does not take into account the geometry of the problem. More precisely, the sufficient implication in (3.1) fails to be true in general. Here is a counterexample.

Consider a stratification as follows

$$\mathcal{M}_1 = \{(x_1, x_2) \in \mathbb{R}^2\ :\ x_1 < 0\ \&\ x_2 > 0\} \quad \mathcal{M}_2 = \{(x_1, x_2) \in \mathbb{R}^2\ :\ x_1 < 0\ \&\ x_2 < 0\},$$

$$\mathcal{M}_3 = \{(x_1, x_2) \in \mathbb{R}^2\ :\ x_1 > 0\ \&\ x_2 < 0\} \quad \mathcal{M}_4 = \{(x_1, x_2) \in \mathbb{R}^2\ :\ x_1 > 0\ \&\ x_2 > 0\},$$

$$\mathcal{M}_5 = (0, +\infty)e_{x_1} \quad \mathcal{M}_6 = (-\infty, 0)e_{x_1} \quad \mathcal{M}_7 = (0, +\infty)e_{x_2} \quad \mathcal{M}_8 = (-\infty, 0)e_{x_2} \quad \mathcal{M}_9 = \{0\}.$$

This stratification satisfies assumptions (H_1), (SH) and (H_D). Take S to be the closed set defined by

$$S = \{(x_1, x_2) \in \mathbb{R}^2\ : x_1^3 \leq x_2^5\},$$

represented in red in Figure 5, and consider the following dynamics

$$F_1(x_1, x_2) = F_2(x_1, x_2) = F_4(x_1, x_2) = F_5(x_1, x_2) = \{(0, 0)\},$$

$$F_6(x_1, x_2) = F_7(x_1, x_2) = F_9(x_1, x_2) = \{(0, 0)\}, \quad F_3(x_1, x_2) = F_8(x_1, x_2) = \{-e_{x_2}\}.$$

Since the proximal normal cone to S at $\bar{x} = (0, 0)$ is equal to $N^p_S(\bar{x}) = \{(0, 0)\}$, the Hamiltonian inequality (3.1) is therefore verified at $\bar{x} = (0, 0)$. Moreover, the inequality (3.1) is also verified, for any other point in $S \setminus \{0\}$ since the dynamics F is reduced to $\{(0, 0)\}$. However, (S, F) is not strongly invariant since $F(0, x_2) = co\{0, -e_{x_2}\}$ if $x_2 \leq 0$, and the trajectory

$$\tilde{Z}(s) = (0, t - s) \in S_{(t,T)}(0, 0)$$

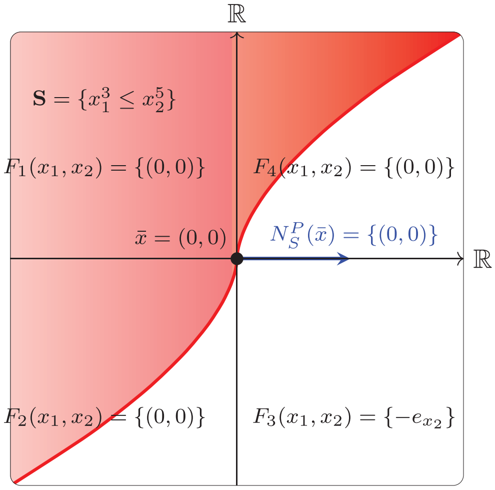

FIGURE 5. Counterexample with a stratification in $\mathbb{R}^N$, with $N = 2$, $n = 4$ and $l = 5$.

is a trajectory of F that starts at $\bar{x} = (0,0) \in S$, but $\tilde{Z}(.) \not\subset S$. In conclusion, in this counterexample the Hamiltonian inequality

$$\forall x \in S,\ \forall \xi \in N_S^p(s), \quad H_{F^\sharp}(x, -\xi) \le 0,$$

is verified. Nonetheless, (S, F) is not strongly invariant (so, the sufficient condition of strong invariance (3.1) is not correct).

In Theorem 3.6, we give a correct characterization of the strong invariance, in the stratified case, using the essential Hamiltonian and by involving, for every $i \in I(x)$, the normal cone $N_{S_i}^p(x)$. Notice that (S, F), defined in the above counterexample, does not satisfy the characterization of strong invariance given in Theorem 3.6. Indeed, we have

$$S_8 := S \cap \overline{\mathcal{M}}_8 = S \cap (-\infty, 0]e_{x_2} = \{(0,0)\}.$$

Therefore, the proximal normal to S_8 at $\bar{x} := (0,0)$ is equal to $\mathbb{R}^2$. Moreover, we have $F_8^\sharp(\bar{x}) = co\{0, -e_{x_2}\} = [0, -e_{x_2}]$. Thus

$$H_{F_8^\sharp}(\bar{x}, -N_{S_8}^p(\bar{x})) \ge \langle -e_{x_2}, -e_{x_2} \rangle = 1 > 0.$$

4. Proof of Theorems 2.11 and 2.12

4.1. Optimal control problem and the value function

In this section, we consider the differential inclusion associated with the set-valued map F:

$$\begin{cases} \dot{y}(s) \in F(y(s)),\ s \in [t, T] \text{ a.e.,} \\ y(t) = x. \end{cases}$$

Since F is u.s.c with nonempty, convex and compact images, then the above differential inclusion admits Lipschitz solutions for any $(t, x) \in [0, T] \times \mathbb{R}^N$. Furthermore, the set of solutions is compact in the topology of uniform convergence ([2], Thm. 1, pp. 60). We denote by $S_{(t,T)}(x)$ the set of solutions of the differential inclusion associated to $F(.)$:

$$S_{(t,T)}(x) := \left\{ y_{(t,x)}(.) \in W^{1,1}([t,T];\mathbb{R}^N) \ : \ \left\{ \begin{array}{l} \dot{y}_{(t,x)}(s) \in F(y_{(t,x)}(s)), \ s \in [t,T], \text{ a.e.,} \\ y(t) = x. \end{array} \right. \right\}.$$

We consider the following Mayer optimal control problem defined for $(t,x) \in [0,T] \times \mathbb{R}^N$ by

$$\left\{ \begin{array}{ll} \inf & \psi(y_{(t,x)}(T)) \\ \text{such that} & \dot{y}_{(t,x)}(s) \in F(y_{(t,x)}(s)), \ s \in [t,T] \\ & y_{(t,x)}(t) = x, \end{array} \right. \tag{4.1}$$

where the infimum is taken over all trajectories $y_{(t,x)}(.) \in S_{(t,T)}(x)$ and it is reached.

Next, we consider the value function associated to the optimal control problem defined on $(t,x) \in [0,T] \times \mathbb{R}^N$ by

$$\vartheta(t,x) = \inf\{ \ \psi(y_{(t,x)}(T)) \ , \ \ y_{(t,x)}(.) \in S_{(t,T)}(x) \ \}.$$

We now proceed to define some properties of the value function.

Definition 4.1. Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function. u is said to enjoy

- **the super-optimality property** if for all $(t,x) \in (0,T] \in \mathbb{R}^N$, there exists $y_{(t,x)}(.) \in S_{(t,T)}(x)$ such that
$$u(t, y_{(t,x)}(t)) \geq u(s, y_{(t,x)}(s)), \ \forall s \in [t,T];$$
- **the sub-optimality property** if for all $(t,x) \in [0,T] \in \mathbb{R}^N$, for all $y_{(t,x)}(.) \in S_{(t,T)}(x)$ we have
$$u(t, y_{(t,x)}(t)) \leq u(s, y_{(t,x)}(s)), \ \forall s \in [t,T].$$

As in the classical case, the value function ϑ satisfies the Dynamic programming principle, which corresponds to the super-optimality and the sub-optimality properties.

Lemma 4.2 ([22])**.** *Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function.*

- *If $u(T,x) \geq \psi(x)$ and u has the super-optimality property, then: $\vartheta(t,x) \leq u(t,x)$ for all $(t,x) \in (0,T] \in \mathbb{R}^N$.*
- *If $u(T,x) \leq \psi(x)$ and u has the sub-optimality property, then: $\vartheta(t,x) \geq u(t,x)$ for all $(t,x) \in (0,T] \times \mathbb{R}^N$.*

The next proposition states that the controllability hypothesis (CH) is a sufficient condition to ensure that the value function is locally Lipschitz continuous.

Proposition 4.3. *Suppose (H_1), (CH) and $(H\psi)$ hold. Then, $\vartheta : [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is locally Lipschitz continuous.*

Proof. From the controllability assumption (CH), there exists a neighborhood of Λ (the interfaces), denoted $V := \Lambda + \varepsilon\mathbb{B}$, and there exists $r > 0$, such that for all $x \in V$, we have $r\mathbb{B} \subset F(x)$.

First, we prove that $\vartheta(t,.)$ is locally Lipschitz. Let $x, z \in \mathbb{R}^N$. We suppose first that $x, z \in V$. Let M be the local supremum bound of F and L_ψ be the Lipschitz constant of ψ in some open ball with radius large enough. Without loss of generality, we suppose $\vartheta(t,x) \geq \vartheta(t,z)$.

Let $y_{t,z}(.) \in S_{(t,T)}(z)$ such that $\vartheta(t,z) = \psi(y_{t,z}(T))$. Set

$$h = \frac{| x - z |}{r} \quad \text{and} \quad \xi(s) = x + r\frac{z - x}{| x - z |}(s - t) \quad \text{for } s \in [t, t + h].$$

We define:

$$\tilde{y}(s) = \begin{cases} \xi(s) & \text{for } s \in [t, t + h] \\ y_{t,z}(s - h) & \text{for } s \in [t + h, T] \end{cases}$$

It is easy to see that $\tilde{y}(.) \in S_{(t,T)}(x)$. If $t + h \leq T$, we get

$$\begin{aligned} \vartheta(t,x) - \vartheta(t,z) \leq \psi(\tilde{y}(T)) - \psi(y_{t,z}(T)) \leq L_\psi |\tilde{y}(T) - y_{t,z}(T)| &= L_\psi |y_{t,z}(T - h) - y_{t,z}(T)| \\ &\leq L_\psi M h \ = \ L_\psi \frac{M}{r}|x - z|. \end{aligned}$$

If $t + h > T$, then we get

$$\begin{aligned} \vartheta(t,x) - \vartheta(t,z) \leq \psi(\tilde{y}(T)) - \psi(y_{t,z}(T)) &\leq L_\psi \left(|\xi(T) - z| + |z - y_{t,z}(T)|\right) \\ &\leq L_\psi \left(|x - z + r\frac{z - x}{| x - z |}(T - t)| + M|T - t|\right) \\ &\leq L_\psi (1 + \frac{r + M}{r})|x - z|. \end{aligned}$$

Thus in all cases we get

$$\vartheta(t,x) - \vartheta(t,z) \leq L_\psi (2 + \frac{M}{r})|x - z|.$$

Suppose now for example $z \notin V$ (we can do the same reasoning on x instead). Then by taking them close enough to each other, there exists $i \in \{1, \ldots, n\}$ such that $x, z \in \mathcal{M}_i$. Let $y_{t,z}(.) \in S_{(t,T)}(z)$ such that $\vartheta(t,z) = \psi(y_{t,z}(T))$. Suppose $y_{t,z}(.)$ crosses the boundary of $\mathcal{M}_i$. Let $t_0 \in [t, T]$ be such that

$$y_{t,z}([t, t_0]) \subset \mathcal{M}_i, \text{ and } y_{t,z}(t_0) \in V.$$

Let $y_{t,x}(.) \in S_{(t,T)}(x)$ such that $y_{t,x}([t, t_0]) \subset \mathcal{M}_i$. We have

$$|y_{t,x}(t_0) - y_{t,z}(t_0)| \leq e^{M t_0}|x - z| \leq e^{MT}|x - z|$$

Since they are both F_i-trajectories on $[t, t_0]$, see Theorem 4.3.11 of [15]. Furthermore, we can also suppose that

$$y_{t,x}(t_0) \in V \cap \mathcal{M}_i.$$

We can always find such a trajectory if x and z are close enough and $y_{t,z}(t_0) \in V \cap \mathcal{M}_i$.

Set $h = \frac{|y_{t,z}(t_0) - y_{t,x}(t_0)|}{r}$ and $\xi(s) = y_{t,x}(t_0) + r\frac{y_{t,z}(t_0) - y_{t,x}(t_0)}{|y_{t,z}(t_0) - y_{t,x}(t_0)|}(s - t_0)$ for $s \in [t_0, t_0 + h]$. We define

$$\tilde{y}(s) = \begin{cases} y_{t,x}(s) & \text{for } s \in [t, t_0] \\ \xi(s) & \text{for } s \in [t_0, t_0 + h] \\ y_{t,z}(s - h) & \text{for } s \in [t_0 + h, T] \end{cases}$$

It is easy to see that $\tilde{y}(.)$ is an F-trajectory. So, arguing in the same way as the previous case, we get

$$\begin{aligned}\vartheta(t,x)-\vartheta(t,z) \le \psi(\tilde{y}(T))-\psi(y_{t,z}(T)) &\le L_\psi|\tilde{y}(T)-y_{t,z}(T)|\\ &\le L_\psi(2+\frac{M}{r})e^{MT}|x-z|.\end{aligned}$$

If the trajectory $y_{t,z}(.)$ does not cross the boundary of $\mathcal{M}_i$, then, it is an F_i-trajectory. Furthermore, we can always find a F-trajectory $y_{t,x}(.)$ that stays in $\mathcal{M}_i$ by the controllability assumption. Indeed one can always choose a trajectory with zero velocity once it reaches the neighborhood V. So $y_{t,x}(.)$ is also an F_i-trajectory. Hence the result follows again from the classical case, see Theorem 4.3.11 of [15]. This finishes the proof of $\vartheta(t,.)$ is locally Lipschitz.

Now we prove that ϑ is locally Lipschitz w.r.t the time variable. Let $x \in \mathbb{R}^N$. Let $t,s \in [0,T]$ such that $t<s$. By the super-optimality property, there exists $y(.) \in S_{(t,T)}(x)$ such that $\vartheta(t,x)=\vartheta(s,y(s))$. Then

$$|\vartheta(t,x)-\vartheta(s,x)| = |\vartheta(s,y(s))-\vartheta(s,x)| \le |\vartheta(s,y(s))-\vartheta(s,y(t))|.$$

Since both $\vartheta(s,.)$ and $y(.)$ are locally Lipschitz, then from the expression above, $\vartheta(.,x)$ is locally Lipschitz. This ends the proof. □

The next proposition shows that F-trajectories are the same as $F^\sharp$-trajectories. This implies that the essential dynamics completely characterize the optimal control problem 4.1.

Proposition 4.4. *([10], Prop. 2.1).*
Let $(t,x) \in [0,T]\times\mathbb{R}^N$ and $y(.) \in S_{(t,T)}(x)$. The following statements are equivalent:
(i) $y(.)$ is a solution of $(DI)_F(t,x)$,
(ii) $y(.)$ is a solution of $(DI)_{F^\sharp}(t,x)$,
(iii) $y(.)$ is a solution of $(DI)_{F_i}(t,x)$ whenever $y(.) \in \mathcal{M}_i$.

Proof. $(iii) \implies (ii) \implies (i)$ is obvious since $F_i(.) \subseteq F^\sharp(.) \subseteq F(.)$. Now, suppose (i) and let $y(.) \in S_{(t,T)}(x)$. For $k \in \{1,\dots,n+l\}$, let $J_k := \{s \in [t,T] \,:\, y(s) \in \mathcal{M}_k\}$. Without loss of generality, we suppose $\mathscr{L}(J_k)>0$ (otherwise there is nothing to prove). We set

$$\tilde{J}_k := \{s \in J_k \,:\, \dot{y}(s) \text{ exists in } F(y(s)) \text{ and } s \text{ is a Lebesgue point of } J_k\}.$$

Clearly $\mathscr{L}(J_k)=\mathscr{L}(\tilde{J}_k)$ ($\mathscr{L}$ stands for the Lebesgue measure). Let $s \in \tilde{J}_k$. So, there exists a sequence $(s_n)_n \subset J_k$ such that $s_n \to s$ and $s_n \neq s$ for all n. Since $y(s_n) \in \mathcal{M}_k$, we have

$$\dot{y}(s) = \lim_{s_n \to s} \frac{y(s_n)-y(s)}{s_n-s} \in \mathcal{T}_{\mathcal{M}_k}(y(s)),$$

which is the required result. □

The above proposition shows in particular that the optimal control problem could be defined using F or $F^\sharp$ or $F_i^\sharp$, $i=1,\dots,n+l$.

4.2. The super-optimality and super-solution property

In this section, we characterize functions that are super-solutions of equation (2.3) with super-optimality property. The characterization using the Hamiltonian H_F is standard in the literature since the set-valued map F satisfies the usual hypotheses (upper semicontinuity with nonempty, convex and compact images). Here, we will prove a more general result. We show that super-solutions are characterized using the Hamiltonians $H_{F_i^\sharp}$,

$i = 1, \dots, n+l$, In the viscosity sense given in Definition 2.6. We recall that for a l.s.c function $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $i = 1, \dots, n+l$, we define the function $u_i : (0,T] \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$u_i \equiv u \text{ on } (0,T] \times \overline{\mathcal{M}}_i \text{ and } u_i \equiv +\infty \text{ elsewhere.}$$

Theorem 4.5. *Suppose (H_1), (SH) and (H_D). Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be l.s.c. The following assertions are equivalent:*

(i) u is a super-solution of (2.3) in the sense of Definition 2.6,

(ii) u satisfies the super-optimality principle.

Proof. The fact that (i) is equivalent to

$$-\theta + H_F(x,\xi) \geq 0 \qquad \forall (\theta,\xi) \in D^- u(t,x), \tag{4.2}$$

is well known since $F(.)$ is u.s.c with nonempty, convex and compact images. For more on this, see Proposition 3.5 of [32] or Chapter 19 of [13]. Moreover, it is obvious from this that $(i) \implies (ii)$ since $H_{F_i^\sharp}(.,.) \leq H_{F_i}(.,.) \leq H_F(.,.)$ and $D^- u(.,.) \subset D^- u_i(.,.)$, for all $i \in [1, n+l]$.

It remains to prove $(ii) \implies (i)$. Let $y(.) : [t,T] \to \mathbb{R}^N$ be a trajectory solution of $(DI)_F(t,x)$ such that the super-optimality property holds in $y(.)$. We claim the following:

Claim: $\exists\, j \in I(x)$ such that there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \to \nu$ and $\nu \in F_j^\sharp(x)$.

Deferring the proof of the claim, let $\phi \in C^1((0,T) \times \mathbb{R}^N)$ such that $u_j - \phi$ attains a local minimum at (t,x) in $(0,T) \times \overline{\mathcal{M}}_j$. For n big enough the super-optimality property gives

$$u_j(t,x) - u_j(t_n, x_n) \geq 0.$$

This inequality combined with the fact that $u_j(t_n,x_n) - \phi(t_n,x_n) \geq u_j(t,x) - \phi(t,x)$ lead to

$$\frac{1}{t_n - t}(\phi(t,x) - \phi(t_n,x_n)) \geq 0.$$

By letting n tend to $+\infty$, we obtain $-\partial_t \phi(t,x) - \langle \nu, \partial_x \phi(t,x) \rangle \geq 0$ and then

$$-\partial_t \phi(t,x) + H_{F_j^\sharp}(x, \partial_x \phi(t,x)) \geq -\partial_t \phi(t,x) - \langle \nu, \partial_x \phi(t,x) \rangle \geq 0.$$

This concludes the proof.

Now we turn our attention to the proof of the claim. We distinguish two cases: either there exists $r > 0$ such that $y([t,t+r])$ stays in one domain $\mathcal{M}_j$ for some $j \in \{1,\dots,n\} \cap I(x)$, almost everywhere, or it touches or crosses the singular set Λ infinitely many times no matter how we are close to x. We begin with the first case: suppose there exists $r > 0$ such that $y([t,t+r]) \subset \mathcal{M}_j$ for some $j \in \{1,\dots,n\} \cap I(x)$ almost everywhere. So, there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \to \nu$.

Notice that $\nu = \lim_{n \to +\infty} \frac{x_n - x}{t_n - t} \in \mathcal{T}_{\overline{\mathcal{M}}_j}(x)$, since $x_n \in \mathcal{M}_j$. It remains to prove that ν belongs to $F_j(x)$. Denote by κ and M respectively the Lipschitz constant of $F_j(.)$ and the Lipschitz constant of $y(.)$. We have

$$\begin{aligned} \nu &= \lim_{n \to +\infty} \frac{1}{t_n - t} \int_{[t,t_n]} \dot{y}(s)\, \mathrm{d}s \\ &\in \lim_{n \to +\infty} \Big(\frac{1}{t_n - t} \int_{[t,t_n]} proj_{F_j(x)}(\dot{y}(s))\, \mathrm{d}s + \frac{\kappa}{t_n - t} \int_{[t,t_n]} |y(s) - x| \mathbb{B}\, \mathrm{d}s \Big) \end{aligned}$$

$$
\begin{aligned}
&\subseteq \lim_{n\to+\infty}\big(F_j(x)+\frac{\kappa M}{t_n-t}\big[\int_{[t,t_n]}(s-t)\,\mathrm{d}s\big]\mathbb{B}\big)\\
&\subseteq \lim_{n\to+\infty}\big(F_j(x)+\kappa M\frac{|t_n-t|}{2}\mathbb{B}\big)=F_j(x).
\end{aligned}
$$

In conclusion, we get $\nu\in F_j(x)\cap\mathcal{T}_{\overline{\mathcal{M}}_j}(x)=F_j^{\sharp}(x)$.

Now we get to the second case. Since $y(.)$ touches or crosses Λ infinitely many times no matter how we are close to x, then there exists $j\in\{n+1,\dots,n+l\}\cap I(x)$, a sequence $(t_n)_n$, $t_n\downarrow t$ and $x_n:=y(t_n)\in\mathcal{M}_j$, so that $\frac{x_n-x}{t_n-t}\to\nu$.

Notice that $\nu=\lim_{n\to+\infty}\frac{x_n-x}{t_n-t}\in\mathcal{T}_{\overline{\mathcal{M}}_j}(x)$, since $x_n\in\mathcal{M}_j$. It remains to prove that ν belongs to $F_j(x)$. For $k=1,\dots,n+l$, we set

$$
J_n^k:=\{\,s\in[t,t+t_n]\;:\;y(s)\in\mathcal{M}_k\,\},\ \mu_n^k:=\mathscr{L}(J_n^k),\ \mathbb{K}(x):=\{k\;:\;\mu_n^k>0,\ \forall n\in\mathbb{N}\},
$$

where we recall that $\mathscr{L}$ is the Lebesgue measure on $\mathbb{R}$. We obviously have $\mathbb{K}(x)\subset I(x)$. Furthermore, up to a subsequence, there exist $0\le\lambda_k\le1$ and $p_k\in\mathbb{R}^N$ such that

$$
\frac{\mu_n^k}{t_n-t}\to\lambda_k,\ \sum_{k\in\mathbb{K}(x)}\lambda_k=1,\ \frac{1}{\mu_n^k}\int_{J_n^k}\dot y(s)\mathrm{d}s\to p_k,\quad \text{as } n\to\infty.
$$

Denote by κ and M respectively the Lipschitz constant of $F_k(.)$ and the Lipschitz constant of $y(.)$, we get

$$
\begin{aligned}
p_k&=\lim_{n\to+\infty}\frac{1}{\mu_n^k}\int_{J_n^k}\dot y(s)\,\mathrm{d}s\\
&\in\lim_{n\to+\infty}\big(\frac{1}{\mu_n^k}\int_{J_n^k}proj_{F_k(x)}(\dot y(s))\,\mathrm{d}s+\frac{\kappa}{\mu_n^k}\int_{J_n^k}|y(s)-x|\mathbb{B}\,\mathrm{d}s\big)\\
&\subseteq\lim_{n\to+\infty}\big(F_k(x)+\frac{\kappa M}{\mu_n^k}\big[\int_{J_n^k}(s-t)\,\mathrm{d}s\big]\mathbb{B}\big)\\
&\subseteq\lim_{n\to+\infty}\big(F_k(x)+\kappa M|t_n-t|\mathbb{B}\big)=F_k(x).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\nu=\lim_{n\to+\infty}\frac{1}{t_n-t}\int_{[t,t_n]}\dot y(s)\,\mathrm{d}s&=\sum_{k\in\mathbb{K}(x)}\lim_{n\to+\infty}\frac{\mu_n^k}{t_n-t}\left[\frac{1}{\mu_n^k}\int_{J_n^k}\dot y(s)\,\mathrm{d}s\right]\\
&\subset co\,\{F_k(x)\;:\;k\in\mathbb{K}(x)\}.
\end{aligned}
$$

Finally, we get

$$
\nu\in co\Big\{F_k(x)\;:\;k\in\mathbb{K}(x)\Big\}\cap\mathcal{T}_{\overline{\mathcal{M}}_j}(x)\subset co\Big\{F_k(x)\;:\;k\in I(x)\Big\}\cap\mathcal{T}_{\overline{\mathcal{M}}_j}(x)=F_j^{\sharp}(x).
$$

This ends the proof of the claim. □

Remark 4.6. By the arguments presented at the beginning of the above proof, it is easy to see that under the same assumptions of Theorem 4.5, the following statements are equivalent:

(i) u satisfies the super-optimality principle,

(ii) u is a super-solution of (2.3) in the sense of Definition 2.6,

(iii) u is a super-solution of (2.2) in the sense of Definition 2.6,
(iv) u verifies inequality (4.2).

Remark 4.7. It was already known from [32], that the inequality (4.2) is equivalent to the super-optimality property if we only use the classical definition of viscosity. The importance of this result lies in the fact that the equivalence is valid even if we take the notion of viscosity stated in Definition 2.6.

4.3. The sub-optimality and sub-solution property

This section aims at establishing the link between the sub-optimality principle and the the sub-solution property of (2.3).

Theorem 4.8. *Suppose (H_1), (SH), (H_D) and (H_{ESS}) hold. Let $u : (0,T] \times \mathbb{R}^N \to \mathbb{R}$ be an u.s.c function. The following assertions are equivalent:*
(i) u satisfies the sub-optimality principle,
(ii) u is a sub-solution of (2.3).

Proof. We prove $(i) \implies (ii)$ first. Let $i \in \{1,\dots,n+l\}$ and $(t,x) \in [0,T] \times \overline{\mathcal{M}}_i$. By Lemma 3.9 of [32], for every $p \in F_i^\sharp(x)$, there exists a C^1 trajectory $y(.)$ defined on some interval $[t, t+\varepsilon]$, with $\varepsilon > 0$, such that $y(t) = x$, $\dot{y}(t) = p$ and $y(.) \subseteq \overline{\mathcal{M}}_i$.

Let $u_i \equiv u$ on $(0,T] \times \overline{\mathcal{M}}_i$ and $u_i \equiv -\infty$ otherwise. Let (θ,ξ) be in $D^+u_i(t,x)$. For any sequence $((t_n,x_n))_n$ such that $(t_n,x_n) \in dom(u_i)$ and $(t_n,x_n) \to (t,x)$, we have

$$\limsup_{n\to\infty} \frac{u(t_n,x_n) - u(t,x) - \theta(t_n - t) - \langle \xi, x_n - x\rangle}{|x_n - x| + |t_n - t|} \leq 0.$$

Setting $x_n = y(t + \frac{\varepsilon}{n})$ and $t_n := t + \frac{\varepsilon}{n}$, we get by sub-optimality of u

$$\frac{-\theta(t_n - t) - \langle \xi, x_n - x\rangle}{|x_n - x| + |t_n - t|} \leq \frac{u(t_n,x_n) - u(t,x) - \theta(t_n - t) - \langle \xi, x_n - x\rangle}{|x_n - x| + |t_n - t|},$$

By letting $n \to \infty$ we get

$$\frac{-\theta - \langle \xi, p\rangle}{|p| + 1} \leq 0 \implies -\theta - \langle \xi, p\rangle \leq 0.$$

Since p is arbitrary, we get the result by taking the supremum over $F_i^\sharp(x)$.

It remains to prove $(ii) \implies (i)$. We define the augmented stratification by

$$M_i := \mathbb{R} \times \mathcal{M}_i \times \mathbb{R}.$$

Furthermore, for all $i = 1,\dots,n+l$, we define $v := -u$ (so v is l.s.c) and we denote by $v_i \equiv v$ on $(0,T] \times \overline{\mathcal{M}}_i$ and $v_i \equiv +\infty$ otherwise. Next, we divide the proof into 2 steps.

Step 1. We show that

$$\forall i \in \{1,\dots,n+l\},\ epi(v_i) = epi(v) \cap \overline{M}_i.$$

Let $(t,x,r) \in epi(v_i)$. So $v_i(t,x) \leq r$. Hence $x \in \overline{\mathcal{M}}_i$ and $v(t,x) \leq r$. Thus we get $(t,x,r) \in epi(v) \cap \overline{M}_i$.

Conversely, if $(t,x,r) \in epi(v) \cap \overline{M}_i$, then $v(t,x) \le r$ and $x \in \overline{\mathcal{M}}_i$. So $v_i(t,x) = v(t,x)$, whence $v_i(t,x) \le r$, which finishes the proof of step 1.

Step 2. (Augmented dynamics). Let us first point out the fact that assertion (ii) is equivalent to

$$-\theta + H_{F_i^\sharp}(x,\nu) \le 0 \quad \text{for all } (t,x) \in (0,T) \times \overline{\mathcal{M}}_i,\ (\theta,\nu) \in -D^- v_i(t,x),$$

since $D^+u_i(t,x) = -D^-(-u_i)(t,x) = -D^-v_i(t,x)$. We establish the following claim

Claim. Let $G_i^\sharp$ be the augmented dynamics defined by

$$G_i^\sharp(t,x,z) := \{1\} \times F_i^\sharp(x) \times \{0\}, \quad \text{for any } (t,x,z) \in \overline{M}_i.$$

If we have

$$-\theta + H_{F_i^\sharp}(x,\nu) \le 0 \quad \text{for all } (t,x) \in (0,T) \times \overline{\mathcal{M}}_i,\ (\theta,\nu) \in -D^- v_i(t,x),$$

Then it holds

$$\sup_{\nu \in G_i^\sharp(t,x,z)} \{ \langle \eta, \nu \rangle \} \le 0 \qquad \forall (t,x,z) \in epi(v_i),\ \eta \in N^p_{epi(v_i)}(t,x,z). \tag{4.3}$$

Let $(t,x,z) \in epi(v_i)$. If $F_i^\sharp(x) = \emptyset$ then the result holds by vacuity. Otherwise, let $(\xi,-\lambda) \in N^p_{epi(v_i)}(t,x,z)$. So we have $\lambda \ge 0$ because $(\xi,-\lambda)$ belongs to the proximal normal cone of the epigraph of v_i. If $\lambda > 0$. Then we have $z = v_i(t,x)$ and there exists $(\theta,\zeta) \in -\partial_p v_i(t,x) \subset -D^-v_i(t,x)$ such that $\xi = (-\lambda\theta, -\lambda\zeta)$. Hence, by Theorem 11.32 of [13], for any $\nu \in G_i^\sharp(t,x,z)$ we have, for some $p \in F_i^\sharp(x)$:

$$\langle (\xi,-\lambda), \nu \rangle = -\lambda(\theta + \langle \zeta, p \rangle) \le \lambda(-\theta + H_{F_i^\sharp}(x,\zeta)) \le 0$$

We take the supremum over ν and we get the result. Now, if $\lambda = 0$, then by Theorem 11.31 of [13], there exist sequences $((t_n,x_n))_n \subseteq [0,T] \times \overline{\mathcal{M}}_i$, $(\xi_n)_n \subseteq \mathbb{R}^{N+1}$ and $(\lambda_n)_n \subseteq (0,\infty)$ such that

$$(t_n,x_n,\lambda_n) \to (t,x,0), \qquad v(t_n,x_n) \to z \qquad \xi_n \to \xi, \qquad \frac{1}{\lambda_n}\xi_n \in -\partial_p v_i(t_n,x_n).$$

Thus the argument above shows that

$$\langle (\xi_n,-\lambda_n), \nu_n \rangle \le 0 \qquad \forall \nu_n \in G_i^\sharp(t_n,x_n,u_i(t_n,x_n)),\ \forall n \in \mathbb{N}.$$

Furthermore, by Hypothesis (H_{ESS}) we have that $G_i^\sharp(.)$ is lower semicontinuous. So for any $\nu \in G_i^\sharp(t,x,z)$, there exists a sequence $(\nu_n)_n \to \nu$ such that $\nu_n \in G_i^\sharp(t_n,x_n,v_i(t_n,x_n))$. By evaluating the last inequality at this sequence and letting $n \to +\infty$, then taking the supremum over ν, we get the result. w

Consequently, since equation (4.3) holds from (ii), then we can apply Theorem 3.6 to the augmented dynamics $G(.) := \{1\} \times F(.) \times \{0\}$, the stratification $\mathbb{R}^{N+2} = \cup_{i=1}^{n+l} M_i$ and the set $S = epi(v)$. Thus we get that $(epi(v), G)$ is strongly invariant. Let $(t,x) \in (0,T) \times \mathbb{R}^N$ and $y(.)$ be a solution of $(DI)_F(t,x)$. So we have that

$$Y(s) = (s, y(s), v(t,y(t)) \qquad s \in [t,T],$$

is a solution of the differential inclusion with the augmented dynamics $G(.)$ and initial condition $(t, y(t), v(t, y(t))) = (t, x, v(t, x)) \in epi(v)$. Thus, by Theorem 3.6, we get

$$(t + h, y(t + h), v(t, y(t)) \in epi(v)$$

for all $h \in [0, T - t]$. Hence

$$v(t + h, y(t + h)) \leq v(t, y(t)) \Longleftrightarrow u(t, y(t)) \leq u(t + h, y(t + h)) \quad (\text{since } u = -v).$$

This ends the proof of $(ii) \Longrightarrow (i)$ and Theorem 4.8. □

4.4. Proof of Theorems 2.11 and 2.12

Proof. (Thm. 2.11). Proposition 4.3 shows that the value function ϑ is locally Lipschitz continuous. Furthermore, we have $\vartheta(T, x) = \psi(x)$. Theorem 4.5 shows that ϑ is a viscosity super-solution since it enjoys the super-optimality property. In addition, Theorem 4.8 shows that ϑ is a viscosity sub-solution since it enjoys the sub-optimality property. Finally, uniqueness comes from the global comparison result in Theorem 2.12. □

Proof. (Thm. 2.12). Let u_1 and u_2 respectively be a l.s.c super-solution and an u.s.c sub-solution of equation (2.3). By Theorem 4.5, we conclude that u_1 satisfies the super-optimality principle which means that for all $(t, x) \in (0, T] \times \mathbb{R}^N$, there exists a trajectory $y(.) \in S_{(t,T)}(x)$ such that

$$u_1(t, x) \geq u_1(T, y(T)).$$

Likewise, by Theorem 4.8, we conclude that u_2 satisfies the sub-optimality principle. Then for the same trajectory $y(.)$ we have

$$\forall\, (t, x) \in (0, T] \times \mathbb{R}^N, \quad u_2(t, x) \leq u_2(T, y(T)).$$

Henceforth, using the fact that $u_2(T, \cdot) \leq u_1(T, \cdot)$, we get the desired result $u_2(t, x) \leq u_1(t, x)$ for any $(t, x) \in (0, T] \times \mathbb{R}^N$. □

5. Stability

Theorem 5.1. *For $i = 1, \ldots, n + l$, let $\left(F_i^j : \overline{\mathcal{M}}_i \rightsquigarrow \mathbb{R}^N\right)_j$ be a sequence of set-valued maps satisfying (SH) and such that $F_i^j \longrightarrow F_i$ w.r.t the Hausdorff metric (i.e. uniform convergence). Let $(v^j : \mathbb{R}^N \to \mathbb{R})_j$ be a sequence of l.s.c functions such that $v^j \to v$ locally uniformly in $\mathbb{R}^N$. Suppose in addition that for all j, v^j is a super-solution of*

$$-\partial_t v^j(t, x) + \max_{i \in I(x)} \{H_{F_i^{j\sharp}}(x, \partial_x v^j(t, x))\} \geq 0 \quad \textit{for all } (t, x) \in (0, T) \times \mathbb{R}^N,$$

in the sense of Definition 2.6. Then v is a super-solution of (2.3).

Proof. Let $(t, x) \in (0, T) \times \mathbb{R}^N$. Using Remark 4.6, it suffices to prove that v is a super-solution of (4.2). Let $\phi \in C^1((0, T) \times \mathbb{R}^N)$ such that $u - \phi$ attains a local minimum at (t, x). Then, there exists $(t^j, x^j) \in (0, T) \times \mathbb{R}^N$ such that $v^j - \phi$ attains local minimum and such that $(t^j, x^j) \to (t, x)$. Since the stratification is finite and v^j is a super-solution of (2.3), then up to a subsequence (not relabeled), there exists $i_0 \in [1, n + l]$ such that for all j, we have

$$-\partial_t \phi(t^j, x^j) + H_{F_{i_0}^{j\sharp}}(x^j, \partial_x \phi(t^j, x^j)) \geq 0.$$

Since $F^{j\sharp}_{i_0}(.) \subseteq F^j_{i_0}(.)$, we get

$$-\partial_t\phi(t^j, x^j) + H_{F^j_{i_0}}(x^j, \partial_x\phi(t^j, x^j)) \geq 0.$$

So by letting j tend to infinity, we get

$$-\partial_t\phi(t, x) + H_{F_{i_0}}(x, \partial_x\phi(t, x)) \geq 0.$$

Finally, since $F_{i_0}(.) \subseteq F(.)$, then we get

$$-\partial_t\phi(t, x) + H_F(x, \partial_x\phi(t, x)) \geq 0,$$

which is the required result by Remark 4.6. □

Theorem 5.2. *For $i = 1, \ldots, n + l$, let $\left(F^j_i : \overline{\mathcal{M}}_i \rightsquigarrow \mathbb{R}^N\right)_j$ be a sequence of set-valued maps satisfying (SH). We denote By*

$$F^{j\sharp}_i(.) = F^j_i(.) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(.).$$

Suppose that $F^{j\sharp}_i \longrightarrow F^\sharp_i$ w.r.t the Hausdorff metric. Let $(v^j : \mathbb{R}^N \to \mathbb{R})_j$ be a sequence of u.s.c functions such that $v^j \to v$ locally uniformly in $\mathbb{R}^N$. Suppose in addition that for all $j \in \mathbb{N}$, v^j is a sub-solution of

$$-\partial_t v^j(t, x) + H_{F^{j\sharp}_i}(x, \partial_x v^j(t, x)) \leq 0 \quad \textit{for all } (t, x) \in (0, T) \times \mathbb{R}^N,\ i \in I(x),$$

in the sense of Definition 2.7. Then v is a sub-solution of (2.3).

Proof. Let $(t, x) \in (0, T) \times \mathbb{R}^N$ and $i \in I(x)$. Let $\phi \in C^1((0, T) \times \mathbb{R}^N)$, such that $u - \phi$ attains a local maximum in $(0, T) \times \overline{\mathcal{M}}_i$, at (t, x). Without loss of generality, we can always suppose that the maximum is strict. Then by Lemma 2.2 of [4] there exists $(t^j, x^j) \in (0, T) \times \overline{\mathcal{M}}_i$, such that $v^j - \phi$ attains local maximum in $(0, T) \times \overline{\mathcal{M}}_i$ at (t^j, x^j), and such that $(t^j, x^j) \to (t, x)$. Since v^j is a sub-solution, we get

$$-\partial_t\phi(t^j, x^j) + H_{F^{j\sharp}_i}(x^j, \partial_x\phi(t^j, x^j)) \leq 0.$$

Now let $\nu \in F^\sharp_i(x)$. Then by the Hausdorff convergence of the sequence $(F^{j\sharp}_i)_j$ there exists a sequence $\nu^j \in F^{j\sharp}_i(x^j)$ such that $\nu^j \to \nu$. Finally, we arrive at

$$-\partial_t\phi(t^j, x^j) + \langle -\nu^j, \partial_x\phi(t^j, x^j)\rangle \leq -\partial_t\phi(t^j, x^j) + H_{F^{j\sharp}_i}(x^j, \partial_x\phi(t^j, x^j)) \leq 0.$$

By letting j tend to infinity, we get

$$-\partial_t\phi(t, x) + \langle -\nu, \partial_x\phi(t, x)\rangle \leq 0.$$

Lastly, since ν is arbitrary, we take the supremum over ν and we get the required result. □

6. General convergence result for monotone schemes

In this section, we aim at studying the convergence of monotone numerical schemes approximating the HJB equation (2.3).

Let $\mathcal{G}^{\Delta x} = \bigcup_{i=1}^{n+l} \mathcal{G}_i^{\Delta x}$ be a spatial grid of $\mathbb{R}^N$ of step Δx, such that each $\mathcal{G}_i^{\Delta x}$ is a discretization of $\overline{\mathcal{M}}_i$ and $\mathcal{G}^{\Delta x}$ is compatible with the stratification $(\mathcal{M}_i)_{i=1,\dots,n+l}$ in the following sense:

$$
\text{(CC):}\begin{cases} (i) \text{ For all } i,j=1,\dots,n+l, \text{ such that } \mathcal{M}_j \subset \overline{\mathcal{M}}_i,\ \mathcal{G}_i^{\Delta x} \text{ and } \mathcal{G}_j^{\Delta x} \text{ coincide on } \mathcal{G}_j^{\Delta x}, \\ (ii)\ \forall R>0,\ \forall i=1,\dots,n+l,\ \lim_{\Delta x\to 0} d_{\mathcal{H}}\Big(\overline{\mathcal{M}}_i \cap \overline{\mathbb{B}}(0,R)\ ,\ \mathcal{G}_i^{\Delta x} \cap \overline{\mathbb{B}}(0,R)\Big) = 0. \end{cases}
$$

Comments on the hypothesis (CC)

Hypothesis (CC)(i) implies that the grid $\mathcal{G}^{\Delta x}$ is divided into $n+l$ subgrids $(\mathcal{G}_i^{\Delta x})_i$ with a partial order relation that ensures compatibility with the stratification. Hypothesis (CC)(ii) asserts that for each $i=1,\dots,n+l$, the subgrid $\mathcal{G}_i^{\Delta x}$ approaches $\overline{\mathcal{M}}_i$ in the the sense of Hausdorff convergence for locally compact sets. Notice that this implies in particular that the points of a subgrid $\mathcal{G}_i^{\Delta x}$ don't have to belong to $\overline{\mathcal{M}}_i$ meaning that we do not require that

$$
\mathcal{G}_i^{\Delta x} \subset \overline{\mathcal{M}}_i.
$$

What is important here is that the grid $\mathcal{G}^{\Delta x}$ is divided into $n+l$ subgrids compatible with the stratification, and each subgrid converges in the Hausdorff sense to its corresponding domain.

We define for any $x \in \mathcal{G}^{\Delta x}$, the index set-valued map of the grid

$$
I_{\mathcal{G}^{\Delta x}}(x) := \{\, i \in \{1,\dots,n+l\}\ :\ x \in \mathcal{G}_i^{\Delta x}\,\}.
$$

Let Δt be a constant time step of a regular grid $\Pi^{\Delta t}$ of $[0,T]$. We denote $h=(\Delta t,\Delta x)$. We consider the following numerical scheme:

$$
\begin{cases} \max_{i\in I_{\mathcal{G}^{\Delta x}}(x_h)} \{\, S_i^h(t_h,x_h,u^h(t_h,x_h),[u^h]_{(t_h,x_h)}\,\} = 0 \quad \text{for } (t_h,x_h)\in(\Pi^{\Delta t}\times\mathcal{G}^{\Delta x})\setminus\{t_h=T\}, \\ u^h(T,x_h)=\psi(x_h), \quad \text{for } (t_h,x_h)\in(\Pi^{\Delta t}\times\mathcal{G}^{\Delta x})\cap\{t_h=T\} \end{cases}
$$

with $u^h : \Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\to\mathbb{R}$ is the approximate solution and $[u^h]_{(t_h,x_h)}$ are all the values of of u^h on $\mathcal{G}^{\Delta x}$ at other points than (t_h,x_h). Each S_i^h, $i=1,\dots,n+l$, is supposed to verify the following hypotheses:

- **Monotonicity** : $S_i^h(s,z,u,[w_1]_{(s,z)}^h) \le S_i^h(s,z,u,[w_2]_{(s,z)}^h)$, if $w_1 \ge w_2$.
- **Stability** :
  - each u^h is bounded on bounded sets of $\mathbb{R}^N$ independently from h, for h small enough, *i.e.* for all $\rho>0$, there exists a $C_\rho>0$, independent of h, such that

$$
|u^h(t_i^h,x_i^h)| \le C_\rho \quad \text{if } (t_i^h,x_i^h) \in \Big([0,T]\times\mathbb{B}(0,\rho)\Big)\cap\Big(\Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\Big).
$$

  - u^h verifies the following inequality on a neighborhood of the interfaces: there exists $r>0$ and $C_r>0$, independent of h, such that

$$
|u^h(t_h,x_h)-u^h(s_h,y_h)| \le C_r(|t_h-s_h|+|x_h-y_h|), \\ \text{for all } (t_h,x_h),(s_h,y_h) \in \Big([0,T]\times(\Lambda+\mathbb{B}(0,r))\Big)\cap\Big(\Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\Big).
$$

- **Consistency** : for all $\phi \in C^1((0,T)\times\mathbb{R}^N)$ and $(t,x)\in(0,T)\times\mathbb{R}^N$, we have

$$\lim_{\substack{\Pi^{\Delta t}\times\mathcal{G}_i^{\Delta x}\ni(s,z)\to(t,x)\\ h\to 0\\ \zeta\to 0}} S_i^h(s,z,\phi(s,z)+\zeta,[\phi+\zeta]^h_{(s,z)}) = -\partial_t\phi(t,x) + H_{F_i^\sharp}(x,\partial_x\phi(t,x)),$$

where $[\phi+\zeta]^h_{(s,z)}$, is a function representing the values of $\phi+\zeta$ on the grid at other points than (s,z).

The following theorem is an extension of the result by Barles and Souganidis [9], to the case of HJB equations defined on stratified domains.

Theorem 6.1. *Suppose that the HJB equation (2.3) admits a continuous viscosity solution u and the comparison principle in Theorem 2.12 holds. Assume that for every $h>0$ small enough, the numerical scheme admits a solution u^h. Assume further that the spatial grid verifies hypothesis **(CC)** and that each S_i^h verifies the monotonicity, stability and consistency hypotheses.*

Then, u^h converges locally uniformly to u on $[0,T]\times\mathbb{R}^N$.

Proof. First, we begin by defining the following functions

$$\overline{u}(t,x) := \limsup_{\substack{\Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\ni(s,z)\to(t,x)\\ h\to 0}} u^h(s,z),\quad \underline{u}(t,x) := \liminf_{\substack{\Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\ni(s,z)\to(t,x)\\ h\to 0}} u^h(s,z).$$

We aim to prove that $\overline{u}=\underline{u}=u$. We already have $\underline{u}\le\overline{u}$. Therefore to prove the result it suffices to prove that $\underline{u}$ is a l.s.c super-solution and $\overline{u}$ is an u.s.c sub-solution of the HJB equation (2.3). We first prove that $\overline{u}$ is a sub-solution.

Let $(t,x)\in(0,T)\times\mathbb{R}^N$. Without loss of generality, we suppose that $x\in\mathbb{B}(0,\rho)$ for some $\rho>0$ big enough and we restrict our analysis on this bounded open set. Let $i\in I(x)$ and let $\phi\in C^1((0,T)\times\mathbb{R}^N)$ such that $\overline{u}_i-\phi$ attains its local maximum at $(t,x)\in(0,T)\times\overline{\mathcal{M}_i}$. We recall that $\bar{u}_i = u$ in $(0,T)\times\overline{\mathcal{M}_i}$ and $u_i\equiv-\infty$ otherwise.

Without loss of generality, we can suppose that

$$\overline{u}_i(t,x)=\phi(t,x),\quad \overline{u}_i(s,z)<\phi(s,z) \text{ if } (s,z)\neq(t,x),$$

$$\phi\ge C_\rho+1 \text{ outside of a neighborhood } \Omega \text{ of } (t,x),\quad \Omega\subsetneq(0,T)\times\mathbb{B}(0,\rho),$$

where C_ρ is defined from the first part of the stability assumption. Furthermore, from the second part of the stability assumption, $\overline{u}$ is Lipschitz continuous in a neighborhood of the interfaces. So we get

$$0=\overline{u}_i(t,x)-\phi(t,x) = \limsup_{\substack{\Pi^{\Delta t}\times\mathcal{G}^{\Delta x}\ni(s,z)\to(t,x)\\ h\to 0}} u^h(s,z)-\phi(s,z) = \limsup_{\substack{\Pi^{\Delta t}\times\mathcal{G}_i^{\Delta x}\ni(s,z)\to(t,x)\\ h\to 0}} u^h(s,z)-\phi(s,z).$$

The second part of the stability assumption is essential here to get the last equality since the limit sup might not be reached from every subsgrid $\mathcal{G}_i^{\Delta x}$. Moreover, outside of Ω, we have $\overline{u}_i-\phi\le-1$. So, there exists $r>0$, such that

$$0\ge u^h(s,z)-\phi(s,z)\ge -1 \text{ for all } (s,z)\in([t-r,t+r]\cap\Pi^{\Delta t})\times(\overline{\mathbb{B}}(0,r)\cap\mathcal{G}_i^{\Delta x})\subset\Omega.$$

So, the maximum of $u^h-\phi$ is attained in the compact set

$$([t-r,t+r]\cap\Pi^{\Delta t})\times(\overline{\mathbb{B}}(0,r)\cap\mathcal{G}_i^{\Delta x})\subset\Omega.$$

Let (t_i^h, x_i^h) be the maximum and (t_i, x_i) the limit when $h \to 0$ of a subsequence not relabeled. we have

$$\lim_{h\to 0} u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h) \geq \limsup_{\substack{\Pi^{\Delta t}\times \mathcal{G}_i^{\Delta x} \ni (s,z)\to(t,x) \\ h\to 0}} u^h(s,z) - \phi(s,z) = \overline{u}_i(t,x) - \phi(t,x) = 0.$$

On the other hand, since $\overline{u}_i - \phi$ is u.s.c, we get

$$0 \geq \overline{u}_i(t_i, x_i) - \phi(t_i, x_i) \geq \lim_{h\to 0} u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h).$$

Thus, we conclude

$$(t_i, x_i) = (t,x), \quad u^h(t_i^h, x_i^h) \to \overline{u}_i(t,x).$$

Let $\zeta_h := u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h)$. We get

$$u^h(t_i^h, x_i^h) = \phi(t_i^h, x_i^h) + \zeta_h, \quad u^h(s,z) \leq \phi(s,z) + \zeta_h, \ (t_i^h, x_i^h) \neq (s,z) \in \Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x}.$$

From the monotonicity of the scheme and u^h being a solution, we get

$$S_i^h(t_i^h, x_i^h, u^h(t_i^h, x_i^h), [\phi + \zeta_h]^h_{(t_i^h, x_i^h)}) \leq S_i^h(t_i^h, x_i^h, u^h(t_i^h, x_i^h), [u^h]_{(t_i^h, x_i^h)}) \leq 0,$$

and by the consistency hypothesis and passing to the limit, we get

$$\lim_{\substack{(t_i^h, x_i^h)\to(t,x) \\ h\to 0 \\ \zeta_h \to 0}} S_i^h(t_i^h, x_i^h, \phi(t_i^h, x_i^h) + \zeta_h, [\phi + \zeta_h]^h_{(t_i^h, x_i^h)}) = -\partial_t \phi(t,x) + H_{F_i^\sharp}(x, \partial_x \phi(t,x)) \leq 0.$$

This is true for any $i \in I(x)$, which ends the proof.

Now we prove that $\underline{u}$ is a super-solution. Following Remark 4.6, it suffices to prove that $\underline{u}$ is a super-solution of (4.2).

Let $(t,x) \in (0,T) \times \mathbb{R}^N$. Let $\phi \in C^1((0,T)\times\mathbb{R}^N)$ such that $\underline{u} - \phi$ attains its local minimum in $(0,T)\times\mathbb{R}^N$ at (t,x) and $\phi(t,x) = \underline{u}(t,x)$. Using the same arguments as in the first part of the proof, we get a sequence $(h_n)_n$ such that

$$h_n \downarrow 0, \quad u^{h_n} - \phi \text{ attains a local maximum at } (t_n, x_n) \in \Pi^{\Delta t} \times \mathcal{G}^{\Delta x}, \text{ and}$$

$$(t_n, x_n) \to (t,x), \quad u^{h_n}(t_n, x_n) \to \underline{u}(t,x).$$

Furthermore, since the stratification is finite and u^{h_n} is a solution to the numerical scheme, there exists a subsequence (not relabeled) of $(h_n)_n$ such that there exists $i_0 \in I_{\mathcal{G}^{\Delta x}}(x_n)$, for all $n \in \mathbb{N}$ and $(x_n)_n \subseteq \mathcal{G}_{i_0}^{\Delta x}$ such that we have

$$\max_{i \in I_{\mathcal{G}^{\Delta x}}(x_n)} S_i^{h_n}(t_n, x_n, u^{h_n}(t_n, x_n), u^{h_n}) = S_{i_0}^{h_n}(t_n, x_n, u^{h_n}(t_n, x_n), u^{h_n}) \geq 0.$$

Let $\zeta_n := u^{h_n}(t_n, x_n) - \phi(t_n, x_n)$. So

$$u^{h_n}(t_n,x_n)=\phi(t_n,x_n)+\zeta_n, \quad u^{h_n}(s,z)\geq \phi(s,z)+\zeta_h,\ (t_i^h,x_i^h)\neq (s,z)\in \Pi^{\Delta t}\times \mathcal{G}^{\Delta x}.$$

From the monotonicity assumption and u^{h_n} being a solution, we get

$$S_{i_0}^{h_n}(t_n,x_n,\phi(t_n,x_n)+\zeta_h,[\phi+\zeta_h]^{h_n}_{(t_n^h,x_n^h)})\geq S_{i_0}^{h_n}(t_n,x_n,u^{h_n}(t_n,x_n),[u^{h_n}]^{h_n}_{(t_n^h,x_n^h)})\geq 0.$$

and by the consistency hypothesis and passing to the limit, we get

$$\lim_{\substack{(t_n,x_n)\to(t,x)\\ h_n\to 0\\ \zeta_n\to 0}} S_{i_0}^{h_n}(t_n,x_n,\phi(t_n,x_n)+\zeta_h,[\phi+\zeta_h]^{h_n}_{(t_n^h,x_n^h)})=-\partial_t\phi(t,x)+\ H_{F_i^\sharp}(x,\partial_x\phi(t,x))\geq 0.$$

By Remark 4.6 and the fact that $F_i^\sharp(.)\subseteq F(.)$, we get the required result.

Finally, with similar arguments as above, we prove that at time $t=T$, $\overline{u}$ (resp. $\underline{u}$) is sub-solution (resp. super-solution) of

$$\min\left(-\partial_t u(t,x)+\max_{i\in I(x)}\{H_{F_i^\sharp}(x,\partial_x u(t,x))\},u(T,x)-\psi(x)\right)\leq 0,$$

resp.

$$\max\Big(-\partial_t u(t,x)+H_F(x,\partial_x u(t,x)),u(T,x)-\psi(x)\Big)\geq 0,$$

for $(t,x)\in(0,T]\times\mathbb{R}^N$. Finally, by the same reasoning as in Theorem 4.7 of [4], we obtain that

$$\overline{u}(T,\cdot)\leq\psi(.)\leq\underline{u}(T,\cdot).$$

In conclusion, $\overline{u}(.,.)=\underline{u}(.,.)=u(.,.)$ and u^h converges locally uniformly to u, which ends the proof. □

Appendix A. Relative wedgeness

This appendix presents the concept of relative wedgeness first introduced in [10]. Let $S\subset\mathbb{R}^N$ be a closed set. S is said to be proximally smooth if there exists $R>0$ such that the projection map $proj_S(.)$ is a singleton on the set $\{\,x\in\mathbb{R}^N\ :\ d_S(x)<R\,\}$. If S is proximally smooth, then its Clarke tangent cone is equal to its Bouligand tangent cone $\mathcal{T}_S(.)$. Clarke tangent cone is always closed and convex.

Now, let $\mathcal{M}$ be a C^2 embedded manifold in $\mathbb{R}^N$ such that $\overline{\mathcal{M}}$ is proximally smooth and let $d=dim(\mathcal{M})$ be its dimension. Then, for every $x\in\overline{\mathcal{M}}$, the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is closed and convex, hence it has a relative interior (in the sense of convex analysis), denoted by $r\text{-}int(\mathcal{T}_{\overline{\mathcal{M}}}(x))$.

The set $\overline{\mathcal{M}}$ is said to be relatively wedged if for every $x\in\overline{\mathcal{M}}$, the dimension of $r\text{-}int(\mathcal{T}_{\overline{\mathcal{M}}}(x))$ (in the sense of convex analysis) is equal to the dimension of $\mathcal{M}$:

$$dim(r\text{-}int(\mathcal{T}_{\overline{\mathcal{M}}}(x))=dim(\mathcal{M})=d.$$

Appendix B. Lower semicontinuity of the essential dynamics

In this section, we give sufficient conditions for Hypothesis (H_{ESS}) to hold. For $i = 1, \dots, n+l$, we recall from Section 2 that the essential dynamics $F_i^\sharp(.)$ defined on $\overline{\mathcal{M}}_i$ is of the form

$$F_i^\sharp(x) = F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x), \quad \forall x \in \overline{\mathcal{M}}_i.$$

We suppose that the dynamics $F_i(.)$ verify Hypotheses (SH), (CH) and (H_D). Let $(\mathcal{M}_i)_{i=1,\dots,n+l}$ be a stratification of $\mathbb{R}^N$ such that any $\overline{\mathcal{M}}_i$ is either vector subspace of $\mathbb{R}^N$ or a half space of a vector subspace of $\mathbb{R}^N$. All Examples 2.1 and 2.2 of a stratification of $\mathbb{R}^N$ verify this condition. Furthermore, it immediately follows that the stratification verifies (H_1). If $\overline{\mathcal{M}}_i$ is a vector subspace of $\mathbb{R}^N$, then we have $\mathcal{M}_i = \overline{\mathcal{M}}_i$. Consequently we get

$$F_i(.) = F_i^\sharp(.), \quad \forall x \in \overline{\mathcal{M}}_i.$$

Therefore, $F_i^\sharp(.)$ is locally Lipschitz continuous. Hence it is l.s.c.

Suppose now that $\overline{\mathcal{M}}_i$ is a half space of a vector subspace of $\mathbb{R}^N$. For simplicity we denote the vector subspace by $\mathbb{E} \subset \mathbb{R}^N$. Since $\overline{\mathcal{M}}_i$ is a convex subset of $\mathbb{E}$, then by Corollary 3.6.13 of [15], the set-valued map

$$\overline{\mathcal{M}}_i \ni x \rightsquigarrow \mathcal{T}_{\overline{\mathcal{M}}_i}(x)$$

is l.s.c as a set-valued map from $\overline{\mathcal{M}}_i$ to $\mathbb{E}$. Furthermore, since $\overline{\mathcal{M}}_i$ is a half space of $\mathbb{E}$, then we have

$$\mathcal{T}_{\overline{\mathcal{M}}_i}(x) = \overline{\mathcal{M}}_i, \quad \forall x \in \overline{\mathcal{M}}_i.$$

Hence, $\mathcal{T}_{\overline{\mathcal{M}}_i}(x)$ is convex with nonempty interior in $\mathbb{E}$ for all $x \in \overline{\mathcal{M}}_i$. On the other hand, by Hypotheses (CH) and (H_D) the set-valued map $x \rightsquigarrow F_i(x)$ is l.s.c as a set-valued map from $\overline{\mathcal{M}}_i$ with images in $\mathbb{E}$ that are convex and have nonempty interior. Therefore, following Theorem B of [28], the set-valued map

$$F_i^\sharp(x) = F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x)$$

is l.s.c as a set-valued map from $\overline{\mathcal{M}_i}$ to $\mathbb{E}$. Whence, $x \rightsquigarrow F_i^\sharp(x)$ is l.s.c as a set-valued map from $\overline{\mathcal{M}}_i$ to $\mathbb{R}^N$.

The condition (CH) is merely a sufficient condition for (H_{ESS}) to hold. Example 2.16 gives a setting where the dynamics do not verify (CH) and (H_{ESS}) still holds.

Appendix C. Proof of invariance theorems

Proof. (**Thm. 3.5**). Since F is an u.s.c set-valued map with convex, compact non-empty images, and since S is a closed set of $\mathbb{R}^N$, it is known that assertion (i) is equivalent to (see for instance [13], Thm. 12.11)

$$H_F(x, \eta) \geq 0 \quad \forall \eta \in N_S^p(x),\ \forall x \in \mathbb{R}^N.$$

Let $x \in \mathbb{R}^N$. We have $F_i^\sharp(.) \subseteq F_i(.) \subseteq F(.)$. We have

$$H_{F_i}(x, \eta_i) \geq H_{F_i^\sharp}(x, \eta_i), \quad \forall \eta_i \in N_{S_i}^p(x). \tag{C.1}$$

From this inequality, we deduce easily the implication $(iii) \implies (ii)$. Moreover, since $N_S^p(x) \subseteq N_{S_i}^p(x)$ for all $i \in I(x)$, then

$$H_F(x, \eta) \geq \max_{i \in I(x)} H_{F_i}(x, \eta) \geq \max_{i \in I(x)} H_{F_i^\sharp}(x, \eta), \quad \forall \eta \in N_S^p(x).$$

Hence, the implication $(ii) \Longrightarrow (i)$ holds. It remains to prove the implication **(i)** $\Longrightarrow$ **(iii)**. Suppose (S, F) is weakly invariant. Let $x \in \mathbb{R}^N$ and $t \in [0, T]$. So there exists a trajectory $y(.)$ solution of $(DI)_F(t, x)$ such that $y(.) \subset S$. We claim the following:

Claim: $\exists\, j \in I(x)$ such that there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \to \nu$ and $\nu \in F_j^\sharp(x)$.

The proof of the claim is the same as the proof of the same claim in Proposition 4.5. With this claim, we are almost done. Indeed let $\eta_j \in N^p_{S_j}(x)$ be such that the proximal normal inequality is satisfied with $\sigma > 0$. we get

$$\langle \nu, \eta_j \rangle = \lim_{n \to +\infty} \big\langle \frac{x_n - x}{t_n - t}, \eta_j \big\rangle \leq \lim_{n \to +\infty} \frac{1}{2\sigma(t_n - t)} |x_n - x|^2 = 0.$$

Thus, we have

$$H_{F_j^\sharp}(x, \eta_j) \geq -\langle \nu, \eta_j \rangle \geq 0,$$

Which is the required result. $\square$

Proof. (**Thm. 3.6**). The implication $(ii) \Longrightarrow (i)$ is proven first. We separate the proof into 3 parts. First we prove the result for every trajectory that lies entirely in one of the domains $\mathcal{M}_i$, $i = 1, \dots, n + l$. Then, we prove the result for every trajectory that does not present any chattering phenomenon, also known as Zeno effect, using an induction argument. Finally we prove the result for every trajectory using Filippov's theorem ([14], Thm. 3.1.6).

Step 1. (Inspired from Theorem 12.15 of [13]). Let $y(.)$ be a trajectory of F such that $y((t, T)) \subset \mathcal{M}_i$ and such that $y(t) = \alpha \in S \cap \overline{\mathcal{M}}_i$. We show that for some $\varepsilon \in (0, T - t)$, we have $y([t, t + \varepsilon]) \subset S$ which is sufficient to conclude.

Let $r > 0$ small enough such that $\mathbb{B}(\alpha, r) \cap \mathcal{M}_i$ is a relative neighborhood in $\mathcal{M}_i$. Let $\kappa > 0$ be The Lipschitz constant of F_i on $\mathbb{B}(\alpha, r)$ and $||F_i|| > 0$ be an upper bound for any velocities that my appear in $\mathbb{B}(\alpha, r)$. So $y(.)$ is Lipschitz continuous on $[t, t + \varepsilon]$. There exists $\varepsilon \in (0, T - t)$ such that

$$\forall\, \tau \in [t, t + \varepsilon], \;\; s \in proj_{S_i}(y(\tau)) \Longrightarrow y(\tau) \in \mathbb{B}(\alpha, r) \cap \mathcal{M}_i, \;\; s \in \mathbb{B}(\alpha, r) \cap \mathcal{M}_i.$$

We define $f(\tau) := d_{S_i}(y(\tau))$. f is Lipschitz continuous on $[t, t + \varepsilon]$. We prove the following Lemma:

Lemma C.1. *$f'(\tau) \leq \kappa f(\tau)$ for almost all $\tau \in (t, t + \varepsilon)$.*

Proof. Let $\tau_* \in (t, t + \varepsilon)$ such that $f'(\tau_*)$ exists, $y'(\tau_*)$ exists and $y'(\tau_*) \in F_i(y(\tau_*))$ (almost all points satisfy those conditions). If $f(\tau_*) = 0$ then f attains a minimum at τ_* and therefore $f'(\tau_*) = 0$ and the inequality holds. Suppose now $f(\tau_*) > 0$ and let $s \in proj_{S_i}(y(\tau_*))$. Then by Proposition 11.29 of [13] we have

$$\eta := \frac{y(\tau_*) - s}{|y(\tau_*) - s|} \in N^p_{S_i}(s).$$

Since $F_i (= F_i^\sharp$ on $\mathcal{M}_i)$ is Lipschitz continuous on $\mathbb{B}(\alpha, r)$ with constant κ, there exists $\nu \in F_i(s)$ such that

$$|y'(\tau_*) - \nu| \leq \kappa |y(\tau_*) - s|.$$

Therefore we get

$$\langle \eta, y'(\tau_*) \rangle = \langle \eta, \nu \rangle + \langle \eta, y'(\tau_*) - \nu \rangle \leq H_{F_i^\sharp}(s, -\eta) + \kappa |y(\tau_*) - s| \leq \kappa |y(\tau_*) - s|,$$

where the last inequality is obtained since $H_{F_i^\sharp}(s,-\eta)\leq 0$ by assumption (ii). Hence, we get

$$f'(\tau_*)=\lim_{\delta\to 0}\frac{d_{S_i}(y(\tau_*+\delta))-d_{S_i}(y(\tau_*))}{\delta}\leq \lim_{\delta\to 0}\frac{|y(\tau_*+\delta)-s|-|y(\tau_*)-s|}{\delta}$$
$$=\langle \eta, y'(\tau_*)\rangle \leq \kappa|y(\tau_*)-s|=kf(\tau_*).$$

□

Since f is Lipschitz, positive and $f(t)=0$ $(y(t)=\alpha\in S_i)$, then by using the Lemma above and Gronwall Lemma ([13], Thm. 6.41), we get that $f\equiv 0$ on $[t,t+\varepsilon]$, which finishes the proof of step 1.

Notice that, every trajectory that lies entirely in $\cup_{i=1}^n\mathcal{M}_i$ also verifies Step 1 since it is a disjoint union.

Step 2. Let $\mathcal{M}$ be a union of subdomains such that $\cup_{i=1}^n\mathcal{M}_i\subseteq\mathcal{M}$ and denote by $\delta_{\mathcal{M}}$ the minimum dimension of the subdomains of $\mathcal{M}$. Let $\mathcal{M}_{k_0}$ be a subdomain such that $\mathcal{M}_{k_0}\subset\overline{\mathcal{M}}\setminus\mathcal{M}$ and its dimension is inferior or equal to $\delta_{\mathcal{M}}$. We show the following proposition:

Proposition C.2. *If we have $(ii)\Longrightarrow(i)$ for every trajectory that lies entirely in $\mathcal{M}$ or lies entirely in $\mathcal{M}_{k_0}$, then $(ii)\Longrightarrow(i)$ holds true for every trajectory that lies in $\mathcal{M}\cup\mathcal{M}_{k_0}$.*

Proof. Let $y(.)\subseteq\mathcal{M}\cup\mathcal{M}_{k_0}$ be a trajectory of F on $[t,T]$ such that $y(t)\in S$. We define

$$J=\{\tau\in[t,T]\ :\ y(\tau)\notin\mathcal{M}_{k_0}\}.$$

The set J is open since $\mathcal{M}_{k_0}$ is of inferior dimension than $\mathcal{M}$, then it is a closed set relative to $\mathcal{M}\cup\mathcal{M}_{k_0}$ (equipped with the inherited topology from $\mathbb{R}^N$). Thus J can be written as a countable union of open intervals in the following way:

$$J=\bigcup_{i=1}^{\infty}(a_i,b_i),$$

such that the open sets (a_i,b_i) are pairwise disjoint and $a_i<b_i\leq a_{i+1}$, $\forall i\geq 1$. Notice that we necessarily have $y(a_i),y(b_i)\in\mathcal{M}_{k_0}$. Set $b_0=t$. First, we prove that

$$\forall\, i\in\mathbb{N},\ y(b_i),y(a_{i+1})\in S \text{ and } \quad y((a_{i+1},b_{i+1}))\subset S.$$

We have $y(b_0)\in S$ by assumption. If $b_0=a_1$ then we have $y(a_1)\in S$. If $b_0<a_1$ then $y(b_0)\in S\cap\mathcal{M}_{k_0}$ and $y((b_0,a_1))\subset\mathcal{M}_{k_0}$ almost everywhere. Hence, by assumption of the proposition, we have $y((b_0,a_1))\subset S$ and therefore $y(a_1)\subset S$ since S is a closed set. Moreover, since $y(a_1)\in S\cap\overline{\mathcal{M}}$ and $y((a_1,b_1))\subset\mathcal{M}$, then by assumption of the proposition, we have that $y((a_1,b_1))\subset S$ and therefore $y(b_1)\in S$ since S is closed. By induction, following the same argument, we get that

$$\forall\, i\in\mathbb{N},\ y(b_i),y(a_{i+1})\in S \text{ and } \quad y((a_{i+1},b_{i+1}))\subset S.$$

It remains to prove $y([t,T]\setminus J)\subset S$. If the set J was equal to a finite union of open intervals then the above argument would have been sufficient to prove that $y([t,T]\setminus J)\subset S$. However, this is not the case for all trajectories $y(.)$. The trajectories $y(.)$ can move in and out of $\mathcal{M}_{k_0}$ infinitely many times exhibiting the phenomenon known as the *Zeno effect*, or can reside in $\mathcal{M}_{k_0}$ for sets of time that have a strictly positive Lebesgue measure but are nowhere dense in $[t,T]$ (think of a Cantor set for example). To deal with this case, we will approximate such trajectories $y(.)$ by ones such that that behave well on the partition J.

We fix $m \geq 1$ and set

$$J_m = \bigcup_{k=1}^{m} (a_k, b_k),$$

Which we can assume to satisfy

$$t = b_0 \leq a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_m < b_m \leq a_{m+1} := T.$$

We choose m large enough such that

$$\mathscr{L}(J \setminus J_m) < \frac{r}{2e^{\kappa T}||F||},$$

with κ being the Lipschitz constant of F_{k_0} and r is equal to

$$r := \inf_{\substack{w \in \overline{\mathcal{M}}_{k_0} \setminus \mathcal{M}_{k_0} \\ s \in [t,T]}} |y(s) - w|,$$

(notice that r is strictly positive and can be infinite). The choice of m is made in such a way to be able to apply Filippov's approximation theorem ([14], Thm. 3.1.6) on manifolds (see [10], Rem. 3.1). 9 We will approximate the arc $y([b_i, a_{i+1}])$, for some $i = 0, \ldots, m$, by trajectories that remain entirely in $\mathcal{M}_{k_0}$. By Filippov's approximation theorem ([14], Thm. 3.1.6) and [10], Prop. 3.2), there exists $z_i(.)$ a trajectory of F_{k_0} on $[b_i, a_{i+1}]$ such that $z_i(b_i) = y(b_i) \in \mathcal{M}_{k_0} \cap S$, $z_i(.) \subset \mathcal{M}_{k_0}$ and

$$||y(.) - z_i(.)||_{L^\infty[b_i, a_{i+1}]} \leq e^{\kappa(a_{i+1} - b_i)} \rho_i \leq 2\, e^{\kappa T}\, ||F|| \varepsilon_i,$$

where we denote $\varepsilon_i = \mathscr{L}(J \cap (b_i, a_{i+1}))$ and

$$\rho_i := \int_{b_i}^{a_{i+1}} d(\dot{y}(s), F_{k_0}(z_i(s)))\mathrm{d}s \leq 2\, ||F|| \varepsilon_i.$$

Since $\varepsilon_i \leq \mathscr{L}(J \backslash J_m)$, we get

$$||y(.) - z_i(.)||_{L^\infty[b_i, a_{i+1}]} \leq 2\, e^{\kappa T}\, ||F|| \mathscr{L}(J \backslash J_m).$$

Furthermore, from the assumption of the proposition we have $z_i(.) \subset S$. Thus we get

$$d_S(y(.)) \leq ||y(.) - z_i(.)||_{L^\infty[b_i, a_{i+1}]} \leq 2\, e^{\kappa T}\, ||F|| \mathscr{L}(J \backslash J_m), \qquad \forall m \geq 1.$$

By letting $m \to \infty$, we have $\mathscr{L}(J \backslash J_m) \to 0$. Therefore $y(.) \subset S$, which is the required result. □

Step 3. From the above Proposition, we deduce that $(ii) \Longrightarrow (i)$ by a simple finite induction argument starting from $\mathcal{M} = \cup_{i=1}^{n} \mathcal{M}_i$ and adding an interface $\mathcal{M}_{k_0} \subset \Lambda$, with $k_0 \in \{n+1, \ldots, n+l\}$, in such a way that decreases the dimension of $\mathcal{M}_{k_0}$ at each iteration.

Now we prove the direct implication $(i) \Longrightarrow (ii)$. For that, we use Lemma 3.9 of [32]. See also Proposition 5.1 and Lemma 5.2 of [10]. Suppose (S, F) is strongly invariant. Let $x \in S \cap \overline{\mathcal{M}}_i$, $\nu \in F_i^\sharp(x)$ and $\eta \in N_{S_i}^p(x)$ such that $|\eta| = 1$ a proximal normal realised at $\sigma > 0$ (from the definition of the proximal normal).

Since $\nu \in F_i^\sharp(x)$, then by Lemma 3.9 of [32], there exists a C^1 trajectory $y(.)$ of $F_i^\sharp$, defined on some interval $[t, t+\varepsilon]$ with $\varepsilon > 0$ such that $y(t) = x$ and $\dot{y}(t) = \nu$ and $y(.) \subseteq \overline{\mathcal{M}}_i$. By the strong invariance hypothesis, we have $y(.) \subseteq S_i$. So we get

$$\langle \nu, \eta \rangle = \lim_{\tau \downarrow t} \langle \frac{y(\tau) - x}{\tau - t}, \eta \rangle \leq \lim_{\tau \downarrow t} \frac{1}{2\sigma(\tau - t)} |y(\tau) - x|^2 = 0.$$

By taking the supremum over $F_i^\sharp(x)$, we obtain the desired result. □

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