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THE BC-METHOD IN MULTIDIMENSIONAL SPECTRAL INVERSE PROBLEM: THEORY AND NUMERICAL ILLUSTRATIONS

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ABSTRACT. This work is devoted to numerical experiments for multidimensional Spectral Inverse Problems. We check the efficiency of the algorithm based on the BC-method, which exploits relations between Boundary Control Theory and Inverse Problems. As a test, the problem for an ellipse is considered. This case is of interest due to the fact that a field of normal geodesics loses regularity on a nontrivial separation set. The main result is that the BC-algorithm works quite successfully in spite of this complication. A theoretical introduction to the BC-method is included.

1. INTRODUCTION

1.1. STATEMENT

The paper deals with an approach to Inverse Problems based on Boundary Control Theory (the so-called "BC-method"; see [5–12]). We consider a spectral variant of this approach.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary $\Gamma \in C^\infty$ and ρ be a positive infinitely smooth function ("density"), $\rho \in C^\infty(\bar{\Omega})$. The operator $L := -\rho^{-1}\Delta$, Dom $L = H^2(\Omega) \cap H^1_0(\Omega)$ acting in the space $\mathcal{H} := L^2(\Omega; \rho dx)$ is selfadjoint. Let $\{\lambda_k\}_{k=1}^\infty; 0 < \lambda_1 < \lambda_2 \leq \ldots$ be its spectrum and $\{\phi_k(\cdot)\}_{k=1}^\infty$ be the eigenfunctions: $L\phi_k = \lambda_k\phi_k$, $(\phi_k, \phi_l) = \delta_k^l$. We denote by $\psi_k := \partial_\nu \phi_k|_\Gamma$, $\psi_k \in C^\infty(\Gamma)$, the traces of normal derivatives (with respect to the outward normal $\nu = \nu(\gamma)$, $\gamma \in \Gamma$). The set of pairs $\{\lambda_k, \psi_k(\cdot)\}_{k=1}^\infty$ is called the *Spectral Data* (SD) of L.

The spectral Inverse Problem (IP) is to recover ρ in Ω, using the given SD.

This statement goes back to the classical papers of M.G. Krein dealing with the one dimensional spectral inverse problem for an inhomogeneous string (see [18]). The multidimensional problem of recovering a potential $q(\cdot)$ in the Schrödinger operator $-\Delta+q$ via its spectral data was first investigated by Yu.M. Berezanskii [13]. In our statement we use the same type of inverse data to recover a density. This problem is more difficult in the following sense. The well known fact is that a second order elliptical operator induces a metric in Ω determined by its principal part. In the case of the Schrödinger

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operator this metric is known (Euclidean), whereas in our case the metric itself has to be found. In other words, we have to recover a main part of the operator in contrast to a recovering the lowest order term.

An important feature distinguishing 1-dim from multidimensional Inverse Problems is that the first are well-posed whereas multidimensional Inverse Problems turn out to be (strongly) ill-posed. In the framework of the BC-method this fact shows up in a different character of controllability of corresponding 1-dim and multidimensional dynamical systems. Namely, the 1-dim systems are exactly controllable whereas the multidimensional systems are only approximately controllable.

1.2. Goals, Contents and Results

As was shown in [5], a density ρ is uniquely determined by the SD. Moreover, a constructive procedure for recovering ρ was proposed. The recovery was first realized numerically in [12]. Let us remark that we did not find other numerical results concerning recovery through the Spectral Data in the multidimensional case in the literature.

Here we continue the numerical testing of the BC-method. By way of a test, the spectral Inverse Problem for an ellipse is considered. In contrast to the case of a circle [12], an ellipse possesses an essentially richer geometry for the field of geodesics starting from a boundary in the normal direction. The field loses regularity on a nontrivial separation set ("cut locus"), on which focusing effects are present. As was shown in [6, 7], this lack of regularity does not in principle hinder the use of the BC-method. To confirm this fact by direct numerical tests is a main goal of this work.

In the first part of the paper (2.1–4.2) an outline of the BC-method is set forth.

A density is recovered by means of the so-called Amplitude Formula (AF), which is a main tool of the method. By its nature the AF has a dynamical origin. It is connected with the dynamical system

$$\begin{cases} \rho\partial_t^2 u - \Delta u = 0, & \text{in } \Omega \times (0, \xi), \\ u|_{t=0} = \partial_t u|_{t=0} = 0, & \\ u|_{\Gamma\times[0,\xi]} = f, & \end{cases}$$

with Dirichlet boundary control f.

The solution $u = u^f(x, t)$ of the problem describes a wave generated in Ω by a boundary source. The basis of the AF is a kind of geometrical optics relation describing the propagation of breaks of wave fields u^f.

The AF allows us to recover the waves u^f in Ω via inverse data, the waves being expressed in terms of natural coordinates related to the ρ-metric. To introduce them we give the geometrical preliminaries in sections 2.1–2.3. The key objects here are the eikonal, semigeodesical coordinates, and a pattern.

A dynamical variant of the AF is derived in 3.1–3.5. The derivation is based on the property of controllability of the dynamical system.

In 4.1–4.2 we reformulate the AF in terms of spectral data, which gives a way to recover the density. The recovery procedure is described in 4.3. It forms the basis of the algorithm which is used for the numerical tests.

In sections 5.1–5.3 we propose a somewhat simpler variant of the algorithm for recovering a density, based on a conjecture related to the inversion of geometrical optics formulas. The conjecture looks natural from the physical point of view, but has not been fully justified up to this moment.

The BC-method was first proposed in [5]. For other approaches to the problem, see [21–23, 26].

The last part (6.1–6.4) is devoted to numerical testing. The experiment consists of three steps. At first the SD for the ellipse are calculated in terms of Mathieu functions. Then the SD are used to recover the field of rays in the ellipse: in the second step a global characteristic (pattern) is recovered, and in the third step the equidistant curves of a boundary are calculated. Results of recovery are demonstrated and discussed.

As a conclusion drawn from the results of the tests, we could say that the BC-method works satisfactorily even in the case of a nonregular field of geodesics. This is a reason to hope for its usefulness in applications.

2. Geometry

This part of the paper concerns some geometrical preliminaries. In sections 2.1–2.3 we introduce the basic objects used in the IP for any smooth compact Riemannian manifold with boundary. These objects are semi-geodesical coordinates (s.-g.c.), a pattern of a manifold, and images. Section 2.3 deals with the particular case of a conformally flat metric $\rho|dx|^2$ determined by a density. We consider the relations between the s.-g.c. (ρ-metric) and the Cartesian coordinates. These relations are applied to solve the IP: we demonstrate a way to recover the density ρ through the images of the Cartesian coordinate functions.

2.1. Semigeodesical coordinates (s.-g.c.)

Let Ω be a compact C^∞ Riemannian manifold with boundary Γ; dim $\Omega = n \geq 2$.

The function

$$\tau(x) := \operatorname{dist}(x, \Gamma), \quad x \in \Omega,$$

is said to be an eikonal and we set $T_* := \max_\Omega \tau(\cdot)$. Eikonal level sets $\Gamma^\xi := \{x \in \Omega \mid \tau(x) = \xi\}$, $0 \leq \xi \leq T_*$, are called *equidistant surfaces* of a boundary Γ. The corresponding family of subdomains $\Omega^\xi := \{x \in \Omega \mid \tau(x) < \xi\}$ is expanding with respect to ξ.

Let l_γ be a geodesic starting from a point $\gamma \in \Gamma$ in the normal direction and let $l_\gamma[0, s]$ be its segment of length $s \geq 0$. The second endpoint of the segment is denoted by $x(\gamma, s) \in l_\gamma$; for $s = 0$ we set $x(\gamma, 0) := \gamma$

A critical length $s = s_*(\gamma)$ is defined by conditions

$$(i)\ \tau\left(x(\gamma, s)\right) = s \text{ for any } 0 \leq s \leq s_*(\gamma);$$
$$(ii)\ \tau\left(x(\gamma, s)\right) < s \text{ for } s > s_*(\gamma).$$

So, if $s \leq s_*(\gamma)$, then the segment $l_\gamma[0, s]$ is the shortest geodesic connecting $x(\gamma, s)$ with Γ, whereas for $s > s_*(\gamma)$ this segment does not minimize a distance between the point and the boundary Γ. The point $x = x(\gamma, s_*(\gamma))$ is said to be *a separation point* on l_γ.

The set

$$\omega := \bigcup_{\gamma \in \Gamma} x(\gamma, s_*(\gamma))$$

is called *a separation set ("cut locus")* of the manifold Ω with respect to its boundary Γ (see [14, 15]). It is a closed set of zero volume:

$$\omega = \bar{\omega}; \quad \text{vol}\ \omega = 0. \tag{2.1}$$

Every point of Ω can be joined with Γ by the shortest normal geodesics. For any $x \in \Omega$ we define *a geodesic projection* $\text{pr}\, x$ as follows $\text{pr}\ x := \{\gamma \in \Gamma \mid \text{dist}(\gamma, x) = \text{dist}(\Gamma, x) = \tau(x)\}$. Thus, $\text{pr}\ x$ is the subset of Γ, which consists of the endpoints of shortest geodesics connecting x with the boundary. If $x \in \Omega \backslash \omega$, then its projection contains a unique point $\gamma(x)$. The pair $(\gamma(x), \tau(x))$ is called *the semigeodesical coordinates* of a point $x \in \Omega \backslash \omega$. Due to property (2.1) the s.-g.c. may be used on Ω "in the large", i.e. almost everywhere [15].

Length and volume elements in s.-g.c. (γ, τ) have the known form

$$ds^2 = d\tau^2 + g_{\mu\nu}(\gamma, \tau)\, d\gamma^\mu d\gamma^\nu;$$

$$d\Omega = \sqrt{\text{det} g_{\mu\nu}}\ d\gamma^1 d\gamma^2 \dots d\gamma^{n-1} d\tau = \beta(\gamma, \tau) d\Gamma d\tau, \tag{2.2}$$

where $\gamma^1, \gamma^2, \dots, \gamma^{n-1}$ are local coordinates on Γ,

$$\beta(\gamma, \tau) = \sqrt{\frac{\det g_{\mu\nu}(\gamma, t)}{\text{det} g_{\mu\nu}(\gamma, 0)}}; \quad \text{and} \quad d\Gamma = \sqrt{\det\ g_{\mu\nu}(\gamma, 0)}\ d\gamma^1 \dots d\gamma^{n-1}$$

is the canonical measure of a boundary; $\beta(\gamma, \tau) > 0$, $\beta(\gamma, 0) \equiv 1$ for $\gamma \in \Gamma$. Let us remark that the function β and surface element $d\Gamma$ do not depend on the local coordinates.

2.2. Pattern

Semigeodesical coordinates induce a canonical correspondence between $\Omega$and a subset of the cylinder $\Gamma \times [0, T_*]$. Let us define the map $i : x \mapsto (\gamma(x), \tau(x))$ from $\bar{\Omega} \backslash \omega$ into the cylinder. The set of values of s.-g.c.

$$\Theta := i(\bar{\Omega} \backslash \omega) \subset \Gamma \times [0, T_*]$$

is said to be *a pattern* of the manifold Ω.

To describe the structure of a pattern it is convenient to introduce a special partition of the boundary. For any fixed $\xi \in [0, T_*]$ the representation

$$\Gamma = \sigma_+^\xi \cup \sigma_\omega^\xi \cup \sigma_-^\xi, \tag{2.3}$$

where $\sigma_+^\xi := \text{pr}\ (\Gamma^\xi \backslash \omega)$, $\sigma_\omega^\xi := \text{pr}\ (\Gamma^\xi \cap \omega)$, $\sigma_-^\xi := \Gamma \backslash (\sigma_+^\xi \cup \sigma_\omega^\xi)$ is valid. The sets σ's in (2.3) possess the following properties, which can be derived from the definitions:

(i) For any positive $\xi < T_\omega := \text{dist}\ (\omega, \Gamma)$ one has $\Gamma^\xi \cap \omega = \emptyset$ what implies $\sigma_+^\xi = \Gamma$, $\sigma_\omega^\xi = \sigma_-^\xi = \emptyset$;

(ii) for $\xi \geq T_\omega$ the sets $\Gamma^\xi \backslash \omega$ are smooth (possibly, not connected) $(n-1)$-dimensional submanifolds in Ω. Thus, the regularity of Γ^ξ may be broken on a cut locus only. The map pr determines a diffeomorphism $\Gamma^\xi \backslash \omega$ on σ_+^ξ;

(iii) the sets σ's may be characterized in terms of (continuous) function $s_*(\cdot)$:

$$\begin{aligned} \sigma_+^\xi &= \{\gamma \in \Gamma \mid s_*(\gamma) > \xi\}, \\ \sigma_\omega^\xi &= \{\gamma \in \Gamma \mid s_*(\gamma) = \xi\}, \\ \sigma_-^\xi &= \{\gamma \in \Gamma \mid s_*(\gamma) < \xi\}. \end{aligned}$$

The sets $\sigma_\pm^\xi$ are open in Γ; σ_ω^ξ is a closed set. Note that the family σ_+^ξ is a decreasing family and that σ_-^ξ is an increasing one: for any $\xi' < \xi''$ one has $\sigma_+^{\xi'} \supset \sigma_+^{\xi''}$, $\sigma_-^{\xi'} \subset \sigma_-^{\xi''}$;

(iv) denoting $\Omega_\perp^\xi := \Omega \backslash \Omega^\xi$, one has the relation

$$\operatorname{dist}(\sigma_-^\xi, \Omega_\perp^\xi) > \xi \tag{2.4}$$

holding true for every ξ satisfying $T_\omega < \xi < T_*$;

(v) the "upper" border θ of a pattern, $\theta := \{(\gamma, \tau) \mid \gamma \in \Gamma, \tau = s_*(\gamma)\}$ has zero measure in $\Gamma \times [0, T_*]$:

$$\text{mes }_{d\Gamma d\tau}\theta = 0. \tag{2.5}$$

The obvious relations

$$\Omega\backslash\omega = \bigcup_{0 \le \xi < T_*} (\Gamma^\xi \backslash \omega), \quad i(\Gamma^\xi\backslash\omega) = \sigma_+^\xi \times \{\tau = \xi\}$$

lead to useful representations

$$\Theta = \bigcup_{0 \le \xi < T_*} \sigma_+^\xi \times \{\tau = \xi\}; \quad \theta = \bigcup_{0 \le \xi \le T_*} \sigma_\omega^\xi \times \{\tau = \xi\}; \tag{2.6}$$

describing a "horizontal" bundle of a pattern and its border. The complement $\Theta^c := (\Gamma \times [0, T_*]) \backslash (\Theta \cup \theta)$ may be represented in the form

$$\Theta^c = \bigcup_{0 \le \xi \le T_*} \sigma_-^\xi \times \{\tau = \xi\}. \tag{2.7}$$

The objects introduced above are illustrated on Fig. 1.

2.3. ρ-metric. Images

Let us return to the case under consideration (see 1.1). A density ρ determines the conformally flat metric

$$ds^2 = \rho |dx|^2,$$

which turns $\Omega \subset \mathbb{R}^n$ into a Riemannian manifold with boundary $\Gamma = \partial\Omega$ and intrinsic distance denoted by dist_ρ. A peculiarity of this situation is that there exists a global coordinate chart on Ω, i.e. the Cartesian coordinates $x^1, ..., x^n$. So, we have two coordinate systems on Ω: there are the s.-g.c. and the Cartesian coordinates. Let us discuss the relation between them.

Considering the eikonal $\tau = \tau(x)$, $\quad x = \{x^1, ..., x^n\}$ as a function of the Cartesian coordinates, one has the well known equality

$$|\nabla_x \tau|^2 = \rho, \quad \text{in } \Omega\backslash\omega. \tag{2.8}$$

Now we introduce the so-called images of functions. This notion will be useful for the IP.

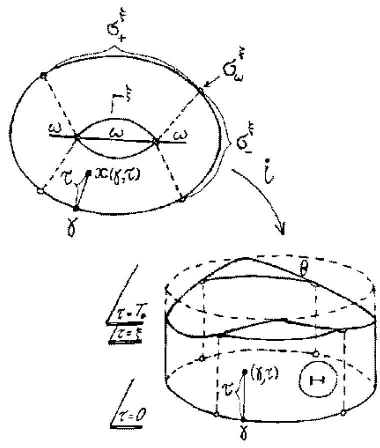

FIGURE 1. Geometrical objects of a Riemanian manifold.

The map $i : x(\gamma,\tau) \mapsto (\gamma,\tau)$ (see sec. 2.2) is bijective from $\Omega\backslash\omega$ to Θ; $x(\gamma,\tau) = i^{-1}((\gamma,\tau))$. For any function $y(\cdot)$ given on Ω we define the function

$$\widetilde{y} = \widetilde{y}(\gamma,\tau) := \begin{cases} \kappa(\gamma,\tau)\beta^{1/2}(\gamma,\tau)y(x(\gamma,\tau)) & (\gamma,\tau)\in\Theta \\ 0 & (\gamma,\tau)\in\Theta^c, \end{cases}$$

where

$$\kappa(\gamma,\tau) := [\rho(x(\gamma,\tau))/\rho(x(\gamma,0))]^{1/2-n/4}\,.$$

The function $\widetilde{y}$ is said to be *image* of y. An image is not defined on the boundary θ of a pattern, but in view of (2.5) $\widetilde{y}$ turns to be defined almost everywhere on $\Gamma\times[0,T_*]$.

2.4. Images and Reconstruction of Density

Let π_j be Cartesian coordinate functions,

$$\pi_j(x) = x^j \quad \text{in } \Omega, \quad j = 1, ..., n;$$

$\widetilde{\pi}_j$ be their images on $\Gamma\times[0,T_*]$. Introduce also the function $\mathbf{1} = \mathbf{1}(x) \equiv 1$, $x\in\Omega$ and its image $\widetilde{\mathbf{1}}$.

Let us explain in advance the roles of $\widetilde{\pi}_j$, $\widetilde{\mathbf{1}}$ in the IP. Suppose, that we can find $\widetilde{\pi}_j$, $\widetilde{\mathbf{1}}$ from the inverse data. Then one can recover the density ρ by the following scheme.

(i). The image $\widetilde{\mathbf{1}}$ determines the pattern:

$$\bar{\Theta} = \operatorname{supp}\widetilde{\mathbf{1}} \subset \Gamma\times[0,T_*].$$

(ii). The obvious relation

$$\frac{\widetilde{\pi}_j(\gamma,\tau)}{\widetilde{\mathbf{1}}(\gamma,\tau)} = \pi_j(x(\gamma,\tau)) = x^j(\gamma,\tau), \quad (\gamma,\tau)\in\Theta$$

permits us to recover the map $i^{-1} : \Theta \mapsto \Omega/\omega$ by the correspondence

$$(\gamma,\tau) \mapsto \{\frac{\widetilde{\pi}_1(\gamma,\tau)}{\widetilde{\mathbf{1}}(\gamma,\tau)}, ..., \frac{\widetilde{\pi}_n(\gamma,\tau)}{\widetilde{\mathbf{1}}(\gamma,\tau)}\},$$

which determines the relations between s.-g.c. and Cartesian coordinates.

(iii). The level surfaces Γ^ξ of the eikonal can be reconstructed by this map:

$$i^{-1} : \{(\gamma,\tau) \in \Theta | \tau = \xi\} \mapsto \Gamma^\xi \backslash \omega, \quad \xi \in (0, T_*).$$

The family $\{\Gamma^\xi\}$ obviously determines the eikonal $\tau(\cdot)$ in Ω/ω.

(iv). The density ρ is recovered in Ω/ω by means of (2.8); it may then be recovered on ω by continuity.

3. Dynamics

In this part of the paper we prepare a main tool of the procedure which solves the IP. It is the so-called Amplitude Formula, which connects the geometrical objects introduced above with a dynamical system determined by the density. It expresses the images of functions via the discontinuities of waves propagating in the system. A basic fact used to derive the Amplitude Formula is controllability of the dynamical system.

3.1. Dynamical System with Boundary Control

Let us consider the initial boundary-value problem

$$\rho \partial_t^2 u - \Delta u = 0, \quad (x,t) \in \Omega \times (0,\xi), \tag{3.1}$$

$$u|_{t=0} = \partial_t u|_{t=0} = 0, \tag{3.2}$$

$$u|_{\Gamma \times [0,\xi]} = f, \tag{3.3}$$

with a "control" $f = f(\gamma, t)$ of class $\mathcal{F}^\xi := L^2(\Gamma \times [0,\xi]; d\gamma d\tau)$ where $d\gamma$ is the Lebesgue measure on Γ. Its (generalized) solution ("wave") $u = u^f(x,t)$ has the following known properties:

(i) the map $f \mapsto u^f$ acts continuously from $\mathcal{F}^\xi$ into $C([0,\xi]; \mathcal{H})$, where $\mathcal{H} := L^2(\Omega; \rho dx)$ (see [19]);

(ii) (finite velocity of wave propagation) for any f one has $u^f(x,t) = 0$ for $t < \tau(x)$, i.e.

$$\text{supp } u^f(\,\cdot\,,\xi) \subset \bar{\Omega}^\xi. \tag{3.4}$$

Thus, Ω^ξ is a subdomain of Ω filled by waves up to a final moment $t = \xi$.

By virtue of *(i), (ii)* the operator $W^\xi : f \mapsto u^f(\,\cdot\,,\xi)$ acts continuously from $\mathcal{F}^\xi$ into $\mathcal{H}^\xi := \{y \in \mathcal{H} \mid \text{supp } y \in \bar{\Omega}^\xi\}$. For any positive $\xi < T_*$ this turns out to be injective:

$$\text{Ker } W^\xi = \{0\}, \ 0 < \xi < T_* \tag{3.5}$$

(see [1, 6]).

3.2. Wave Shaping and Controllability

Let us formulate a problem of Boundary Control Theory. Can one select a control f so as to obtain a prescribed shape of wave u^f? More precisely, the problem is to find a control $f \in \mathcal{F}^\xi$ such that

$$u^f(\,\cdot\,,\xi) = a(\,\cdot\,) \text{ in } \Omega \tag{3.6}$$

for given $a \in \mathcal{H}^\xi$ (see [2, 3, 25]). It is clear that this problem is equivalent to the equation

$$W^\xi f = a. \tag{3.7}$$

The relation (3.5) implies uniqueness of its solution for any $\xi < T_*$.

The set

$$\mathcal{U}^\xi := \text{Ran } W^\xi = \{u^f(\,\cdot\,,\xi) \mid f \in \mathcal{F}^\xi\}$$

is called reachable (in time ξ); $\mathcal{U}^\xi \subset \mathcal{H}^\xi$ for any $\xi > 0$ by virtue of (3.4).

The structure of $\mathcal{U}^\xi$ and properties of reachable sets provide the subject matter for Boundary Control Theory. The same problems turn out to be principal ones for the BC-method. The following result concerning problem (3.6) plays a central role in IP.

Theorem 3.1. *For any $\xi > 0$ the equality*

$$\text{clos } \mathcal{U}^\xi = \mathcal{H}^\xi. \tag{3.8}$$

is valid.

In Control Theory, the property (3.8) is called (approximate) controllability of the system (3.1)–(3.3). In accordance with it any function $a \in \mathcal{H}^\xi$ may be approximated by waves $u^f(\,\cdot\,,\xi)$.

Remark 3.2. The proof of (3.8) is based on the fundamental Holmgren's Uniqueness Theorem. Over a period of years there was a gap in the papers devoted to the BC-method connected with a lack of generalization to the case of nonanalytical ρ. Recent progress in this direction stimulated by L. Robbiano [24] and L. Hörmander [17] was crowned by a remarkable result of D. Tataru [27], which settled the question.

In addition to (3.8) it would be interesting to note that $\mathcal{U}^\xi \neq \mathcal{H}^\xi$ for $\xi < T_*$ (see [1]). This leads to unboundedness of $\left[W^\xi\right]^{-1}$. So in the multidimensional case the Wave Shaping Problem turns out to be ill-posed. In its turn, this leads to the ill-posedness of the multidimensional IP.

Let us discuss equality (3.8) in more detail. Introduce the (orthogonal) projector G^ξ in $\mathcal{H}$ onto $\mathcal{H}^\xi$. It cuts off functions on subdomain Ω^ξ filled by waves:

$$(G^\xi a)(x) = \begin{cases} a(x) & x \in \Omega^\xi \\ 0 & x \in \Omega^\xi_\perp \end{cases},$$

where $\Omega^\xi_\perp := \Omega \backslash \Omega^\xi$. Define also the "wave" projector P^ξ in $\mathcal{H}$ onto clos $\mathcal{U}^\xi$. The projector G^ξ is of geometrical origin connected with the ρ-metric in Ω, whereas the projector P^ξ is an intrinsic object of the dynamical system (3.1)–(3.3). Obviously, both are determined by density ρ, but the coincidence

$$P^\xi = G^\xi, \quad \xi > 0, \tag{3.9}$$

following from (3.8) is a deep result of Boundary Control Theory.

Let $G^\xi_\perp = \mathbf{Id}_\mathcal{H} - G^\xi$, $\quad P^\xi_\perp = \mathbf{Id}_\mathcal{H} - P^\xi$ be the complement projectors. As a corollary of (3.9) one has the equalities

$$\left(G^\xi_\perp a\right)(x) = \left(P^\xi_\perp a\right)(x) = \begin{cases} 0 & x \in \Omega^\xi \\ a(x) & x \in \Omega^\xi_\perp \end{cases} \tag{3.10}$$

for any $\xi > 0$.

One further corollary of (3.8) is the following. If $\{f_p\} \subset \mathcal{F}^\xi$ is an arbitrary complete system of controls, i.e.

$$\operatorname{clos}_{\mathcal{F}^\xi} \text{ Lin Span } \{f_p\}_{p=1}^\infty = \mathcal{F}^\xi,$$

then the relation

$$\operatorname{clos}_\mathcal{H} \text{ Lin Span } \{W^\xi f_p\}_{p=1}^\infty = \mathcal{H}^\xi, \tag{3.11}$$

is valid for any fixed $\xi > 0$.

Property (3.5) allows us to equip the space of controls $\mathcal{F}^\xi$ with a new scalar product

$$(f,g)_{\Phi^\xi} := (u^f(\,\cdot\,,\xi), u^g(\cdot,\xi))_\mathcal{H} = \left(W^\xi f, W^\xi g\right)_\mathcal{H}, \quad 0 < \xi < T_*.$$

A completion of $\mathcal{F}^\xi$ with respect to the norm $\|f\|_{\Phi^\xi} := (f,f)^{1/2}_{\Phi^\xi}$, which we denote by Φ^ξ, may be considered as a space of generalized controls. The space Φ^ξ turns out to be a rather exotic object containing distributions of arbitrarily high order. What is more, elements of Φ^ξ are not, generally speaking, distributions in L. Schwartz sense [1].

By definition of the norm $\|\cdot\|_{\Phi^\xi}$ the operator $\mathcal{W}^\xi : \Phi^\xi \mapsto \mathcal{H}^\xi$, Dom $\mathcal{W}^\xi = \mathcal{F}^\xi \subset \Phi^\xi$, $\mathcal{W}^\xi f = W^\xi f$, acts isometrically and may be extended to the whole space Φ^ξ by continuity as a unitary operator. For any fixed $\xi < T_*$, $a \in \mathcal{H}^\xi$ a solution of (3.6), (3.7) in the class of generalized controls is

$$f = \left[\mathcal{W}^\xi\right]^{-1} a \in \Phi^\xi. \tag{3.12}$$

The procedures solving the IP use the relations of the Geometrical Optics describing a propagation of breaks of wave fields in dynamical systems. Below we demonstrate these relations.

3.3. Propagation of Discontinuities in the System with Boundary Control and Geometrical Optics

Let the control f in (3.3) be a smooth function, $f \in C^\infty(\Gamma \times [0,\xi])$ and set $f(\cdot,t) \equiv 0$ for $t < 0$. In applications (acoustics, geophysics etc.) its value $f(\gamma,0)$, $\gamma \in \Gamma$ is called the *onset* of f. So, if $f(\cdot,0) \neq 0$, then control has a break at the initial moment $t = 0$.

It is well known that a discontinuous control generates a discontinuous wave, which has a break propagating along bicharacteristics of equation (3.1). At the final moment $t = \xi$ the break reaches a surface Γ^ξ. Its value ("amplitude") is determined by the onset of control and may be derived by means of geometrical optics (see [16, 28]): if $x = x(\gamma,\xi) \in \Gamma^\xi \backslash \omega$, $0 < \xi < T_*$, then

$$\lim_{\tau\to\xi-0} u^f(x(\gamma,\tau),\xi) = \left(\frac{\rho(x(\gamma,\xi))}{\rho(x(\gamma,0))}\right)^{n/4-1/2} \beta^{-1/2}(\gamma,\xi)f(\gamma,0), \quad \gamma\in\sigma_+^\xi. \tag{3.13}$$

The equality $u^f(x(\gamma,\xi+0),\xi)=0$ follows from (3.4). It gives a reason to call Γ^ξ a forward front of the wave $u^f(\,\cdot\,,\xi)$.

The formula (3.13) leads to the following useful relation. Let f, g be the smooth controls having the onsets $f(\cdot,0)$ and $g(\cdot,0)$, $g(\cdot,0)\neq 0$. Then the equality

$$\frac{f(\gamma,0)}{g(\gamma,0)} = \frac{\left(W^\xi f\right)(x(\gamma,\xi-0))}{\left(W^\xi g\right)(x(\gamma,\xi-0))}, \quad \gamma\in\sigma_+^\xi \tag{3.14}$$

is valid. Thus the break of a wave at its forward front is proportional to onset of a control.

3.4. Breaks and Geometrical Optics in Free Dynamics

Along with (3.13) we need a similar result concerning propagation of breaks in a dynamical system with a switched off control. Passing through a domain towards the boundary, these breaks carry information about the density. Amplitudes of the breaks link geometrical and dynamical parameters of the system.

The problem

$$\rho\partial_t^2 v - \Delta v = 0, \quad x\in\Omega, \quad t>0, \tag{3.15}$$

$$v|_{t=0}=0,\ \partial_t v|_{t=0} = y\in\mathcal{H}, \tag{3.16}$$

$$v(\cdot,t)|_\Gamma = 0, \tag{3.17}$$

describes an evolution of a dynamical system without control. Its (generalized) solution $v=v^y(x,t)$ possesses the known properties:

(i) for any $T>0$ function v^y belongs to the Sobolev class $H^1(\Omega\times(0,T))$;

(ii) the map $\mathcal{O}: y\mapsto \partial_\nu v^y|_{\Gamma\times[0,T]}$ acts continuously from $\mathcal{H}$ into $L^2(\Gamma\times[0,T])$ for any $T>0$ (see [20]), where ν is an outward normal on Γ;

(iii) (finite velocity of wave propagation) the solution $v^y(x,t)$ vanishes identically for (x,t) such that t is less than distance in ρ-metric between x and $\operatorname{supp} y$. In particular, if $\operatorname{supp} y\subset\Omega_\perp^\xi=\Omega\backslash\Omega^\xi$, then by virtue of (2.4) $\partial_\nu v^y(\gamma,t)\equiv 0$ for any (γ,t) belonging to some vicinity of the set $\sigma_-^\xi\times\{t=\xi\}$ on $\Gamma\times\{t\geq 0\}$. This result has a clear dynamical sense. Points $\gamma\in\sigma_-^\xi\subset\Gamma$ are separated from the initial perturbation by a distance more than ξ. That is why the wave generated by the perturbation reaches σ_-^ξ later than $t=\xi$.

The analog of (3.13) for the free system has the following form. For arbitrary $a\in C^\infty(\Omega)$ and fixed $\xi<T_*$ let us put in initial condition (3.16)

$$y=\left(G_\perp^\xi a\right)(x) := \begin{cases} 0 & x\in\Omega^\xi \\ a(x) & x\in\Omega_\perp^\xi \end{cases}$$

and denote a corresponding solution by v_ξ^a. The Cauchy Data $\{0, G_\perp^\xi a\}$ turns out to be discontinuous on Γ^ξ. A break of Data generates a break of wave, propagating along bicharacteristics of (3.15) and reaching the boundary Γ at the moment $t=\xi$. This break interacts not with the whole boundary Γ

but with its "illuminated zone" σ_+^ξ only. Indeed, $\operatorname{dist}_\rho(\sigma_+^\xi, \Omega_\perp^\xi) = \xi$ and the wave v_ξ^a is in time to reach $\sigma_+^\xi \subset \Gamma$. At the same time, the "shadow zone" σ_-^ξ is not covered by wave up to the moment $t = \xi$, since it is located far from $\Omega_\perp^\xi$ (see *(iii)*). Geometrical optics permits us to calculate an amplitude of the break observed on Γ:

$$\lim_{t \to \xi + 0} \left(\partial_\nu v_\xi^a\right)(\gamma, t) = \begin{cases} \kappa(\gamma,\xi)\beta^{1/2}(\gamma,\xi)a(x(\gamma,\xi)) & (\gamma,\xi) \in \sigma_+^\xi \times \{t = \xi\} \\ 0 & (\gamma,\xi) \in \sigma_-^\xi \times \{t = \xi\} \end{cases} \tag{3.18}$$

(definition of κ see in sec. 2.3).[1]

Let us remark that Geometrical Optics is not applicable for points $(\gamma, \xi) \in \sigma_\omega^\xi \times \{t = \xi\}$ lying on the border θ of the pattern (see (2.6)).

3.5. Amplitude Formula

Let us consider the left-hand side of (3.18) as a function of $(\gamma, \xi) \in \Gamma \times [0, T_*]$. Recalling, the representations (2.6), (2.7), and the definition of images, one can write (3.18) in the form of

$$\lim_{t \to \xi + 0} (\partial_\nu v_\xi^a)(\gamma, t) = \widetilde{a}(\gamma, \xi).$$

Indeed, the right-hand side of (3.18) is defined everywhere on $\Gamma \times [0, T_*]$ except the border θ of the pattern. In view of (3.7) it is not essential to define functions from $L^2(\Gamma \times [0, T_*])$.

Returning to the notations of 3.4 and using (3.10), we obtain

$$\lim_{t \to \xi + 0} \left(\mathcal{O} G_\perp^\xi a\right)(\gamma, t) = \widetilde{a}(\gamma, \xi) \ \text{ for almost all } (\gamma, \xi) \in \Gamma \times [0, T_*].$$

Using the property of controllability in the form (3.9) one can substitute $G_\perp^\xi$ for $P_\perp^\xi$:

$$\lim_{t \to \xi + 0} \left(\mathcal{O} P_\perp^\xi a\right)(\gamma, t) = \widetilde{a}(\gamma, \xi) \ \text{ for almost all } (\gamma, \xi) \in \Gamma \times [0, T_*]. \tag{AF}$$

This relation was first proposed in [6, 7]. It gives a dynamical representation of images of functions $a \in C^\infty(\Omega)$ via wave breaks propagating in the free system (3.15)–(3.17).

Setting $a = \mathbf{1}$ and $a = \pi_j$ in AF leads to the equalities

$$\lim_{t \to \xi + 0} \left(\mathcal{O} P_\perp^\xi \mathbf{1}\right)(\gamma, t) = \widetilde{\mathbf{1}}(\gamma, \xi), \tag{3.19}$$

$$\lim_{t \to \xi + 0} \left(\mathcal{O} P_\perp^\xi \pi_j\right)(\gamma, t) = \widetilde{\pi}_j(\gamma, \xi). \tag{3.20}$$

4. Inverse Problem

Here we apply the AF to solve the IP. We demonstrate that the left-hand sides of (3.19), (3.20) may be expressed via Spectral Data. In accordance with the results of section 2.4 this enables us to recover the density.

[1]in the paper [12] the factor κ is missed by mistake, which has no effects on the final results. Note that $\kappa \equiv 1$ in dimension $n = 2$.

4.1. Space $\widehat{\mathcal{H}}$. Spectral representation

Let $U : \mathcal{H} \mapsto \widehat{\mathcal{H}} := l^2$, $Uy = \widehat{y} := \{y^k\}_{k=1}^{\infty}$, $y^k = (y, \phi_k)_{\mathcal{H}}$ be the Fourier Transform corresponding to the operator L (see 1.1). The following objects in $\widehat{\mathcal{H}}$ are needed for the IP.

(i) Spectral representations of the functions $\mathbf{1}$, π_j may be found with the help of integration by parts:

$$\begin{array}{l} \widehat{\mathbf{1}} = U\mathbf{1} = \{\mathbf{1}^k\}_{k=1}^{\infty}, \quad \mathbf{1}^k = \frac{1}{\lambda_k} \int_\Gamma d\gamma \psi_k(\gamma); \\ \widehat{\pi}_j = U\pi_j = \{\pi_j^k\}_{k=1}^{\infty}, \quad \pi_j^k = \frac{1}{\lambda_k} \int_\Gamma d\gamma \pi_j(\gamma)\psi_k(\gamma);\ j = 1, 2, \ldots, n. \end{array} \tag{4.1}$$

(ii) The Fourier method for the problem (3.1)–(3.3) gives a spectral representation of a solution

$$\begin{array}{l} u^f(x,\xi) = \sum_{k=1}^{\infty} y^k c_k^f(\xi)\phi_k(x),\ c_k^f(\xi)\phi_k(x) \\ = \int_{\Gamma\times[0,T]} d\gamma dt \left[\frac{\sin\sqrt{\lambda_k}(\xi-t)}{\sqrt{\lambda_k}}\psi_k(\gamma)\right] f(\gamma,t) = (s_k^\xi, f)_{\mathcal{F}^\xi}; \\ s_k^\xi = s_k^\xi(\gamma,t) = \frac{\sin\sqrt{\lambda_k}(t)}{\sqrt{\lambda_k}}\psi_k(\gamma), \end{array}$$

or, in other words,

$$\widehat{u^f(\,\cdot\,,\xi)} = Uu^f(\,\cdot\,,\xi) = \{(s_k^\xi, f)_{\mathcal{F}^\xi}\}_{k=1}^{\infty}, \tag{4.2}$$

(see, for instance, [20]). Thus the Spectral Data determine the map $\widehat{W^\xi}$: $f \mapsto \widehat{\mathcal{H}}$, $\widehat{W^\xi} f := \widehat{u^f(\,\cdot\,,\xi)}$; $\widehat{W^\xi} = UW^\xi$ and the "reachable set"

$$\widehat{\mathcal{U}^\xi} := \operatorname{Ran} \widehat{W^\xi} = U\mathcal{U}^\xi \subset \widehat{\mathcal{H}}.$$

(iii) The projector $\widehat{P^\xi}$ in $\widehat{\mathcal{H}}$ onto clos $\widehat{\mathcal{U}^\xi}$, $\widehat{P^\xi} = UP^\xi U^{-1}$ is also determined by SD. Using any complete system of controls $\{f_p\}$ (see (3.11)) one can construct $\widehat{P^\xi}$ as the projector in $\widehat{\mathcal{H}}$ onto the subspace clos Linear Span $\{\widehat{W^\xi} f_p\}_{p=1}^{\infty}$.

(iv) Evolution of the system (3.15), (3.16) may also be described in spectral terms. The Fourier method leads to the well known representation

$$\widehat{v^y}(t) = Uv^y(\cdot,t) = \left\{\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}} y^k\right\}_{k=1}^{\infty}.$$

The map $\widehat{\mathcal{O}} := \mathcal{O}U^{-1} : \widehat{y} \mapsto \partial_\nu v^y|_{\Gamma\times\{t\ge 0\}}$ acts as

$$\left(\widehat{\mathcal{O}}\widehat{y}\right)(\gamma,t) = \sum_{k=1}^{\infty} y^k s_k^\xi = \sum_{k=1}^{\infty} y^k \left[\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\psi_k(\gamma)\right]. \tag{4.3}$$

By virtue of continuity of $\mathcal{O}$, this series converges in $L^2(\Gamma\times(0,T))$ for any $T > 0$.

4.2. Amplitude Formula in Terms of SD

Since

$$\mathcal{O}P_\perp^\xi a = \mathcal{O}U^{-1}UP_\perp^\xi U^{-1}Ua = \widehat{\mathcal{O}}\widehat{P^\xi}_\perp \widehat{a}$$

(where $\widehat{P^\xi}_\perp := \mathbf{Id}_{\widehat{\mathcal{H}}} - \widehat{P^\xi}$), the AF may be transformed into the form

$$\lim_{t\to\xi+0} \left(\widehat{\mathcal{O}}\widehat{P^\xi}_\perp \widehat{a}\right)(\gamma,t) = \widetilde{a}(\gamma,\xi) \text{ for almost all } (\gamma,\xi) \in \Gamma\times[0,T_*], \tag{$\widehat{AF}$}$$

which is adequate for a spectral representation.

The same transformation may be used in equalities (3.19), (3.20). We write them in a final form convenient for the Inverse Problem (see (3.18)):

$$\lim_{t\to\xi+0}\left(\widehat{O}\widehat{P^{\xi}}_{\perp}\widehat{\mathbf{1}}\right)(\gamma,t)=\widetilde{\mathbf{1}}(\gamma,\xi)$$

$$=\left\{\begin{array}{cc}\kappa(\gamma,\xi)\beta^{1/2}(\gamma,\xi) & (\gamma,\xi)\in\sigma_{+}^{\xi}\times\{t=\xi\}\\ 0 & (\gamma,\xi)\in\sigma_{-}^{\xi}\times\{t=\xi\}\end{array}\right., \tag{4.4}$$

and

$$\lim_{t\to\xi+0}\left(\widehat{O}\widehat{P^{\xi}}_{\perp}\widehat{\pi}_j\right)(\gamma,t)=\widetilde{\pi}(\gamma,\xi)$$

$$=\left\{\begin{array}{cc}\kappa(\gamma,\xi)\beta^{1/2}(\gamma,\xi)\pi_j(x(\gamma,\xi)) & (\gamma,\xi)\in\sigma_{+}^{\xi}\times\{t=\xi\}\\ 0 & (\gamma,\xi)\in\sigma_{-}^{\xi}\times\{t=\xi\}.\end{array}\right. \tag{4.5}$$

As was shown in 4.1, the limits in (4.4), (4.5) are completely determined by the Spectral Data. That is why $\widehat{\text{(AF)}}$ works in the IP permitting us to visualize the images of elements of $\mathcal{H}$.

4.3. Recovery Procedure

Let us recall that the problem is to recover a density ρ in Ω from given SD $\{\lambda_k,\psi_k(\cdot)\}_{k=1}^{\infty}$. In essence, the procedure of solving of the IP consists of two steps. The first is to construct all the objects entering the left-hand side of amplitude formulas (4.4), (4.5) via Spectral Data and to find images $\widetilde{\mathbf{1}}$, $\widetilde{\pi}_j$ of the unit and the coordinate functions. These images determine a correspondence between semigeodesical and Cartesian coordinates on Ω. In the second step, using the s.-g.c., one can recover the eikonal in Ω, and then find the density. Below we present the procedure in more detail.

(i) Find elements $\widehat{\mathbf{1}}$, $\widehat{\pi}_j$ in $\widehat{\mathcal{H}}=l^2$ by means of (4.1);

(ii) Find projections $\widehat{P}^{\xi}\widehat{\mathbf{1}}$, $\widehat{P}^{\xi}\widehat{\pi}_j$ on the subspace clos $\widehat{\mathcal{U}}^{\xi}$ formed by spectral representations $\widehat{u}^f(\xi)$ of waves. To construct these projections we choose a complete system of controls $\{f_p\}_{p=1}^{\infty}$ in $\mathcal{F}^{\xi}$, and find the system $\widehat{u}^{f_p}(\xi)\}_{p=1}^{\infty}$ in $\widehat{\mathcal{U}}^{\xi}$ (see (4.2)). Using the complete system in clos $\widehat{\mathcal{U}}^{\xi}$, one can find the projections. Then find the complementary projections $\widehat{P}_{\perp}^{\xi}\widehat{\mathbf{1}}$, $\widehat{P}_{\perp}^{\xi}\widehat{\pi}_j$ entering (4.4), (4.5).

(iii) Find the operator $\widehat{O}$ in (4.4), (4.5) by means of its representation (4.3).

All the objects in the left-hand side of (4.4), (4.5) have now been found. Using these formulas we determine images $\widetilde{\mathbf{1}}$, $\widetilde{\pi}_j$.

(iv) Recover the pattern Θ as the support of the image $\widetilde{\mathbf{1}}$ (see 2.4).

Using the images $\widetilde{\mathbf{1}}$, $\widetilde{\pi}_j$ on Θ, one can recover surfaces Γ^{ξ} in Ω, the eikonal, and, at last, the density ρ in accordance with the scheme (i)–(iv), in 3.1.

The Inverse Problem is solved.

5. Procedure II

There exists another variant of the recovery procedure based on the "front formula" (3.13). To use this variant we should accept a conjecture, which has not been justified rigorously as yet.

5.1. Onset Conjecture

Let us return to the Wave Shaping Problem (3.6), (3.7) and consider the cases $a = \mathbf{1}^\xi$ and $a = \pi_j^\xi$, where

$$\mathbf{1}^\xi(x) := G^\xi \mathbf{1} = \begin{cases} 1 & x \in \Omega^\xi \\ 0 & x \in \Omega^\xi_\perp \end{cases} \quad ; \quad \pi_j^\xi(x) := G^\xi \pi_j = \begin{cases} \pi_j(x) & x \in \Omega^\xi \\ 0 & x \in \Omega^\xi_\perp \end{cases} .$$

By virtue of (3.9) one has

$$\mathbf{1}^\xi = P^\xi \mathbf{1}, \quad \pi_j^\xi = P^\xi \pi_j .$$

Let q^ξ, p_j^ξ be the (generalized) solution of the problem given by (3.12):

$$q^\xi = \left[\mathcal{W}^\xi\right]^{-1} P^\xi \mathbf{1}; \quad p_j^\xi = \left[\mathcal{W}^\xi\right]^{-1} P^\xi \pi_j . \tag{5.1}$$

As to properties of controls q^ξ, p_j^ξ, the only fact known is that they belong to a very wide class Φ^ξ (see 3.2)[2]. To use them for the recovering of a density we accept the following conjecture.

Onset Conjecture. *For any $\xi \in (0, T_*)$ and for arbitrary open subset $\sigma \subset \sigma^\xi_+$ there exists a vicinity (on $\Gamma \times \{t \geq 0\}$) of the set $\sigma \times \{t = 0\}$, such that controls q^ξ, p_j^ξ are smooth functions in the vicinity. In particular, onsets $q^\xi(\cdot, 0)$, $p_j^\xi(\cdot, 0)$, are the smooth functions on σ.*

Let us note that for small T we have $\sigma^\xi_+ = \Gamma$ and the smoothness of controls q^ξ, p_j^ξ is the corollary of the conjecture. We could say in addition that the conjecture is true for the case of a ball $\Omega = \{x \in \mathbb{R}^n \mid |x| = 1\}$, $\rho \equiv 1$.

5.2. Onset Formula

The conjecture accepted above permits us to use q^ξ, p_j^ξ in the following way.

First, q^ξ, p_j^ξ must be expressed in spectral terms. We introduce the (isometrical) operator $\widehat{\mathcal{W}^\xi} : \Phi^\xi \mapsto \widehat{\mathcal{H}}$, $\widehat{\mathcal{W}^\xi} := U\mathcal{W}^\xi$. For any control $f \in \mathcal{F}^\xi \subset \Phi^\xi$ one has

$$\widehat{\mathcal{W}^\xi} f = U\mathcal{W}^\xi f = UW^\xi f = \widehat{W^\xi} f .$$

Since the operator $\widehat{W^\xi}$ is determined by the SD (see *(ii)*, sec. 4.1), the same is true for its extension $\widehat{\mathcal{W}^\xi}$. Therefore, the relations

$$q^\xi = \left[\widehat{\mathcal{W}^\xi}\right]^{-1} \widehat{P^\xi} \widehat{\mathbf{1}}; \quad p_j^\xi = \left[\widehat{\mathcal{W}^\xi}\right]^{-1} \widehat{P^\xi} \widehat{\pi}_j , \tag{5.2}$$

[2] If $T_\omega < \xi < T_*$, then q^ξ, p_j^ξ do not belong to $\mathcal{F}^\xi = L^2(\Gamma \times [0, \xi])$ (G. Lebeau; private communication).

following from (5.1), give a representation appropriate for the Inverse Problem.

In accordance with their definitions, the solutions q^ξ, p_j^ξ satisfy

$$\left(\mathcal{W}^\xi q^\xi\right)(\cdot,\xi) = \mathbf{1}^\xi, \quad \left(\mathcal{W}^\xi p_j^\xi\right)(\cdot,\xi) = \pi_j^\xi.$$

Consequently, the equality (3.14) together with the Onset Conjecture leads to an "Onset Formula":

$$\frac{p_j^\xi(\gamma,0)}{q^\xi(\gamma,0)} = \pi_j(x(\gamma,\xi)), \quad \gamma\in\sigma,\ 0<\xi<T_*,\ j=1,2,\ldots,n. \tag{5.3}$$

Its left-hand side may be found through the SD by means of (5.2).

5.3. Recovery of Density

Let us describe briefly the second procedure.

(i) Using the procedure of 4.3, find the pattern Θ.

(ii) Find controls q^ξ, p_j^ξ via (5.2).

(iii) Determine the coordinate functions by (5.3).

(iv) Recover the density via the coordinate functions (see 2.4).

6. Numerical Testing

The first experiment to test the numerical opportunities of the BC-method was realized by V.B. Filippov in [12]. It dealt with 2-dim spectral IP for a nonhomogeneous circle. To prepare the Spectral Data for a smooth nonconstant density, a conformal automorphism of the circle with $\rho\equiv 1$ was used. A peculiarity of this case is that the picture of normal geodesics is trivial, i.e., the same as in the homogeneous circle with $\rho = 1$. For further experiments an ellipse was chosen. On the one hand it admits a separation of variables permitting us to find the SD efficiently. On the other hand an ellipse possesses a richer geometry of geodesics. It has a nontrivial "cut locus", where focusing effects are present. In this concluding part of the paper we describe the numerical experiment for an ellipse.

The numerical algorithm used for testing exactly follows the procedures described in 4.3 and 5.3. The pattern Θ and the family of equidistant curves (wave fronts) Γ^ξ were reconstructed. As was shown above, these objects determine a density, therefore, we do not recover the density $\rho\equiv 1$ itself. The results show that the algorithms work satisfactorily, permitting us to recover the picture of an intrinsic geometry in a domain.

6.1. Preparing the Data

As a test, we consider the spectral problem for the operator L in the ellipse $\Omega = \left\{(x,y)\mid x^2/a^2+y^2/b^2<1\right\}\subset\mathbb{R}^2$, $a=2$, $b=1$ with the density $\rho\equiv 1$. In this case the cut locus is the segment $[-1/2,1/2]$ on the x-axis. Therefore $T_\omega = \operatorname{dist}(\omega,\Gamma) = 1/2$, i.e., for $\tau>1/2$ the field of normal geodesics loses regularity. The time T_* needed for waves to fill the whole domain is equal to 1.

The SD are found by separation of variables in terms of ordinary and modified Mathieu functions (see [4], chapter 16); the subset $\{\lambda_k,\psi_k(\cdot)\}_{k=1}^N$ of SD , $N\approx 1000$, was used.

To check the accuracy of the found SD we use the equalities

$$\|\widehat{\mathbf{1}}\|^2_{l^2} = \|\mathbf{1}\|^2_{L^2(\Omega)} = \text{mes}\ \Omega = \pi ab = 2\pi.$$

and similar equalities for coordinate functions $\widehat{\pi}_j$. This accuracy grows with the number N (see the table below).

N	error of $\|\widehat{1}\|_{l^2}$	error of $\|\widehat{\pi}_1\|_{l^2}$	error of $\|\widehat{\pi}_2\|_{l^2}$
100	3.6%	5.8%	7.9%
400	1.8%	2.6%	3.9%
960	1.1%	1.5%	2.5%

6.2. Spectral Representation of Projections $\widehat{P}^\xi\widehat{\mathbf{1}}$ and $\widehat{P}^\xi\widehat{\pi}_j$. Basis of Controls

A central point of the algorithm is to calculate the projections $\widehat{P}^\xi\widehat{\mathbf{1}}$, $\widehat{P}^\xi\widehat{\pi}_j$. Let $\{f_p\}_{p=1}^\infty$ be the complete system of controls in $\mathcal{F}^\xi$ (see *(ii)* in 4.3); let $\{\widehat{u}^{f_p}(\xi)\}_{p=1}^\infty$ be the spectral representation of the waves generated by these controls (see (4.2)). We seek the projections in the approximate form corresponding to the finite part of this system:

$$\widehat{P}^\xi\widehat{\mathbf{1}} \approx \sum_{p=1}^{M} d_p^{(0)}\widehat{u^{f_p}(\,\cdot\,,\xi)}; \quad \widehat{P}^\xi\widehat{\pi}_j \approx \sum_{p=1}^{M} d_p^{(j)}\widehat{u^{f_p}(\,\cdot\,,\xi)}.$$

To find the coefficients $d_p^{(j)}$, linear systems with the Gram matrix $\mathcal{G}$

$$\mathcal{G}_{lp} = \left(\widehat{u^{f_l}(\,\cdot\,,\xi)},\ \widehat{u^{f_p}(\,\cdot\,,\xi)}\right)_{l^2} = \sum_{k=1}^{N}(s_k^\xi, f_l)_{\mathcal{F}^\xi}(s_k^\xi, f_p)_{\mathcal{F}^\xi}$$

must be solved.

This linear system has the form

$$\mathcal{G}d^{(j)} = h^{(j)}, \quad d^{(j)} = \{d_p^{(j)}\}_{p=1}^M, \tag{6.1}$$

where the right-hand sides are

$$h^{(0)} = \{(\widehat{\mathbf{1}}, \widehat{u}^{f_p})_{l^2}\}_{p=1}^M, \quad h^{(j)} = \{(\widehat{\pi}_j, \widehat{u}^{f_p})_{l^2}\}_{p=1}^M.$$

A main difficulty is that the condition number of the matrix $\mathcal{G}$ grows rapidly with M. This corresponds to the ill-posedness of both the Wave Shaping Problem and the Inverse Problem. The calculations are also complicated by a slow convergence of the spectral expansions (4.1), (4.2) (see the table in the previous section).

The solutions of the system give the controls q^ξ, p_j^ξ:

$$q^\xi(\cdot,t) \approx \sum_1^M d_p^{(0)} f_p(\cdot,t), \quad p_j^\xi(\cdot,t) \approx \sum_1^M d_p^{(j)} f_p(\cdot,t), \tag{6.2}$$

which are used in the recovery procedure based on the Onset Formula (see sec. 5.3).

To simulate a complete system of controls $\{f_j\}$ (see (3.11)), we use the functions

$$f_{lr}(\gamma,t)=\chi_l^{\xi}(t)\zeta_r(\gamma),\quad l=1,\dots,Q,\ r=-R,-R+1,\dots,R,$$

where $t\in[0,\xi)$, $\gamma\in[0,2\pi)$ and

$$\chi_l^{\xi}(t)=\begin{cases}1 & \frac{(l-1)\xi}{Q}<t<\frac{l\xi}{Q}\\ 0 & \text{else}\end{cases}\quad\text{and}\quad \zeta_r(\gamma)=\begin{cases}\cos r\gamma & r\ge 0\\ \sin r\gamma & r<0.\end{cases}$$

Let $\Delta t:=\xi/Q$ be the duration of the "impulse" χ_l^{ξ}. The following complication arises in calculations. On the one hand, to obtain more precise approximations (6.2) one needs to decrease Δt (i.e., increase Q). On the other hand, the spectral expansions (4.2) for waves generated by short impulses converge more slowly when Δt decreases. A compromise value of Δt was found experimentally, $\Delta t\approx 0.1$ ($Q\approx 10\xi$). The number of angular harmonics $2R+1$ did not exceed 65. The total number $(2R+1)Q$ of controls f_{lr} used in the testing was bounded by 360.

6.3. Recovery of Pattern

The pattern Θ is recovered layer by layer in accordance with the representation (2.6). For fixed $\xi\in(0,1)$, we find the values $\widetilde{\mathbf{1}}(\cdot,\xi)$ of the image, calculating the limit in the left-hand side of (4.4). By doing so, we recover the set

$$\bar\sigma_+^{\xi}=\operatorname{supp}\widetilde{\mathbf{1}}(\cdot,\xi)$$

on Γ. Varying ξ, we recover the pattern by (2.6).

Typical results of calculations of $\widetilde{\mathbf{1}}(\cdot,\xi)$ are demonstrated in Fig. 2–4 for $\xi=0.1$, $\xi=0.3$, and $\xi=0.8$ (for the quarter of the ellipse). The boundary Γ of the ellipse is parametrized by $\gamma\in[0,2\pi)$.

Figures 2 and 3 corresponding to $\xi<T_\omega$ demonstrate the absence of a "shadow zone": $\sigma_+^{\xi}=\Gamma$, $\sigma_-^{\xi}=\emptyset$ (see *(i)* in 2.2).

An appearance of a shadow zone is demonstrated on Fig. 4, for $\xi=0.8>T_\omega$.

The parts of sets σ_+^{ξ} and σ_-^{ξ} lying in the first quadrant correspond to intervals $0.81<\gamma<\pi/2$ and $0<\gamma<0.81$. The calculations give $\sigma_+^{\xi}\approx(0.84,\pi/2)$ and, consequently, $\sigma_-^{\xi}\approx(0,0.84)$. Let us remark that the point $\gamma\in\sigma_\omega^{\xi}$ separating the shadow zone from the illuminated one was determined by means of an auxiliary fitting algorithm.

A final result for reconstruction of the pattern is given on Fig. 5.

To obtain Fig. 5, the subset $\{\lambda_k,\psi_k(\cdot)\}_{k=1}^{N}$, $N\sim 1000$, of the SD was used. The results show that the algorithm detects the presence of shadow zones σ_-^{ξ}. Thus the BC-algorithm enables us to reveal the effect of a breakdown of regularity of a normal geodesic field from the analysis of the Spectral Data. It is interesting to note that the effect may be detected even for $N\sim 40$. A rather good picture of the pattern may be obtained already for $N\sim 100$.

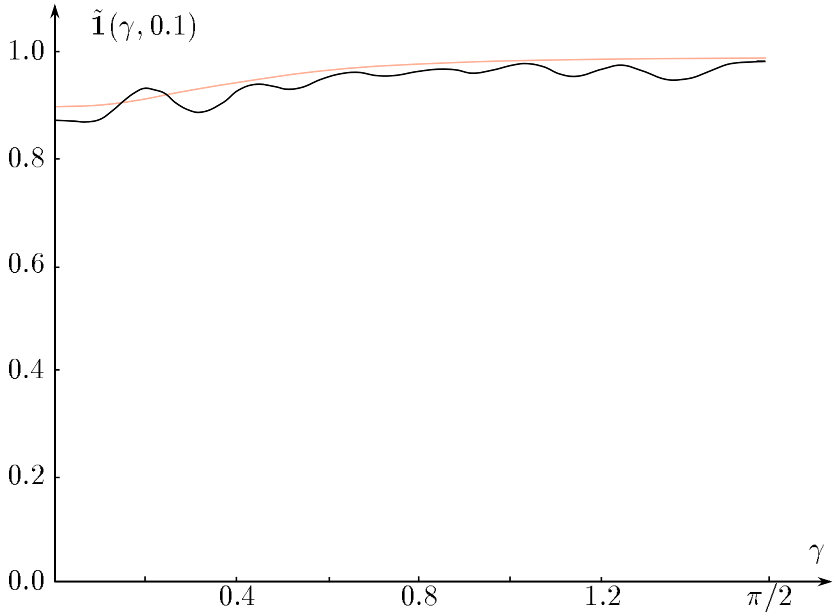

FIGURE 2

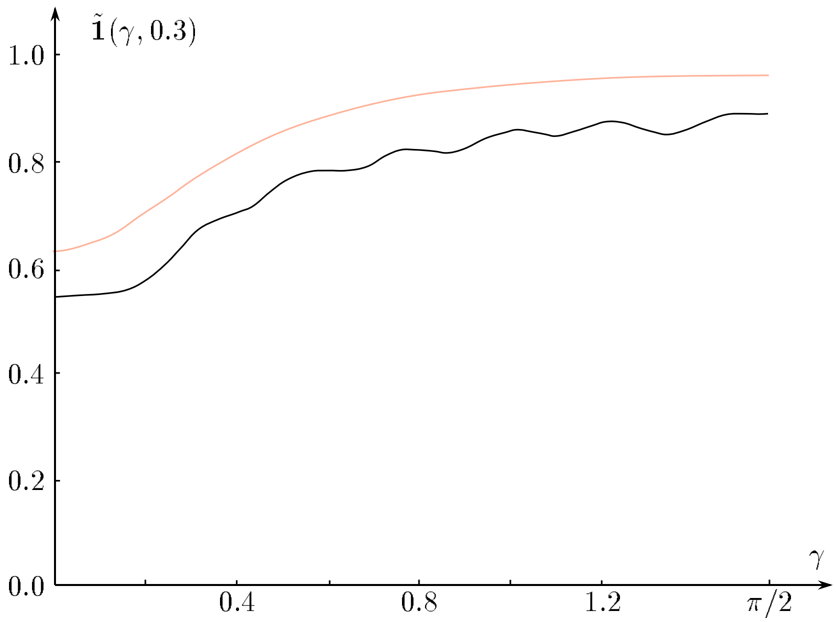

FIGURE 3

6.4. RECOVERY OF Γ^ξ

Both of the procedures for recovering the family $\{\Gamma^\xi\}$ (see 4.3, 5.3) were tested for $\xi = 0.1, 0.2, \dots, 0.8$. The first procedure, based on the Amplitude Formulas (4.4), (4.5), gives the results presented in Fig. 6.
The second variant, using the Onset Formula (5.3), gives the following picture (Fig. 7).

Comparing the results one can note that the AF seems to be more efficient for calculations. On the other hand, the algorithm based on the Onset Formula is simpler to program and takes less computational time. The

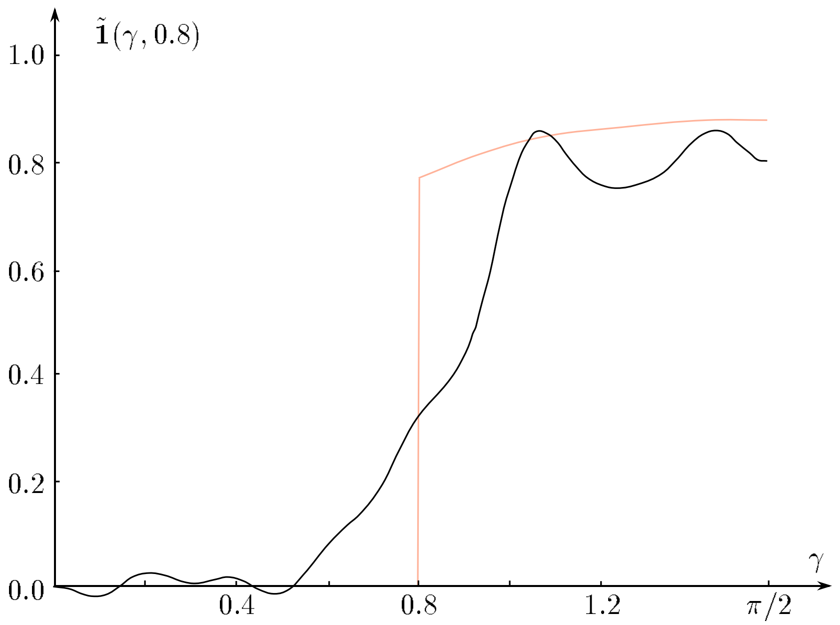

FIGURE 4

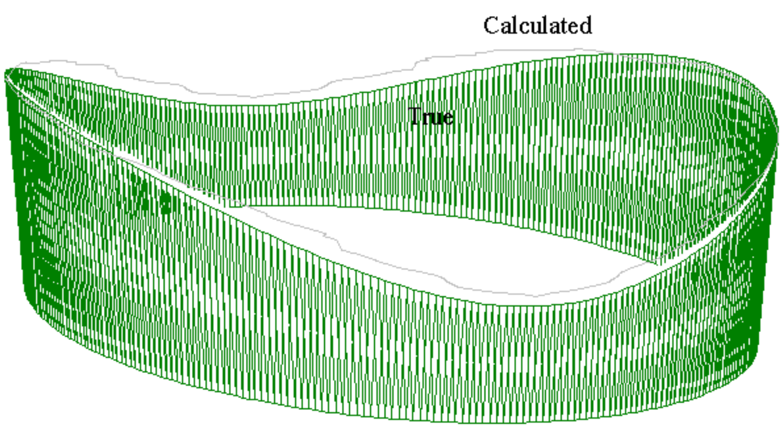

FIGURE 5

recovery turns out to be more difficult near the focusing points $x = \pm 1/2$, $y = 0$ belonging to the cut locus ω.

The time needed to recover Γ^{ξ} varies from a few seconds (for small $\xi = 0.1 - 0.3$ using the Onset Formula) to 10 min. (for $\xi = 0.6 - 0.9$ using the Amplitude Formula).

We would like to emphasize that we used the BC-algorithm in its "pure" form. Additional regularizations could essentially improve the results.

The numerical implementation of the algorithm was performed in Fortran 77 on an IBM PS/1 486DX 33Mhz computer.

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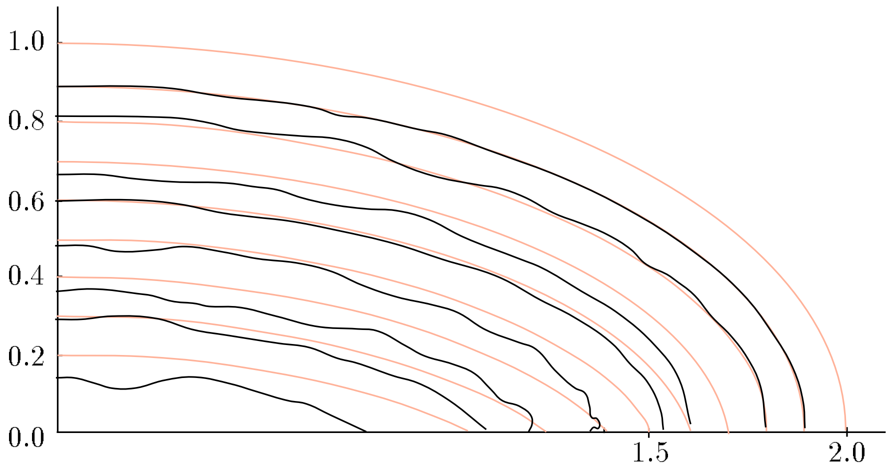

FIGURE 6. Recovery of Γ^ξ, $\xi = 0.1, 0.1, \ldots, 0.8$; the Amplitude Formula.

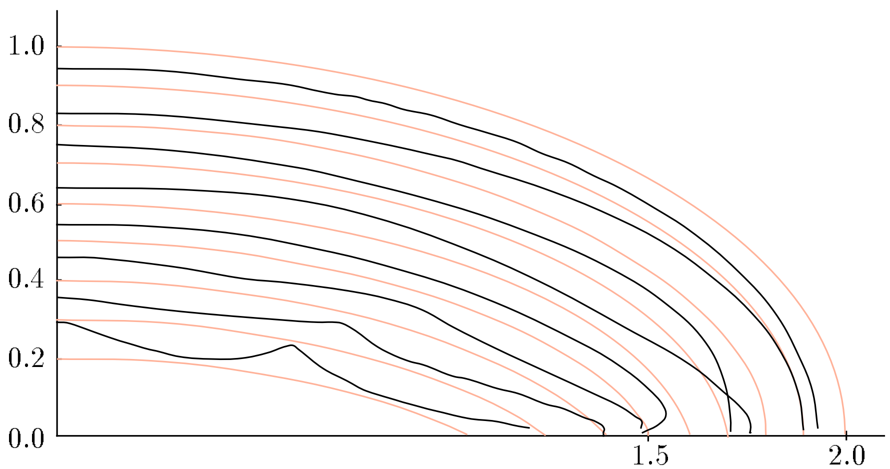

FIGURE 7. Recovery of Γ^ξ, $\xi = 0.1, 0.1, \ldots, 0.8$; the Onset Formula.

his program [12]. It was used as a test to debug our program. *We thank the referees, whose advices help us to improve the presentation.*

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