

REPLICATOR DYNAMICS AS THE LARGE POPULATION LIMIT OF A DISCRETE MORAN PROCESS IN THE WEAK SELECTION REGIME: A PROOF VIA EULERIAN SPECIFICATION

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Abstract. We study the large population limit of a multi-strategy discrete-time Moran process in the weak selection regime. We show that the replicator dynamics is interpreted as the large-population limit of the Moran process. This result is obtained by interpreting the discrete process in its Eulerian specification, proving a compactness result in the Wasserstein space of probability measures for the law of the proportions of strategies, and passing to the limit in the continuity equation that describes the evolution of the proportions.

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1. INTRODUCTION

Understanding the behavior of large populations of interacting agents is crucial in a variety of scientific fields, including biology [1, 2], sociology [3, 4], ecology [5, 6], artificial intelligence [7], and economics [8]. Evolutionary game theory provides the mathematical framework to model and analyze these dynamics, in which agents are typically endowed with strategies. These are rationally selected by each agent according to complex mechanisms, often influenced by the interplay of the individuals among themselves and with the environment, and based on an underlying optimality principle.

In this context, the replicator dynamics accomplishes the task of modeling the evolution of strategies in the population by subjecting the probability of reproduction to their fitness: The higher the fitness, the higher the chance of being selected to reproduce. More specifically, letting $\mathcal{U} = \{u_1, \dots, u_M\}$ be the set of strategies and $\lambda(t) = (\lambda_{u_1}(t), \dots, \lambda_{u_M}(t))^T$ be the vector collecting the proportions of agents with strategies $u_1, \dots, u_M$ at time t , and given a fixed payoff matrix $A = (a_{ij}) \in \mathbb{R}^{M \times M}$, the fitness of strategy u_i at time t is modeled as (see [9], Chapter 8)

$$(A\lambda(t))_i - (A\lambda(t)) \cdot \lambda(t) = \sum_{j=1}^M a_{ij} \lambda_{u_j}(t) - \sum_{\ell=1}^M \sum_{j=1}^M a_{\ell j} \lambda_{u_j}(t) \lambda_{u_\ell}(t), \quad \text{for } i = 1, \dots, M.$$

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Interpreting a_{ij} as the payoff of an agent with strategy u_i interacting with an agent with strategy u_j , the formula above measures the performance of the selected strategy u_i compared to the average performance of all the strategies. The replicator dynamics encodes the Darwinian principle that if u_i outperforms the average, then its fitness is positive and its spreading within the population is favored. Vice versa, strategies which underperform are progressively suppressed. In formulae, the replicator dynamics evolves according to the system of ODEs

$$\begin{aligned} \frac{d}{dt} \lambda_{u_i}(t) &= \lambda_{u_i}(t) ((A\lambda(t))_i - (A\lambda(t)) \cdot \lambda(t)) \\ &=: b_i(\lambda(t)), \end{aligned} \quad t \in (0, T), \quad \text{for } i = 1, \dots, M, \quad (1.1)$$

subject to a given initial condition $\lambda(0) = \lambda^0$. The criterion followed by individual agents to select their strategies is latent in (1.1), which describes the continuous-time evolution of the proportions of strategies in an averaged fashion.

The objective of this paper is to provide the mathematical framework to derive (1.1) as a mean-field limit of a discrete stochastic process modeling this evolutionary mechanism from the point of view of individual agents and their pairwise interactions.

Before describing in detail this process, we recall the classical *Moran process* [10], a prototypical example of discrete stochastic process used in evolutionary biology to model natural selection in finite populations with two strategies (alleles, in biology) competing for dominance. Letting $M = 2$, the set of strategies becomes $\mathcal{U} = \{u_1, u_2\}$. We denote by N the number of individuals in the population, and by N_{u_i} the number of individuals with strategy u_i at a given time. If no fitness affects the evolution (*i.e.*, a neutral mutation scenario), the Moran process is a discrete-time Markov chain for N_{u_1} with uniform birth-death transition probabilities given by:

$$\begin{aligned} \mathbb{P}(\text{"}N_{u_1} \text{ increases by 1"}\text{)} &= \frac{N_{u_1}}{N} \frac{N_{u_2}}{N} = \frac{N_{u_1}}{N} \frac{N - N_{u_1}}{N} \\ &= \mathbb{P}(\text{"a } u_1\text{-individual is chosen for reproduction"}) \mathbb{P}(\text{"a } u_2\text{-individual is chosen to die"}), \\ \mathbb{P}(\text{"}N_{u_1} \text{ decreases by 1"}\text{)} &= \frac{N_{u_2}}{N} \frac{N_{u_1}}{N} = \frac{N - N_{u_1}}{N} \frac{N_{u_1}}{N} \\ &= \mathbb{P}(\text{"a } u_2\text{-individual is chosen for reproduction"}) \mathbb{P}(\text{"a } u_1\text{-individual is chosen to die"}); \end{aligned} \quad (1.2)$$

by complementarity, one obtains the probability that N_{u_1} remains unchanged.

When fitness of strategies is introduced, the Moran process is modified to account for the fact that the higher the fitness of a strategy, the higher the probability that one of its bearers is selected to reproduce. More precisely, letting f_{u_i} be the fitness of individuals with strategy u_i , the reproduction probabilities in (1.2) are replaced by

$$\begin{aligned} \mathbb{P}(\text{"a } u_1\text{-individual is chosen for reproduction"}) &= \frac{N_{u_1} f_{u_1}}{N_{u_1} f_{u_1} + N_{u_2} f_{u_2}}, \\ \mathbb{P}(\text{"a } u_2\text{-individual is chosen for reproduction"}) &= \frac{N_{u_2} f_{u_2}}{N_{u_1} f_{u_1} + N_{u_2} f_{u_2}}. \end{aligned} \quad (1.3)$$

Let us discuss the case where a payoff matrix $A \in \mathbb{R}^{2 \times 2}$ is used to describe interactions. An individual carrying strategy u_1 interacts with another individual chosen randomly uniformly among the others in the population and has an expected payoff

$$\pi_{u_1} = a_{11} \frac{N_{u_1} - 1}{N - 1} + a_{12} \frac{N_{u_2}}{N - 1}.$$

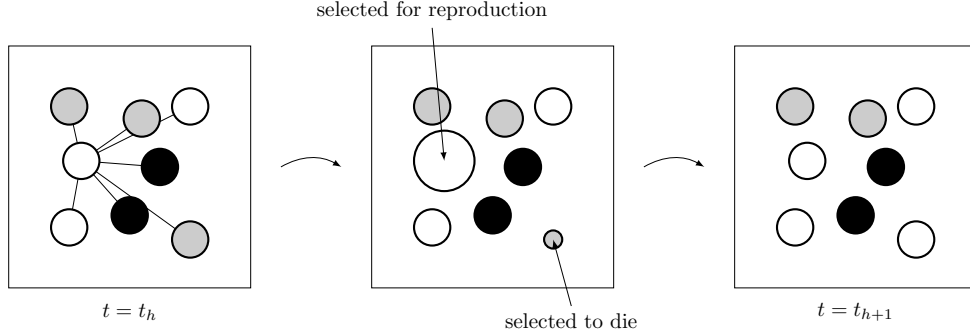


FIGURE 1. Graphical representation of one time step of a Moran process. In the picture, we have $N = 8$ agents and $M = 3$ strategies, u_1 (white circle), u_2 (grey circle), u_3 (black circle). At time t_h , for each agent a fitness is calculated in terms of the expected payoff due to the interaction with an agent sampled randomly uniformly in the population. The fitness is used to determine the probability of reproduction. In the figure, an agent with strategy u_1 has been chosen for reproduction (biggest circle), and an agent with strategy u_2 has been chosen to die (smallest circle). The generation is updated at time t_{h+1} accordingly.

Analogously, for an individual carrying strategy u_2 , the expected payoff is

$$\pi_{u_2} = a_{21} \frac{N_{u_1}}{N-1} + a_{22} \frac{N_{u_2} - 1}{N-1}.$$

A common choice [11] is to define the fitness of individuals with strategy u_i as a convex combination of the expected payoff and the constant 1, *i.e.*,

$$f_{u_i} = (1 - w) + w\pi_{u_i}, \quad \text{for } i = 1, 2,$$

where $w \in [0, 1]$ is a weight that interpolates between the case of neutral selection $w = 0$, giving (1.2), and the case of strong selection, $w = 1$, where the fitness is solely determined by the payoffs. The regime $w \approx 0$ is typically referred to as the *weak selection* regime, where the payoffs provide a small contribution to the fitness.

In this paper, we consider a generalization of the Moran process to a population with strategies $\mathcal{U} = \{u_1, \dots, u_M\}$, $M \geq 2$, see Figure 1. In this case, the probability that an individual with strategy u_i is chosen for reproduction (as in (1.3)) is given by

$$\mathbb{P}\left(\text{"a } u_i\text{-individual is chosen for reproduction"}\right) = \frac{N_{u_i} f_{u_i}}{\sum_{j=1}^M N_{u_j} f_{u_j}}, \quad (1.4)$$

where N_{u_i} is the number of individuals with strategy u_i in the population and f_{u_i} is the fitness of individuals with strategy u_i defined *via* the convex combination

$$f_{u_i} = (1 - w) + w\pi_{u_i}, \quad \text{for } i = 1, \dots, M. \quad (1.5)$$

The expected payoff of an individual with strategy u_i interacting with another individual chosen uniformly at random in the population is now given by

$$\pi_{u_i} = \sum_{\substack{j=1 \\ j \neq i}}^M a_{ij} \frac{N_{u_j}}{N-1} + a_{ii} \frac{N_{u_i} - 1}{N-1}, \quad \text{for } i = 1, \dots, M. \quad (1.6)$$

In this paper we study rigorously the asymptotic behavior of this generalized Moran process by considering:

- the large population limit $N \rightarrow +\infty$;
- the discrete-time to continuous-time limit; letting τ be the time step of the process, this corresponds to $\tau \rightarrow 0$;
- the weak selection regime $w \rightarrow 0$.

From our analysis it emerges that the relation between the three limits have to be taken simultaneously and with a precise relation among the parameters to obtain a nontrivial and meaningful result. To describe this relation between these parameters, we introduce sequences $N_k \rightarrow +\infty$, $\tau_k \rightarrow 0$, and $w_k \rightarrow 0$.

Our main result can be summarized as follows: Assuming that

$$N_k \sim \tau_k^{-\alpha}, \quad w_k \sim \tau_k^\beta, \quad \text{for some } \alpha, \beta > 0, \quad \alpha + \beta = 1, \quad \alpha > \frac{1}{2}, \quad (1.7)$$

the replicator dynamics describes the limit as $k \rightarrow \infty$ of the generalized Moran process modeled by (1.4)–(1.6).

Previous works in the literature addressed the problem of deriving the replicator dynamics from discrete stochastic processes, see, *e.g.*, [12–18]. The results in [13–15, 17–19] are aligned with the scaling assumptions (1.7). In fact, they are not merely technical, as they determine the PDE governing the limit dynamics. In particular, if $\alpha + \beta > 1$, then the result is trivial (the evolution is constant). If $\alpha + \beta < 1$, a different time scaling would be necessary. Finally, if $\alpha = 1/2$, then higher order terms have to be considered in the limit dynamics [14, 15, 17, 18].

We describe now more in detail the results obtained in this paper. The discrete process is fully described by the stochastic evolution of the vector $\lambda^k(t_h)$ collecting the proportions of strategies at discrete times $t_h = h\tau_k$, for $h = 0, \dots, k$, see Section 3. Noticing that the components of $\lambda^k(t_h)$ sum to one, we can interpret $\lambda^k(t_h)$ as a random point in the $(M - 1)$ -dimensional simplex $\Delta^{M-1} \subset \mathbb{R}^M$ (recall that M is number of available strategies). The discrete path $\lambda^k(t_h)$ is suitably interpolated to obtain a continuous path $\lambda^k: [0, T] \rightarrow \Delta^{M-1}$, see (4.1). The asymptotic behavior of the discrete process as $k \rightarrow \infty$ is obtained in terms of the limit of the law $\Lambda_t^k \in \mathcal{P}(\Delta^{M-1})$ of the random vector $\lambda^k(t)$ (see (4.2)); for every k , the map $t \mapsto \Lambda_t^k \in \mathcal{P}(\Delta^{M-1})$ is a continuous path in the space of probability measures on the simplex Δ^{M-1} . In Proposition 4.4, we show that this path solves, in the sense of distributions,

$$\partial_t \Lambda_t^k + \operatorname{div}(b \Lambda_t^k) \approx 0, \quad (\text{with initial condition}), \quad (1.8)$$

a continuity equation driven by the velocity field $b = (b_1, \dots, b_M)^\top$ with components defined in (1.1). We turn the reader's attention to the symbol $\approx$ in (1.8), meaning that the equation holds in an approximate sense, up to an error term vanishing as $k \rightarrow \infty$ under the assumptions (1.7).

A crucial technical step in our result is showing compactness of the sequence of paths $t \mapsto \Lambda_t^k$, that guarantees a meaningful limit as $k \rightarrow \infty$ to be taken in (1.8). Since the probability measures Λ_t^k have finite first moment (see Rem. 5.1), it is natural to endow $\mathcal{P}(\Delta^{M-1})$ with the 1-Wasserstein distance $\mathcal{W}_1$. As we illustrate in Remark 5.2, straightforward estimates for the discrete stochastic process seem not to guarantee that the paths $t \mapsto \Lambda_t^k$ satisfy equi-continuity properties in the Wasserstein distance, ruling out an immediate application of available compactness results for paths of probability measures. For this reason, we prove compactness in Theorem 5.3 by deriving a more refined estimate and exploiting it in an argument that follows the lines of the proof of the original Arzelà–Ascoli theorem. Once compactness is established, we can pass to the limit in (1.8) and obtain that $\mathcal{W}_1(\Lambda_t^k, \Lambda_t) \rightarrow 0$ as $k \rightarrow \infty$ for every $t \in [0, T]$, where $\Lambda_t \in \mathcal{P}(\Delta^{M-1})$ is the solution to

$$\partial_t \Lambda_t + \operatorname{div}(b \Lambda_t) = 0, \quad (\text{with initial condition}). \quad (1.9)$$

We recognize that the latter equation is the Eulerian specification of the replicator dynamics (1.1), see Section 6.

We conclude this introduction by remarking that the technique in this paper allows us to obtain the limit result with no regularity ansatz on the distribution of proportions Λ_t^k . The approach we follow is inspired by

[20–22], where the equivalence of the Eulerian and Lagrangian notions of solution to (1.9) is obtained *via* the *superposition principle* [21], Section 5.2 (see also [23], Thm. 8.2.1 and [24]).

2. NOTATION AND BASIC DEFINITIONS

In this section we collect some basic definitions and notation that will be used throughout the paper.

Probability

Hereafter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

If X is a random variable, we usually adopt the notation $\hat{X}$ to indicate a fixed realization of the random variable X .

The expectation of a random variable X is denoted by $\mathbb{E}[X]$.

We denote by $X_{\#}\mathbb{P}$ the pushforward of the probability measure $\mathbb{P}$ through the random variable X , which defines a probability measure on the range of X .

1-Wasserstein distance

Let $(\mathcal{M}, d)$ be a complete, separable metric space, equipped with its Borel σ -algebra. We let $\mathcal{P}(\mathcal{M})$ denote the set of Borel probability measures on $\mathcal{M}$. The 1-Wasserstein distance between two probability measures $\mu, \nu \in \mathcal{P}(\mathcal{M})$ is defined by

$$\mathcal{W}_1(\mu, \nu) = \inf_{\gamma} \int_{\mathcal{M} \times \mathcal{M}} d(x, y) \, d\gamma(x, y),$$

where the infimum is taken over all probability measures $\gamma \in \mathcal{P}(\mathcal{M} \times \mathcal{M})$ with marginals μ and ν .

We consider the subset of $\mathcal{P}(\mathcal{M})$ consisting of probability measures with finite first moment, *i.e.*,

$$\mathcal{P}_1(\mathcal{M}) := \left\{ \mu \in \mathcal{P}(\mathcal{M}) : \int_{\mathcal{M}} d(x_0, x) \, d\mu(x) < +\infty \right\},$$

for some (and hence any) $x_0 \in \mathcal{M}$. The 1-Wasserstein distance is a metric on $\mathcal{P}_1(\mathcal{M})$.

We recall the Kantorovich's duality formula for the 1-Wasserstein distance [25], Remark 6.5:

$$\mathcal{W}_1(\mu, \nu) = \sup \left\{ \int_{\mathcal{M}} \psi \, d\nu - \int_{\mathcal{M}} \psi \, d\mu : \psi \text{ 1-Lipschitz} \right\}.$$

Finally, we recall the following fact: If $\mathcal{W}_1(\mu_n, \mu) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\int_{\mathcal{M}} \psi \, d\mu_n \rightarrow \int_{\mathcal{M}} \psi \, d\mu,$$

for every $\psi \in C(\mathcal{M})$ such that $|\psi(x)| \leq C(1 + d(x, x_0))$ for some $C > 0$ and some $x_0 \in \mathcal{M}$.

3. THE DISCRETE STOCHASTIC PROCESS

In this section we describe in detail the multi-strategy Moran process studied in this paper. Each subsection is dedicated to a specific aspect of the process.

Time

We fix an equi-spaced discretization of step τ_k of a time interval $[0, T]$ given by

$$0 = t_0 < t_1 < \dots < t_k = T, \quad \tau_k := \frac{T}{k}, \quad t_h = h\tau_k, \quad \text{for } h = 0, \dots, k.$$

Population set

We consider a population of $N \geq 2$ agents and we identify agents with elements $n \in \mathcal{A} := \{1, \dots, N\}$. We will be interested in the limit behavior of the population size $N \rightarrow \infty$. Later on, we will consider a sequence of population sizes of the form

$$N_k \sim \tau_k^{-\alpha}, \quad \text{for some } \alpha > 0.$$

In this section, we stick to the notation N for the population size, not to overload the notation.

Strategies

Each agent is allowed to choose a strategy in a set $\mathcal{U} = \{u_1, \dots, u_M\}$ ¹. The strategy chosen by agent $n \in \mathcal{A}$ at the discrete time t_h , for $h \in \{0, \dots, k\}$, is represented by a random variable $S_n(t_h): \Omega \rightarrow \mathcal{U}$. At each discrete time t_h , the state of the system is described by the random N -tuple

$$S(t_h) = (S_1(t_h), \dots, S_N(t_h)): \Omega \rightarrow \mathcal{U}^N.$$

For $i \in \{1, \dots, M\}$, we let $\mathcal{A}_{u_i}(t_h): \Omega \rightarrow 2^{\mathcal{A}}$ be the random subset of agents with strategy u_i at discrete time t_h given by

$$\mathcal{A}_{u_i}(t_h) := \{n \in \mathcal{A} : S_n(t_h) = u_i\}.$$

Accordingly, we let $N_{u_i}(t_h) := \#\mathcal{A}_{u_i}(t_h): \Omega \rightarrow \{0, 1, \dots, N\}$ be the random variable given by the number of agents with strategy u_i at discrete time t_h . We remark that

$$\sum_{i=1}^M N_{u_i}(t_h) = N. \tag{3.1}$$

Accordingly, for $i \in \{1, \dots, M\}$, we let $\lambda_{u_i}(t_h) := \frac{N_{u_i}(t_h)}{N}: \Omega \rightarrow \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$ be the proportion of agents with strategy u_i . We introduce the random vector collecting all proportions:

$$\lambda(t_h) := (\lambda_{u_1}(t_h), \dots, \lambda_{u_M}(t_h))^\top: \Omega \rightarrow [0, 1]^M.$$

We will refer to $\lambda(t_h)$ as the *proportions* at time t_h . In fact, by (3.1), $\lambda(t_h)$ takes values in the $(M-1)$ -dimensional simplex

$$\Delta^{M-1} = \left\{ \lambda = (\lambda_1, \dots, \lambda_M)^\top \in [0, 1]^M : \sum_{i=1}^M \lambda_i = 1 \right\}.$$

It follows that $\lambda(t_h)$ can be identified with the empirical probability measure on the strategy set $\mu(t_h) := \sum_{i=1}^M \lambda_{u_i}(t_h) \delta_{u_i} \in \mathcal{P}(\mathcal{U})$.

¹When $M = 2$, the strategies u_1 and u_2 are typically called the *mutant* and the *resident*.

Later on, we will consider a sequence $N_k \rightarrow +\infty$ and the the proportions $\lambda(t_h)$ will depend on k and will be denoted by $\lambda^k(t_h)$. In this section, we stick to the notation $\lambda(t_h)$, not to overload the notation.

If not explicitly specified, when a realization $S(t_h) = \hat{S}$ of the random state of the system $S(t_h)$ is observed, we will write $\mathcal{A}_{u_i}(t_h) = \hat{\mathcal{A}}_{u_i}$, $N_{u_i}(t_h) = \hat{N}_{u_i}$, and $\lambda(t_h) = \hat{\lambda}$ for the corresponding realizations of the random strategy sets, random number of agents, and random proportions, respectively.

Initial conditions

The process starts with an initial random state $S(0) = (S_1(0), \dots, S_N(0)): \Omega \rightarrow \mathcal{U}^N$. This fixes an initial proportion $\lambda(0) = (\lambda_{u_1}(0), \dots, \lambda_{u_M}(0))^\top \in \Delta^{M-1}$. We assume that the initial proportions $\lambda(0)$ are distributed according to a given law $\Lambda_0 \in \mathcal{P}(\Delta^{M-1})$. This means that

$$\Lambda_0 = \lambda(0)_\# \mathbb{P} \in \mathcal{P}(\Delta^{M-1}).$$

Later on, we will consider a sequence of initial laws $\Lambda_0^k \in \mathcal{P}(\Delta^{M-1})$.

Payoff matrix

The evolution of the strategies in the population will be described in terms of a payoff matrix $A = (a_{ij}) \in \mathbb{R}^{M \times M}$, where a_{ij} is the payoff of an agent with strategy u_i when interacting with an agent with strategy u_j . We shall assume that the entries of A are non-negative, *i.e.*, $a_{ij} \geq 0$ for all $i, j \in \{1, \dots, M\}$. The payoff matrix is represented in the table below.

1 \backslash 2			
	u_1	$\dots$	u_M
u_1	a_{11}	$\dots$	a_{1M}
u_2	a_{21}	$\dots$	a_{2M}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_M	a_{M1}	$\dots$	a_{MM}

At each time step t_h , an agent $n \in \mathcal{A}$ with strategy $u_i \in \mathcal{U}$ interacts with a different agent $n' \neq n$ chosen uniformly at random in $\mathcal{A} \setminus \{n\}$. Specifically, given that the observed strategies at time t_h are $S(t_h) = \hat{S}$, there are two possibilities for the second agent:

- for $j \neq i$, agent n' has strategy u_j with probability $\frac{\hat{N}_{u_j}}{N-1} = \frac{N}{N-1} \hat{\lambda}_{u_j}$;
- for $j = i$, agent n' has strategy u_i with probability $\frac{\hat{N}_{u_i}-1}{N-1} = \frac{N}{N-1} \hat{\lambda}_{u_i} - \frac{1}{N-1}$.

As a consequence, agent n has an average payoff (depending on on the strategy u_i of agent n , the proportions $\hat{\lambda}$, and N) given by

$$\pi_{u_i}(\hat{\lambda}) = \frac{N}{N-1} \sum_{j=1}^M a_{ij} \hat{\lambda}_{u_j} - \frac{1}{N-1} a_{ii} = \frac{N}{N-1} (A\hat{\lambda})_i - \frac{1}{N-1} a_{ii}, \quad (3.2)$$

where $(A\hat{\lambda})_i$ denotes the i -th component of the matrix-vector product $A\hat{\lambda}$.

Remark 3.1. The assumption that $a_{ij} \geq 0$ for all $i, j \in \{1, \dots, M\}$ is not restrictive, but it simplifies the exposition. If some entries of the payoff matrix A are negative, one can add a matrix C with entries all equal to a suitable constant $c > 0$ to obtain a new payoff matrix $A' = A + C$. Working with this new payoff matrix

A' leads, in the limit, to the following replicator dynamics:

$$\begin{aligned}
\frac{d}{dt}\lambda_{u_i}(t) &= \lambda_{u_i}(t)((A'\lambda(t))_i - (A'\lambda(t)) \cdot \lambda(t)) \\
&= \lambda_{u_i}(t)((A+C)\lambda(t))_i - ((A+C)\lambda(t)) \cdot \lambda(t) \\
&= \lambda_{u_i}(t)((A\lambda(t))_i - (A\lambda(t)) \cdot \lambda(t) + c - c) \\
&= \lambda_{u_i}(t)((A\lambda(t))_i - (A\lambda(t)) \cdot \lambda(t)),
\end{aligned}$$

where we used that $C\lambda(t) = (c, \dots, c)^\top \in \mathbb{R}^M$. This shows an invariance with respect to the addition of a constant matrix C to the payoff matrix A .

Fitness

Let $w \in [0, 1]$ be a weight, whose precise value will be specified later. Given an agent $n \in \mathcal{A}$ with strategy $u_i \in \mathcal{U}$, and given that the observed strategies are $S(t_h) = \hat{S}$, we define the *fitness* of agent n at time t_h as the convex combination

$$f_{u_i}(\hat{\lambda}) = (1 - w) + w\pi_{u_i}(\hat{\lambda}). \quad (3.3)$$

The weight w is a parameter that determines the importance of the payoff in the fitness, as it interpolates between two extreme cases:

- For $w = 0$, the fitness is constant and equal to 1: This corresponds to a population where the agents are indifferent to the payoffs;
- For $w = 1$, the fitness is equal to the payoff.

The regime $w \rightarrow 0$ is typically referred to as the *weak selection* regime. We are interested in this regime, hence, in the rest of the paper, we will consider a sequence of weights $w_k \rightarrow 0$ of the form

$$w_k \sim \tau_k^\beta, \quad \text{for some } \beta > 0.$$

Note that for w_k small enough, the fitness $f_{u_i}(\hat{\lambda})$ is non-negative.

Replication

At each new time step t_{h+1} , an agent $n \in \mathcal{A}$ is chosen to reproduce its strategy, depending on the state of the system at the time t_h .

The probability that an agent $n \in \mathcal{A}$ with strategy u_i is chosen to reproduce its strategy depends on its fitness. We explain this precisely in the following. Let $R(t_{h+1}): \Omega \rightarrow \mathcal{A}$ denote the random variable representing the agent chosen to reproduce its strategy at time t_{h+1} . Let $S(t_h) = \hat{S}$ be an observed realization of the random state of the system $S(t_h)$ at the previous time step. Then

$$\mathbb{P}(R(t_{h+1}) = n \mid S(t_h) = \hat{S}) = cf_{u_i}(\hat{\lambda}) \quad \text{for every } n \in \hat{\mathcal{A}}_{u_i}.$$

The value c (depending on the status of the system) is chosen so that

$$\sum_{\ell=1}^M \sum_{n \in \hat{\mathcal{A}}_{u_\ell}} \mathbb{P}(R(t_{h+1}) = n \mid S(t_h) = \hat{S}) = 1,$$

hence

$$1 = \sum_{\ell=1}^M \sum_{n \in \mathcal{A}_{u_\ell}(t_h)} c f_{u_\ell}(\hat{\lambda}) = c \sum_{\ell=1}^M \# \hat{\mathcal{A}}_{u_\ell} f_{u_\ell}(\hat{\lambda}) = cN \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda}).$$

It follows that

$$\begin{aligned} \mathbb{P}(R(t_{h+1}) \in \hat{\mathcal{A}}_{u_i} \mid S(t_h) = \hat{S}) &= \sum_{n \in \mathcal{A}_{u_i}} \mathbb{P}(R(t_{h+1}) = n \mid S(t_h) = \hat{S}) \\ &= N \hat{\lambda}_{u_i} \frac{1}{N \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda})} f_{u_i}(\hat{\lambda}) = \frac{\hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda})}{\sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda})}. \end{aligned} \quad (3.4)$$

Introducing the notation

$$\bar{f}(\hat{\lambda}) = \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda}), \quad (3.5)$$

to denote the average fitness of the population with proportions $\hat{\lambda}$, we conclude that

$$\mathbb{P}(R(t_{h+1}) = n \mid S(t_h) = \hat{S}) = \frac{1}{\bar{f}(\hat{\lambda})} f_{u_i}(\hat{\lambda}) \quad \text{for every } n \in \hat{\mathcal{A}}_{u_i}.$$

Abandoned strategy

At each new time step t_{h+1} , after the selection $R(t_{h+1}) = n$, an agent $n' \in \mathcal{A}$ (not necessarily distinct from the agent n) is chosen uniformly at random in $\mathcal{A}$ to abandon its strategy, independently from $R(t_{h+1})$. Letting $D(t_{h+1}): \Omega \rightarrow \mathcal{A}$ denote the random variable (independent from $R(t_{h+1})$) representing the “dead” agent chosen to abandon its strategy at time t_{h+1} , we have

$$\mathbb{P}(D(t_{h+1}) = n') = \frac{1}{N} \quad \text{for every } n' \in \mathcal{A}.$$

In particular, if $S(t_h) = \hat{S}$ is an observed realization of the random state of the system $S(t_h)$ at time t_h , then the probability that an agent with strategy u_j is chosen to abandon its strategy is given by

$$\mathbb{P}(D(t_{h+1}) \in \hat{\mathcal{A}}_{u_j} \mid S(t_h) = \hat{S}) = \frac{\# \hat{\mathcal{A}}_{u_j}}{N} = \hat{\lambda}_{u_j}, \quad (3.6)$$

for $j = 1, \dots, M$.

Transition probabilities

Let us fix $i, j \in \{1, \dots, M\}$ (possibly $i = j$). Let us assume that at discrete time t_h we have observed a state $S(t_h) = \hat{S}$. Let us compute the probability that the proportions transition from $\lambda(t_h) = \hat{\lambda}$ to $\lambda(t_{h+1}) = \hat{\lambda} + \frac{1}{N}(e_i - e_j)$, where $e_i, e_j \in \mathbb{R}^M$ are vectors of the standard basis of $\mathbb{R}^M$. Writing more explicitly the updated proportions, we have that

$$\hat{\lambda} + \frac{1}{N}(e_i - e_j) = \left(\hat{\lambda}_{u_1}, \dots, \underbrace{\hat{\lambda}_{u_i} + \frac{1}{N}}_{\text{place } i}, \dots, \underbrace{\hat{\lambda}_{u_j} - \frac{1}{N}}_{\text{place } j}, \dots, \hat{\lambda}_{u_M} \right),$$

i.e., they are the proportions where the number of agents with strategy u_i increases by 1 while the number of agents with strategy u_j decreases by 1. This means that, given $\{\lambda(t_h) = \hat{\lambda}\}$, the event $\{\lambda(t_{h+1}) = \hat{\lambda} + \frac{1}{N}(e_i - e_j)\}$ occurs if and only if an agent n with strategy u_i is selected for replication and an agent n' with strategy u_j is chosen to abandon its strategy. In formulas, by independence, (3.4) and (3.6), we have that

$$\begin{aligned} \mathbb{P}\left(\lambda(t_{h+1}) = \hat{\lambda} + \frac{1}{N}(e_i - e_j) \mid S(t_h) = \hat{S}\right) &= \mathbb{P}(\{R(t_{h+1}) \in \hat{\mathcal{A}}_{u_i}\} \cap \{D(t_{h+1}) \in \hat{\mathcal{A}}_{u_j}\} \mid S(t_h) = \hat{S}) \\ &= \mathbb{P}(R(t_{h+1}) \in \hat{\mathcal{A}}_{u_i} \mid S(t_h) = \hat{S}) \mathbb{P}(D(t_{h+1}) \in \hat{\mathcal{A}}_{u_j} \mid S(t_h) = \hat{S}) = \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j}. \end{aligned} \quad (3.7)$$

The probability that the proportions remain unchanged is given by²

$$\begin{aligned} \mathbb{P}(\lambda(t_{h+1}) = \hat{\lambda} \mid S(t_h) = \hat{S}) &= \sum_{i=1}^M \mathbb{P}(\{R(t_{h+1}) \in \hat{\mathcal{A}}_{u_i}\} \cap \{D(t_{h+1}) \in \hat{\mathcal{A}}_{u_i}\} \mid S(t_h) = \hat{S}) \\ &= \sum_{i=1}^M \mathbb{P}(R(t_{h+1}) \in \hat{\mathcal{A}}_{u_i} \mid S(t_h) = \hat{S}) \mathbb{P}(D(t_{h+1}) \in \hat{\mathcal{A}}_{u_i} \mid S(t_h) = \hat{S}) \\ &= \sum_{i=1}^M \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_i} = \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \hat{\lambda}_{u_i}^2 f_{u_i}(\hat{\lambda}). \end{aligned}$$

We concluded the description of the discrete stochastic process. We are interested in the large-population limit of the process as $N \rightarrow \infty$ and the time step $\tau_k \rightarrow 0$. Before proceeding with the analysis, we introduce the main tool that we will use to study the limit of the process.

4. EULERIAN SPECIFICATION OF THE DISCRETE STOCHASTIC PROCESS

In this section we introduce a probability measure Λ_t^k that we will use to describe the discrete stochastic process with an Eulerian point of view. After deriving some properties of this measure, we will show that Λ_t^k is an approximate solution of a suitable continuity equation. This will also allow us to study the limit of the process as $N \rightarrow \infty$ and $\tau_k \rightarrow 0$.

Sequences N_k and w_k

From now on, we will consider

$$N_k \sim \tau_k^{-\alpha}, \quad \text{for some } \alpha > 0, \quad w_k \sim \tau_k^\beta, \quad \text{for some } \beta > 0,$$

²This can be also computed *via* the complementary probability

$$\begin{aligned} \mathbb{P}(\lambda(t_{h+1}) = \hat{\lambda} \mid S(t_h) = \hat{S}) &= 1 - \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M \mathbb{P}\left(\lambda(t_{h+1}) = \hat{\lambda} + \frac{1}{N}(e_i - e_j) \mid S(t_h) = \hat{S}\right) \\ &= 1 - \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j} = 1 - \sum_{i=1}^M \sum_{j=1}^M \frac{\hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j}}{\sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda})} + \frac{\sum_{i=1}^M \hat{\lambda}_{u_i}^2 f_{u_i}(\hat{\lambda})}{\sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda})} \\ &= 1 - \sum_{i=1}^M \hat{\lambda}_{u_i} + \frac{\sum_{i=1}^M \hat{\lambda}_{u_i}^2 f_{u_i}(\hat{\lambda})}{\sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda})} = \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \hat{\lambda}_{u_i}^2 f_{u_i}(\hat{\lambda}). \end{aligned}$$

where we recall that $\tau_k = \frac{T}{k}$ is the time step of the process. Since we are interested in the limit behavior of the process as $k \rightarrow +\infty$, we will stress the dependence on k of $\lambda^k(t_h)$, *i.e.*, the proportions at time t_h in the process described in Section 3 with population size N_k and time step τ_k .

Piecewise affine interpolation

First of all we extend the discrete stochastic process to all times $t \in [0, T]$ by defining $\lambda^k(t): \Omega \rightarrow \Delta^{M-1}$ by piecewise affine interpolation, *i.e.*, we set

$$\lambda^k(t) = \lambda^k(t_h) + \frac{t - t_h}{\tau_k} (\lambda^k(t_{h+1}) - \lambda^k(t_h)) \in \Delta^{M-1}, \quad \text{for } t \in [t_h, t_{h+1}], \quad (4.1)$$

for $h = 0, \dots, k-1$. For every realization $\omega \in \Omega$ of the probability space, we obtain a continuous path $\lambda^k(\cdot)(\omega) \in C([0, T], \Delta^{M-1})$. Hence, we have defined a random continuous path

$$\lambda^k(\cdot): \Omega \rightarrow C([0, T], \Delta^{M-1}).$$

Distribution of proportions

We define the probability measure on continuous paths

$$\Lambda^k := \lambda^k(\cdot)_\# \mathbb{P} \in \mathcal{P}(C([0, T], \Delta^{M-1})),$$

where $\lambda^k(\cdot)_\# \mathbb{P}$ denotes the pushforward of $\mathbb{P}$ through the random continuous path $\lambda^k(\cdot)$. The probability measure Λ^k describes how the evolution of the random proportions $\lambda^k(t)$ is distributed over the space of continuous paths.

For every $t \in [0, T]$, we consider the evaluation map on continuous paths, *i.e.*,

$$\text{ev}_t: C([0, T], \Delta^{M-1}) \rightarrow \Delta^{M-1}, \quad \lambda \mapsto \text{ev}_t(\lambda) = \lambda(t).$$

With this map, we define the probability measure

$$\Lambda_t^k = (\text{ev}_t)_\# \Lambda^k = (\text{ev}_t)_\# \lambda^k(\cdot)_\# \mathbb{P} = (\text{ev}_t \circ \lambda^k(\cdot))_\# \mathbb{P} = \lambda^k(t)_\# \mathbb{P} \in \mathcal{P}(\Delta^{M-1}). \quad (4.2)$$

This probability measure can be regarded as the distribution of the random proportions $\lambda^k(t)$ at a given time t .

Remark 4.1. If one adopts the identification of Δ^{M-1} with the set of probability measures on $\mathcal{U}$, one can interpret Λ_t^k as an element of $\mathcal{P}(\mathcal{P}(\mathcal{U}))$.

Piecewise constant interpolation

We introduce the notation $\bar{\Lambda}_t^k$ for the piecewise constant curve defined by

$$\bar{\Lambda}_t^k = \Lambda_{t_h}^k \quad \text{for } t \in [t_h, t_{h+1}), \quad \bar{\Lambda}_T^k = \Lambda_T^k. \quad (4.3)$$

In the next lemma we show that $\bar{\Lambda}_t^k$ and Λ_t^k are close in the 1-Wasserstein distance as $k \rightarrow +\infty$.

Lemma 4.2. *Let Λ_t^k be as in (4.2) and $\bar{\Lambda}_t^k$ be as in (4.3). We have that $\mathcal{W}_1(\Lambda_t^k, \bar{\Lambda}_t^k) \rightarrow 0$ as $k \rightarrow +\infty$ for every $t \in [0, T]$.*

Proof. If $t = T$ there is nothing to prove. Let us fix $t \in [t_h, t_{h+1})$ for some $h \in \{0, \dots, k-1\}$ and let us fix $\psi: \Delta^{M-1} \rightarrow \mathbb{R}$ a 1-Lipschitz function. By the definition of Λ^k and $\bar{\Lambda}^k$, we have that

$$\begin{aligned} & \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_t^k(\lambda) - \int_{\Delta^{M-1}} \psi(\lambda) d\bar{\Lambda}_t^k(\lambda) \\ &= \int_{\Delta^{M-1}} \psi(\lambda) d(\Lambda_t^k - \Lambda_{t_h}^k)(\lambda) = \int_{\Delta^{M-1}} \psi(\lambda) d(\lambda^k(t) - \lambda^k(t_h))_{\#} \mathbb{P}(\lambda) \\ &= \mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(t_h))] \leq \mathbb{E}[|\lambda^k(t) - \lambda^k(t_h)|]. \end{aligned} \quad (4.4)$$

By (4.1), we have that

$$\mathbb{E}[|\lambda^k(t) - \lambda^k(t_h)|] = \frac{t - t_h}{\tau_k} \mathbb{E}[|\lambda^k(t_{h+1}) - \lambda^k(t_h)|] \leq \mathbb{E}[|\lambda^k(t_{h+1}) - \lambda^k(t_h)|].$$

To estimate the right-hand side, we apply the law of total probability and take the sum over all possible states of the system at time t_h :

$$\mathbb{E}[|\lambda^k(t_{h+1}) - \lambda^k(t_h)|] = \sum_{\hat{S}} \mathbb{E}[|\lambda^k(t_{h+1}) - \hat{\lambda}| \mid S(t_h) = \hat{S}] \mathbb{P}(S(t_h) = \hat{S}). \quad (4.5)$$

Given $\lambda^k(t_h) = \hat{\lambda}$, the proportions $\lambda^k(t_{h+1})$ either remain unchanged or one component of the proportions increases by $\frac{1}{N_k}$ and another decreases by $\frac{1}{N_k}$. Hence,

$$\mathbb{E}[|\lambda^k(t_{h+1}) - \hat{\lambda}| \mid S(t_h) = \hat{S}] \leq \frac{\sqrt{2}}{N_k}.$$

By (4.4)–(4.5), we infer that

$$\int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_t^k(\lambda) - \int_{\Delta^{M-1}} \psi(\lambda) d\bar{\Lambda}_t^k(\lambda) \leq \frac{\sqrt{2}}{N_k}, \quad \text{for every 1-Lipschitz function } \psi.$$

Taking the supremum over all 1-Lipschitz functions ψ , by Kantorovich's duality we obtain that $\mathcal{W}_1(\Lambda_t^k, \bar{\Lambda}_t^k) \leq \frac{\sqrt{2}}{N_k} \rightarrow 0$ as $k \rightarrow +\infty$, concluding the proof. $\square$

Continuity equation

To describe the Eulerian evolution of the process, we introduce the vector field $b = (b_1, \dots, b^M)^\top: \Delta^{M-1} \rightarrow \mathbb{R}^M$ defined by

$$b_i(\hat{\lambda}) := \hat{\lambda}_{u_i}((A\hat{\lambda})_i - (A\hat{\lambda}) \cdot \hat{\lambda}), \quad \text{for } i = 1, \dots, M. \quad (4.6)$$

Remark 4.3. The vector field b is tangent to the simplex Δ^{M-1} , i.e., $b(\hat{\lambda}) \in \text{Tan}(\Delta^{M-1}, \hat{\lambda})$ for every $\hat{\lambda} \in \Delta^{M-1}$. Indeed, recalling that $e_1 + \dots + e_M$ is the normal to Δ^{M-1} at every $\hat{\lambda}$, we have that

$$\sum_{i=1}^M b_i(\hat{\lambda}) = \sum_{i=1}^M \hat{\lambda}_{u_i}((A\hat{\lambda})_i - (A\hat{\lambda}) \cdot \hat{\lambda}) = (A\hat{\lambda}) \cdot \hat{\lambda} - (A\hat{\lambda}) \cdot \hat{\lambda} = 0.$$

We have the following result, in which we show that, in a suitable sense, Λ^k solves is an approximate distributional solution to the continuity equation

$$\partial_t \Lambda_t^k + \operatorname{div}(b \Lambda_t^k) \approx 0.$$

Proposition 4.4. *Let Λ_t^k be as in (4.2) and $\bar{\Lambda}_t^k$ be as in (4.3). Let $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^M)$. Then*

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt + \int_0^T \int_{\Delta^{M-1}} D\phi(t, \lambda) b(\lambda) d\bar{\Lambda}_t^k(\lambda) dt = - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0^k(\lambda) + \tilde{\rho}_k,$$

where

$$\tilde{\rho}_k = \mathcal{O}\left(\frac{w_k}{\tau_k N_k^2} \|D\phi\|_\infty\right) + \mathcal{O}\left(\frac{w_k^2}{\tau_k N_k} \|D\phi\|_\infty\right) + \mathcal{O}\left(\frac{1}{\tau_k N_k^2} \|D^2\phi\|_{L^\infty}\right).$$

Before proving the proposition, we need a technical estimate.

Lemma 4.5. *Let Λ_t^k be as in (4.2) and $\bar{\Lambda}_t^k$ be as in (4.3). Let $\psi \in C_c^\infty(\mathbb{R}^M)$. Then*

$$\mathbb{E}[D\psi(\lambda^k(t_h))[\lambda^k(t_{h+1}) - \lambda^k(t_h)]] = \frac{w_k}{N_k} \int_{\Delta^{M-1}} D\psi(\lambda) b(\lambda) d\Lambda_{t_h}^k(\lambda) + \rho_k,$$

for $h = 0, \dots, k-1$, where

$$\rho_k = \mathcal{O}\left(\frac{w_k}{N_k^2} \|D\psi\|_\infty\right) + \mathcal{O}\left(\frac{w_k^2}{N_k} \|D\psi\|_\infty\right).$$

Proof. By the law of total probability:

$$\mathbb{E}[D\psi(\lambda^k(t_h))[\lambda^k(t_{h+1}) - \lambda^k(t_h)]] = \sum_{\hat{S}} \mathbb{E}[D\psi(\hat{\lambda})[\lambda^k(t_{h+1}) - \hat{\lambda}] \mid S(t_h) = \hat{S}] \mathbb{P}(S(t_h) = \hat{S}). \quad (4.7)$$

Then, by (3.7), we obtain that

$$\begin{aligned} & \mathbb{E}[D\psi(\hat{\lambda})[\lambda^k(t_{h+1}) - \hat{\lambda}] \mid S(t_h) = \hat{S}] \\ &= \sum_{i=1}^M \sum_{j=1}^M D\psi(\hat{\lambda}) \left[\frac{1}{N_k} (e_i - e_j) \right] \mathbb{P}\left(\lambda^k(t_{h+1}) = \hat{\lambda} + \frac{1}{N_k} (e_i - e_j) \mid S(t_h) = \hat{S}\right) \\ &= \frac{1}{N_k} \sum_{i=1}^M \sum_{j=1}^M (\partial_i \psi(\hat{\lambda}) - \partial_j \psi(\hat{\lambda})) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j}. \end{aligned} \quad (4.8)$$

Using the fact that $\sum_{j=1}^M \hat{\lambda}_{u_j} = 1$, we infer that

$$\frac{1}{N_k} \sum_{i=1}^M \sum_{j=1}^M \partial_i \psi(\hat{\lambda}) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j} = \frac{1}{N_k} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}),$$

while, by the definition of $\bar{f}(\hat{\lambda})$ in (3.5), we have that

$$\begin{aligned} \frac{1}{N_k} \sum_{i=1}^M \sum_{j=1}^M \partial_j \psi(\hat{\lambda}) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j} &= \frac{1}{N_k} \sum_{j=1}^M \partial_j \psi(\hat{\lambda}) \hat{\lambda}_{u_j} \sum_{i=1}^M \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) = \frac{1}{N_k} \sum_{j=1}^M \partial_j \psi(\hat{\lambda}) \hat{\lambda}_{u_j} \\ &= \frac{1}{N_k} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \hat{\lambda}_{u_i}. \end{aligned}$$

Inserting these two expressions in (4.8), we deduce that

$$\begin{aligned} \mathbb{E}[\mathbf{D}\psi(\hat{\lambda})[\lambda^k(t_{h+1}) - \hat{\lambda}] \mid S(t_h) = \hat{S}] &= \frac{1}{N_k} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) - \frac{1}{N_k} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \hat{\lambda}_{u_i} \\ &= \frac{1}{N_k} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \frac{1}{\bar{f}(\hat{\lambda})} \hat{\lambda}_{u_i} (f_{u_i}(\hat{\lambda}) - \bar{f}(\hat{\lambda})). \end{aligned} \tag{4.9}$$

On the one hand, by (3.5), (3.3), and (3.2) we have that

$$\begin{aligned} \bar{f}(\hat{\lambda}) &= \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} f_{u_\ell}(\hat{\lambda}) = \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} (1 - w_k + w_k \pi_{u_\ell}(\hat{\lambda})) = 1 - w_k + w_k \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} \pi_{u_\ell}(\hat{\lambda}) \\ &= 1 - w_k + w_k \sum_{\ell=1}^M \hat{\lambda}_{u_\ell} \left(\frac{N_k}{N_k - 1} (A\hat{\lambda})_\ell - \frac{1}{N_k - 1} a_{\ell\ell} \right) \\ &= 1 - w_k + w_k \frac{N_k}{N_k - 1} (A\hat{\lambda}) \cdot \hat{\lambda} - \frac{w_k}{N_k - 1} \text{diag}(A) \cdot \hat{\lambda}, \end{aligned}$$

where $\text{diag}(A) \in \mathbb{R}^M$ denotes the vector with the diagonal elements of A . On the other hand, by (3.3)

$$f_{u_i}(\hat{\lambda}) = 1 - w_k + w_k \pi_{u_i}(\hat{\lambda}) = 1 - w_k + w_k \frac{N_k}{N_k - 1} (A\hat{\lambda})_i - \frac{w_k}{N_k - 1} a_{ii}.$$

We substitute these expressions in (4.9) to obtain that

$$\begin{aligned} \mathbb{E}[\mathbf{D}\psi(\hat{\lambda})[\lambda^k(t_{h+1}) - \hat{\lambda}] \mid S(t_h) = \hat{S}] &= \frac{w_k}{N_k} \frac{N_k}{N_k - 1} \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \hat{\lambda}_{u_i} \left((A\hat{\lambda})_i - (A\hat{\lambda}) \cdot \hat{\lambda} \right) \\ &\quad + \frac{w_k}{N_k} \frac{1}{N_k - 1} \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \hat{\lambda}_{u_i} \left(\text{diag}(A) \cdot \hat{\lambda} - a_{ii} \right) \\ &= \frac{w_k}{N_k} \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) b_i(\hat{\lambda}) + \frac{w_k}{N_k} R_{1,k}(\hat{\lambda}) \\ &= \frac{w_k}{N_k} \frac{1}{\bar{f}(\hat{\lambda})} \mathbf{D}\psi(\hat{\lambda}) b(\hat{\lambda}) + \frac{w_k}{N_k} R_{1,k}(\hat{\lambda}), \end{aligned} \tag{4.10}$$

where

$$R_{1,k}(\hat{\lambda}) = \frac{1}{N_k - 1} \frac{1}{\bar{f}(\hat{\lambda})} \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) \hat{\lambda}_{u_i} \left((A\hat{\lambda})_i - (A\hat{\lambda}) \cdot \hat{\lambda} + \text{diag}(A) \cdot \hat{\lambda} - a_{ii} \right). \quad (4.11)$$

Let us estimate $R_{1,k}$. To bound $\frac{1}{\bar{f}(\hat{\lambda})}$, we set

$$g_k(\hat{\lambda}) := -1 + \frac{N_k}{N_k - 1} (A\hat{\lambda}) \cdot \hat{\lambda} - \frac{1}{N_k - 1} \text{diag}(A) \cdot \hat{\lambda}.$$

We start by observing that

$$|g_k(\hat{\lambda})| \leq c_0, \quad \text{for every } \hat{\lambda} \in \Delta^{M-1}, \quad k \geq 1, \quad (4.12)$$

for some constant $c_0 > 0$ depending on A . This implies that

$$\frac{1}{\bar{f}(\hat{\lambda})} \leq c_1, \quad \text{for every } \hat{\lambda} \in \Delta^{M-1}, \quad k \text{ large enough}, \quad (4.13)$$

for some constant $c_1 > 0$, where $k \geq 1$ large enough means, *e.g.*, $w_k \leq \frac{1}{2c_0}$.

We use (4.13) to estimate the remainder $R_{1,k}$ in (4.11) by

$$|R_{1,k}(\hat{\lambda})| \leq C_1 \|\text{D}\psi\|_{L^\infty} \frac{1}{N_k}, \quad \text{for every } \hat{\lambda} \in \Delta^{M-1}, \quad k \text{ large enough}, \quad (4.14)$$

where the constant $C_1 > 0$ depends on A and c_1 .

We recast the term $\frac{1}{\bar{f}(\hat{\lambda})}$ in (4.10) in a more convenient form. Exploiting once more (4.12), from

$$\frac{1}{\bar{f}(\hat{\lambda})} = \frac{1}{1 + w_k g_k(\hat{\lambda})} = 1 - \frac{w_k g_k(\hat{\lambda})}{1 + w_k g_k(\hat{\lambda})},$$

we also deduce that

$$\left| \frac{1}{\bar{f}(\hat{\lambda})} - 1 \right| \leq \frac{w_k g_k(\hat{\lambda})}{|1 + w_k g_k(\hat{\lambda})|} \leq c_2 w_k, \quad \text{for every } \hat{\lambda} \in \Delta^{M-1}, \quad k \text{ large enough}, \quad (4.15)$$

for some constant $c_2 > 0$, where $k \geq 1$ large enough means, *e.g.*, $w_k \leq \frac{1}{2c_0}$.

Then we recast (4.10) as

$$\mathbb{E}[\text{D}\psi(\hat{\lambda})[\lambda^k(t_{h+1}) - \hat{\lambda}] \mid S(t_h) = \hat{S}] = \frac{w_k}{N_k} \left(\text{D}\psi(\hat{\lambda})b(\hat{\lambda}) + R_{1,k}(\hat{\lambda}) + R_{2,k}(\hat{\lambda}) \right), \quad (4.16)$$

where

$$R_{2,k}(\hat{\lambda}) = \left(\frac{1}{\bar{f}(\hat{\lambda})} - 1 \right) \sum_{i=1}^M \partial_i \psi(\hat{\lambda}) b_i(\hat{\lambda}).$$

By (4.15), we estimate the remainder $R_{2,k}$ by

$$|R_{2,k}(\hat{\lambda})| \leq C_2 \|D\psi\|_{L^\infty} w_k, \quad \text{for every } \hat{\lambda} \in \Delta^{M-1}, \text{ } k \text{ large enough,} \quad (4.17)$$

where the constant $C_2 > 0$ depends on A and c_2 .

We have concluded the estimate. Indeed, by (4.7), (4.16), (4.14), and (4.17), we deduce that

$$\begin{aligned} \mathbb{E}[D\psi(\lambda^k(t_h))[\lambda^k(t_{h+1}) - \lambda^k(t_h)]] &= \frac{w_k}{N_k} \sum_{\hat{S}} \left(D\psi(\hat{\lambda})b(\hat{\lambda}) + R_{1,k}(\hat{\lambda}) + R_{2,k}(\hat{\lambda}) \right) \mathbb{P}(S(t_h) = \hat{S}) \\ &= \frac{w_k}{N_k} \left(\mathbb{E}[D\psi(\lambda^k(t_h))b(\lambda^k(t_h))] + \mathbb{E}[R_{1,k}(\lambda^k(t_h))] + \mathbb{E}[R_{2,k}(\lambda^k(t_h))] \right) \\ &= \frac{w_k}{N_k} \left(\mathbb{E}[D\psi(\lambda^k(t_h))b(\lambda^k(t_h))] + \mathcal{O}\left(\frac{\|D\psi\|_{L^\infty}}{N_k}\right) + \mathcal{O}(\|D\psi\|_{L^\infty} w_k) \right). \end{aligned}$$

We note that

$$\begin{aligned} \mathbb{E}[D\psi(\lambda^k(t_h))b(\lambda^k(t_h))] &= \int_{\Omega} D\psi(\lambda^k(t_h)(\omega))b(\lambda^k(t_h)(\omega)) d\mathbb{P}(\omega) = \int_{\Delta^{M-1}} D\psi(\lambda)b(\lambda) d\lambda^k(t_h)_{\#} \mathbb{P}(\lambda) \\ &= \int_{\Delta^{M-1}} D\psi(\lambda)b(\lambda) d\Lambda_{t_h}^k(\lambda), \end{aligned}$$

hence

$$\begin{aligned} \mathbb{E}[D\psi(\lambda^k(t_h))[\lambda^k(t_{h+1}) - \lambda^k(t_h)]] &= \frac{w_k}{N_k} \left(\int_{\Delta^{M-1}} D\psi(\lambda)b(\lambda) d\Lambda_{t_h}^k(\lambda) \right. \\ &\quad \left. + \mathcal{O}\left(\frac{\|D\psi\|_{L^\infty}}{N_k}\right) + \mathcal{O}(\|D\psi\|_{L^\infty} w_k) \right). \end{aligned} \quad (4.18)$$

This concludes the proof of the lemma. $\square$

We are now in a position to prove Proposition 4.4.

Proof of Proposition 4.4. Let us fix $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^M)$. By (4.1), for a.e. realization $\omega \in \Omega$, we have that

$$\begin{aligned} \frac{d}{dt}(\phi(t, \lambda^k(t)(\omega))) &= \partial_t \phi(t, \lambda^k(t)(\omega)) + D\phi(t, \lambda^k(t)(\omega)) \left[\frac{d}{dt} \lambda^k(t)(\omega) \right] \\ &= \partial_t \phi(t, \lambda^k(t)(\omega)) + D\phi(t, \lambda^k(t)(\omega)) \left[\frac{\lambda^k(t_{h+1})(\omega) - \lambda^k(t_h)(\omega)}{\tau_k} \right] \end{aligned} \quad (4.19)$$

for every $t \in (t_h, t_{h+1})$. We take the expectation and we integrate in t the above equality. The left-hand side in (4.19) reads

$$\begin{aligned} \int_{t_h}^{t_{h+1}} \frac{d}{dt} \int_{\Omega} \phi(t, \lambda^k(t)(\omega)) d\mathbb{P}(\omega) dt &= \int_{\Omega} \phi(t_{h+1}, \lambda^k(t_{h+1})(\omega)) d\mathbb{P}(\omega) - \int_{\Omega} \phi(0, \lambda^k(0)(\omega)) d\mathbb{P}(\omega) \\ &= \int_{\Delta^{M-1}} \phi(t_{h+1}, \lambda) d\Lambda_{t_{h+1}}^k(\lambda) - \int_{\Delta^{M-1}} \phi(t_h, \lambda) d\Lambda_{t_h}^k(\lambda). \end{aligned}$$

The right-hand side in (4.19) reads

$$\begin{aligned} & \int_{t_h}^{t_{h+1}} \int_{\Omega} \partial_t \phi(t, \lambda^k(t)(\omega)) d\mathbb{P}(\omega) dt + \int_{t_h}^{t_{h+1}} \int_{\Omega} D\phi(t, \lambda^k(t)(\omega)) \left[\frac{\lambda^k(t_{h+1})(\omega) - \lambda^k(t_h)(\omega)}{\tau_k} \right] d\mathbb{P}(\omega) dt \\ &= \int_{t_h}^{t_{h+1}} \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt + \frac{1}{\tau_k} \int_{t_h}^{t_{h+1}} \mathbb{E}[D\phi(t, \lambda^k(t)) [\lambda^k(t_{h+1}) - \lambda^k(t_h)]] dt. \end{aligned}$$

We now estimate the term $\mathbb{E}[D\phi(t, \lambda^k(t)) [\lambda^k(t_{h+1}) - \lambda^k(t_h)]]$. To do so, we use (4.1) and Taylor's formula to write a.s. in Ω

$$\begin{aligned} D\phi(t, \lambda^k(t)) [\lambda^k(t_{h+1}) - \lambda^k(t_h)] &= D\phi\left(t, \lambda^k(t_h) + (\lambda^k(t) - \lambda^k(t_h))\right) [\lambda^k(t_{h+1}) - \lambda^k(t_h)] \\ &= D\phi(t, \lambda^k(t_h)) [\lambda^k(t_{h+1}) - \lambda^k(t_h)] + R(t, \lambda^k(t_h), \lambda^k(t_{h+1})), \end{aligned}$$

where

$$\begin{aligned} & R(t, \lambda^k(t_h), \lambda^k(t_{h+1})) \\ &= \frac{t - t_h}{\tau_k} \int_0^1 D^2\phi\left(t, \lambda^k(t_h) + r(\lambda^k(t) - \lambda^k(t_h))\right) [\lambda^k(t_{h+1}) - \lambda^k(t_h), \lambda^k(t_{h+1}) - \lambda^k(t_h)] dr. \end{aligned}$$

Since t is fixed, we apply Lemma 4.5 to $\psi = \phi(t, \cdot)$ to get that

$$\begin{aligned} & \mathbb{E}[D\phi(t, \lambda^k(t_h)) [\lambda^k(t_{h+1}) - \lambda^k(t_h)]] \\ &= \frac{w_k}{N_k} \int_{\Delta^{M-1}} D\phi(t, \lambda) b(\lambda) d\Lambda_{t_h}^k(\lambda) + \mathcal{O}\left(\frac{w_k}{N_k^2} \|D\psi\|_{\infty}\right) + \mathcal{O}\left(\frac{w_k^2}{N_k} \|D\psi\|_{\infty}\right). \end{aligned}$$

Moreover, we use the law of total probability to estimate

$$\mathbb{E}[R(t, \lambda^k(t_h), \lambda^k(t_{h+1}))] = \sum_{\hat{S}} \mathbb{E}[R(t, \hat{\lambda}, \lambda^k(t_{h+1})) \mid S(t_h) = \hat{S}] \mathbb{P}(S(t_h) = \hat{S}),$$

and

$$\mathbb{E}[R(t, \hat{\lambda}, \lambda^k(t_{h+1})) \mid S(t_h) = \hat{S}] \leq \|D^2\phi\|_{L^{\infty}} \frac{2}{N_k^2} = \mathcal{O}\left(\frac{\|D^2\phi\|_{L^{\infty}}}{N_k^2}\right),$$

from which we deduce that

$$\mathbb{E}[R(t, \lambda^k(t_h), \lambda^k(t_{h+1}))] = \mathcal{O}\left(\frac{\|D^2\phi\|_{L^{\infty}}}{N_k^2}\right). \quad (4.20)$$

Combining (4.19)–(4.20), we infer that

$$\begin{aligned} & \int_{\Delta^{M-1}} \phi(t_{h+1}, \lambda) d\Lambda_{t_{h+1}}^k(\lambda) - \int_{\Delta^{M-1}} \phi(t_h, \lambda) d\Lambda_{t_h}^k(\lambda) \\ &= \int_{t_h}^{t_{h+1}} \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt + \frac{w_k}{\tau_k N_k} \int_{t_h}^{t_{h+1}} \int_{\Delta^{M-1}} D\phi(t, \lambda) b(\lambda) d\Lambda_{t_h}^k(\lambda) \\ &\quad + \mathcal{O}\left(\frac{w_k}{N_k^2} \|D\psi\|_{\infty}\right) + \mathcal{O}\left(\frac{w_k^2}{N_k} \|D\psi\|_{\infty}\right) + \mathcal{O}\left(\frac{1}{N_k^2} \|D^2\phi\|_{L^{\infty}}\right). \end{aligned}$$

Summing the above equality over $h = 0, \dots, k-1$, we obtain that

$$\begin{aligned} - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0^k(\lambda) &= \int_{\Delta^{M-1}} \phi(T, \lambda) d\Lambda_T^k(\lambda) - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0^k(\lambda) \\ &= \int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt + \frac{w_k}{\tau_k N_k} \int_{\Delta^{M-1}} D\phi(t, \lambda) b(\lambda) d\bar{\Lambda}_t^k(\lambda) \\ &\quad + \mathcal{O}\left(\frac{w_k}{\tau_k N_k^2} \|D\psi\|_\infty\right) + \mathcal{O}\left(\frac{w_k^2}{\tau_k N_k} \|D\psi\|_\infty\right) + \mathcal{O}\left(\frac{1}{\tau_k N_k^2} \|D^2\phi\|_{L^\infty}\right). \end{aligned}$$

This concludes the proof of the proposition. □

5. CONTINUOUS-TIME LARGE POPULATION LIMIT

We are in a position to prove the main result in this paper, in which we characterize the limit of the measures Λ^k as $k \rightarrow +\infty$.

First of all, we need to establish a compactness result for the sequence Λ^k . For this, we start with a preliminary observation.

Remark 5.1. Let Λ_t^k be as in (4.2). We have that:

- (i) $\Lambda_t^k \in \mathcal{P}_1(\Delta^{M-1})$ for every $t \in [0, T]$;
- (ii) $(\mathcal{P}_1(\Delta^{M-1}), \mathcal{W}_1)$ is a compact metric space.

For (i), we fix $e_1 \in \Delta^{M-1}$ and we obtain, by boundedness of Δ^{M-1} , that

$$\int_{\Delta^{M-1}} |e_1 - \lambda| d\Lambda_t^k(\lambda) \leq C \int_{\Delta^{M-1}} d\Lambda_t^k(\lambda) \leq C,$$

for some constant $C > 0$. In fact, the estimate is uniform in $t \in [0, T]$, N , and τ_k .

Fact (ii) is a known result, since Δ^{M-1} is compact. See, *e.g.*, [25], Remark 6.19.

Remark 5.2. If the curves $t \in [0, T] \mapsto \Lambda_t^k \in \mathcal{P}_1(\Delta^{M-1})$ were equicontinuous (for example if they were equi-Lipschitz continuous), then one could apply Arzelà–Ascoli’s theorem to extract a subsequence converging to a continuous curve $t \in [0, T] \mapsto \Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$. However, straightforward estimates coming from the discrete stochastic process do not seem to provide equicontinuity. Indeed, notice that we can obtain the rough estimate

$$|\lambda^k(t_{h+1}) - \lambda^k(t_h)| \leq \frac{\sqrt{2}}{N_k}, \quad \text{almost surely.} \tag{5.1}$$

From this, we deduce that for every Lipschitz continuous function ψ with Lipschitz constant ≤ 1 , we have that

$$\begin{aligned} \int_{\Delta^{M-1}} \psi(\lambda) d(\Lambda_{t_{h+1}}^k - \Lambda_{t_h}^k)(\lambda) &= \int_{\Omega} \psi(\lambda^k(t_{h+1})) - \psi(\lambda^k(t_h)) d\mathbb{P} = \mathbb{E}[\psi(\lambda^k(t_{h+1})) - \psi(\lambda^k(t_h))] \\ &\leq \mathbb{E}[|\lambda^k(t_{h+1}) - \lambda^k(t_h)|] \leq \frac{\sqrt{2}}{N_k} = \sqrt{2}\tau_k^\alpha. \end{aligned}$$

Owing to Kantorovich’s duality formula, we obtain that

$$\mathcal{W}_1(\Lambda_{t_{h+1}}^k, \Lambda_{t_h}^k) \leq \sqrt{2}\tau_k^\alpha.$$

It seems that this estimate is not enough to deduce equicontinuity. Note that, iterating it yields

$$\mathcal{W}_1(\Lambda_t^k, \Lambda_s^k) \leq C \frac{|t-s|}{\tau_k} \frac{1}{N_k} \leq C |t-s| \frac{1}{\tau_k^{1-\alpha}},$$

an evidently bad estimate for equicontinuity purposes.

In the proof of the next theorem we avoid using (5.1), replacing it with (5.8) below. However, this requires to provide a proof in the spirit of Arzelà–Ascoli’s compactness result from scratch.

Theorem 5.3. *Let $N_k \sim \tau_k^{-\alpha}$, $w_k = \tau_k^\beta$ and assume that $\alpha, \beta > 0$ and $\alpha + \beta = 1$, $\alpha > \frac{1}{2}$. Let Λ_t^k be as in (4.2) and $\bar{\Lambda}_t^k$ be as in (4.3). Then there exists a subsequence independent of t (that we do not relabel) and a Lipschitz curve $t \in [0, T] \mapsto \Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$ such that $\mathcal{W}_1(\Lambda_t^k, \Lambda_t) \rightarrow 0$ as $k \rightarrow +\infty$ for every $t \in [0, T]$.*

Proof. Before proving the compactness result, we need to establish a crucial estimate, i.e., (5.8) below. We begin with an estimate for $s \leq t$ satisfying $s, t \in [t_h, t_{h+1}]$ for some $h \in \{0, \dots, k-1\}$. Let us fix $\psi \in C_c^\infty(\mathbb{R}^M)$. By the definition of Λ^k , we get that

$$\begin{aligned} \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_t^k(\lambda) - \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_s^k(\lambda) &= \int_{\Delta^{M-1}} \psi(\lambda) d(\Lambda_t^k - \Lambda_s^k)(\lambda) \\ &= \int_{\Delta^{M-1}} \psi(\lambda) d(\lambda^k(t) - \lambda^k(s))_{\#} \mathbb{P}(\lambda) = \int_{\Omega} (\psi(\lambda^k(t)(\omega)) - \psi(\lambda^k(s)(\omega))) d\mathbb{P}(\omega) \\ &= \mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(s))]. \end{aligned} \quad (5.2)$$

By Taylor’s formula, we have that

$$\begin{aligned} \psi(\lambda^k(t)) - \psi(\lambda^k(t_h)) &= D\psi(\lambda^k(t_h))[\lambda^k(t) - \lambda^k(t_h)] + R(\lambda^k(t_h), \lambda^k(t)), \\ \psi(\lambda^k(s)) - \psi(\lambda^k(t_h)) &= D\psi(\lambda^k(t_h))[\lambda^k(s) - \lambda^k(t_h)] + R(\lambda^k(t_h), \lambda^k(s)), \end{aligned} \quad (5.3)$$

where the remainder is given by

$$R(\lambda^k(t_h), \lambda^k(t)) = \int_0^1 (1-r) D^2\psi(\lambda^k(t_h) + r(\lambda^k(t) - \lambda^k(t_h))) [\lambda^k(t) - \lambda^k(t_h), \lambda^k(t) - \lambda^k(t_h)] dr,$$

with the same formula with s in place of t for $R(\lambda^k(t_h), \lambda^k(s))$. Subtraction of the two equations in (5.3) and (4.1) yields

$$\begin{aligned} \psi(\lambda^k(t)) - \psi(\lambda^k(s)) &= D\psi(\lambda^k(t_h))[\lambda^k(t) - \lambda^k(s)] + R(\lambda^k(t_h), \lambda^k(t)) - R(\lambda^k(t_h), \lambda^k(s)) \\ &= \frac{t-s}{\tau_k} D\psi(\lambda^k(t_h))[\lambda^k(t_{h+1}) - \lambda^k(t_h)] + R(\lambda^k(t_h), \lambda^k(t)) - R(\lambda^k(t_h), \lambda^k(s)). \end{aligned} \quad (5.4)$$

Let us estimate the expectation of $R(\lambda^k(t_h), \lambda^k(t))$ and $R(\lambda^k(t_h), \lambda^k(s))$. We show the estimate for $R(\lambda^k(t_h), \lambda^k(t))$; the estimate for $R(\lambda^k(t_h), \lambda^k(s))$ is completely analogous. By the law of total probability, we have that

$$\mathbb{E}[R(\lambda^k(t_h), \lambda^k(t))] = \sum_{\hat{S}} \mathbb{E}[R(\hat{\lambda}, \lambda^k(t)) \mid S(t_h) = \hat{S}] \mathbb{P}(S(t_h) = \hat{S}). \quad (5.5)$$

Then, using that $\lambda^k(t) - \hat{\lambda} = \frac{t-t_h}{\tau_k}(\lambda^k(t_{h+1}) - \hat{\lambda})$, we estimate

$$\begin{aligned}
& \mathbb{E}[R(\hat{\lambda}, \lambda^k(t)) \mid S(t_h) = \hat{S}] \\
&= \mathbb{E}\left[\int_0^1 (1-s) D^2\psi(\hat{\lambda} + s(\lambda^k(t) - \hat{\lambda})) [\lambda^k(t) - \hat{\lambda}, \lambda^k(t) - \hat{\lambda}] ds \mid S(t_h) = \hat{S}\right] \\
&= \sum_{i=1}^M \sum_{j=1}^M \int_0^1 (1-r) D^2\psi\left(\hat{\lambda} + r \frac{t-t_h}{\tau_k} \frac{1}{N_k} (e_i - e_j)\right) \left[\frac{t-t_h}{\tau_k} \frac{1}{N_k} (e_i - e_j), \frac{t-t_h}{\tau_k} \frac{1}{N_k} (e_i - e_j)\right] \\
&\quad \times \mathbb{P}\left(\lambda^k(t_{h+1}) = \hat{\lambda} + \frac{1}{N_k} (e_i - e_j) \mid S(t_h) = \hat{S}\right) dr \\
&\leq \sum_{i=1}^M \sum_{j=1}^M \|D^2\psi\|_{L^\infty} \frac{2}{N_k^2} \frac{1}{f(\hat{\lambda})} \hat{\lambda}_{u_i} f_{u_i}(\hat{\lambda}) \hat{\lambda}_{u_j} = \frac{2}{N_k^2} \|D^2\psi\|_{L^\infty}.
\end{aligned}$$

Inserting this estimate in (5.5), we obtain that

$$\mathbb{E}[R(\lambda^k(t_h), \lambda^k(t))] \leq \frac{2}{N_k^2} \|D^2\psi\|_{L^\infty} = \mathcal{O}\left(\frac{\|D^2\psi\|_{L^\infty}}{N_k^2}\right). \quad (5.6)$$

We now exploit the estimate obtained in Lemma 4.5. Taking the expectation in (5.4) and using (4.18) and (5.6), we infer that

$$\begin{aligned}
\mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(s))] &= \frac{t-s}{\tau_k} \left(\frac{w_k}{N_k} \int_{\Delta^{M-1}} D\psi(\lambda) b(\lambda) d\Lambda_{t_h}^k(\lambda) \right. \\
&\quad \left. + \frac{w_k}{N_k} \mathcal{O}\left(\frac{\|D\psi\|_{L^\infty}}{N_k}\right) + \frac{w_k}{N_k} \mathcal{O}(\|D\psi\|_{L^\infty} w_k) \right) + \mathcal{O}\left(\frac{\|D^2\psi\|_{L^\infty}}{N_k^2}\right) \\
&\leq C|t-s| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{N_k^2}.
\end{aligned} \quad (5.7)$$

Let us now fix generic $s \leq t$ in $[0, T]$. Let us denote by $h' \in \{0, \dots, k-1\}$ the index such that $s \in [t_{h'}, t_{h'+1})$ and by $h'' \in \{h', \dots, k-1\}$ the index such that $t \in [t_{h''}, t_{h''+1})$ (if $t = T$, we set $h'' = k-1$). We estimate the difference $\mathbb{E}[\psi(\lambda^k(t)) - \lambda^k(s)]$ by concatenating the local inequality in (5.7) in the intervals $[t_{h'}, t_{h'+1}), \dots, [t_{h''}, t_{h''+1})$:

$$\begin{aligned}
& \mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(s))] \\
&\leq \mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(t_{h''}))] + \sum_{h=h'+1}^{h''-1} \mathbb{E}[\psi(\lambda^k(t_{h+1})) - \lambda^k(t_h)] + \mathbb{E}[\psi(\lambda^k(t_{h'+1})) - \psi(\lambda^k(s))] \\
&\leq C|t-t_{h''}| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{N_k^2} \\
&\quad + \sum_{h=h'+1}^{h''-1} \left(C|t_{h+1} - t_h| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{N_k^2} \right) \\
&\quad + C|t_{h'+1} - s| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{N_k^2} \\
&\leq C|t-s| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C(h'' - h' + 1) \frac{\|D^2\psi\|_{L^\infty}}{N_k^2}.
\end{aligned}$$

We remark that

$$h''\tau_k - h'\tau_k = t_{h''} - t_{h'} \leq t - s + \tau_k \implies h'' - h' + 1 \leq \frac{|t-s|}{\tau_k} + 2 \leq \frac{C}{\tau_k}.$$

It follows that

$$\mathbb{E}[\psi(\lambda^k(t)) - \psi(\lambda^k(s))] \leq C|t-s| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{\tau_k N_k^2}.$$

Inserting this estimate in (5.2), we have shown that

$$\begin{aligned} & \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_t^k(\lambda) - \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_s^k(\lambda) \\ & \leq C|t-s| \frac{w_k}{N_k \tau_k} \left(\|b\|_{L^\infty} + \frac{1}{N_k} + w_k \right) \|D\psi\|_{L^\infty} + C \frac{\|D^2\psi\|_{L^\infty}}{\tau_k N_k^2}, \end{aligned} \quad (5.8)$$

for every $s, t \in [0, T]$ and every $\psi \in C_c^\infty(\mathbb{R}^M)$.

We are now in a position to prove the compactness result. Let us fix a countable dense set $D \subset [0, T]$. We exploit the compactness of $(\mathcal{P}_1(\Delta^{M-1}), \mathcal{W}_1)$ in Remark 5.1–ii to extract, by a diagonal argument, a subsequence independent of $q \in D$ (that we do not relabel) such that Λ_q^k converges in $\mathcal{W}_1$ to some $\Lambda_q \in \mathcal{P}_1(\Delta^{M-1})$ for every $q \in D$. We now extend the convergence to the whole interval $[0, T]$. Let us fix $q, q' \in D$. Recalling the assumptions on N_k and w_k , we have that

$$\frac{w_k}{N_k \tau_k} \sim \frac{\tau_k^\beta}{\tau_k \tau_k^{-\alpha}} = \tau_k^{\alpha+\beta-1} = 1, \quad \frac{1}{\tau_k N_k^2} \sim \frac{1}{\tau_k \tau_k^{-2\alpha}} = \tau_k^{2\alpha-1} \rightarrow 0.$$

Then we use the convergences $\mathcal{W}_1(\Lambda_q^k, \Lambda_q) \rightarrow 0$ and $\mathcal{W}_1(\Lambda_{q'}^k, \Lambda_{q'}) \rightarrow 0$ as $k \rightarrow +\infty$ to pass to the limit in (5.8) with $s = q$ and $t = q'$ and obtain that

$$\int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_q(\lambda) - \int_{\Delta^{M-1}} \psi(\lambda) d\Lambda_{q'}(\lambda) \leq C|q - q'| \|b\|_{L^\infty} \|D\psi\|_{L^\infty}, \quad (5.9)$$

for every $\psi \in C_c^\infty(\mathbb{R}^M)$.

We can generalize (5.9) to 1-Lipschitz competitors ψ by an approximation argument. Indeed, let $\psi: \Delta^{M-1} \rightarrow \mathbb{R}$ be a 1-Lipschitz function. By Kirszbraun's extension theorem, we can extend ψ to a 1-Lipschitz function (not relabeled) $\psi: \mathbb{R}^M \rightarrow \mathbb{R}$. We can then approximate ψ with a family of smooth functions $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^M)$ obtained by convolution with a mollifier and a multiplication by a compactly supported cut-off function identically equal to 1 on a neighborhood of Δ^{M-1} . Since $\psi_\varepsilon \rightarrow \psi$ locally uniformly, we pass to the limit in (5.8) with ψ_ε in place of ψ and obtain that (5.9) holds for every 1-Lipschitz function ψ .

Taking the supremum in (5.9) over all 1-Lipschitz functions ψ , we obtain that

$$\mathcal{W}_1(\Lambda_q, \Lambda_{q'}) \leq C|q - q'|. \quad (5.10)$$

This implies that for $q \rightarrow t$ and $q' \rightarrow t$, with $q, q' \in D$, we have that $\mathcal{W}_1(\Lambda_q, \Lambda_{q'}) \rightarrow 0$, i.e., $\{\Lambda_q \mid q \rightarrow t, q \in D\}$ is a Cauchy sequence in $(\mathcal{P}_1(\Delta^{M-1}), \mathcal{W}_1)$. Since $(\mathcal{P}_1(\Delta^{M-1}), \mathcal{W}_1)$ is a complete metric space, we deduce that there exists a limit $\Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$ such that $\mathcal{W}_1(\Lambda_q, \Lambda_t) \rightarrow 0$ as $q \rightarrow t$.

We observe that $\mathcal{W}_1(\Lambda_t^k, \Lambda_t) \rightarrow 0$ as $k \rightarrow +\infty$ for every $t \in [0, T]$. For, given $q \rightarrow t, q \in D$, we have that

$$\mathcal{W}_1(\Lambda_t^k, \Lambda_t) \leq \mathcal{W}_1(\Lambda_t^k, \Lambda_q^k) + \mathcal{W}_1(\Lambda_q^k, \Lambda_q) + \mathcal{W}_1(\Lambda_q, \Lambda_t),$$

and the right-hand side converges to 0 as $k \rightarrow +\infty$ and then $q \rightarrow t$.

Finally, passing to the limit in (5.10) for $q \rightarrow t$ and $q' \rightarrow s$, we obtain that

$$\mathcal{W}_1(\Lambda_t, \Lambda_s) \leq C|t - s|,$$

for every $s, t \in [0, T]$, i.e., $\Lambda \in C([0, T]; \mathcal{P}_1(\Delta^{M-1}))$ is Lipschitz continuous.

This concludes the proof of the proposition. $\square$

We are ready to prove the main result of this paper.

Theorem 5.4. *Let $N_k \sim \tau_k^{-\alpha}$, $w_k = \tau_k^\beta$ and assume that $\alpha, \beta > 0$ and $\alpha + \beta = 1$, $\alpha > \frac{1}{2}$. Let Λ_t^k be as in (4.2) and $\bar{\Lambda}_t^k$ be as in (4.3). Let $t \in [0, T] \mapsto \Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$ be the Lipschitz curve obtained in Theorem 5.3 (we do not relabel the subsequence). Then Λ_t is a distributional solution to*

$$\partial_t \Lambda_t + \operatorname{div}(b \Lambda_t) = 0, \quad \text{in } (0, T) \times \Delta^{M-1}, \quad (5.11)$$

with initial condition Λ_0 , i.e.,

$$\int_0^T \int_{\Delta^{M-1}} (\partial_t \phi(t, \lambda) + D\phi(t, \lambda)b(\lambda)) d\Lambda_t(\lambda) dt = - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0(\lambda),$$

for every $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^M)$.

Proof. Let us fix $\phi \in C_c^\infty((-\infty, T) \times \mathbb{R}^M)$. By the assumptions on N_k and w_k and by Proposition 4.4, we have that

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt + \int_0^T \int_{\Delta^{M-1}} D\phi(t, \lambda)b(\lambda) d\bar{\Lambda}_t^k(\lambda) dt = - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0^k(\lambda) + \tilde{\rho}_k,$$

where

$$\tilde{\rho}_k \sim \tau_k^{\alpha+\beta-1} \mathcal{O}\left(\frac{\|D\psi\|_\infty}{N_k}\right) + \tau_k^{\alpha+\beta-1} \mathcal{O}(\|D\psi\|_\infty w_k) + \tau_k^{2\alpha-1} \mathcal{O}(\|D^2\psi\|_{L^\infty}) \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

Moreover, by Theorem 5.3, we have that

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t^k(\lambda) dt \rightarrow \int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t(\lambda) dt, \quad \text{as } k \rightarrow +\infty,$$

and, exploiting also Lemma 4.2,

$$\int_0^T \int_{\Delta^{M-1}} D\phi(t, \lambda)b(\lambda) d\bar{\Lambda}_t^k(\lambda) dt \rightarrow \int_0^T \int_{\Delta^{M-1}} D\phi(t, \lambda)b(\lambda) d\Lambda_t(\lambda) dt, \quad \text{as } k \rightarrow +\infty.$$

We have shown that

$$\int_0^T \int_{\Delta^{M-1}} \partial_t \phi(t, \lambda) d\Lambda_t(\lambda) dt + \int_0^T \int_{\Delta^{M-1}} D\phi(t, \lambda)b(\lambda) d\Lambda_t(\lambda) dt = - \int_{\Delta^{M-1}} \phi(0, \lambda) d\Lambda_0(\lambda),$$

concluding the proof of the theorem. $\square$

Remark 5.5. In the next section we recall that the continuity equation (5.11) has a unique solution. This implies that the limit curve $t \in [0, T] \mapsto \Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$ is independent of the subsequence extracted in Theorem 5.3. Hence, *a posteriori*, the convergence $\mathcal{W}_1(\Lambda_t^k, \Lambda_t) \rightarrow 0$ as $k \rightarrow +\infty$ holds for the whole sequence $k \rightarrow +\infty$.

6. REPLICATOR DYNAMICS

We conclude this section by recalling the connection between the continuity equation

$$\partial_t \Lambda_t + \operatorname{div}(b\Lambda_t) = 0, \quad \text{in } (0, T) \times \Delta^{M-1},$$

and the replicator dynamics. This will explain in which sense the replicator dynamics is a mean-field limit of the finite-population stochastic process described in Section 3.

The replicator dynamics is governed by the following system of ODEs:

$$\begin{cases} \frac{d}{dt} \lambda_{u_i}(t) = \lambda_{u_i}(t) ((A\lambda(t))_i - (A\lambda(t)) \cdot \lambda(t)), & i = 1, \dots, M, \quad t \in (0, T), \\ \lambda(0) = \lambda_0. \end{cases}$$

In a more compact form, using the vector field $b: \Delta^{M-1} \rightarrow \mathbb{R}^M$ introduced in (4.6),

$$\begin{cases} \frac{d}{dt} \lambda(t) = b(\lambda(t)), & t \in (0, T), \\ \lambda(0) = \lambda_0. \end{cases}$$

Let us denote by $\Psi: (0, T) \times \Delta^{M-1} \rightarrow \Delta^{M-1}$ the flow of the vector field b , *i.e.*, $\Psi(t, \lambda_0)$ is the solution of the ODE system above with initial condition λ_0 .

A first connection between the continuity equation and the replicator dynamics is given by the following result. This fact is known, see, *e.g.*, [23], Lemma 8.1.6.

Proposition 6.1. *Let $\Lambda_0 \in \mathcal{P}_1(\Delta^{M-1})$ and let*

$$\Lambda_t := \Psi(t, \cdot)_{\#} \Lambda_0 \in \mathcal{P}_1(\Delta^{M-1}), \quad t \in [0, T]. \quad (6.1)$$

Then Λ_t is a distributional solution to the continuity equation

$$\partial_t \Lambda_t + \operatorname{div}(b\Lambda_t) = 0, \quad \text{in } (0, T) \times \Delta^{M-1},$$

with initial condition Λ_0 .

In fact, (6.1) characterizes the solution of the continuity equation. See also [23], Proposition 8.1.8.

Theorem 6.2. *Let $\Lambda_0 \in \mathcal{P}_1(\Delta^{M-1})$. Let $t \in [0, T] \mapsto \Lambda_t \in \mathcal{P}_1(\Delta^{M-1})$ be a narrowly continuous path, being a distributional solution to the continuity equation*

$$\partial_t \Lambda_t + \operatorname{div}(b\Lambda_t) = 0, \quad \text{in } (0, T) \times \Delta^{M-1}.$$

Then

$$\Lambda_t = \Psi(t, \cdot)_{\#} \Lambda_0, \quad t \in [0, T].$$

The two results above explain the connection between the continuity equation and the replicator dynamics.

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DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

REFERENCES

- [1] J.M. Smith and G.R. Price, The logic of animal conflict. *Nature* **246** (1973) 15–18.
- [2] P.D. Taylor and L.B. Jonker, Evolutionarily stable strategies and game dynamics. *Math. Biosci.* **40** (1978) 145–156.
- [3] B. Düring, P. Markowich, J.-F. Pietschmann and M.-T. Wolfram, Boltzmann and Fokker–Planck equations modelling opinion formation in the presence of strong leaders. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **465** (2009) 3687–3708.
- [4] G. Toscani, Kinetic models of opinion formation. *Commun. Math. Sci.* **4** (2006) 481–496.
- [5] M. Dorigo and C. Blum, Ant colony optimization theory: a survey. *Theoret. Comput. Sci.* **344** (2005) 243–278.
- [6] J. Kennedy, Particle swarm optimization, in Encyclopedia of Machine Learning, edited by C. Sammut and G.I. Webb. Springer US, Boston, MA (2010) 760–766.
- [7] J.H. Holland, Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control, and Artificial Intelligence. University of Michigan Press, Ann Arbor (1975).
- [8] R. Carmona, Applications of mean field games in financial engineering and economic theory, in Mean Field Games. Agent Based Models to Nash Equilibria, edited by F. Delarue. AMS (2021) 165–218.
- [9] J. Hofbauer and K. Sigmund, Evolutionary Games and Population Dynamics. Cambridge University Press, Cambridge (1998).
- [10] P.A.P. Moran, The Statistical Process of Evolutionary Theory/ Clarendon Press, Oxford (1962).
- [11] M.A. Nowak, Evolutionary Dynamics: Exploring the Equations of Life. Belknap Press (2006).
- [12] T. Börgers and R. Sarin, Learning through reinforcement and replicator dynamics. *J. Econ. Theory* **77** (1997) 1–14.
- [13] F.A.C.C. Chalub, L. Monsaingeon, A.M. Ribeiro and M.O. Souza, Gradient flow formulations of discrete and continuous evolutionary models: a unifying perspective. *Acta Appl. Math.* **171** (2021) 24.
- [14] F.A.C.C. Chalub and M.O. Souza, From discrete to continuous evolution models: a unifying approach to drift-diffusion and replicator dynamics. *Theoret. Popul. Biol.* **76** (2009) 268–277.
- [15] F.A.C.C. Chalub and M.O. Souza, The frequency-dependent Wright–Fisher model: diffusive and non-diffusive approximations. *J. Math. Biol.* **68** (2014) 1089–1133.
- [16] C. Hilbe, Local replicator dynamics: A simple link between deterministic and stochastic models of evolutionary game theory. *Bull. Math. Biol.* **73** (2011) 2068–2087.
- [17] A. Traulsen, J.C. Claussen and C. Hauert, Coevolutionary dynamics: from finite to infinite populations. *Phys. Rev. Lett.* **95** (2005) 238701.

- [18] A. Traulsen, J.C. Claussen and C. Hauert, Coevolutionary dynamics in large, but finite populations. *Phys. Rev. E* **74** (2006) 011901.
- [19] J.-B. Casteras and L. Monsaingeon, Hidden dissipation and convexity for Kimura equations. *SIAM J. Math. Anal.* **55** (2023) 7361–7398.
- [20] S. Almi, M. Morandotti and F. Solombrino, A multi-step Lagrangian scheme for spatially inhomogeneous evolutionary games. *J. Evol. Equ.* **21** (2021) 2691–2733.
- [21] L. Ambrosio, M. Fornasier, M. Morandotti and G. Savaré, Spatially inhomogeneous evolutionary games. *Commun. Pure Appl. Math.* **LXXIV** (2021) 1353–1402.
- [22] M. Morandotti and F. Solombrino, Mean-field analysis of multipopulation dynamics with label switching. *SIAM J. Math. Anal.* **52** (2020) 1427–1462.
- [23] L. Ambrosio, N. Gigli and G. Savaré, Gradient flows in metric spaces and in the space of probability measures. *Lectures in Mathematics*, 2nd edn. ETH Zürich, Birkhäuser Verlag, Basel (2008).
- [24] L. Ambrosio and D. Trevisan, Well-posedness of Lagrangian flows and continuity equations in metric measure spaces. *Anal. PDE* **7** (2014) 1179–1234.
- [25] C. Villani, *Optimal Transport: Old and New*. Springer, Berlin (2008).



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