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HOMOTOPY TRUST-REGION METHOD FOR PHASE-FIELD APPROXIMATIONS IN PERIMETER-REGULARIZED BINARY OPTIMAL CONTROL

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Abstract. We consider optimal control problems that have binary-valued control input functions and a perimeter regularization. We develop and analyze a trust-region algorithm that solves a sequence of subproblems in which the regularization term and the binarity constraint are relaxed by a non-convex energy functional. We show how the parameter that controls the distinctiveness of the resulting phase field can be coupled to the trust-region radius updates and be driven to zero over the course of the iterations in order to obtain convergence to points that satisfy a first-order optimality condition of the limit problem under suitable regularity assumptions. Finally, we highlight and discuss the assumptions and restrictions of our approach and provide the first computational results for a motivating application in the field of control of acoustic waves in dissipative media.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ be a bounded Lipschitz domain. We are interested in solving optimal control problems of the following form:

$$\begin{cases} & \min_{u,w} \; J(u) + \gamma C_0 P_\Omega(w^{-1}(\{1\})), \\ \text{with} & \\ & w \in W^0_{\mathrm{ad}} \coloneqq \mathrm{BV}(\Omega, \{0,1\}), \quad u \in \mathcal{U}, \\ \text{subject to} & \\ & u = S(w). \end{cases} \tag{P}$$

In (P), J is the principal part of the objective and depends on $u \in \mathcal{U}$, where $\mathcal{U}$ is the state space of an underlying (initial) boundary-value problem for a partial differential equation (PDE). Further, w is the control input of

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the PDE and may only attain the values 0 and 1, which is reflected by our notation $\mathrm{BV}(\Omega, \{0,1\})$ for the space of functions of bounded variation on Ω that are $\{0,1\}$-valued almost everywhere (a.e.).

We assume that the PDE is uniquely solvable for any admissible control function w and we denote the solution map by $S : W_{\mathrm{ad}}^0 \to \mathcal{U}$. The optimization problem is regularized with the term $\gamma C_0 P_\Omega(w^{-1}(\{1\}))$ with positive constants $C_0 > 0$ and $\gamma > 0$ and the perimeter of the level set $w^{-1}(\{1\})$ in Ω, that is, the function P_Ω gives the perimeter of a set in Ω. We recall that for our binary setting, $P_\Omega(w^{-1}(\{1\})) = \mathrm{TV}(w)$ if $\mathrm{TV} : L^1(\Omega) \to [0,\infty]$ is the total variation seminorm on Ω.

Optimization problems of the form (P) arise, for example, in acoustics in the context of optimal focusing of ultrasound waves and are particularly relevant in acoustic imaging [1] and applications of high-intensity ultrasound waves [2]. Focusing can be achieved through the use of acoustic lenses, which represent regions with different acoustic material properties (*e.g.*, the speed of sound). Thus, problems of optimal focusing can generally be treated with algorithmic approaches from optimal control and shape and topology optimization. Excellent overviews on such methods can be found in [3, 4]. We refer to [5] for a rigorous study of an optimal control problem for a linear wave equation with the control in the propagation speed. We also mention the works [6, 7], which use a phase-field method to solve the problem of optimal acoustic focusing by reframing it as an optimal control problem with the control in the phase-field function.

In this work, we combine a trust-region algorithm, see [8–10] for recent work on trust-region algorithms in topology optimization, with ideas from phase-field models. Instead of replacing (P) with a phase-field model, we replace the perimeter functional $C_0 P(w^{-1}(\{1\}))$ by the Ginzburg–Landau energy E_ε in the subproblems generated by the trust-region algorithm and drive $\varepsilon \searrow 0$, where ε controls the distinctiveness or non-binarity of the phase field. More precisely, for $\varepsilon > 0$, E_ε is defined by

$$E_\varepsilon(w) \coloneqq \begin{cases} \displaystyle\int_\Omega \frac{\varepsilon}{2}|\nabla w|^2 + \frac{1}{\varepsilon}\Psi(w)\,\mathrm{d}x & \text{if } w \in H^1(\Omega), \\ \infty & \text{else,} \end{cases}$$

where Ψ is the superposition operator that is defined for all $w \in L^1(\Omega)$ and almost every (a.ee.) $x \in \Omega$ by

$$\Psi(w)(x) \coloneqq \begin{cases} w(x)(1 - w(x)) & \text{if } w(x) \in [0,1], \\ \infty & \text{else.} \end{cases}$$

Consequently, the feasible set of our subproblems is relaxed to $W_{\mathrm{ad}}^\varepsilon \coloneqq H^1(\Omega; [0,1])$, the space of weakly differentiable $L^2(\Omega)$-functions with weak derivative in $L^2(\Omega)$ that are $[0,1]$-valued a.e.

To the best of our knowledge, there is not much work on the question of how to drive $\varepsilon \searrow 0$ in optimization algorithms in order to obtain beneficial properties for the limit. We start from the recent work on a trust-region algorithm [9, 11] that allows for non-trivial geometric and thus topological changes of the level sets of the control functions. We study a modification of this approach, where the perimeter regularization is replaced by the Ginzburg–Landau energy and ε is driven to zero over the course of the iterations. In this way, the smoothness allows for a relatively straightforward consistent discretization of the regularization term. In contrast, a consistent numerical analysis for the subproblems proposed in [9] is difficult because established strategies for the discretization of the perimeter regularizer, see, for example, [12–14], are not directly applicable, as they use convex duality and that the projection of controls to piecewise constant functions on a given grid yields a feasible function. Both properties are not available for (P) or induced subproblems that do not relax the binarity constraint as in [9].

1.1. Contributions

We develop a homotopy trust-region algorithm that solves subproblems, in which the main part of the objective $J \circ S$ is linearized and the regularizer is approximated with the Ginzburg–Landau energy functional

E_ε. We assume that the trust-region subproblems can be solved exactly and limit points of our iterates admit a certain regularity condition, see the assumptions of Theorem 4.12, which can be interpreted as a constraint qualification when relating the approach to nonlinear programming algorithms. Under these conditions, we prove convergence of subsequences of the iterates to so-called L-stationary points of the limit problem (P), where L-stationarity is the first-order optimality condition from [9].

Our proof uses a compactness result that requires boundedness of the Ginzburg–Landau energies for $\varepsilon \searrow 0$ for (a subsequence of) the iterations of our algorithm. The boundedness is not trivial to establish because the Ginzburg–Landau energy may increase when ε is reduced. We establish this property by showing how to construct a suitable competitor that eventually becomes feasible for the trust-region subproblems and then implies reductions of the objective whenever the Ginzburg–Landau energy exceeds a certain threshold before the next reduction of ε is triggered.

Since our trust-region subproblems are non-convex, we also show how to integrate convex subproblems' solves into the algorithm in order to accelerate it, while our convergence analysis only hinges on the non-convex solves so far. This is due to the fact that we do not know how Cauchy points, which usually guide the convergence analysis of trust-region algorithms, could be constructed from the purely local information that is given by the violation of L-stationarity condition.

Consequently, our convergence analysis requires three relatively strong assumptions, which open up ample possibilities for future research on their alleviation and handling and thus improving the analyzed algorithm. First, we need to solve a class of non-convex subproblems to global optimality in order to guarantee the asymptotic properties. Second, we need an additional regularity condition (constraint qualification) on the (reduced) boundaries $\partial^* \bar{w}^{-1}(\{1\})$ of the level sets of our limit points. Third, the regularity assumptions on S that are required by the algorithm currently have a gap to the regularity that we can assert for our motivating application. While we focus on the functionals E_ε introduced above in this work, we believe that our results can be carried over to other families of energy functionals that approximate the perimeter functional, see, for example, [15, 16].

1.2. Structure of the remainder

After providing some guidance on our notation, we formally introduce our problem and recap the L-stationarity concept from [9] in Section 2. We prove the L-stationarity in slightly more generality than is done in [9] so that only the first derivatives are involved. We then introduce and describe the analyzed algorithm as well as its subproblems in Section 3, where we also provide a Γ-convergence type result for the trust-region subproblems when driving $\varepsilon \searrow 0$. In Section 4, we show our main results, the asymptotic boundedness and L-stationarity of subsequences of our iterates. In Section 5, we verify our regularity assumptions on S for an elliptic PDE and show how far we can currently get for a linear acoustic equation when the design variable enters the propagation speed coefficient. Moreover, we provide a discretization of the acoustic equation and show some preliminary computational results of our algorithm in this case, where we resort to convex subproblem solves. Finally, we provide a brief conclusion in Section 6.

1.3. Notation and assumptions

We assume throughout that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. In the context of acoustic waves, $T > 0$ denotes the final propagation time, which is given and fixed. We use $H^1(\Omega)^*$ to denote the dual of $H^1(\Omega)$. When denoting norms on Bochner spaces, we often omit the temporal domain; that is, $\|\cdot\|_{L^p(L^q(\Omega))}$ denotes the norm on $L^p(0,T;L^q(\Omega))$. Note that we also replace $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^1(\Omega)}$ by $\|\cdot\|_{H^1}$ in several of our proofs when there are no ambiguities in the interest of a clean and compact presentation. We use $x \lesssim y$ to denote $x \le Cy$, where $C > 0$ is a generic constant. For a set of finite perimeter (Caccioppoli set) E, we denote its reduced boundary by $\partial^* E$. We recall that for a Caccioppoli set E with $\{0,1\}$-valued indicator function χ_E, the reduced boundary $\partial^* E$ is given as the set of points $x \in \operatorname{supp} D\chi_E$ such that

$$\nu_E(x) \coloneqq \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$$

exists and satisfies $\|\nu_E(x)\| = 1$. Here, $D\chi_E$ denotes the weak derivative of χ_E. We refer to Section 15 in [17] for more details. For a set A, δ_A denotes its $\{0,\infty\}$-valued indicator function. We write $\mathcal{H}^{d-1}$ for the $d-1$-dimensional Hausdorff measure on $\mathbb{R}^d$.

2. Abstract problem formulation and optimality conditions

For the statement and analysis of the asymptotics of our algorithm, we replace $J(u)$ and $u = S(w)$ in (P) by the reduced objective $j(w) \coloneqq J(S(w))$, which gives the following abstract and handy problem formulation for $C_0 > 0$:

$$\begin{aligned} \min_{w \in L^2(\Omega)} \quad & j(w) + \gamma C_0 P_\Omega(w^{-1}(\{1\})) \\ \text{s.t.} \quad & w(x) \in \{0,1\} \text{ for a.e. } x \in \Omega. \end{aligned} \tag{PA}$$

The following existence result for minimizers is well known.

Proposition 2.1. *Let $j : L^2(\Omega) \to \mathbb{R}$ be lower semi-continuous and bounded below. Then (PA) admits a minimizer.*

Proof. The claim follows from the direct method of calculus of variations, where the necessary compactness comes from the fact that bounded subsets of W^0_{ad} (with respect to the perimeter functional) are compact in $L^1(\Omega)$, see, for instance, [11, 18]. □

With the help of local variations (smooth perturbations of the essential boundary) of the level sets of minimizers of (PA), one can prove first-order optimality conditions for (PA). To this end, it is necessary to bound the remainder term of the Taylor expansion of the perimeter functional [9, 11, 17, 19], which is possible, for example, by means of the following assumption.

Assumption 2.2 (Relaxation of Assumption 4.3 in [9])**.** Let $j : L^1(\Omega) \to \mathbb{R}$ be continuously Fréchet differentiable.

As in [9, 11], we note that the requirement that the differentiability is with respect to the $L^1(\Omega)$-norm on the domain in Assumption 2.2 implies a regularity enhancement inside F. This is a typical assumption in the presence of binary- or discrete-valued control functions because the linear part of Taylor's expansion of F can only be estimated with respect to the L^1-norm difference and the L^1-norm difference is the same as the squared L^2-norm for binary-valued functions. We introduce the tailored concept of first-order optimality below.

Proposition 2.3. *Let Assumption 2.2 hold and $\nabla_R\, j(\bar w) \in C(\bar\Omega)$ for some $\bar w \in W^0_{\mathrm{ad}}$. If $\bar w$ is a local minimizer, that is*

$$j(\bar w) + \gamma C_0 P_\Omega(\bar w^{-1}(\{1\})) \le j(w) + \gamma C_0 P_\Omega(w^{-1}(\{1\}))$$

for all $w \in W^0_{\mathrm{ad}}$ with $\|w - \bar w\|_{L^1(\Omega)} \le r$ for some $r > 0$, then $\bar w$ and $E = \bar w^{-1}(\{1\})$ satisfy

$$\int_{\partial^* E \cap \Omega} -\nabla_R\, j(\bar w)(\phi \cdot n_E) \,\mathrm{d}\mathcal{H}^{d-1} = \gamma C_0 \int_{\partial^* E \cap \Omega} \operatorname{div}_E \phi \,\mathrm{d}\mathcal{H}^{d-1} \tag{2.1}$$

for all $\phi \in C_c^\infty(\Omega, \mathbb{R}^d)$*, where* $\operatorname{div}_E \phi : \partial^* E \to \mathbb{R}$ *is the* boundary divergence of ϕ *on* E*, that is*

$$\operatorname{div}_E \phi(x) \coloneqq \operatorname{div} \phi(x) - n_E(x) \cdot \nabla \phi(x) n_E(x)$$

for $x \in \partial^* E$*,* n_E *is the outer normal on* $\partial^* E$*, and* $\nabla_R\, j(\bar w) \in L^2(\Omega)$ *is the Riesz representative of* $j'(\bar w)$*.*

Proof. We closely follow the arguments in [9]. Let the local variation $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be defined by $f_t \coloneqq I + t\phi$ for a smooth and compactly supported vector field $\phi \in C_c^\infty(\Omega;\mathbb{R}^d)$. Here, $\varepsilon > 0$ is small enough such that $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ is a family of diffeomorphisms. Moreover, $\{x \in \mathbb{R}^d \,|\, f_t(x) \neq x\} \subset K$ holds for some compact set $K \subset \Omega$ and all $t \in (-\varepsilon, \varepsilon)$, see [9], Proposition 3.2. We define the induced transformation of the $\{0, 1\}$-valued function $\bar w$ by means of f_t as

$$f_t^\# \bar w \coloneqq \chi_{f_t(\bar w^{-1}(\{1\}))} = \begin{cases} 1 \text{ if } x \in f_t(\bar w^{-1}(\{1\})) \\ 0 \text{ else} \end{cases}$$

for $t \in (-\varepsilon, \varepsilon)$. This means that $f_t^\# \bar w$ is the $\{0, 1\}$-valued function whose level sets are given by the images of the level sets of $\bar w$ under f_t. In other words, we obtain a corresponding perturbation of the level sets of $\bar w$, which translates to a perturbation of the interface between the level sets. Our condition (2.1) arises below from local optimality because no such perturbation can be able to improve the objective.

Specifically, the function $t \mapsto J(f_t^\# \bar w) + \gamma C_0 P_\Omega((f_t^\# \bar w)^{-1}(\{1\}))$ is differentiable at $t = 0$, the first-order optimality condition

$$\frac{\mathrm{d}}{\mathrm{d}t} J(f_t^\# \bar w) + \gamma C_0 P_\Omega((f_t^\# \bar w)^{-1}(\{1\}))\Big|_{t=0} = 0$$

holds by virtue of Fermat's theorem, the assumed optimality of $\bar w$ in an L^1-ball of radius r, and the estimate

$$\|f_t^\# \bar w - \bar w\|_{L^1} \leq \bar C |t| P_\Omega((f_t^\# \bar w)^{-1}(\{1\})) \tag{2.2}$$

that holds for some $\varepsilon_0 > 0$ and $\bar C > 0$ for all $t \in (-\varepsilon_0, \varepsilon_0)$. Estimate (2.2) is shown in Lemma 3.8 in [9] (use $\operatorname{TV}(f_t^\# \bar w) = P_\Omega((f_t^\# \bar w)^{-1}(\{1\}))$).

The Fréchet differentiability of J implies

$$\frac{\mathrm{d}}{\mathrm{d}t} J(f_t^\# \bar w)\Big|_{t=0} = \lim_{t \searrow 0} \frac{(\nabla_R J(\bar w), f_t^\# \bar w - \bar w)_{L^2}}{t} + \frac{o(\|f_t^\# \bar w - \bar w\|_{L^1})}{t} = \frac{\mathrm{d}}{\mathrm{d}t} (\nabla J(v), f_t^\# \bar w)_{L^2}\Big|_{t=0},$$

where the second identity follows from the estimate (2.2). In combination with Theorem 17.5 in [17], the function $t \mapsto J(f_t^\# \bar w) + \gamma C_0 P_\Omega((f_t^\# \bar w)^{-1}(\{1\}))$ is differentiable at $t = 0$.

Consequently, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} J(f_t^\# \bar w)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} (\nabla J(v), f_t^\# \bar w)_{L^2}\Big|_{t=0} \tag{2.3}$$

and the claim follows from Lemmas 3.3 and 3.5 in [9]. ☐

Definition 2.4 (Def. 4.4 in [9])**.** Let Assumption 2.2 hold and $\nabla_R\, j(\bar w) \in C(\bar\Omega)$ for some $\bar w \in W^0_{\mathrm{ad}}$. Then we call $\bar w$ *L-stationary* if it satisfies (2.1).

L-stationarity is a first-order condition for local optimality for (PA) at points, where the Riesz representative $\nabla_R\, j(w)$ is also a continuous function, which we recall in the following proposition. (2.1) means that the negative gradient of the first part of the objective is the distributional mean curvature of the boundary of the level set

$w^{-1}(\{1\})$ after scaling by $\gamma^{-1}C_0^{-1}$, see Remark 17.7 in [17]. Using the *second variation of the perimeter*, a stronger optimality condition for (P) that involves the *principal curvatures* of the boundaries of the level sets can likely be shown under the assumption of additional boundary regularity at the local minimizer. In the case $\Omega \subset \mathbb{R}$, which is not covered in this work, L-stationarity was improved to a second-order type optimality condition under the assumption that $\nabla_R j(\bar{w})$ is in C^1 in [19], Section 3.2.

3. Homotopy trust-region algorithm

We develop a trust-region algorithm for the optimization of (P), where we follow the abstract algorithm design in [9, 11]. We introduce trust-region subproblems for the sharp interface problem and the phase-field approximation in Section 3.1, which are related to each other with a result of Γ-convergence-type. Then, we introduce the algorithm that solves trust-region subproblems and drives the interface parameter to zero over the course of its iterations in Section 3.2.

3.1. Trust-region subproblems

We consider three trust-region subproblems. The first arises from the sharp-interface formulation:

$$\text{TR}(\bar{w}, g, \Delta) \coloneqq \begin{cases} \min\limits_{w \in L^2(\Omega)} & (g, w - \bar{w})_{L^2(\Omega)} + \gamma C_0 P_\Omega(w^{-1}(\{1\})) - \gamma C_0 P_\Omega(\bar{w}^{-1}(\{1\})) \\ \quad\text{s.t.} & \|w - \bar{w}\|_{L^1(\Omega)} \le \Delta \text{ and } w(x) \in \{0, 1\} \text{ for a.e. } x \in \Omega. \end{cases} \tag{TR}$$

The inputs of (TR) are a (partial) linearization point $\bar{w}$, a function g, typically the gradient of the reduced objective $\nabla_R\, j(\bar{w})$ or an approximation, and a trust-region radius Δ. This subproblem is analyzed in [9, 11] (choose $V = \{0, 1\}$, $\alpha = \gamma$, and $C_0 = 1$ therein and observe that $\text{TV}(w) = P_\Omega(w^{-1}(\{1\}))$). A distinct feature of (TR) is that the regularization term is considered exactly and not replaced by a model function.

After discretization, an instance of class (TR) becomes a finite-dimensional integer linear program [11]. It is, however, not known if the discretizations can be solved efficiently for $d \ge 2$. Moreover, binary-valued functions are usually modeled as piecewise constant functions. Therefore any discretization of the domain Ω into finitely many cells restricts the geometry of the level sets of binary-valued ansatz functions. Specifically, the boundaries of their level sets can only follow the boundaries of the discretization cells. This may lead to a gap between the value of the perimeter functional for a discretized approximant and the perimeter functional for the function that is approximated that is bounded below. A detailed example with visualization is given in Example 2.10 and Figure 1 in [20]. Our analysis of the trust-region algorithm introduced below solves a sequence of subproblems that approximate (TR).

Specifically, the second trust-region subproblem arises by replacing the perimeter regularization $C_0 P_\Omega(w^{-1}(\{1\}))$ with the Ginzburg–Landau energy $E_\varepsilon(w)$, which yields

$$\text{TR}_\varepsilon(\bar{w}, g, \Delta) \coloneqq \begin{cases} \min\limits_{w \in L^2(\Omega)} & (g, w - \bar{w})_{L^2(\Omega)} + \gamma E_\varepsilon(w) - \gamma E_\varepsilon(\bar{w}) \\ \quad\text{s.t.} & \|w - \bar{w}\|^2_{L^2(\Omega)} \le \Delta \text{ and } w(x) \in [0, 1] \text{ for a.e. } x \in \Omega. \end{cases} \tag{TR$_\varepsilon$}$$

We note that we impose the trust region in (TR$_\varepsilon$) with the (smooth) squared L^2-norm instead of the L^1-norm. This does not make a difference for binary-valued functions, however, because the difference of two binary-valued functions is $\{-1, 0, 1\}$-valued so that the squared L^2-norm coincides with the L^1-norm of the difference. While instances of (TR$_\varepsilon$) are non-convex optimization problems, there are well-established discretizations available for them that can, in principle, be solved to ε-global optimality with successive approximations of the non-convexities and a branch-and-bound strategy, albeit this is clearly computationally very expensive.

Since this trust-region subproblem is hard to solve due to the non-convexity of the term Ψ, we propose to replace many of its solves by solves of the following trust-region subproblem that replaces the non-convex term in $E_\varepsilon(w)$ by its linearization at $\bar w$ and is convex.

$$\mathrm{TR}^c_\varepsilon(\bar w, g, \Delta) \coloneqq \begin{cases} \min\limits_{w\in L^2(\Omega)} & \left(g + \frac{\gamma}{\varepsilon}(1-2\bar w), w - \bar w\right)_{L^2(\Omega)} \\ & \quad + \gamma\frac{\varepsilon}{2}\|\nabla w\|^2_{L^2(\Omega)} - \gamma\frac{\varepsilon}{2}\|\nabla \bar w\|^2_{L^2(\Omega)} \\ \text{s.t.} & \|w-\bar w\|^2_{L^2(\Omega)} \le \Delta \text{ and } w(x)\in[0,1] \text{ for a.e. } x\in\Omega. \end{cases} \tag{$\mathrm{TR}^c_\varepsilon$}$$

We briefly state that the trust-region subproblems (TR), (TR_ε), and ($\mathrm{TR}^c_\varepsilon$) admit minimizers under mild assumptions.

Proposition 3.1. *Let $\gamma > 0$, $\varepsilon > 0$, $g \in L^2(\Omega)$, and $\Delta > 0$.*

1. *If $\bar w \in H^1(\Omega)$ and $\bar w(x) \in [0,1]$ for a.e. $x\in\Omega$, then $\mathrm{TR}_\varepsilon(\bar w, g, \Delta)$ admits a minimizer in $H^1(\Omega)$.*
2. *If $\bar w \in H^1(\Omega)$ and $\bar w(x) \in [0,1]$ for a.e. $x\in\Omega$, then $\mathrm{TR}^c_\varepsilon(\bar w, g, \Delta)$ admits a minimizer in $H^1(\Omega)$.*
3. *If $\bar w(x) \in \{0,1\}$ for a.e. $x\in\Omega$ and $P_\Omega(\bar w^{-1}(\{1\})) < \infty$, then $\mathrm{TR}(\bar w, g, \Delta)$ admits a minimizer in W^0_{ad}.*

In all of these situations, the minimal objective value is less than or equal to zero.

Proof. The first and second existence claims follow by using the direct method of calculus of variations. The existence of an infimal value in $\mathbb{R}$ follows, for example, from the L^∞-bounds, which yields a bound on the first term in the objective in both cases and a bound on the non-convex term in E_ε, as well as the coercivity of the term $\frac{\varepsilon}{2}\|\nabla w\|^2_{L^2(\Omega)}$ with respect to $H^1(\Omega)$ under the boundedness in $L^2(\Omega)$. Then, the limit of a weakly converging infimal sequence is a minimizer because of the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, which yields continuity of the first term and the non-convex term in E_ε in the first claim. Finally, the term $\frac{\varepsilon}{2}\|\nabla w\|^2_{L^2(\Omega)}$ is lower semi-continuous with respect to weak convergence in $H^1(\Omega)$.

The third existence claim follows with the direct method of calculus of variations when the feasible set of $\mathrm{TR}(\bar w, g, \Delta)$ is equipped with the weak-* topology of functions of bounded variations, see [11], Proposition 2.3.

Finally, the feasibility of $\bar w$ with objective value zero gives the non-positivity of the minimal objective value for all of the three trust-region subproblems. □

L-stationarity in the sense of Definition 2.4 also holds for minimizers of (TR), which is given below.

Proposition 3.2. *Let $g \in C(\bar\Omega)$. Let $\Delta > 0$. Let $w \in L^1(\Omega)$ satisfy $w(x) \in \{0,1\}$ for a.e. $x \in \Omega$ and $P_\Omega(w^{-1}(\{1\})) < \infty$. Let w be optimal for $\mathrm{TR}(w, g, \Delta)$ in an L^1-ball of radius r around w for some $r > 0$, then w is L-stationary.*

Proof. This follows from Proposition 2.3 with the choice $j(w) \coloneqq (g, w)_{L^2}$, $w \in L^2(\Omega)$, after observing that the optimality of w in an L^1-ball of radius r also implies optimality in an L^1-ball of radius r_0 for some $r_0 \le \Delta$, see also the proof of Proposition 5.5 in [9]. □

In [9], the convergence analysis of the trust-region algorithm therein uses that, for a fixed trust-region radius, the minimizers of the trust-region subproblems for a converging (sub)sequence of iterates converge to a minimizer to the limit problem. This then leads to a sufficient decrease for iterates that are close enough to the limit if the limit is not stationary. This work will use a similar argument and we thus re-establish this convergence of minimizers for our situation, where the regularization term changes with $\varepsilon^n \searrow 0$ over the course of the iterations $n \to \infty$. The intended result that minimizers of (TR_ε) converge to minimizers (TR), follows from the following Γ-convergence considerations. To this end, we consider a sequence $(v^n)_n \subset W^0_{\mathrm{ad}}$ and $v \in L^1(\Omega)$, a sequence $(g^n)_n \subset L^2(\Omega)$ and $g \in L^2(\Omega)$, $\varepsilon^n \subset (0,\infty)$, and $\Delta > 0$.

For $n \in \mathbb{N}$, we define the functionals $T^n : \{w \in L^1(\Omega)\,|\, \|w\|_{L^\infty(\Omega)} \le 1\} \to \mathbb{R}$ by

$$T^n(w) \coloneqq (g^n, w - v^n)_{L^2(\Omega)} + \gamma E_{\varepsilon^n}(w) - \gamma E_{\varepsilon^n}(v^n) + \delta_{[0,\Delta]}(\|w - v^n\|^2_{L^2(\Omega)})$$

for $w \in L^1(\Omega)$ and the limit functional $T : \{w \in L^1(\Omega) \,|\, \|w\|_{L^\infty(\Omega)} \le 1\} \to \mathbb{R}$ by

$$T(w) \coloneqq (g, w - v)_{L^2} + \gamma C_0 P_\Omega(w^{-1}(\{1\})) - \gamma C_0 P_\Omega(v^{-1}(\{1\})) + \delta_{\{0\}}(|w^{-1}((0,1))|) + \delta_{[0,\Delta]}(\|w - v\|_{L^1})$$

for $w \in L^1(\Omega)$.

Theorem 3.3. *Let $(v^n)_n$ and v satisfy $\sup_{n\in\mathbb{N}} \|v^n\|_{L^\infty(\Omega)} \le 1$, $v^n \to v$ in $L^1(\Omega)$, and $E_{\varepsilon^n}(v^n) \to C_0 P_\Omega(v^{-1}(\{1\}))$ for $n \to \infty$ with $v(x) \in \{0,1\}$ for a.e. $x \in \Omega$. Let $(g^n)_n$ and g satisfy $g^n \rightharpoonup g$ in $L^2(\Omega)$. Let $\varepsilon^n \searrow 0$ and $\Delta > 0$. Then the functionals T^n Γ-converge to T with respect to convergence in $L^1(\Omega)$.*

Proof. We extend the arguments from the proof of Theorem 5.2 in [9], where the perimeter regularization is present in the functionals T^n instead of the Ginzburg–Landau energy. We prove the lower bound inequality first and the upper bound inequality second.

In order to prove the lower bound inequality, let $w^n \to w$ in $L^1(\Omega)$ and $\sup_{n\in\mathbb{N}} \|w^n\|_{L^\infty} \le 1$. Then we obtain $w^n \to w$ in $L^2(\Omega)$ and thus $(g^n, w^n - v^n)_{L^2} \to (g, w - v)_{L^2}$. We also have $\gamma E_{\varepsilon^n}(v^n) \to \gamma C_0 P_\Omega(v^{-1}(\{1\}))$ by assumption. If $\|w^{n_k} - v^{n_k}\|^2_{L^2} \le \Delta$ holds for a subsequence (indexed by k), we obtain $\|w - v\|_{L^1} \le \Delta$ by virtue of the triangle inequality and the fact that $\|f\|_{L^1} = \|f\|^2_{L^2}$ if f is $\{0,1\}$-valued. Combining these considerations with the analysis of E_{ε^n} with respect to convergence in $L^1(\Omega)$ in Proposition 1 in [21] yields the lower bound inequality

$$T(w) \le \liminf_{n\to\infty} T^n(w^n).$$

Note that Proposition 1 in [21] also asserts $w(x) \in \{0,1\}$ a.e. if $\liminf_{n\to\infty} E_\varepsilon(w^n) < \infty$.

The upper bound inequality is immediate if $T(w) = \infty$ holds with the choice $w^n = w$ for all $n \in \mathbb{N}$. Let $w \in L^1(\Omega)$ with $T(w) < \infty$ be given. We distinguish the cases $\|v - w\|_{L^1} < \Delta$ and $\|v - w\|_{L^1} = \Delta$. In the case $\|v - w\|_{L^1} =: \tilde{\Delta} < \Delta$, we apply Lemma 1 in [21], which provides open and bounded subsets of $\mathbb{R}^d$ with smooth boundary F_k so that $P_\Omega(F_k) \to P_\Omega(w^{-1}(\{1\}))$ and $|(F_k \cap \Omega) \triangle w^{-1}(\{1\})| \to 0$. For each such set F_k, Proposition 2 in [21] provides functions $w^n_k \to \chi_{F_k\cap\Omega}$ in $L^1(\Omega)$ with $\|w^n_k\|_{L^\infty} \le 1$ for all $n \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \gamma E_{\varepsilon^n}(w^n_k) = \gamma C_0 P_\Omega(F_k).$$

By choosing a suitable diagonal sequence, which we denote by w^n, we obtain

$$\limsup_{n\to\infty} \gamma E_{\varepsilon^n}(w^n) = \gamma C_0 P_\Omega(w^{-1}(\{1\})).$$

Moreover, we obtain

$$\|w^n - v^n\|^2_{L^2} \le \|w^n - v^n\|_{L^1} \le \|v - v^n\|_{L^1} + \|w - w^n\|_{L^1} + \tilde{\Delta}, \tag{3.1}$$

where the first inequality follows from $|w^n(x) - v^n(x)| \le 1$, thereby implying $|w^n(x) - v^n(x)|^2 \le |w^n(x) - v^n(x)|$, for a.e. $x \in \Omega$. Because of $v^n \to v$ and $w^n \to w$ in $L^1(\Omega)$ and $\tilde{\Delta} < \Delta$, the right hand side in (3.1) is bounded by Δ for all large enough n.

We finish the proof by considering the case $\|v - w\|_{L^1} = \Delta$. This implies that there is a set $D = \{x \in \Omega \,|\, w(x) = 1 - v(x)\}$ that has strictly positive d-dimensional Lebesgue measure. We assume without loss of generality that $v(x) = 1$ a.e. in D and obtain

$$P_\Omega(D) = P_\Omega\left(v^{-1}(\{1\}) \cap w^{-1}(\{0\})\right).$$

Because D has strictly positive d-dimensional Lebesgue measure, D has a point $y \in D$ of density 1, that is

$$\lim_{r \searrow 0} \frac{|D \cap B_r(y)|}{|B_r(y)|} = 1.$$

We deduce that there exists $r_0 > 0$ such that

$$\overline{B_{r_0}(y)} \subset \Omega \quad \text{and} \quad |D \cap B_r(y)| \geq \frac{1}{2}|B_r(y)| \text{ for all } 0 < r < r_0.$$

In particular, $B_{r_0}(y)$ has strictly positive distance to the boundary $\partial\Omega$.

Let $0 < \delta < r_0$. We define for $x \in \Omega$:

$$w^\delta(x) \coloneqq \begin{cases} w(x) \text{ if } x \in \Omega \setminus B_\delta(y), \\ v(x) \text{ else} \end{cases}$$

so that

$$\|w^\delta - v\|_{L^1} \leq \|w - v\|_{L^1} - \frac{1}{2}\delta^d |B_1| = \Delta - \frac{1}{2}\delta^d |B_1|. \tag{3.2}$$

We approximate $(w^\delta)^{-1}(\{1\})$ with an open and bounded set F^δ that has a smooth boundary and define $\tilde{w}^\delta \coloneqq \chi_{F^\delta \cap \Omega}$ such that

$$\|w^\delta - \tilde{w}^\delta\|_{L^1} \leq \frac{1}{4}\delta^d |B_1| \text{ and} \tag{3.3}$$

$$|P_\Omega(F^\delta) - P_\Omega((w^\delta)^{-1}(\{1\}))| < \delta \frac{1}{2C_0}, \tag{3.4}$$

which is possible because of Lemma 2 in [21]. Next, we consider an approximation as is asserted in Proposition 2 in [21] for $\tilde{w}^\delta$. In particular, there exists a large enough $n_\delta \in \mathbb{N}$ such that for all $n \geq n_\delta$ there exists a $[0,1]$-valued function $\tilde{w}_n^\delta \in H^1(\Omega)$ such that

$$\begin{aligned} |E_{\varepsilon^n}(\tilde{w}_n^\delta) - C_0 P_\Omega((w^\delta)^{-1}(\{1\}))| &\leq |E_{\varepsilon^n}(\tilde{w}_n^\delta) - C_0 P_\Omega((\tilde{w}^\delta)^{-1}(\{1\}))| \\ &\quad + |C_0 P_\Omega((\tilde{w}^\delta)^{-1}(\{1\})) - C_0 P_\Omega((w^\delta)^{-1}(\{1\}))| \\ &< \delta, \end{aligned} \tag{3.5}$$

$$\|w^\delta - \tilde{w}_n^\delta\|_{L^1} \leq \|w^\delta - \tilde{w}_\delta\|_{L^1} + \|\tilde{w}^\delta - \tilde{w}_n^\delta\|_{L^1} \leq \frac{\delta^d}{4}|B_1|, \text{ and} \tag{3.6}$$

$$\|v^n - v\|_{L^1} \leq \frac{1}{4}\delta^d |B_1|. \tag{3.7}$$

We combine (3.2), (3.6), and (3.7) in order to obtain

$$\begin{aligned} \|v^n - \tilde{w}_n^\delta\|_{L^2}^2 &\leq \|v^n - v\|_{L^1} + \|v - w^\delta\|_{L^1} + \|w^\delta - \tilde{w}_n^\delta\|_{L^1} \\ &\leq \|v^n - v\|_{L^1} + \Delta - \frac{\delta^d}{4}|B_1| \leq \Delta. \end{aligned} \tag{3.8}$$

Next, we consider the asymptotics of $E_{\varepsilon^n}(\tilde{w}_n^\delta)$. First, we obtain by the construction of w^δ, the σ-additivity of $\mathcal{H}^{d-1}$, the finiteness of all involved terms, and the triangle inequality that

$$\begin{aligned}
&\big|C_0 P_\Omega\big((w^\delta)^{-1}(\{1\})\big) - C_0 P_\Omega\big(w^{-1}(\{1\})\big)\big| \\
&\quad = C_0\Big|\mathcal{H}^{d-1}\big((\Omega\setminus\overline{B_\delta(y)})\cap\partial^* w^{-1}(\{1\})\big) + \mathcal{H}^{d-1}\big(\partial B_\delta(x)\cap\partial^*(w^\delta)^{-1}(\{1\})\big) \\
&\qquad + \mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* v^{-1}(\{1\})\big) - \mathcal{H}^{d-1}\big(\Omega\cap\partial^* w^{-1}(\{1\})\big)\Big| \\
&\quad = C_0\Big|\mathcal{H}^{d-1}\big(\partial B_\delta(y)\cap\partial^*(w^\delta)^{-1}(\{1\})\big) + \mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* v^{-1}(\{1\})\big) \\
&\qquad - \mathcal{H}^{d-1}\big(\partial B_\delta(y)\cap\partial^* w^{-1}(\{1\})\big) - \mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* w^{-1}(\{1\})\big)\Big| \\
&\quad \le C_0\left(2\mathcal{H}^{d-1}\big(\partial B_\delta(y)\big) + \mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* v^{-1}(\{1\})\big) + \mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* w^{-1}(\{1\})\big)\right) \\
&\quad \le \delta^{d-1}c_1 + C_0\mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* v^{-1}(\{1\})\big) + C_0\mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* w^{-1}(\{1\})\big)
\end{aligned} \tag{3.9}$$

for some $c_1 > 0$. In combination with (3.5), the triangle inequality gives

$$\begin{aligned}
&|C_0 P_\Omega(w^{-1}(\{1\})) - E_{\varepsilon^n}(\tilde{w}_n^\delta)| \\
&\qquad \le \delta + \delta^{d-1}c_1 + C_0\mathcal{H}^{d-1}\big(B_\delta(y)\cap\partial^* v^{-1}(\{1\})\big) + C_0\mathcal{H}^{d-1}(B_\delta(y)\cap\partial^* w^{-1}(\{1\})).
\end{aligned} \tag{3.10}$$

We observe that the restricted measures $\mathcal{H}^{d-1}(\,\cdot\,\cap\partial^* w^{-1}(\{1\}))$ and $\mathcal{H}^{d-1}(\,\cdot\,\cap\partial^* v^{-1}(\{1\}))$ are finite Radon measures so that the last two terms tend to zero for $\delta\searrow 0$. Thus, to conclude the argument, we consider a sequence $\delta^k \searrow 0$ and define $n_k \coloneqq n_{\delta^k}$. We define for $k\in\mathbb{N}$

$$w^{n_k} \coloneqq \tilde{w}_{n_k}^{\delta^k}$$

and obtain by virtue of (3.8) and (3.10) that

$$T(w) = \lim_{k\to\infty} T^{n_k}(w^{n_k}),$$

which implies the desired upper-bound inequality. □

Remark 3.4. The assumption that the regularization terms converge, specifically $E_{\varepsilon^n}(v^n) \to C_0 P_\Omega(v^{-1}(\{1\}))$, may seem restrictive at first. It corresponds to assuming strict (or intermediate) convergence of v^n in the space of functions of bounded variation if we were considering convergence in BV instead of a phase-field approach. In [9], it is shown that this property can be ensured for iterates produced by a trust-region algorithm with a reset strategy for the trust-region radius. Our algorithm will provide this feature as well.

Unfortunately, the result above does not hold for the convex problem ($\mathrm{TR}_\varepsilon^c$), which can be seen by means of the example below. Because of this, our convergence analysis hinges on solutions to the subproblems (TR_ε) and the subproblems ($\mathrm{TR}_\varepsilon^c$) only provide a means to accelerate the algorithm.

Example 3.5. Let $w = 1$ and $\bar{w} = 1$ and $\Delta = |\Omega|$, implying that we can neglect the trust-region constraint in ($\mathrm{TR}_\varepsilon^c$). Let $g^n = g \in L^2(\Omega)$, $v^n = v$ for all $n\in\mathbb{N}$. Let $\varepsilon^n \searrow 0$. We choose $w^n \to w$ in $L^2(\Omega)$ with $\limsup_{n\to\infty}\gamma E_{\varepsilon^n}(w^n) < \infty$ such that $\varepsilon^n \in o(\|w - w^n\|_{L^1})$. Then

$$\begin{aligned}
&(g, w-\bar{w})_{L^2} + \gamma C_0 P_\Omega(w^{-1}(\{1\})) - \gamma C_0 P_\Omega(\bar{w}^{-1}(\{1\})) \\
&\qquad = 0 > -\infty = \liminf_{n\to\infty} \underbrace{\frac{\gamma}{\varepsilon^n}\int_\Omega -(w^n - w)\,\mathrm{d}x}_{\to-\infty}
\end{aligned}$$

$$= \liminf_{n\to\infty} \left(g + \frac{\gamma}{\varepsilon}(1-2\bar{w}), w^n - \bar{w} \right)_{L^2(\Omega)} + \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\bar{w}).$$

This implies that the lower bound inequality cannot hold for the objective functional of ($\mathrm{TR}_\varepsilon^c$).

3.2. Algorithm statement

The homotopy trust-region algorithm is given in Algorithm 1 and proceeds as follows. For a given ε, it repeatedly solves convex trust-region subproblems of the form ($\mathrm{TR}_\varepsilon^c$) and accepts candidate iterates $\tilde{w}^n$ as new iterates w^{n+1} if the acceptance criterion in ln. 8 is satisfied. The trust-region radius Δ^n is doubled on acceptance of the computed candidate step and halved on rejection. The trust-region radius is not increased over the value Δ^0, which stays constant over the course of the algorithm.

If the trust-region radius falls below $\underline{\Delta}^n$, then the trust-region radius is reset to Δ^0 and Algorithm 1 invokes solves of the non-convex trust-region subproblems (TR_ε) for successively halved trust-region radii until a computed iterate can be accepted or the trust-region radius falls below $\underline{\Delta}^n$ again. In the former case, the trust-region radius is reset, and Algorithm 1 returns to solving trust-region subproblems of the form ($\mathrm{TR}_\varepsilon^c$). In the latter case, the trust-region is reset as well, $\underline{\Delta}^n$ is divided by two, and the regularization parameter ε^n of the Ginzburg–Landau energy is divided by $r > 4$. Then Algorithm 1 also returns to solving instances of type ($\mathrm{TR}_\varepsilon^c$).

In this way, Algorithm 1 ensures that a closer approximation of the perimeter regularizer $C_0 P_\Omega$ by means of a tighter Ginzburg–Landau energy is only triggered if Algorithm 1 cannot improve the objective further with steps larger than $\underline{\Delta}^n$ and that are computed by means of the non-convex trust-region subproblem (TR_ε). We will see that this procedure allows us to relate a first-order sufficient decrease with respect to (PA) / (TR) to an eventual acceptance of a solution of (TR_ε) in the analysis of the asymptotics of Algorithm 1 without having the problem that the algorithm does not progress towards sharp interface limits because it produces infinitely many iterates for one ε^n.

This, in turn, allows us to prove the convergence of iterates produced by Algorithm 1 to L-stationary points of (PA). We have failed to prove the convergence of iterates produced by Algorithm 1 to L-stationary points of (PA) when relying on subproblem solves of type ($\mathrm{TR}_\varepsilon^c$), which are much less expensive after discretization due to their convexity. However, we have added the preferred solution of instances of ($\mathrm{TR}_\varepsilon^c$) in order to show how to accelerate the solution process of many of the trust-region subproblems and, thereby, the overall algorithm if better results cannot be had in the future.

For the sufficient-decrease condition in Algorithm 1 ln. 8, we consider $\rho \in (0,1)$ and choose the acceptance criterion

$$\mathrm{ared}(w^n, \tilde{w}^n, \varepsilon^n) \geq \rho\, \mathrm{pred}(w^n, \Delta^n, \varepsilon^n) \tag{3.11}$$

with the *actual reduction* ared of the objective that is defined by

$$\mathrm{ared}(w, v, \varepsilon) \coloneqq j(w) + \gamma E_\varepsilon(w) - j(v) - \gamma E_\varepsilon(v)$$

and the *predicted reduction* pred that is defined by the solution of the trust-region subproblem as

$$\mathrm{pred}(w, \Delta, \varepsilon) \coloneqq - \begin{cases} \text{minimal objective value of } \mathrm{TR}_\varepsilon^c(w, \nabla_R\, j(w), \Delta) \text{ if } \text{cvx\_flag} = 1, \\ \text{minimal objective value of } \mathrm{TR}_\varepsilon(w, \nabla_R\, j(w), \Delta) \text{ if } \text{cvx\_flag} = 0. \end{cases}$$

4. Asymptotics of Algorithm 1

In order to analyze the limits of iterates generated by Algorithm 1, we need to ensure their existence and feasibility (that is, their binarity) for (PA), which is the purpose of Section 4.1. We continue by proving that the

Algorithm 1 Homotopy Trust-region Algorithm.

Input: j sufficiently regular, $\rho \in (0,1)$,
Input: $\Delta^0 > 0$, $\varepsilon^0 > 0$, $r > 4$, $\rho > \kappa^0 > 0$, $\underline{\Delta}^0 \in (0, \Delta^0)$,
Input: $w^0 \in H^1(\Omega) \cap \{w \in L^\infty(\Omega) \,|\, w(x) \in \{0,1\} \text{ a.e.}\}$.

cvx_flag $\leftarrow 1$
for $n = 0, \dots$ **do**
  if cvx_flag $= 1$ **then**
    $\tilde{w}^n \leftarrow$ minimizer of $\mathrm{TR}^c_\varepsilon(w^n, \nabla_R j(w^n), \Delta^n)$
  else
    $\tilde{w}^n \leftarrow$ minimizer of $\mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^n)$
  end if
  if sufficient decrease (3.11) and $\mathrm{ared}(w^n, \tilde{w}^n, \varepsilon^n) > \kappa^n \underline{\Delta}^n$ **then**
    $w^{n+1} \leftarrow \tilde{w}^n$
    $\Delta^{n+1} \leftarrow \min\{\Delta^0, 2\Delta^n\}$
    cvx_flag $\leftarrow 1$
  else
    $w^{n+1} \leftarrow w^n$
    $\Delta^{n+1} \leftarrow \Delta^n/2$
  end if
  if $\Delta^{n+1} < \underline{\Delta}^n$ and cvx_flag $= 1$ **then**
    cvx_flag $\leftarrow 0$
    $\Delta^{n+1} \leftarrow \Delta^0$
    $\underline{\Delta}^{n+1} \leftarrow \underline{\Delta}^n$
    $\varepsilon^{n+1} \leftarrow \varepsilon^n$
    $\kappa^{n+1} \leftarrow \kappa^n$
  else if $\Delta^{n+1} < \underline{\Delta}^n$ **then**
    cvx_flag $\leftarrow 1$
    $\Delta^{n+1} \leftarrow \Delta^0$
    $\underline{\Delta}^{n+1} \leftarrow \underline{\Delta}^n/2$
    $\varepsilon^{n+1} \leftarrow \varepsilon^n/r$
    $\kappa^{n+1} \leftarrow \kappa^n/2$
  else
    $\underline{\Delta}^{n+1} \leftarrow \underline{\Delta}^n$
    $\varepsilon^{n+1} \leftarrow \varepsilon^n$
    $\kappa^{n+1} \leftarrow \kappa^n$
  end if
end for

limits produced by Algorithm 1 are L-stationary for (PA) in the sense of Definition 2.4 under certain regularity conditions in Section 4.2.

4.1. Boundedness of iterates and compactness

A standard way of proving boundedness of the iterates produced by optimization algorithms and in turn weak-* convergence of a subsequence is to combine the boundedness of the objective from below with descent properties of the produced sequence of iterates. In our situation, descent of the objective values is only guaranteed for a fixed value of ε^n so that the objective value may increase when ε^n is reduced and thus the objective term itself changes from one iteration to the next. We recover convergence of subsequences of the iterates in $L^1(\Omega)$ by combining a compactness result with competitor constructions that use sufficient descent properties of the algorithm during the iterations when ε^n is not changed to ensure that the objective values have to remain bounded despite the changes of the objective term itself. We briefly state the compactness result due to [22], on sequences w^ε that are bounded with respect to the functionals $j + \gamma E_\varepsilon$ for $\varepsilon \searrow 0$, which guarantees the convergence of a subsequence of iterates produced by Algorithm 1 to a binary-valued limit under boundedness of $(E_\varepsilon(w^\varepsilon))_\varepsilon$.

Proposition 4.1. *Let $\varepsilon \searrow 0$, $(w^\varepsilon)_\varepsilon \subset H^1(\Omega)$ with $0 \le w^\varepsilon(x) \le 1$ for a.e. $x \in \Omega$. Let there exist $C > 0$ such that $j(w^\varepsilon) + \gamma E_\varepsilon(w^\varepsilon) \le C$ for all ε. Then w^ε has a subsequence that converges in $L^p(\Omega)$ for all $p \in [1,\infty)$ and the $L^p(\Omega)$-limit is $\{0,1\}$-valued and satisfies $P_\Omega(w^{-1}(\{1\})) \le \liminf_{\varepsilon\to 0} E_\varepsilon(w^\varepsilon)$.*

Proof. The existence of an $L^1(\Omega)$-limit follows from the compactness result [22], Theorem 4.1 and the uniform boundedness of F on $\{w \in L^\infty(\Omega)\,|\, 0 \le w(x) \le 1 \text{ for a.e. } x \in \Omega\}$. The L^∞-bounds on the w^ε also yield convergence in $L^p(\Omega)$ for all $p \in [1,\infty)$ and the lower semi-continuity is due to [21], Proposition 1. □

In order to leverage Proposition 4.1 in the convergence analysis of Algorithm 1, we need the existence of a bounded subsequence. We pose this as an assumption below and then proceed by asserting it under appropriate assumptions.

Assumption 4.2. For $k \in \mathbb{N}$, let $n(k) \in \mathbb{N}$ be the last iteration such that $\varepsilon^{n(k)} = \varepsilon^0 r^{-k}$. We assume that the subsequence $(w^{n(k)})_k$ of the iterates $(w^n)_n$ produced by Algorithm 1 is bounded.

Assumption 4.2 is well defined if $n(k)$ exists for all k, which we prove in Lemma 4.3 below.

Lemma 4.3. *Let $n_0 \in \mathbb{N}$ be the first iteration of Algorithm 1 with $\varepsilon^n = \varepsilon^{n_0}$. Then there are only finitely many iterations $n \ge n_0$ such that $\varepsilon^n = \varepsilon^{n_0}$.*

Proof. We note that the iterations $n \ge n_0$ with $\varepsilon^n = \varepsilon^{n_0}$ also satisfy $\kappa^n = \kappa^{n_0}$. Because the objective $j + \gamma E_{\varepsilon^n}$ is bounded below, there can only be finitely many iterates that are accepted in Algorithm 1 ln. 8 and thus only finitely many iterations, in which cvx_flag is set to one. The trust-region radius is halved if an iterate is not accepted, so that cvx_flag is set to zero eventually and not set to one anymore afterward. Since there is no accepted iterate anymore, the trust-region radius is halved with cvx_flag $= 0$ until the condition in Algorithm 1 ln. 22 is met and ε^n is halved. This happens after finitely many iterations because $\underline{\Delta}^n > 0$ holds inductively. □

The boundedness of the iterates that allows to apply Proposition 4.1 can be ensured under boundedness assumptions on j and its gradient, which we prove below.

After the auxiliary result Lemma 4.4, Proposition 4.5 shows the boundedness of the sequence of iterates for the case that the reset trust-region radius Δ^0 is so large that all of Ω is contained in the trust region after the reset occurs. In particular, the zero function is feasible and satisfies $\gamma E_\varepsilon(0) = 0$. Consequently, if the penalty parameter becomes small, the regularization dominates the objective in the subproblem in the event of an unbounded sequence so that the function 0 is a competitor that reduces the subproblem solution to lead to a contradiction. Then Proposition 4.6 shows the boundedness for smaller values of Δ^0. The proof is based on the same general idea as Proposition 4.5 but the function 0 may not be feasible and therefore, we provide a more involved construction and estimates for a feasible point of the trust-region subproblems for the initial trust-region radius Δ^0 that satisfies our needs.

Lemma 4.4. *Algorithm 1 produces a sequence of iterates with non-increasing objective values over subsequent iterations during which ε^n is not altered.*

Proof. The iterate w^{n+1} coincides with the previous iterate w^n if the computed update $\tilde{w}^n$ is rejected. Thus, under the condition $\varepsilon^{n+1} = \varepsilon^n$, the objective value only changes if a step is accepted. Specifically, it changes by $\operatorname{ared}(w^n, w^{n+1}, \varepsilon^n)$. Because Proposition 3.1 asserts that $\operatorname{pred}(w^n, \Delta^n, \varepsilon^n)$ is non-negative, the sufficient decrease condition (3.11) and Algorithm 1 ln. 8 yield the claim. □

Proposition 4.5. *Let j and $\nabla_R j$ be bounded on the unit ball in $L^\infty(\Omega)$. Let $(w^n)_n \subset \{w \in H^1(\Omega)\,|\, 0 \le w(x) \le 1$ for a.e. $x \in \Omega\}$ be the sequence of iterates produced by Algorithm 1. Let $\Delta^0 \ge |\Omega|$. Then there exist $C > 0$ and a subsequence $(w^{n_k})_k \subset (w^n)_n$ such that*

$$\sup_{k\in\mathbb{N}} j(w^{n_k}) + \gamma E_{\varepsilon^{n_k}}(w^{n_k}) < C < \infty.$$

Proof. We define constants for uniform bounds on the value of j and the first term of the objective of the trust-region subproblem:

$$\begin{aligned} C_j &\coloneqq \sup\{|j(w)|\,|\, w \in L^2(\Omega) \text{ and } \|w\|_{L^\infty} \le 1\} \text{ and} \\ C_g &\coloneqq \sup\{(\nabla_R\, j(w), d)_{L^2}\,|\, w, d \in L^2(\Omega) \text{ and } \|w\|_{L^\infty} \le 1, \|d\|_{L^\infty} \le 1\}. \end{aligned}$$

Because of the assumption $\Delta^0 \ge |\Omega|$, the trust-region constraint is not present in (or does not restrict) the trust-region subproblems of the form $\mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$ and $\mathrm{TR}^c_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$, which implies that all $w \in H^1(\Omega)$ with $0 \le w(x) \le 1$ for a.e. $x \in \Omega$ are feasible with finite objective value for the trust-region subproblems. We obtain

$$\gamma E_{\varepsilon^n}(w^n) + C_g \ge \mathrm{pred}(w^n, \Delta^0, \varepsilon^n) \ge \gamma E_{\varepsilon^n}(w^n) - C_g$$

because 0 is feasible and $E_{\varepsilon^n}(0) = 0$ and for any solution $\bar{w}^n \in \arg\min \mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$ and cvx_flag $= 1$ that

$$\mathrm{ared}(w^n, \bar{w}^n, \varepsilon^n) \ge \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\bar{w}^n) - 2C_j \ge \gamma E_{\varepsilon^n}(w^n) - 2C_g - 2C_j,$$

where the second inequality follows from the optimality of $\bar{w}^n$, which gives

$$(\nabla_R\, j(w^n), \bar{w}^n - w^n)_{L^2} + \gamma E_{\varepsilon^n}(\bar{w}^n) \le -(\nabla_R\, j(w^n), w^n)_{L^2}$$

and thus

$$-2C_g \le -\gamma E_{\varepsilon^n}(\bar{w}^n).$$

Consequently, if $\gamma E_{\varepsilon^n}(w^n) \ge C_E \coloneqq (1-\rho)^{-1}\left((2+\rho)C_g + 2C_j\right)$, we have

$$\frac{\mathrm{ared}(w^n, \bar{w}^n, \varepsilon^n)}{\mathrm{pred}(w^n, \Delta^0, \varepsilon^n)} \ge \frac{\gamma E_{\varepsilon^n}(w^n) - 2C_g - 2C_j}{\gamma E_{\varepsilon^n}(w^n) + C_g} \ge \rho. \tag{4.1}$$

Lemma 4.4 and the design of Algorithm 1 give that the objective can only increase if $\varepsilon^n = \varepsilon^{n-1}/r$. In this case, the trust-region radius was reduced below $\underline{\Delta}^{n-1}$, implying that $w^n = w^{n-1}$ by design of Algorithm 1.

This means: if there is some iteration such that

$$j(w^n) + \gamma E_{\varepsilon^n}(w^n) \ge j(w^n) + C_E > j(w^n) + \gamma E_{\varepsilon^{n-1}}(w^n)$$

and thus also $\gamma E_{\varepsilon^n}(w^n) \ge C_E$ for some $n \in \mathbb{N}$, then Lemmas 4.3 and 4.4 give that there is some $k \in \mathbb{N}$ such that $\varepsilon^n = \ldots = \varepsilon^{n+k}$

$$j(w^n) + \gamma E_{\varepsilon^n}(w^n) \ge \ldots \ge j(w^{n+k}) + \gamma E_{\varepsilon^n}(w^{n+k})$$

with cvx_flag $= 1$ and in the next iteration $n+k+1$ we have

$$\varepsilon^{n+k+1} = \varepsilon^{n+k} \text{ and } \Delta^{n+k+1} = \Delta^0 \text{ and cvx\_flag} = 1.$$

Then (4.1) implies that $\bar{w}^{n+k+1}$ is accepted and the objective is reduced by at least $C_E - 2C_g - 2C_j$. Such iterations occur consecutively at least until the value of $\gamma E_{\varepsilon^{n+k+1}}(w^{n+k+1})$ falls below C_E because the algorithm reduces the objective until it sets cvx_flag to 1 and the trust-region radius to Δ^0 after further finitely many iterations, in which case (4.1) implies a further acceptance and subsequent reduction of the objective by at least $C_E - 2C_g - 2C_j$.

Because $j(w^n) \le C_j$ holds for all $n \in \mathbb{N}$, the considerations above yield that there is a subsequence, indexed by k, such that $j(w^{n_k}) + \gamma E_{\varepsilon^{n_k}}(w^{n_k}) \le C_j + C_E$. □

Proposition 4.6. *Let Ω be convex. Let j and $\nabla_R j$ be bounded on the unit ball in $L^\infty(\Omega)$. Let $(w^n)_n \subset \{w \in H^1(\Omega) \,|\, 0 \le w(x) \le 1$ for a.e. $x \in \Omega\}$ be the sequence of iterates produced by Algorithm 1. Then there exist $C > 0$ and a subsequence $(w^{n_k})_k \subset (w^n)_n$ such that*

$$\sup_{k\in\mathbb{N}} j(w^{n_k}) + \gamma E_{\varepsilon^{n_k}}(w^{n_k}) < C < \infty.$$

Proof. We consider the case $\Delta^0 < |\Omega|$ and define the ratio $R := \left\lceil \frac{|\Omega|}{\Delta^0} \right\rceil + 1$. Without loss of generality, that is, after possibly applying a suitable translation to Ω, we assume that $\inf\{x_1 \,|\, x \in \Omega\} = 0$. Moreover, we define $\ell := \sup\{x_1 \,|\, x \in \Omega\}$. Then, by virtue of Fubini's theorem, we obtain

$$|\Omega| = \int_0^\ell \lambda_{\mathbb{R}^{d-1}}(\Omega_t)\,\mathrm{d}t \text{ with } \Omega_t = \{(a + te_1)_{2,\ldots,d} \,|\, a \in \{e_1\}^\perp \text{ and } a + te_1 \in \Omega\} \text{ for } t \in [0,\ell]$$

and where $\lambda_{\mathbb{R}^{d-1}}$ denotes the Lebesgue measure on $\mathbb{R}^{d-1}$. The function $s \mapsto \int_0^s \lambda_{\mathbb{R}^{d-1}}(\Omega_t)\,\mathrm{d}t$ is continuous and we thus obtain $0 = s_0 \le s_a < s_b \le s_L = \ell$ so that

$$|S| = \int_{s_a}^{s_b} \lambda_{\mathbb{R}^{d-1}}(\Omega_t)\,\mathrm{d}t = \frac{|\Omega|}{R}$$

for a slice $S := \bigcup_{t=s_a}^{s_b} \Omega_t$. All such slices S have a finite perimeter because we may estimate their perimeter from above by $P_{\mathbb{R}^d}(\Omega) + 2(\operatorname{diam}\Omega)^{d-1}$, which is finite. All such slices S are also convex because they are intersections of two half-spaces with the convex set Ω. We will approximate χ_S smoothly along e_1 following the ideas in [21, 23]. Then, we will use this approximation in order to construct a competitor that is feasible for the trust-region subproblem and reduces the objective value substantially enough to contradict that the sequence of iterates produced by Algorithm 1 is unbounded with respect to the objective values.

To this end, we first analyze S independently of its position along e_1. Our required choices for S will be defined later. We construct a set $\hat{S}$ from S by extending it in the directions in $\{e_1\}^\perp$ in order to obtain a cylinder with positive distance of its edges to Ω and then smoothing the edges. By making this extension large enough, these properties can be established independently of the specific choice of S. This set satisfies $\hat{S} \cap \Omega = S$, $\mathcal{H}^{d-1}(\partial\hat{S} \cap \partial\Omega) = 0$, and

$$P_\Omega(\hat{S}) = \mathcal{H}^{d-1}(\partial\hat{S} \cap \Omega) = \mathcal{H}^{d-1}(\partial S \cap \Omega) = P_\Omega(S). \tag{4.2}$$

The situation is sketched in Figure 1. To this end, we define the signed distance function as

$$h(x) := \begin{cases} \operatorname{dist}(x, \partial\hat{S} \cap \Omega) \text{ if } x \in \hat{S} \\ -\operatorname{dist}(x, \partial\hat{S} \cap \Omega) \text{ else} \end{cases}$$

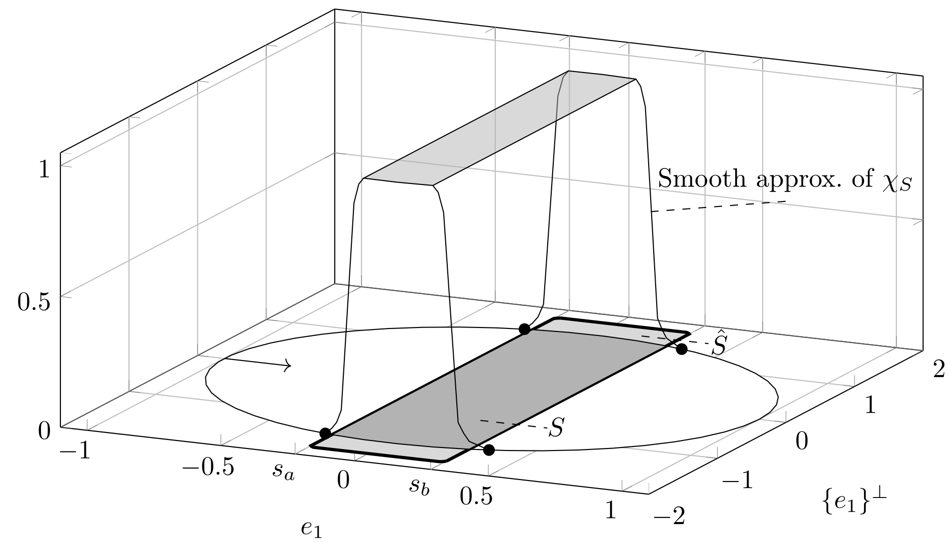


FIGURE 1. Positioning of S (dark grey) with s_a and s_b inside Ω (white ellipse) along e_1, extended set $\hat{S}$ (light gray, extending to the exterior of Ω), and smooth approximation of χ_S.

We approximate the jump function on $\mathbb{R}$, $\chi_{(-\infty,0)}$, smoothly with a function q that solves the ordinary differential equation

$$-q''(s) + W'(q(s)) = 0 \text{ for } s \in \mathbb{R}$$

with

$$q'(s) = \sqrt{2W(q(s))} > 0,$$

where $W(q) = q(1-q)$. To make the solution unique, we set $q(0) = \frac{1}{2}$. Let $\tau > 0$ be fixed but arbitrary and let $a = \frac{q'(\tau)^2}{4q(\tau)}$ and $b = 2\frac{q(\tau)}{q'(\tau)}$. We define

$$q_\tau(s) \coloneqq \begin{cases} 0 \text{ if } s \in (-\infty, -\tau - b], \\ \tilde{q}_\tau(s) \text{ if } s \in (-\tau - b, -\tau), \\ q(s) \text{ if } s \in [-\tau, \tau], \\ 1 - \tilde{q}_\tau(-s) \text{ if } s \in (\tau, \tau + b), \\ 1 \text{ if } s \in [\tau + b, \infty), \end{cases}$$

where $\tilde{q}_\tau(s) = a(s - (-\tau - b))^2$ is a parabola with vertex at $(-\tau - b)$ such that $q_\tau \in C^2(\mathbb{R})$ and $\operatorname{supp} q'_\tau = [-\tau - b, \tau + b]$. We define $v_{\delta,\tau} \coloneqq q_\tau \circ (h/\delta)$ and obtain $\frac{\delta}{2}|\nabla v_{\delta,\tau}(x)|^2 = \frac{1}{2\delta}(q'_\tau(h(x)/\delta))^2$ because the chain rule gives $\nabla v_{\delta,\tau}(x) = \frac{1}{\delta} q'_\tau(h(x)/\delta)\nabla h(x)$ and $\nabla h(x)$ exists and satisfies $\|\nabla h(x)\| = 1$ a.e. Moreover, we can prove the following assertion.

Claim:

$$
\begin{aligned}
I_{\delta,\tau} &:= \int_S \frac{\delta}{2}|\nabla v_{\delta,\tau}(x)|^2 + \frac{1}{\delta}W(v_{\delta,\tau}(x))\,\mathrm{d}x \\
&\to \left(\int_{\mathbb{R}} \frac{1}{2}q_\tau'(s)^2 + W(q_\tau(s))\,\mathrm{d}s\right) P_\Omega(S) \text{ for } \delta \searrow 0.
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
I_{\delta,\tau}^c &:= \int_{\Omega\setminus S} \frac{\delta}{2}|\nabla v_{\delta,\tau}(x)|^2 + \frac{1}{\delta}W(v_{\delta,\tau}(x))\,\mathrm{d}x \\
&\to \left(\int_{\mathbb{R}} \frac{1}{2}q_\tau'(s)^2 + W(q_\tau(s))\,\mathrm{d}s\right) P_\Omega(S) \text{ for } \delta \searrow 0.
\end{aligned}
\tag{4.4}
$$

Proof of Claim: Because the length of the support interval of $s \mapsto q_\tau(s/\delta)$ shrinks linearly with δ we can assume that δ is small enough such that the length of the support is strictly smaller than the halved distance between the two half-spaces that cut S and $\hat{S}$ into two pieces of width $(s_b - s_a)/2$. We combine this insight with Fubini's theorem and obtain

$$
\begin{aligned}
I_{\delta,\tau} &= \int_{\{e_1\}^\perp} \int_{-\infty}^{\frac{s_a+s_b}{2}} \chi_S(y+te_1)\left(\frac{1}{2\delta}q_\tau'(h(y+te_1)/\delta)^2 + \frac{1}{\delta}W(q_\tau(h(y+te_1)/\delta))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\quad + \int_{\{e_1\}^\perp} \int_{\frac{s_a+s_b}{2}}^{\infty} \chi_S(y+te_1)\left(\frac{1}{2\delta}q_\tau'(h(y+te_1)/\delta)^2 + \frac{1}{\delta}W(q_\tau(h(y+te_1)/\delta))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&= \int_{\{e_1\}^\perp} \int_0^{\frac{s_b-s_a}{2}} \chi_S(y+s_a+te_1)\left(\frac{1}{2\delta}q_\tau'(t/\delta)^2 + \frac{1}{\delta}W(q_\tau(t/\delta))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\quad + \int_{\{e_1\}^\perp} \int_0^{\frac{s_b-s_a}{2}} \chi_S(y+s_b-te_1)\left(\frac{1}{2\delta}q_\tau'(t/\delta)^2 + \frac{1}{\delta}W(q_\tau(t/\delta))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&= \int_{\{e_1\}^\perp} \int_0^{\frac{s_b-s_a}{2\delta}} \chi_S(y+s_a+t\delta e_1)\left(\frac{1}{2}q_\tau'(t)^2 + W(q_\tau(t))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\quad + \int_{\{e_1\}^\perp} \int_0^{\frac{s_b-s_a}{2\delta}} \chi_S(y+s_b-t\delta e_1)\left(\frac{1}{2}q_\tau'(t)^2 + W(q_\tau(t))\right)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&= \int_{-\infty}^{\infty} \int_{\{e_1\}^\perp} \chi_S(y+s_a+t\delta e_1)\,\mathrm{d}\mathcal{H}^{d-1}(y)\left(\frac{1}{2}q_\tau'(t)^2 + W(q_\tau(t))\right)\mathrm{d}t \\
&\quad + \int_{-\infty}^{\infty} \int_{\{e_1\}^\perp} \chi_S(y+s_b-t\delta e_1)\,\mathrm{d}\mathcal{H}^{d-1}(y)\left(\frac{1}{2}q_\tau'(t)^2 + W(q_\tau(t))\right)\mathrm{d}t
\end{aligned}
$$

for all δ small enough such that $\frac{s_b-s_a}{2\delta} > \tau + b$, which implies $q_\tau'(t) = 0$ and $W(q_\tau(t)) = 0$ for all $t > \frac{s_b-s_a}{2\delta}$. Note that we have used that h is defined with respect to $\hat{S}$ in order to obtain the second equality. For $t \in [0, \tau + b]$, we consider the term

$$
f_\delta(t) := \int_{\{e_1\}^\perp} \chi_S(y+s_b-t\delta e_1) + \chi_S(y+s_b-t\delta e_1)\,\mathrm{d}\mathcal{H}^{d-1}(y).
$$

Again, because h is defined with respect to $\hat{S}$ and not S, we observe for all δ small enough that

$$f_\delta(t) = \mathcal{H}^{d-1}(\{x \in S \,|\, h(x) = \delta t\}) = \mathcal{H}^{d-1}(\Omega \cap \{x \in \hat{S} \,|\, h(x) = \delta t\}).$$

Moreover, $\hat{S}$ satisfies the assumptions of Lemma 3 in [21], from which we thus obtain

$$f_\delta(t) \to \mathcal{H}^{d-1}(\partial \hat{S} \cap \Omega) \underset{(4.2)}{=} P_\Omega(S),$$

for $\delta \searrow 0$. Moreover, $\mathcal{H}^{d-1}(\{x \in \Omega \,|\, h(x) = \delta t\}) \le \operatorname{diam}(\Omega)^{d-1}$ holds for all t due to the axis-parallel construction of S and $\hat{S}$. Thus we may employ Lebesgue's dominated convergence theorem in order to obtain (4.3). A very similar chain of arguments and the equality $P_\Omega(S) = P_\Omega(\Omega \setminus S)$ yield (4.4). Thus, the claim consisting of (4.3) and (4.4) is proven.

Next, we define

$$\begin{aligned}
&C_j \coloneqq \sup\{|j(w)| \,|\, w \in L^2(\Omega) \text{ and } \|w\|_{L^\infty} \le 1\} \text{ and} \\
&C_g \coloneqq \sup\{(\nabla_R\, j(w), d)_{L^2} \,|\, w, d \in L^2(\Omega) \text{ and } \|w\|_{L^\infty} \le 1, \|d\|_{L^\infty} \le 1\}
\end{aligned}$$

and proceed with a contradictory argument towards the overall claim that there is a subsequence of iterates, which is bounded with respect to the objective. First, Lemma 4.3 and the definition of R give for any arbitrary but fixed $\tau > 0$ that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ it holds that

$$2 \operatorname{diam}(\Omega)^{d-1}(\tau + b)\varepsilon^n + \frac{|\Omega|}{R} \le \Delta^0. \tag{4.5}$$

If there is no bounded subsequence, we obtain for all $M > 0$ that there is $n_M \ge n_0$ so that for all $n \ge n_M$

$$j(w^n) + \gamma E_{\varepsilon^n}(w^n) \ge M.$$

We can assume wlog that n_M is the first such iteration, implying $\Delta^{n_M} = \Delta^0$, because the objective can only increase in the event of a reduction of ε. The inequality gives

$$\gamma E_{\varepsilon^n}(w^n) \ge M - C_j$$

for all $n \ge n_M$. Now, we consider finitely many sets $S_1, \ldots, S_{2R-1}$ as starting points for possible competitors that are constructed as follows. We define

$$S_i = \bigcup_{t = s_a^i}^{s_b^i} \Omega_t$$

for all $i \in \{1, \ldots, 2R-1\}$. For $i \in \{1, 3, \ldots, 2R-1\}$, we choose the scalars $0 \le s_a^i < s_b^i \le \ell$ such that $s_a^{i+2} = s_b^i$ and $|S_i| = |\Omega|/R$ as above so that they decompose Ω into R pieces of equal Lebesgue measure. Moreover, we construct for the remaining $R-1$ sets S_i, $i \in \{2, 4, \ldots, 2R-2\}$, we choose scalars $0 \le s_a^i < s_b^i \le \ell$ such that S_i is *centered* around s_a^{i+1}, that is $(s_a^i + s_b^i)/2 = s_a^{i+1}$ and again $|S_i| = |\Omega|/R$.

For any such set $S = S_i$, $i \in \{1, \ldots, 2R-1\}$, we make the following considerations. We define $\tilde{S}^n \coloneqq \operatorname{supp} v_{\varepsilon^n,\tau} \cap \Omega$. Then

$$\|\tilde{w}^n - w\|_{L^2}^2 \le \|\tilde{w}^n - w\|_{L^1} = \|v_{\varepsilon^n,\tau}\|_{L^1} \le |\tilde{S}^n| \le |S| + 2\operatorname{diam}(\Omega)^{d-1}(\tau + b)\varepsilon^n + \frac{|\Omega|}{R} \underset{(4.5)}{\le} \Delta^0$$

and in turn we obtain that the function $\tilde{w}^n \coloneqq (1 - v_{\varepsilon^n,\tau})w^n$ is feasible for $\mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$ for all $n \geq n_M$, where the third inequality follows due to the axis-parallel construction of S, which implies $\mathcal{H}^{d-1}(\{x \in \Omega \,|\, h(x) = t\}) \leq \operatorname{diam}(\Omega)^{d-1}$ for all t. Moreover, we define $T^n \coloneqq \{x \in \Omega \,|\, v_{\varepsilon^n,\tau}(x) = 1\}$. Then $|\tilde{S}^n \setminus T^n| \leq \varepsilon^n 2(\tau + b) \operatorname{diam}(\Omega)^{d-1}$ and the function $\tilde{w}^n$ coincides with w^n outside $\tilde{S}^n$, is zero inside T^n. It transitions smoothly from w^n to zero in $\tilde{S}^n \setminus T^n$. Because a reduction of $\tau > 0$ leaves (4.5) intact for $n \geq n_M$, for any of the finitely many choices of S, we can choose some $\tau > 0$ small enough so that $|T^n| \geq |\Omega|/(R+1)$ holds for the corresponding T^n. By choosing the minimal of these τ, this inequality holds uniformly over the possible choices for S. By possibly decreasing τ further, we also obtain that, independently of the choice of S, the corresponding sets T^n cover Ω.

For all $n \geq n_M$, we can do the following. Because $|T^n| \geq |\Omega|/(R+1)$ does not depend on the specific choice of S, we can choose the set $S = S_i$, $i \in \{1, \ldots, 2R-1\}$, such that

$$\frac{\varepsilon^n}{2}\|\nabla w\|^2_{L^2(T^n)} + \frac{1}{\varepsilon^n}\int_{T^n} w^n(1 - w^n)\,\mathrm{d}x \geq \frac{1}{R+1}E_{\varepsilon^n}(w^n) \tag{4.6}$$

holds for all $n \geq n_M$. Such a choice can be made because the induced options for T^n (by the choice of S) cover Ω. Then

$$\begin{aligned} E_{\varepsilon^n}(\tilde{w}^n) = \frac{\varepsilon^n}{2}\|\nabla w^n\|^2_{L^2(\Omega\setminus\tilde{S}^n)} + \frac{1}{\varepsilon^n}\int_{\Omega\setminus\tilde{S}^n} w^n(1 - w^n)\,\mathrm{d}x& \\ + \underbrace{\frac{\varepsilon^n}{2}\|\nabla \tilde{w}^n\|^2_{L^2(\tilde{S}^n\setminus T^n)} + \frac{1}{\varepsilon^n}\int_{\tilde{S}^n\setminus T^n} \tilde{w}^n(1 - \tilde{w}^n)\,\mathrm{d}x}_{=:A^n}&. \end{aligned}$$

Claim: For all n large enough, it holds that

$$E_{\varepsilon^n}(w^n) - E_{\varepsilon^n}(\tilde{w}^n) \geq \frac{1}{2(R+1)}E_{\varepsilon^n}(w^n). \tag{4.7}$$

Proof of Claim: The product rule yields

$$\begin{aligned} A^n \leq \frac{\varepsilon^n}{2}\|\nabla w^n\|^2_{L^2(\tilde{S}^n\setminus T^n)} + \frac{1}{\varepsilon^n}\int_{\tilde{S}^n\setminus T^n} w^n(1 - w^n)\,\mathrm{d}x& \\ + \underbrace{\frac{\varepsilon^n}{2}\|1 - v_{\tau,\varepsilon^n}\|^2_{L^2(\tilde{S}^n)} + \frac{1}{\varepsilon^n}\int_{\tilde{S}^n}(1 - v_{\tau,\varepsilon^n})v_{\tau,\varepsilon^n}\,\mathrm{d}x}_{=:B^n}&, \end{aligned} \tag{4.8}$$

where the third and fourth term satisfy

$$\begin{aligned} B^n &= \frac{\varepsilon^n}{2}\|\nabla v_{\tau,\varepsilon^n}\|^2_{L^2(\tilde{S}^n)} + \frac{1}{\varepsilon^n}\int_{\tilde{S}^n} v_{\tau,\varepsilon^n}(1 - v_{\tau,\varepsilon^n})\,\mathrm{d}x \\ &= I_{\varepsilon^n,\tau} + I^c_{\varepsilon^n,\tau} \leq 3\left(\int_{\mathbb{R}} \frac{1}{2}q_\tau'(s)^2 + W(q_\tau(s))\,\mathrm{d}s\right)\left(\operatorname{diam}(\Omega)^{d-1} + P_{\mathbb{R}^d}(\Omega)\right) =: C_S \end{aligned}$$

for all large $n \geq \tilde{n}$ for some $\tilde{n} \in \mathbb{N}$ by means of (4.3) and (4.4). Again, the uniform $\tilde{n} \in \mathbb{N}$ exists because there are only finitely many possible realizations of S considered and in turn only finitely many sequences $(I_{\varepsilon^n,\tau})_n$ and $(I^c_{\varepsilon^n,\tau})_n$ need to be taken into account.

By possibly choosing $\tilde{n}$ larger, we obtain for all $n \geq \max\{n_M, \tilde{n}\}$ that

$$B^n \leq C_S \leq \frac{1}{2(R+1)} \left(\frac{\varepsilon^n}{2} \|\nabla w^n\|^2_{L^2(T^n)} + \frac{1}{\varepsilon^n} \int_{T^n} w^n(1-w^n)\,\mathrm{d}x \right). \tag{4.9}$$

We combine our insights so that we compute and estimate by means of (4.6), (4.8), and (4.9):

$$\begin{aligned} E_{\varepsilon^n}(w^n) - E_{\varepsilon^n}(\tilde{w}^n) &= \frac{\varepsilon^n}{2} \|\nabla w^n\|^2_{L^2(\tilde{S}^n)} + \frac{1}{\varepsilon^n} \int_{\tilde{S}^n} w^n(1-w^n)\,\mathrm{d}x - A^n \\ &\geq \frac{\varepsilon^n}{2} \|\nabla w\|^2_{L^2(T^n)} + \frac{1}{\varepsilon^n} \int_{T^n} w^n(1-w^n)\,\mathrm{d}x - B^n \\ &\geq \frac{1}{2(R+1)} E_{\varepsilon^n}(w^n), \end{aligned}$$

which proves the claim (4.7).

We continue to prove the overall assertion. To this end, let $\bar{w}^n$ solve $\mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$. As in the proof of Proposition 4.5, we obtain

$$\begin{aligned} \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\bar{w}^n) + C_g &\geq \mathrm{pred}(w^n, \Delta^0, \varepsilon^n) \text{ and} \\ \mathrm{ared}(w^n, \bar{w}^n, \varepsilon^n) &\geq \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\bar{w}^n) - 2C_j. \end{aligned}$$

Moreover, the optimality of $\bar{w}^n$ and the feasibility of $\tilde{w}^n$ for $\mathrm{TR}_\varepsilon(w^n, \nabla_R j(w^n), \Delta^0)$ give

$$d^n \coloneqq \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\bar{w}^n) \geq \gamma E_{\varepsilon^n}(w^n) - \gamma E_{\varepsilon^n}(\tilde{w}^n) - 2C_g \geq \frac{1}{2(R+1)} \gamma E_{\varepsilon^n}(w^n) - 2C_g.$$

We obtain that

$$\frac{\mathrm{ared}(w^n, \bar{w}^n, \varepsilon^n)}{\mathrm{pred}(w^n, \Delta^0, \varepsilon^n)} \geq \frac{d^n - 2C_j}{d^n + C_g} \geq \rho$$

if $d^n \geq (1-\rho)^{-1}(\rho C_g + 2C_j)$, which holds in particular if $\gamma E_{\varepsilon^n}(w^n) \geq 2(R+1)((1-\rho)^{-1}(\rho C_g + 2C_j) + 2C_g) =: C_E$. Moreover, if n is large enough, this also implies that

$$\mathrm{ared}(w^n, \bar{w}^n, \varepsilon^n) \geq \frac{1}{2(R+1)} \gamma E_{\varepsilon^n}(w^n) - 2C_g - 2C_j \geq \frac{\rho}{1-\rho} C_g > \kappa^n \underline{\Delta}^n.$$

Now, we apply the same argument as in Proposition 4.5. Thus there exists $n_0 \in \mathbb{N}$ such that whenever for $n \geq n_0$ the situation $\gamma E_{\varepsilon^n}(w^n) \geq C_E$ occurs in the first iteration after ε^n was decreased and the trust-region radius reset to Δ^0 for $n \geq n_0$, the next step is accepted and ε^n will not be reduced further before $\gamma E_{\varepsilon^n}(w^n) < C_E$ holds. Consequently, there exists a subsequence, indexed by k, which abides the bound $\gamma E_{\varepsilon^{n_k}}(w^{n_k}) < C_E$ for all $k \in \mathbb{N}$, which implies the claim. ☐

Remark 4.7. The constructions in Propositions 4.5 and 4.6 together with the fact that the objective decreases monotonically during subsequent iterations during which ε^n is not altered implies that the boundedness assumption of the iterates $\varepsilon^{n(k)}$ in Assumption 4.2 are satisfied under the respective prerequisites of Propositions 4.5 and 4.6.

4.2. Asymptotic L-stationarity

We are ready to prove the asymptotics of Algorithm 1 under Assumptions 2.2 and 4.2. We first prove that the approximating iterates of limit points also have subsequences so that the values of the corresponding regularizers converge to the value of the regularizer in the limit, that is a multiple of the perimeter of the level set to the value 1 in Ω. Second we prove an auxiliary result that asserts that we can find a function that satisfies the sufficient decrease condition (3.11) using the actual reduction ared of the unregularized objective values and the predicted reduction pred given by the unregularized trust-region subproblem (TR). Finally, we prove the L-stationarity of the limit points.

Lemma 4.8. *Let Assumptions 2.2 and 4.2 hold. Let $\bar{w}$ be a limit point in the sense of Proposition 4.1 of the subsequence $(w^{n(k)})_k$ with $n(k)$ defined as in Assumption 4.2 for all $k \in \mathbb{N}$. Let $\bar{w}^{-1}(\{1\})$ admit an extension A outside of Ω, that is to $\mathbb{R}^d$, such that $\partial^* A$ is a compact, smooth hypersurface and satisfies $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* \Omega) = 0$. Then $\liminf_k E_{\varepsilon^{n(k)}}(w^{n(k)}) = C_0 P_\Omega(\bar{w}^{-1}(\{1\}))$.*

Proof. Proposition 4.1 implies that there is a subsequence of $(w^{n(k)})_k$ that converges in $L^1(\Omega)$ and because of the $L^\infty(\Omega)$-bounds also in $L^2(\Omega)$. In order to prove the claim, we restrict to this subsequence, denote its index by $n(k)$ as well in the interest of a cleaner presentation, and prove the claim for this subsequence. This is sufficient because Proposition 4.1 already gives $\liminf_k E_{\varepsilon^{n(k)}}(w^{n(k)}) \geq C_0 P_\Omega(\bar{w}^{-1}(\{1\}))$.

Because $n(k)$ is the last iteration such that $\varepsilon^{n(k)} = \varepsilon^0 r^{-k}$, it is clear that cvx_flag $= 0$ and the minimizers $\tilde{w}^{n(k)-j}$ of $\mathrm{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j})$ were rejected in Algorithm 1 ln. 8 for all $j \in \{0, \dots, k\}$, which implies that

$$\operatorname{ared}(w^{n(k)}, \tilde{w}^{n(k)-j}, \varepsilon^{n(k)}) \leq \kappa^0 \underline{\Delta}^0 2^{-2k} \tag{4.10}$$

$$\text{or } \operatorname{ared}(w^{n(k)}, \tilde{w}^{n(k)-j}, \Delta^0 2^{-j}) < \rho \operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j}, \varepsilon^{n(k)}) \tag{4.11}$$

We proceed with a proof by contradiction and assume that $\liminf_k \gamma E_{\varepsilon^{n(k)}}(w^{n(k)}) - P_\Omega(\bar{w}^{-1}(\{1\})) > \delta$ holds for some $\delta > 0$. We show that neither of the inequalities above can hold for sufficiently large values of k so that a step must have been accepted and thus a subsequence with these properties cannot exist. We start by showing that (4.11) cannot hold for large enough j and k (depending on j) and then infer that (4.10) cannot hold as well.

For all $\eta > 0$ there exists $j_0 \in \mathbb{N}$ such that for all $k > j_0$, all $j \in \{j_0, \dots, k\}$, and all w that are feasible for $\mathrm{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j})$, we obtain the estimates

$$\begin{aligned}\operatorname{ared}(w^{n(k)}, w, \varepsilon^{n(k)}) &= j(w^{n(k)}) - j(w) + \gamma E_{\varepsilon^{n(k)}}(w^{n(k)}) - \gamma E_{\varepsilon^{n(k)}}(w)\\ &\geq \gamma E_{\varepsilon^{n(k)}}(w^{n(k)}) - \gamma E_{\varepsilon^{n(k)}}(w) - \eta\end{aligned}$$

and

$$\left|\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j}, \varepsilon^{n(k)}) - \gamma E_{\varepsilon^{n(k)}}(w^{n(k)}) + \gamma E_{\varepsilon^{n(k)}}(w)\right| = |(\nabla_R\, j(w^{n(k)}), w^{n(k)} - w)_{L^2}| \leq \eta,$$

which give

$$\frac{\operatorname{ared}(w^{n(k)}, \tilde{w}^{n(k)}, \varepsilon^{n(k)})}{\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j_0}, \varepsilon^{n(k)})} \geq \frac{\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j_0}, \varepsilon^{n(k)}) - 2\eta}{\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j_0}, \varepsilon^{n(k)})}.$$

In order to continue the estimate, we consider an appropriate feasible approximation of $\bar{w}$ and distinguish two cases, specifically whether $\partial^* \bar{w}^{-1}(\{1\}) \cap \Omega = \emptyset$ or not.

If $\partial^* \bar{w}^{-1}(\{1\}) \cap \Omega = \emptyset$, then $\bar{w}(x) = 0$ a.e. or $\bar{w}(x) = 1$ a.e., which gives that $\bar{w}$ is feasible for the subproblems $\mathrm{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j_0})$ for all k large enough. We choose $\eta = \delta \cdot \frac{1-\rho}{3}$, in particular $3\eta < \delta$, and obtain for all $k > j_0$ that

$$\frac{\operatorname{ared}(w^{n(k)}, \tilde{w}^{n(k)}, \varepsilon^{n(k)})}{\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j_0}, \varepsilon^{n(k)})} \geq \frac{\delta - 3\eta}{\delta} = \rho$$

because $\bar{w}$ is feasible, implying $\operatorname{pred}(w^{n(k)}, \Delta^0 2^{-j_0}, \varepsilon^{n(k)}) \geq \delta - \eta$, and the function $x \mapsto \frac{x-2\eta}{x}$ is monotonically increasing for $x \geq 2\eta$. This implies that the step is accepted latemost for the trust-region radius $\Delta^0 2^{-j_0}$ (depending on η but not on k for $k > j_0$), implying that (4.11) cannot hold for $k > j_0$. For the accepted step $\tilde{w}$ for the trust-region radius $\Delta^0 2^{-\tilde{j}}$ with $\tilde{j} \leq j_0$, we know that

$$\operatorname{ared}(w^{n(k)}, \tilde{w}, \varepsilon^{n(k)}) \geq \rho \operatorname{pred}(w^{n(k)}, \Delta^0 2^{-\tilde{j}}, \varepsilon^{n(k)}) \geq \rho\,(\delta - \eta) \geq \frac{2\rho}{3}\delta,$$

where the second inequality follows from the optimality of $\tilde{w}$ for the trust-region subproblem. Consequently, (4.10) cannot hold for large enough k. This implies that the sequence $(n(k))_k$ cannot exist under the assumption that $\liminf_k \gamma E_{\varepsilon^{n(k)}}(w^{n(k)}) - P_\Omega(\bar{w}^{-1}(\{1\})) > \delta$ for some $\delta > 0$.

If $\partial^* \bar{w}^{-1}(\{1\}) \cap \Omega \neq \emptyset$, then $\bar{w} \notin H^1(\Omega)$ so that $E_{\varepsilon^{n(k)}}(\bar{w}) = \infty$ and we need an additional approximation in order to obtain a feasible approximation. The properties of the assumed extension A of $\bar{w}^{-1}(\{1\})$ imply that we can apply the reverse approximation argument from Proposition 2 in [21], which gives that there are functions $w_{\varepsilon^{n(k)}}$, $k \in \mathbb{N}$, such that $E_{\varepsilon^{n(k)}}(\bar{w}_{\varepsilon^{n(k)}}) \to C_0 P_\Omega(\bar{w}^{-1}(\{1\}))$. As part of its proof (see p. 135 in [21]), it is shown that the functions satisfy

$$\|\bar{w}_{\varepsilon^{n(k)}} - \bar{w}\|_{L^1(\Omega)} \leq (\varepsilon^{n(k)})^{\frac{1}{2}} C(\bar{w}), \tag{4.12}$$

where $C(\bar{w})$ is a constant that depends on $\bar{w}$. Because $r > 4$ we have that $\varepsilon^{n(k)} \in o((\underline{\Delta}^{n(k)})^2)$ by means of the update rules for ε^n and $\underline{\Delta}^n$. Consequently, by choosing $\eta = \delta \cdot \frac{1-\rho}{3}$, we obtain again a corresponding j_0 and for all k large enough we obtain that $\bar{w}_{\varepsilon^{n(k)}}$ is feasible for the trust-region subproblem $\mathrm{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j_0})$ (for example by choosing $j_0 - 1$ for the j_0 that could be chosen for $\bar{w}$). Now, we obtain

$$\begin{aligned} &E_{\varepsilon^{n(k)}}(w^{n(k)}) - E_{\varepsilon^{n(k)}}(\bar{w}_{\varepsilon^{n(k)}}) \\ &\quad \geq E_{\varepsilon^{n(k)}}(w^{n(k)}) - C_0 P_\Omega(w^{-1}(\{1\})) - |E_{\varepsilon^{n(k)}}(\bar{w}_{\varepsilon^{n(k)}}) - C_0 P_\Omega(w^{-1}(\{1\}))| \geq \frac{\delta}{2} \end{aligned}$$

for all k large enough. Then, using $\frac{\delta}{2}$ instead of δ for the adjustment of η, the same reasoning as for $\partial^* \bar{w}^{-1}(\{1\}) \cap \Omega = \emptyset$ applies in order to refute the existence of the claimed sequence $(n(k))_k$. ☐

Remark 4.9. If the boundary regularity assumption of the extension in Lemma 4.8 is dropped, a further approximation of $\bar{w}^{-1}(\{1\})$ by sets with smooth boundary is necessary in order to apply Proposition 2 from [21]. Unfortunately, the constant in the estimate (4.12) then depends on these approximants and might increase for every approximant, thereby preventing us from controlling the convergence rate of the left hand side of (4.12). Consequently, we have not been able to prove the claim in this case.

Remark 4.10. In Algorithm 1 22, $\underline{\Delta}^n$ is divided by two while ε^n is divided by $r > 4$ when Δ^{n+1} falls below $\underline{\Delta}^n$. This sufficient to guarantee $\varepsilon^{n(k)} \in o((\underline{\Delta}^{n(k)})^2)$ in the proof of Lemma 4.8. While we have used these update rules in the interest of a clean presentation, other update rules that guarantee $\varepsilon^{n(k)} \in o((\underline{\Delta}^{n(k)})^2)$ would be feasible as well.

The following auxiliary lemma is a variant of Lemma 6.2 in [9], where we assume the relaxed condition Assumption 2.2 instead of twice continuous differentiability.

Lemma 4.11. *Let Assumption 2.2 hold. Let $\rho \in (0,1)$. Let $w \in L^1(\Omega)$ satisfy $w(x) \in \{0,1\}$ for a.e. $x \in \Omega$ and $P_\Omega(w^{-1}(\{1\})) < \infty$. Let $\nabla_R j(w) \in C(\bar{\Omega})$. Let $\Delta^k \searrow 0$. Let $\tilde{w}^k$ minimize $\mathrm{TR}(w, \nabla_R j(w), \Delta^k)$ for $k \in \mathbb{N}$. Then at least one of the following statements holds true.*

1. *The function w satisfies the L-stationarity conditions (2.1).*
2. *There exists $k_0 \in \mathbb{N}$ such that the objective value of $\mathrm{TR}(w, \nabla_R\, j(w), \Delta^{k_0})$ at w^{k_0} is zero.*
3. *There exists $k_0 \in \mathbb{N}$ such that*

$$\begin{aligned} \delta^{k_0} &:= j(w) + \gamma C_0 P(w^{-1}(\{1\})) - j(\tilde{w}^{k_0}) - \gamma C_0 P((\tilde{w}^{k_0})^{-1}(\{1\})) \\ &\geq \rho(\nabla_R\, j(w), w - \tilde{w}^{k_0}) + \gamma C_0 P(w^{-1}(\{1\})) - \gamma C_0 P((\tilde{w}^{k_0})^{-1}(\{1\})) =: p^{k_0}. \end{aligned} \tag{4.13}$$

Proof. We closely follow the lines of the proof of Lemma 6.2 in [9]. Let $k \in \mathbb{N}$. The objective of $\mathrm{TR}(w, \nabla_R\, j(w), \Delta^k)$, the negative of the right hand side in (4.13), is zero when inserting w itself. Consequently, if the optimal solution $\tilde{w}^k$ has objective value zero (Outcome 2), then w minimizes $\mathrm{TR}(w, \nabla_R\, j(w), \Delta^k)$. Proposition 3.2 implies that w is L-stationary (Outcome 1). Thus it remains to show that Outcome 3 when Outcome 1 does not, that is we show (4.13) for some k_0.

Because w is not L-stationary, there exists $\phi \in C_c^\infty(\Omega; \mathbb{R}^d)$ with $\operatorname{supp} \phi \neq \emptyset$ and $\eta > 0$ such that

$$\int_{\partial^* E \cap \Omega} -\nabla_R\, j(w)(\phi \cdot n_E)\, \mathrm{d}\mathcal{H}^{d-1} - \gamma C_0 \int_{\partial^* E \cap \Omega} \operatorname{div}_E \phi\, \mathrm{d}\mathcal{H}^{d-1} > \eta.$$

Let $(f_t)_{t \in (-\tau_0, \tau_0)}$ be defined by $f_t := I + t\phi$ for $\phi \in C_c^\infty(\Omega, \mathbb{R}^d)$ and $t \in (-\tau_0, \tau_0)$ for some $\tau_0 > 0$, see also Definition 3.1 and Proposition 3.2 in [9]. We define $f_t^\# w := \chi_{f_t(w^{-1}(\{1\}))}$. Then as in the proof of Lemma 6.2 in [9], we obtain for all $k \in \mathbb{N}$ sufficiently large that there is $\tau^k > 0$ such that $f_t^\# w$ is feasible for $\mathrm{TR}(w, \nabla_R\, j(w), \Delta^k)$ for all $t \in [0, \tau^k]$ and $\|f_{\tau^k}^\# w - w\|_{L^1} = \Delta^k$.

Assumption 2.2 yields

$$\delta^k = p^k + o(\|\tilde{w}^k - w\|_{L^1}).$$

Consequently, we obtain for all k large enough and all $t \in (0, \tau^k]$:

$$\begin{aligned} \delta^k &\geq \rho p^k - (1-\rho)\left((\nabla_R\, j(w), f_{\tau^k}^\# w - w)_{L^2} + \alpha\, \mathrm{TV}(f_{\tau^k}^\# w) - \alpha\, \mathrm{TV}(w)\right) + o(\|\tilde{w}^k - w\|_{L^1}) \\ &\geq \rho p^k + (1-\rho)\left(\tau^k \eta + o(\tau^k)\right) + o(\|\tilde{w}^k - w\|_{L^1}) \\ &\geq \rho p^k + (1-\rho)\left(\tau^k \eta + o(\tau^k)\right) + o(\tau^k). \end{aligned}$$

Here, the first inequality holds because $\tilde{w}^k$ minimizes $\mathrm{TR}(w, \nabla_R\, j(w), \Delta^k)$, the second inequality holds by virtue of Lemmas 3.3 and 3.5 in [9], and the third holds because

$$\|\tilde{w}^k - w\|_{L^1} \leq \Delta^k = \|f_{\tau^k}^\# w - w\|_{L^1} \leq \kappa \tau^k$$

where the existence of $\kappa > 0$, which is independent of k, is asserted by Lemma 3.8 in [9]. The term $(1-\rho)\tau^k\eta$, which is positive, is larger than the $o(\tau^k)$ terms for all k large enough so that (4.13) holds for some $k_0 \in \mathbb{N}$. ☐

We are ready to prove the asymptotics of our algorithm.

Theorem 4.12. *Let Assumptions 2.2 and 4.2 hold. Let $\bar{w}$ be a limit point in the sense of Proposition 4.1 of the subsequence $(w^{n(k)})_k$ with $n(k)$ defined as in Assumption 4.2. Let $\nabla_R\, j(\bar{w}) \in C(\bar{\Omega})$ and $\bar{w}^{-1}(\{1\})$ admit an extension A outside of Ω, that is to $\mathbb{R}^d$, such that $\partial^* A$ is a compact, smooth hypersurface and satisfies $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* \Omega) = 0$. Then $\bar{w}$ is L-stationary.*

Proof. We consider a subsequence of the iterates $(n(k))_k$, which we denote by the same symbol for ease of notation, as is asserted by Assumption 4.2 such that

$$E_{\varepsilon^{n(k)}}(w^{n(k)}) \to C_0 P_\Omega(\bar{w}^{-1}(\{1\})). \tag{4.14}$$

By virtue of the update rules for $\underline{\Delta}^n$ in Algorithm 1 and the well-definedness of the sequence $(n(k))_k$ it follows that for all $j \in \mathbb{N}$ there is k_j so that for all $k \geq k_j$ it holds that $\Delta^0 2^{-j} > \underline{\Delta}^{n(k)}$ and the solution of $\text{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j})$ is not accepted in Algorithm 1 ln. 8.

For any such j, we employ Theorem 3.3 to deduce that the minimizers $w^{n(k),*}$ returned by the subproblem solves of $\text{TR}_\varepsilon(w^{n(k)}, \nabla_R\, j(w^{n(k)}), \Delta^0 2^{-j})$ in Algorithm 1 ln. 6 converge (after possibly restricting to subsequences and after invoking the compactness result from [22], see Prop. 4.1) to minimizers w^* of $\text{TR}(\bar{w}, \nabla_R\, j(\bar{w}), \Delta^0 2^{-j})$ in $L^1(\Omega)$ for $k \to \infty$:

$$T(w^*) \leq \liminf_{k\to\infty} T^{n(k)}(w^{n(k),*}) \leq \limsup_{k\to\infty} T^{n(k)}(w^{n(k),r}) \to T(w^*),$$

where the $w^{n(k),r}$ denote the elements of the reverse approximating sequence that exist by virtue of the upper bound inequality of the Γ-convergence result in Theorem 3.3.

Consequently, the predicted reductions for the returned minimizers of $\text{TR}_\varepsilon(w^{n(k)}, \nabla_R j(w^{n(k)}), \Delta^0 2^{-j})$ converge to the predicted reduction of the limit problem $\text{TR}(\bar{w}, \nabla_R j(\bar{w}), \Delta^0 2^{-j})$, which in combination with the continuity properties of ∇j yields

$$\gamma E_{\varepsilon^{n(k)}}(w^{n(k),*}) \to C_0 P_\Omega((w^*)^{-1}(\{1\})).$$

In turn, we imply convergence of the induced actual reductions as well.

This implies that if there is $s \in (\rho, 1)$ such that

$$\frac{j(\bar{w}) + \gamma C_0 P(\bar{w}^{-1}(\{1\})) - j(w^*) - \gamma C_0 P((w^*)^{-1}(\{1\}))}{(\nabla_R\, j(\bar{w}), \bar{w} - w^*) + \gamma C_0 P(\bar{w}^{-1}(\{1\})) - \gamma C_0 P((w^*)^{-1}(\{1\}))} \geq s, \tag{4.15}$$

then there is $\bar{k}_j \geq k_j$ such that for all $k \geq \bar{k}_j$ it follows that

$$\frac{\text{ared}(w^{n(k)}, w^{n(k),*}, \varepsilon^{n(k)})}{\text{pred}(w^{n(k)}, \Delta^0 2^{-j}, \varepsilon^{n(k)})} \geq \rho,$$

which would contradict the definition of $n(k)$ and in turn $w^{n(k)}$.

If $\bar{w}$ is not L-stationary, then Lemma 4.11 (3) implies that (4.15) holds for some large enough j, which leads to said contradiction and thus we conclude that $\bar{w}$ is L-stationary. ☐

Remark 4.13. We do not see how Algorithm 1 could possibly ensure that its limit points $\bar{w}$ satisfy $\nabla_R\, j(\bar{w}) \in C(\bar{\Omega})$ or the regularity of the extension of $\bar{w}^{-1}$ by choice of its parameters, which we thus keep as true regularity assumptions. While $\nabla_R\, j(\bar{w}) \in C(\bar{\Omega})$ can be verifiable *a priori* by means of the regularity properties of the underlying PDE, we note that we do not know at the moment if and how the assumption in Theorem 4.12 that the extension A of $\bar{w}^{-1}$ has a reduced boundary $\partial^* A$ that is a compact, smooth hypersurface and satisfies $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* \Omega) = 0$ can be checked in practice. Therefore, it has a similar role as so-called constraint qualifications in nonlinear programming, where some regularity on the constraints (in our case on the integrality restriction) needs to be satisfied in order to obtain stationarity or convergence to stationary points.

We believe that the assumption may turn out to be realistic in many settings, as it is common in optimization and in particular optimal control that the optimality conditions allow to improve regularity of stationary points.

5. Applications of the algorithmic framework

We discuss two classes of problems. In Section 5.1, we consider the case of elliptic equations and briefly argue that non-standard regularity theory allows to certify Assumptions 2.2 and 4.2. In Section 5.2 we provide a numerical setup and computational results for the elliptic equation. In Section 5.3, we consider a more challenging class of coefficient-control problems for acoustic wave equations. Here, we point out the gaps between our regularity assumptions. In Section 5.4, we give a numerical setup for the acoustic wave equations and provide some computational results in order to illustrate the algorithm albeit outside of the scope of its assumptions.

5.1. Source control of elliptic equations

We first discuss the developed optimization framework for the control of a source term in an elliptic equation. Let $\Omega = (0, 2)^2$ and $\nu > 0$. Consider

$$\begin{cases} -\nu\Delta u + u = Bw + f & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for some bounded linear operator $B : L^1(\Omega) \to L^1(\Omega)$, $f \in L^1(\Omega)$, and $w \in L^1(\Omega)$ is the given control. We further assume that the adjoint $B^* : L^\infty(\Omega) \to L^\infty(\Omega)$ is also bounded as a mapping $B^* : C(\bar{\Omega}) \to C(\bar{\Omega})$.

By means of [24], Theorems 1 and 3 and [25], Theorem 4.6, there exists $q > 2$ such that

$$\|u\|_{W^{1,q}(\Omega)} \le c_1\|w\|_{W^{-1,q}(\Omega)} + c_2\|f\|_{W^{-1,q}(\Omega)}$$

for some constants c_1, $c_2 > 0$. Because of the dense embeddings $W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$ for $q > 2$, we obtain $L^1(\Omega) \hookrightarrow \mathcal{M}(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the space of Radon measures by means of the Riesz representation theorem.

Let j in (PA) be $j = J \circ S$, where $S : L^1(\Omega) \to L^2(\Omega)$ is the solution operator of (5.1) and $J : L^2(\Omega) \to \mathbb{R}$ be continuously differentiable. The corresponding boundary-value problem for the adjoint equation reads

$$\begin{cases} -\nu\Delta p + p = -J'(S(w)) & \text{in } \Omega, \\ \qquad\quad p = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

and inherits the regularity properties of (5.1) and we obtain $\nabla_R\, j(w) = B^* p \in C(\bar{\Omega})$. Let $B_{L^\infty(\Omega)}$ be the unit ball of $L^\infty(\Omega)$. Then $B_{L^\infty(\Omega)}$ with respect to all L^s-norms, $s \ge 1$, and in turn $S(B_{L^\infty(\Omega)})$ is bounded in $W^{1,q}(\Omega)$ for the $q > 2$ from above. Consequently, $S(B_{L^\infty(\Omega)})$ is also bounded in $H^1(\Omega)$ so that $S(B_{L^\infty(\Omega)})$ is a compact subset of $L^2(\Omega)$. Thus because $J : L^2(\Omega) \to \mathbb{R}$ is continuous, $J \circ S$ is bounded on $B_{L^\infty(\Omega)}$. Because $J : L^2(\Omega) \to \mathbb{R}$ is even continuously differentiable, we obtain that the right hand side of (5.2) is bounded in $L^2(\Omega)$ and in turn p is bounded in $W^{1,q}(\Omega)$ and $C(\bar{\Omega})$ too. Together with the assumptions on B^*, we obtain that $\nabla_R\, j$ is bounded in $L^\infty(\Omega)$ and $C(\bar{\Omega})$.

Combining these considerations, we obtain the following proposition, which proves that Assumption 2.2 is satisfied as well as the prerequisites of Proposition 4.5 and Proposition 4.6, implying that Assumption 4.2 is satisfied for the execution of Algorithm 1.

Proposition 5.1. *Let $S : L^1(\Omega) \to L^2(\Omega)$ be the solution operator of (5.1) and $J : L^2(\Omega) \to \mathbb{R}$ be continuously differentiable. Let j in (PA) be $j = J \circ S$. Then $j = J \circ S$ is continuously differentiable as a mapping $j : L^1(\Omega) \to \mathbb{R}$ and j and $\nabla_R\, j$ are bounded on the unit ball of $L^\infty(\Omega)$. Moreover, $\nabla_R\, j(w) \in C(\bar{\Omega})$ for all $w \in B_{L^\infty(\Omega)}$.*

Consequently, we satisfy all assumptions *a priori* except the constraint qualification-type assumption that any limit point $\bar{w}$ produced by the algorithm satisfies that $\bar{w}^{-1}(\{1\})$ admits an extension A outside of Ω, that is to $\mathbb{R}^d$, such that $\partial^* A$ is a compact, smooth hypersurface and $\mathcal{H}^{d-1}(\partial^* A \cap \partial^* \Omega) = 0$.

5.2. Numerical experiments for the source control problem

We provide an example in Section 5.2.1, where a global minimizer of the problem (P) is known exactly. We briefly describe the discretization of the state and adjoint PDE in Section 5.2.2. Our algorithmic setup and the experiments are described in Section 5.2.3. We describe the results in Section 5.2.4.

5.2.1. Exact solution

We construct a globally optimal solution (u^*, w^*) to the relaxed problem

$$\begin{aligned} \min_{u,w}\ & \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \gamma C_0 P_\Omega(w^{-1}(\{1\})) \\ \text{subject to }\ & -\nu\Delta u + u = w + f \text{ in } \Omega, \\ & u = 0 \text{ on } \partial\Omega, \\ & w(x) \in \mathbb{R} \text{ for a.e. } x \in \Omega \end{aligned} \tag{5.3}$$

such that w^* is $\{0,1\}$-valued and $\partial^* {w^*}^{-1}(\{1\})$ is a compact, smooth hypersurface that is completely contained in Ω. This implies that u^* is also a solution to (P) with the choices $J(u) \coloneqq \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)}$ and $S(w) \coloneqq (-\nu\Delta + I)^{-1}(w+f)$ and satisfies the additional constraint qualification-type assumption of Theorem 4.12. Regarding the constants, we choose $\nu = 10^{-2}$ and $\gamma = 10^{-2}C_0^{-1}$ and note that $C_0 = \int_0^1 \sqrt{2s(1-s)}\,\mathrm{d}s = \frac{\pi}{4\sqrt{2}}$ for our choice E_ε, which can be deduced by following the analysis in [21], p. 132 and observing that choosing $W(x) = 2x(1-x)$ therein corresponds to our setting (after a transformation of $\frac{\varepsilon}{2}$ to $\sqrt{\varepsilon}$). We follow the ideas of [26] and construct u_d, f, u^*, and p^* such that $w^* = \chi_{B_{0.5}((1,1)^T)}$, u^*, and p^* satisfy the following optimality system of (5.3), see also Theorem 3 in [27] and Theorem 2.6 in [28] for the derivation of optimality conditions,

$$-\nu\Delta u + u = w + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.4a}$$

$$-\nu\Delta p + p = u - u_d \text{ in } \Omega, \quad p = 0 \text{ on } \partial\Omega, \tag{5.4b}$$

$$\int_\Omega p\,\mathrm{d}x = 0 \tag{5.4c}$$

$$(-p, w)_{L^2} = \gamma C_0 \,\mathrm{TV}(w), \tag{5.4d}$$

$$(-p, v)_{L^2} \le \gamma C_0 \,\mathrm{TV}(v) \text{ for all } v \in \mathrm{BV}(\Omega). \tag{5.4e}$$

Following [26], we first construct a function $\psi \in C_c^3((0,1))$ that satisfies $\psi(0.5) = 1$ and $0 \le \psi(s) \le 1$ for all $s \in (0,1)$. This then allows to define

$$\phi(x) = \begin{cases} -\psi(\|x-(1,1)^T\|)\frac{x-(1,1)^T}{\|x-(1,1)^T\|} & \text{if } \|x-(1,1)^T\| \in \operatorname{supp}\psi, \\ 0 & \text{else.} \end{cases}$$

In order to meet all conditions, we solve a linear system for the coefficients of a degree 8 polynomial ansatz for ψ. Then we choose $p^* = \gamma C_0 \operatorname{div}\phi$, which allows to verify (5.4d) and (5.4e) for the choice $w = w^*$. Next, we choose $u^* \in C_0^2(\Omega)$ as $u^*(x) = -2x_1^2(2-x_1)^2 x_2^2(2-x_2)^2$ and in turn $f = -\nu\Delta u^* + u^* - w^*$ to satisfy (5.4a) and $u_d = u^* - (-\nu\Delta p^* + p^*)$ to satisfy (5.4b). Finally, (5.4c) holds by means of Gauß–Green formula and the fact that ϕ has compact support.

5.2.2. Discretization of the state and adjoint problems

We discretize Ω into 64×64 squares, which are subdivided into two triangles each. We choose CG1 ansatz functions for u, p, and w. We use the `FEniCSx` library [29–31] to assemble and solve the PDE (5.4a) and its adjoint (5.4b). In order to provide (approximations of) u_d and f, we implement the exact solution from Section 5.2.1 using `sympy` [32] and interpolate it into CG1 functions.

5.2.3. Setup

Regarding the implementation of Algorithm 1, we only solve instances of the convex subproblems ($\mathrm{TR}^c_\varepsilon$) in order to keep the implementation and computational effort manageable, where we note that solving quadratic programs with one negative eigenvalue is already NP-hard in general [33]. We execute our algorithm on a laptop computer with an Intel(R) Core(TM) i7-11850H CPU (2.50 GHz) and 64 GB RAM. We solve the discretized subproblems with Gurobi 10.0.0 [34]. We initialize the algorithm with the choices $\Delta^0 = 1$, $\rho = 10^{-4}$, $\kappa^0 = 10^{-8}$, $\underline{\Delta}^0 = 2.44 \times 10^{-4}$, and $r = 5$. We initialize the homotopy with the different values $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}, 8 \times 10^{-3}\}$ and for each of these six values for ε^0 with 21 different initial controls w^0. Specifically, we choose eleven constant initial controls $w^0(x) = i \times 10^{-1}$ for $i \in \{0, \ldots, 10\}$ and $w^0(x) = \sin(0.25 i\pi x_1)^2 \sin(0.25 i\pi x_2)^2$ for $i \in \{1, \ldots, 10\}$. This gives a total of 126 runs of Algorithm 1.

Our fixed CG1 ansatz limits the gradient of w and, consequently, E_ε cannot be expected to be reliable for very small values of ε^n. In particular, $\|\nabla w^n\|^2_{L^2}$ cannot become larger and $\int_\Omega \Psi(w^n)$ cannot become smaller if a phase transition from 0 to 1 already happens inside one grid cell. We have observed such effects when ε^n was reduced below 8×10^{-3} and therefore always consider the last accepted step for $\varepsilon^n = 8 \times 10^{-3}$ as the last iterate of the homotopy but note that in all cases, the algorithm did accept at most one further iteration for the next smaller value $\varepsilon^n = 1.6 \times 10^{-3}$ with the L^2-norm of the step always being below 10^{-10}.

5.2.4. Results

All 126 runs produced sequences of iterates so that the nodal values in the CG1 coefficient vector were almost binary. The largest non-binarity of the coefficients over all choices of w^0 and for $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}\}$ was 3.37×10^{-10} while the largest final non-binarity for $\varepsilon^0 = 8 \times 10^{-2}$ was 1.08×10^{-7}.

For the choices $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}\}$, all sequences of produced iterates for the corresponding 105 runs have approximately the same final iterate w_f and corresponding state and adjoint u_f and p_f, which are close to the CG1 interpolations of the exact solution $\mathcal{I}w^*$ and corresponding $\mathcal{I}u^*$ and $\mathcal{I}p^*$. Specifically, we obtain for all cases

$$\|w_f - \mathcal{I}w^*\|_{L^2(\Omega)} = 3.12 \times 10^{-2}, \quad \|u_f - \mathcal{I}u^*\|_{L^2(\Omega)} = 3.68 \times 10^{-3}, \quad \|p_f - \mathcal{I}p^*\|_{L^2(\Omega)} = 2.21 \times 10^{-3}$$

after rounding to three significant digits. In line with the fact that $\mathcal{I}w^*$ is not optimal for the discretized problems, we observe that the objective value $J(S(w_f)) + \gamma E_{8\cdot 10^{-3}}(w_f)$ is always slightly smaller than $J(S(\mathcal{I}w^*)) + \gamma E_{8\cdot 10^{-3}}(\mathcal{I}w^*)$. We have validated for two runs that these errors decrease when choosing a finer grid. To give a visual impression of this global convergence behavior, we pick the choice $w^0(x) = \sin(2.5\pi x_1)^2 \sin(2.5\pi x_2)^2$ and plot the L^2-distance $\|w^n - \mathcal{I}w^*\|_{L^2}$ over the course of the iterations n of Algorithm 1 for the choices $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}\}$ in Figure 2.

The 21 runs for the choice $\varepsilon^0 = 8.0 \times 10^{-3}$ produced only very few iterations (between 1 and 8) and did not yield final iterates close to $\mathcal{I}w^*$ but generally a binary-valued function that was close to the initialization. While it is difficult to come to a definite conclusion about the reasons here since discretization and numerical precision effects may also be involved here, we believe that this can mainly be attributed the fact that we only use the convex trust-region subproblems, which imply a high penalty proportionally to $\frac{1}{\varepsilon}$ on the change of a nodal value that is already close to 1 to 0 or vice versa, see ($\mathrm{TR}^c_\varepsilon$) although the difference may be very small or even zero in the non-convex formulation.

For the choices $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}\}$, where the final iterates were close to the globally optimal solution of the limit problem and who are thus comparable, the number of iterations and run times for the execution of Algorithm 1 decreased when starting with a smaller value of ε^0. We have tabulated the mean and average iteration numbers as well as the average run times in Table 1 for $\varepsilon^0 \in \{25, 5, 1, 2 \times 10^{-1}, 4 \times 10^{-2}\}$.

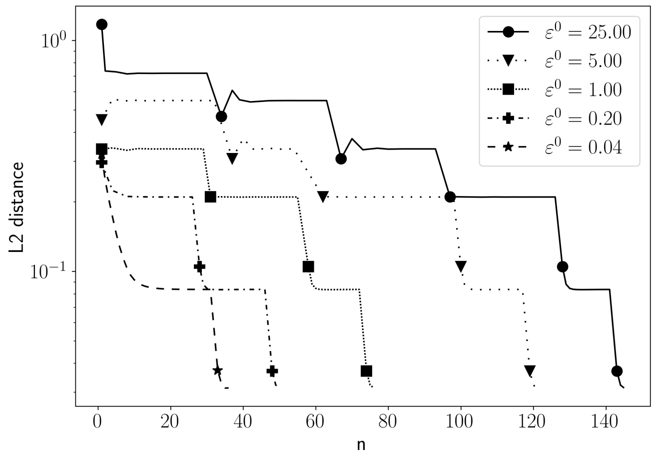


FIGURE 2. Global convergence of the iterates produced by Algorithm 1 indicated by the distance $\|w^n - \mathcal{I}w^*\|_{L^2(\Omega)}$ for the different values of ε^0. The locations of the marker symbols indicate the first iteration of an accepted iterate for a new (reduced) value of ε^n.

TABLE 1. Average and median number of iterations and average run time of the executions of Algorithm 1 over the different initializations w^0 for the different values of ε^0.

ε^0	25	5	1	2×10^{-1}	4×10^{-2}
Average iteration number	146.76	109.81	76.81	52	37.24
Median iteration number	147	109	76	52	33
Average run time [s]	593	458	325	213	147

5.3. Control of the propagation speed in a linear wave equation

Let now $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ be a bounded Lipschitz-regular domain. We next consider the control of the propagation speed in a damped wave equation. This problem falls outside of the developed theoretical framework; however, as shown later, the algorithm nevertheless performs well in numerical tests. More precisely, we consider the following problem:

$$\begin{cases} u_{tt} - \operatorname{div}(a(w)\nabla u) - b\Delta u_t = f \quad \text{on} & \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} & \partial\Omega \times (0,T), \\ (u, u_t)_{|t=0} = (u_0, u_1). \end{cases} \tag{5.5}$$

In the context of acoustics, the wave equation given above describes propagation of sound waves through thermoviscous media. Here $u = u(x,t)$ represents the acoustic pressure, $a(w)$ corresponds to the speed of sound squared (controlled by w), and $b > 0$ is the so-called sound diffusivity. We assume that

$$a(w) = c^2(1 + w(x)),$$

where $c > 0$ is the reference value for the speed of sound in the medium and

$$w \in W_{\mathrm{ad}} = \{w \in L^\infty(\Omega) : \ \underline{w} \le w \le \overline{w} \quad \text{a.e. in } \Omega\} \tag{5.6}$$

for some $\overline{w}$, $\underline{w}$, such that $1+\underline{w}>0$. This setting is similar to the one of [5], which studied coefficient-control of the undamped wave equation (*i.e.*, with $b=0$ in (5.5)). The objective of the optimization process is to reach the desired pressure distribution in a focal area of interest $D\subset\Omega$:

$$J(u)=\frac{1}{2}\int_0^T\int_D(u-u_{\mathrm{d}})^2\,\mathrm{d}x\mathrm{d}s$$

for a given $u_{\mathrm{d}}\in L^2(0,T;L^2(\Omega))$. The admissible space for u is

$$\begin{aligned}\mathcal{U}=\{u\in L^\infty(0,T;H^1(\Omega)):\, &u_t\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\ &u_{tt}\in L^2(0,T;H^1(\Omega)^*)\}.\end{aligned}\tag{5.7}$$

Control of a strongly damped (nonlinear) wave equation has been considered in [6] using a phase-field approach. We can can adapt the theoretical results from [6] to the present simpler setting in a straightforward manner.

Proposition 5.2 (Well-posedness of the state problem)**.** *Given $T>0$ and $b>0$, let $f\in L^2(0,T;H^1(\Omega)^*)$, and $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$. Then for every $w\in W_{\mathrm{ad}}$, there is a unique $u\in\mathcal{U}$, which solves*

$$\int_\Omega u_{tt}(t)v\,\mathrm{d}x+\int_\Omega(a(w)\nabla u(t)+b\nabla u_t(t))\cdot\nabla v\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x\tag{5.8}$$

a.e. in time for all $v\in H^1(\Omega)$, with $(u,u_t)|_{t=0}=(u_0,u_1)$. Furthermore, this solution satisfies the following energy estimate:

$$\begin{aligned}&\|u_t\|^2_{L^\infty(L^2(\Omega))}+\|\nabla u\|^2_{L^\infty(L^2(\Omega))}+\|\nabla u_t\|^2_{L^2(L^2(\Omega))}\\ &\quad\le C_1\exp(C_2T)(\|u_0\|^2_{H^1(\Omega)}+\|u_1\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(H^1(\Omega)^*)}),\end{aligned}\tag{5.9}$$

where $C_{1,2}>0$ do not depend on w.

Proof. Unique solvability follows, for example, by [35], Theorem 1, Chapter 5. Estimate (5.9) follows by testing the problem with $v=u_t(t)\in H^1(\Omega)$ and a straightforward modification of the energy arguments in, *e.g.*, [6], Proposition 3.1. Note that from (5.9) and the PDE, we also have the following bound:

$$\begin{aligned}\|u_{tt}\|_{L^2(H^1(\Omega)^*)}&\lesssim\|\nabla u\|_{L^2(L^2(\Omega))}+\|\nabla u_t\|_{L^2(L^2(\Omega))}+\|f\|_{L^2(H^1(\Omega)^*)}\\ &\le C_1\exp(C_2T)\left(\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|f\|_{L^2(H^1(\Omega)^*)}\right);\end{aligned}$$

see, *e.g.* [36], Chapter 7 for similar arguments. □

Thanks to the above well-posedness result, the operator $S:W_{\mathrm{ad}}\to\mathcal{U}$ is well-defined. The next result established the differentiability of the control-to-state mapping.

Proposition 5.3. *Under the assumptions of Proposition 5.2, let $\tilde{u}$ and u be the solutions of the state problem corresponding to the controls $\tilde{w}$ and w, respectively. Then*

$$\|\tilde{u}-u\|_{\mathcal{U}}\le C(u_0,u_1,T)\|\tilde{w}-w\|_{L^\infty(\Omega)}.\tag{5.10}$$

Proof. As the difference $\bar{u}=\tilde{u}-u$ solves

$$\int_\Omega\bar{u}_{tt}(t)v\,\mathrm{d}x+\int_\Omega(a(w)\nabla\bar{u}(t)+b\nabla\bar{u}_t(t))\cdot\nabla v\,\mathrm{d}x=-\int_\Omega(a(\tilde{w})-a(w))\nabla\tilde{u}\cdot\nabla v\,\mathrm{d}s,$$

for all $v \in H^1(\Omega)$ a.e. in time, with zero initial data, the statement follows by testing the above weak form with $v = \bar{u}_t(t) \in H^1(\Omega)$, integrating over time, and using the fact that $\tilde{u} \in \mathcal{U}$. $\square$

Proposition 5.4. *Under the assumptions of Proposition 5.2, the control-to-state operator $S : W_{\mathrm{ad}} \to \mathcal{U}$ is well-defined. Furthermore, it is Fréchet differentiable and its directional derivative at w in the direction of $\xi \in L^\infty(\Omega)$ is given by $S'(w)\xi = u^{*,\xi}$, where $u^{*,\xi}$ is the unique solution of*

$$\int_\Omega u_{tt}^{*,\xi}(t) v \,\mathrm{d}x + \int_\Omega (a(w)\nabla u^{*,\xi}(t) + b\nabla u_t^{*,\xi}(t)) \cdot \nabla v \,\mathrm{d}x = -c^2 \int_\Omega \xi \nabla S(w)(t) \cdot \nabla v \,\mathrm{d}x \tag{5.11}$$

a.e. in time for all $v \in H^1(\Omega)$, with $(u^{*,\xi}, u_t^{*,\xi})|_{t=0} = (0,0)$. Furthermore, it holds

$$\|u^{*,\xi}\|_{\mathcal{U}} \leq C\,(T)\, \|\xi\|_{L^\infty(\Omega)} (\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(H^1(\Omega)^*)}). \tag{5.12}$$

Proof. The proof of the statement follows similarly to [6], Proposition 4.2 and Theorem 4.1; the main difference is that we are in a lower-order setting compared to [6] in terms of the regularity of data (on the other hand, we are working with a simpler equation). We provide some details here as they are relevant for discussing this problem in the context of the theoretical framework of the paper. Given $\xi \in L^\infty(\Omega)$, let

$$r = S(w + \xi) - S(w) - u^{*,\xi}.$$

Note that the well-posedness of (5.11) follows by Proposition 5.2 with the right-hand side set to $f = \mathrm{div}(c^2 \xi \nabla u) \in L^2(0,T; H^1(\Omega)^*)$. We can see r as the solution of the following problem:

$$\int_\Omega r_{tt} v \,\mathrm{d}x + \int_\Omega (a(w)\nabla r + b\nabla r_t) \cdot \nabla v \,\mathrm{d}x = -\int_\Omega c^2 \xi \,(S(w+\xi) - S(w)) \cdot \nabla v \,\mathrm{d}x \tag{5.13}$$

for all test functions $v \in H^1(\Omega)$, a.e. in time, supplemented by zero initial conditions. To show that

$$\|r\|_X = o(\|\xi\|_{L^\infty(\Omega)}) \quad \text{ as } \|\xi\|_{L^\infty(\Omega)} \to 0.$$

we choose $v = r_t(t) \in H^1(\Omega)$ in (5.13) and estimate the resulting right-hand side as follows:

$$\begin{aligned} &-\int_\Omega c^2 \xi \nabla\,(S(w+\xi) - S(w)) \cdot \nabla r_t \,\mathrm{d}x \\ &\lesssim \|\xi\|_{L^\infty(\Omega)} \|\nabla(S(w+\xi) - S(w))\|_{L^2(\Omega)} \|\nabla r_t\|_{L^2(\Omega)} \end{aligned} \tag{5.14}$$

a.e. in time. The statement then follows by the continuity of the mapping S established in Proposition 5.3. $\square$

Let $w \in W_{\mathrm{ad}}$ and let $u = S(w) \in \mathcal{U}$ be the corresponding state. On account of Proposition 5.2, we can introduce the reduced functional $j(w) = J(S(w))$.

Proposition 5.5. *Under the assumptions of of Proposition 5.2, the reduced functional $j : W_{\mathrm{ad}} \to \mathbb{R}$ is Fréchet differentiable.*

Proof. The proof follows similarly to [6], Proposition 5.2 and we thus again omit the details. $\square$

Note that we have

$$j'(w)\xi = \int_0^T \int_\Omega (u - u_{\mathrm{d}}) u^{*,\xi} \,\mathrm{d}x\mathrm{d}s \tag{5.15}$$

with $u^{*,\xi}$ given by (5.11). The adjoint problem is given by

$$\int_\Omega p_{tt} v \,\mathrm{d}x + \int_\Omega (a(w)\nabla p(t) - b\nabla p_t)\cdot\nabla v \,\mathrm{d}x = \int_\Omega (u-u_\mathrm{d})v \,\mathrm{d}x\mathrm{d}s, \tag{5.16}$$

for all $v \in H^1(\Omega)$ a.e. in time, with $(p,p_t)|_{t=T} = (0,0)$. After time reversal $t \mapsto p(T-t)$, its unique solvability follows by Proposition 5.2 together with the bound

$$\|p\|_{\mathcal{U}} \le C(T)\|u-u_\mathrm{d}\|_{L^2(L^2(\Omega))}.$$

Using the adjoint problem, we then find

$$j'(w)\xi = -\int_0^T\int_\Omega a'(w)\xi\nabla u\cdot\nabla p \,\mathrm{d}x\mathrm{d}s. \tag{5.17}$$

Looking at (5.14), it does not seem feasible in this setting to have an estimate involving only $\xi \in L^q(\Omega)$ for some $q < \infty$ so as to come closer to satisfying Assumption 2.2. Thus this acoustic problem does not fit into our set of assumptions for the algorithmic framework so far. As a possible alleviation, one may want to introduce a regularization of the input by means of a mollification as is proposed in [37]. However, the continuity with respect to L^∞ is not enough to pass to the limit in the mollification. We intend to use non-standard elliptic regularity theory to improve on the properties of j' in the future. Nevertheless, we demonstrate in the next section that the proposed algorithm still shows a sensible output when applied to this setting.

5.4. Numerical experiments for the acoustic field

We next wish to provide a qualitative assessment of the algorithm in practice. To this end, we first describe the used discretization in Section 5.4.1 as well as the setup and our implementation of Algorithm 1 in Section 5.4.2. The numerical results are provided in Section 5.4.3.

5.4.1. Discretization of the state and adjoint problems

We follow the numerical approach of [5] and adapt the implementation used there[1] to incorporate strong damping in the wave equation. More precisely, we employ continuous piecewise linear finite elements in space and time to discretize the state and adjoint problems. We briefly discuss the extension of the numerical discretization in [5] to the strongly damped acoustic setting (that is, having $b > 0$ in (5.5)).

Let $\mathcal{T}_h = \{T\}$ be a mesh consisting of triangles or tetrahedra T with the mesh size h and let $D_h \subset H^1(\Omega)\cap C(\overline{\Omega})$ be the corresponding finite element space consisting of continuous piecewise linear functions. We further define the finite element space $D_\tau \subset H^1(0,T) \subset C[0,T]$ of continuous piecewise linear functions associated with the uniform discretization $0 = t_0 < t_1 < \ldots < t_{N_\tau} = T$, where $\tau = t_{i+1} - t_i$, $i \in \{0,\ldots,N_\tau - 1\}$. Let $\{e_i\}$ be the basis (hat) functions of D_τ.

Let $\nu := (h,\tau)$ and $D_\nu := D_h \otimes D_\tau$. We look for the approximate solution $u_\nu \in D_\nu$, which satisfies

$$\begin{aligned}\int_0^T\int_\Omega \Big\{ -\partial_t u_\nu \partial_t v - (\sigma - \frac{1}{6})\tau^2 a(w_h)\nabla\partial_t u_\nu\cdot\nabla\partial_t v + a(w_h)\nabla u_\nu\cdot\nabla v \\ + b\nabla\partial_t u_\nu\cdot\nabla v\Big\}\,\mathrm{d}x\mathrm{d}s = \int_\Omega u_1 v(0)\,\mathrm{d}x + \int_0^T\int_\Omega f v\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{5.18}$$

[1] https://github.com/clason/tvwavecontrol

for all $v \in D_\nu$ with $v(T) = 0$, and $u_\nu(0) = P_0 u_0$, where P_0 is the $L^2(\Omega)$ projection operator defined by

$$(P_0 u_0, \varphi)_{L^2(\Omega)} = (u_0, \varphi)_{L^2(\Omega)} \quad \text{for all} \quad \varphi \in D_h. \tag{5.19}$$

In (5.18), $\sigma > 0$ denotes the stabilization parameter; *cf.* [5], Definition 5.1. Problem (5.18) can be restated as a time-stepping scheme using the fact that

$$u_\nu(t) = \begin{cases} \dfrac{t_i - t}{t_i - t_{i-1}} u_h(t_{i-1}) + \dfrac{t - t_{i-1}}{t_i - t_{i-1}} u_h(t_i), & t \in [t_{i-1}, t_i], \\ \dfrac{t_{i+1} - t}{t_{i+1} - t_i} u_h(t_i) + \dfrac{t - t_i}{t_{i+1} - t_i} u_h(t_{i+1}), & t \in [t_i, t_{i+1}], \end{cases}$$

and expressing the test function as $v = \sum_{i=0}^{N_\tau - 1} e_i(t)\phi_i(x)$. Indeed, let $u_h^0 = P_0 u_0$. We first compute u_h^1 using the fact that it solves

$$\begin{aligned} \left(\frac{u_h^1 - u_h^0}{\tau}, \phi\right)_{L^2(\Omega)} &+ \tau(a(w_h)\nabla(\sigma u_h^1 + (\frac{1}{2} - \sigma)u_h^0), \nabla\phi)_{L^2(\Omega)} \\ &+ \frac{b}{2}(\nabla(u_h^1 - u_h^0), \nabla\phi)_{L^2(\Omega)} = (u_1, \phi)_{L^2(\Omega)} + \left(\int_0^{t_1} f e_0, \phi\right)_{L^2(\Omega)} \end{aligned} \tag{5.20}$$

for all $\phi \in D_h$. Then given (u_h^{i-1}, u_h^i), we can compute u_h^{i+1} since it solves

$$\begin{aligned} \left(\frac{u_h^{i+1} - 2u_h^i + u_h^{i-1}}{\tau}, \phi\right)_{L^2(\Omega)} &+ \tau\left(a(w_h)\nabla(\sigma u_h^{i+1} + (1 - 2\sigma)u_h^i + \sigma u_h^{i-1}, \nabla\phi\right)_{L^2(\Omega)} \\ &+ \frac{b}{2}\left(\nabla(u_h^{i+1} - u_h^{i-1}), \nabla\phi\right)_{L^2(\Omega)} = \left(\int_{t_{i-1}}^{t_{i+1}} f e_i \,\mathrm{d}s, \phi\right)_{L^2(\Omega)} \end{aligned} \tag{5.21}$$

for $1 \le i \le N_\tau - 1$ and all $\phi \in D_h$.

Concerning the adjoint problem, discrete adjoint state $p_\nu = \sum_{i=1}^{N_\tau} p_h^i(x)e_i(t) \in D_\nu$ satisfies the following problem:

$$\begin{aligned} \int_0^T \int_\Omega \Big\{ -\partial_t v \partial_t p_\nu - (\sigma - \frac{1}{6})\tau^2 a(w_h)\nabla\partial_t v \cdot \nabla\partial_t p_\nu + a(w_h)\nabla v \cdot \nabla p_\nu \\ + b\nabla\partial_t v \cdot \nabla p_\nu \Big\} \,\mathrm{d}x\mathrm{d}s = \int_0^T \int_\Omega (u_\nu - u_\mathrm{d}) v \,\mathrm{d}x\mathrm{d}s \end{aligned} \tag{5.22}$$

for all $v \in D_\nu$ with $v(0) = 0$, and $p_\nu(T) = 0$. This problem can be reformulated as a time-stepping scheme analogously to the state equation. Indeed, starting from $p_h^{N_\tau} = 0$, we compute $p_h^{N_\tau - 1}$ using the fact that it solves

$$\begin{aligned} -\left(\frac{p_h^{N_\tau} - p_h^{N_\tau - 1}}{\tau}, \phi\right)_{L^2(\Omega)} &+ \tau(a(w_h)\nabla(\sigma p_h^{N_\tau - 1} + (\frac{1}{2} - \sigma)p_h^{N_\tau}), \nabla\phi)_{L^2(\Omega)} \\ &+ \frac{b}{2}(\nabla(p_h^{N_\tau} + p_h^{N_\tau - 1}), \nabla\phi)_{L^2(\Omega)} = \int_{N_\tau - 1}^{N_\tau} \int_\Omega (u_\nu - u_\mathrm{d}) e_{N_\tau} \,\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{5.23}$$

Given (p_h^{i+1}, p_h^i), we can compute p_h^{i-1} using the fact that

$$\begin{aligned}\left(\frac{p_h^{i+1} - 2p_h^i + p_h^{i-1}}{\tau}, \phi\right)_{L^2(\Omega)} &+ \tau\left(a(w_h)\nabla(\sigma p_h^{i+1} + (1-2\sigma)p_h^i + \sigma p_h^{i-1}, \nabla\phi\right)_{L^2(\Omega)} \\ &- \frac{b}{2}\left(\nabla(p_h^{i+1} - p_h^{i-1}), \nabla\phi\right)_{L^2(\Omega)} = \left(\int_{t_{i-1}}^{t_{i+1}} (u_\nu - u_{\mathrm{d}})e_i \,\mathrm{d}s, \phi\right)_{L^2(\Omega)}\end{aligned} \tag{5.24}$$

for $1 \le i \le N_\tau - 1$ and all $\phi \in D_h$. The considerations concerning the properties of the discrete control-to-state operator follow analogously to Section 5.1 in [5], so we omit them here.

5.4.2. Setup

We consider a two-dimensional rectangular spatial domain $\Omega = (-1,1) \times (-1,2)$ that are discretized into 96×96 squares, which are subdivided into two triangles each and a time horizon from 0 to $T = 5$ that is discretized into 256 intervals. We choose $c^2 = 20$ and $b = 1.25 \cdot 10^{-2}$. We choose CG1 elements as ansatz functions for the state vector as well as the control vector. We use the `FEniCSx` library [29–31] and the space-time discretization from [5] described in the section above in order to solve the PDE and its adjoint. We use a tracking-type functional and choose $\gamma = 7.5 \cdot 10^{-6}$ in the objective. In order to approximate the L^1-norm in the trust-region subproblems, we compute the L^1-norm of the function that is obtained by reflecting the negative node values (our ansatz has a nodal basis) at the origin. We choose the acoustic source term f as the Ricker wavelet as in [5] and set $\sigma = 0.25$.

Regarding Algorithm 1, as in Section 5.2, we only solve instances of the convex subproblems ($\mathrm{TR}_\varepsilon^c$). We initialize $\varepsilon^0 = 1$, $\Delta^0 = 1.5$, $\rho = 10^{-4}$, $\kappa^0 = 10^{-8}$, and $\underline{\Delta}^0 = 1.14 \cdot 10^{-5}$. The initial control w^0 is the constant function with the value 0.5.

We execute our algorithm on a laptop computer with an Intel(R) Core(TM) i7-11850H CPU (2.50 GHz) and 64 GB RAM. We solve the discretized subproblems with Gurobi 10.0.0 [34]. We stop the algorithm when no futher progress is made, that is ε^n is reduced without any step having been accepted for the previous value of ε^n.

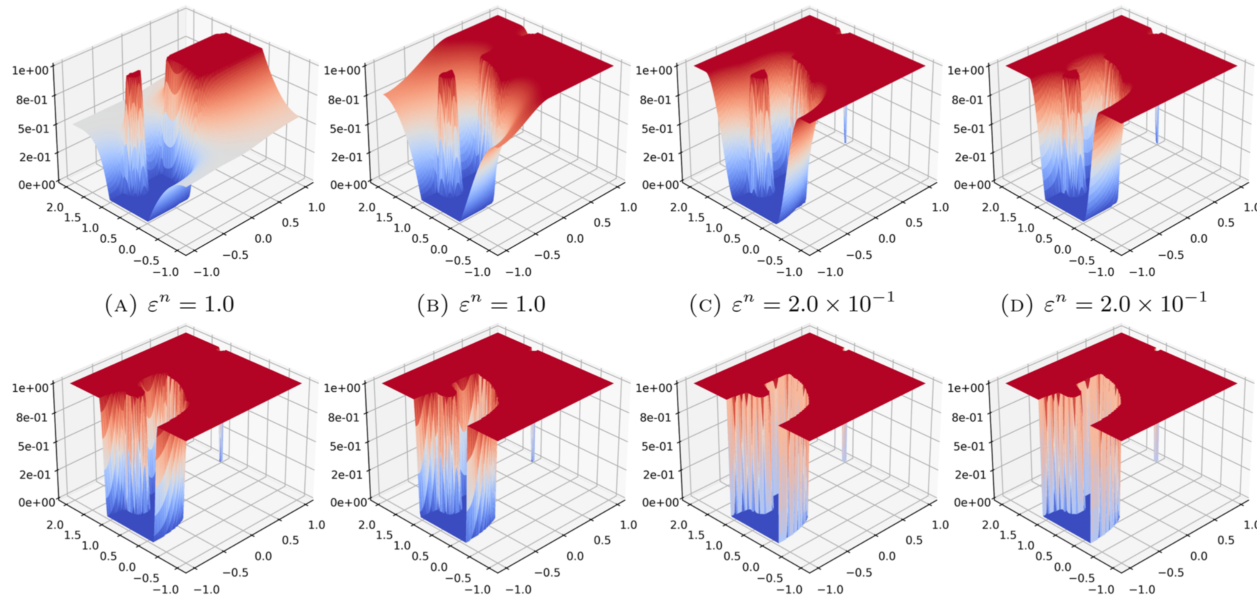
5.4.3. Results

The execution of our implementation requires 174 iterations, of which 62 are accepted, which are broken down for the different values of ε^n in Table 2. The number of accepted iterations is between 11 and 27 for the first three values of ε^n drops to 5 for the fourth value. For the fifth value $\varepsilon^n = 1.6 \cdot 10^{-3}$, the trust-region immediately contracts because it is already close to stationarity for ε^n and no improvement can be made from last accepted iterate for the previous value of ε^n. This is closely linked to the fact that we only use convex subproblems and the iterate is already very close to being binary-valued. Hence the objective of the convex subproblem exhibits a high penalty that is proportional to $\frac{1}{\varepsilon}$ on changes of the nodal values from 0 to 1 or vice versa. In other words, the objective gradient points to the outside of the feasible set, implying the stationarity. A further reduction of ε^n lead to the same result for this test case. We also note that handling the coefficients in the trust-region subproblem becomes more difficult for smaller values of ε^n due to the opposed scaling effect of ε on the two terms in E_ε. The running time of the execution of the algorithm was 2 hours and 18 minutes.

While it is difficult to measure instationarity with respect to (P) directly, we compute the predicted reduction of the solved trust-region subproblems ($\mathrm{TR}_\varepsilon^c$) for the reset trust-region radius Δ^0 for the initial and final iterate for each of the values of ε^n in order to have a surrogate and get some impression on the behavior of the algorithm. We observe a decreasing (but not monotonic) trend for the initial predicted reduction over the different values of ε^n. The predicted reduction drops the first to the second value of ε by almost three orders of magnitude and then remains relatively constant over the next two reductions of ε and then drops almost by another order of magnitude.

TABLE 2. Number of accepted iterations and initial predicted reduction for the reset trust-region radius $\Delta^0 = 1.5$ for different values of ε^n.

ε^n	Accepted iterates	Predicted reduction for $\Delta = 1.5$ Initial	Final
1.0	19	1.162×10^{-2}	4.10×10^{-8}
2.0×10^{-1}	11	7.687×10^{-6}	4.99×10^{-9}
4.0×10^{-2}	27	1.139×10^{-5}	2.62×10^{-9}
8.0×10^{-3}	5	8.962×10^{-6}	1.27×10^{-8}
1.6×10^{-3}	–	9.023×10^{-7}	9.023×10^{-7}



(A) $\varepsilon^n = 1.0$ (B) $\varepsilon^n = 1.0$ (C) $\varepsilon^n = 2.0 \times 10^{-1}$ (D) $\varepsilon^n = 2.0 \times 10^{-1}$

(E) $\varepsilon^n = 4.0 \times 10^{-2}$ (F) $\varepsilon^n = 4.0 \times 10^{-2}$ (G) $\varepsilon^n = 8.0 \times 10^{-3}$ (H) $\varepsilon^n = 8.0 \times 10^{-3}$

FIGURE 3. First (a, c, e, g) and last accepted iterate (b, d, f, h) over the decreasing values of ε^n.

As one expects from gradient-based optimization algorithms, the value of the predicted reduction is much smaller for the final iterate for the four values of ε for which iterates are accepted. We tabulate these values in the Table 2 too.

Over the course of the iterations, the binarity of the (accepted) iterates w^n increases significantly and a sharp interface emerges. We have visualized this in Figure 3, where we display the first and last accepted iterates for each of the values ε^n.

6. CONCLUSION

We have provided an algorithmic phase-field approach to a class of optimal control problems with a restriction to binary control functions with perimeter regularization. We have shown that the algorithmic idea of [9, 11] can be transferred to the phase-field setting and a similar convergence theory can be obtained, where the L-stationarity of the limits requires an additional geometric assumption, see Theorem 4.12, which we believe cannot be dropped easily without replacement and which we interpret as a constraint qualification in an analogy to nonlinear programming. Crucially, we have derived a relationship between the reduction of the interface parameter and the reduction of the lower bound on the trust-region radius that in turn triggers reduction of the interface parameter.

As our applications and computational examples demonstrate, this work motivates research on several aspects of numerical optimization of problems of type (P) by means of the analyzed algorithmic framework. We highlight a few of them below.

Assumption 2.2 requires that the reduced objective $j = J \circ S$ is Fréchet differentiable as a function from $L^1(\Omega)$ to $\mathbb{R}$. We cannot satisfy this assumption for our motivating application and only get differentiability with respect to $L^\infty(\Omega)$ so far, see Section 5.3. This is not satisfactory because for any two binary-valued functions that differ on a set of strictly positive Lebesgue measure, their $L^\infty(\Omega)$-difference is one. This motivates research on approximations of the PDE so that the assumption can be satisfied as well as research on improving the Assumption 2.2 so that differentiability is only required with respect to some $L^p(\Omega)$, $p > 1$.

A crucial step in our convergence analysis is Theorem 3.3, in which we prove Γ-convergence of the non-convex trust-region subproblems for a fixed trust-region radius. Example 3.5 shows that this result does not hold in the same way for the convex trust-region subproblems. Therefore, in order to reduce the computational burden that is required for convergence guarantees, we find it important to seek alternatives to the statement of Theorem 3.3 in the convergence analysis that would allow us to work with convex subproblems. An alternative avenue is of course to develop efficient solution strategies for the non-convex subproblems.

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