Correction to "PARTIAL EXACT CONTROLLABILITY
AND EXPONENTIAL STABILITY IN
HIGHER-DIMENSIONAL LINEAR THERMOELASTICITY"

WEIJIU LIU

In [1], as a consequence of (3.16), we were assuming that

\[ \|e^t(t)\| \leq e^{-\alpha_0 T} E(v, T), \]

in lines 5 and 6 on page 41. But, the correct inequality is

\[ \|e^t(t)\| \leq 2e^{-\alpha_0 |T-t|} E(v, T). \]

Consequently, the proof of Theorem 2.3 is not correct. We give here a corrected proof that needs however an additional smallness condition on \( \alpha_0 \).

We acknowledge Aissa Guesmia for pointing out this mistake to us.

We now give a corrected and weaker version of Theorem 2.3 with a complete proof. We adopt the notation and numbering of formulas, Theorems and Definitions from [1].

**THEOREM 2.3.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be given by (1.4) and (1.5), respectively, satisfying (2.13). Assume that the function \( a(x) \) satisfies (2.19) or (2.20). Suppose that

\[ \alpha \beta < \frac{\omega}{2\sigma} \]

Let \( T_0 \) be large enough so that

\[ 2e^{2\omega|\omega(T_0)|} < 1 - \frac{4\alpha^2 \beta^2 \pi^2 \omega^2}{\sigma^2}, \]

and \( T \geq T_0 \), where \( \omega \) is the constant in Theorem 2.1. Then for any \((u_0, u_1) \in W\), there exists a boundary control function \( \phi(x, T) \) with

\[ \phi \in (L^2(S_{\omega}))^n \]

such that the solution of (2.30) satisfies (2.31). Moreover, there exists a positive constant \( c \), independent of \((u_0, u_1)\), such that

\[ \|e^t(t)\| \leq c \|\phi, u_1\|_W. \]
Proof of Theorem 2.3. The main idea of the proof is first to construct a linear operator $A$ by using the Lamé system and the system of thermoelasticity, and then to show that $I - A$ is an isomorphism by using the uniform stabilization of these systems. Given $(v^0, v^1) \in W$, we consider the Lamé system

\begin{align}
&
\begin{align}
\rho \frac{\partial v}{\partial t} + (\lambda + \mu) \text{div}(v) v + \alpha \nabla \psi = -\alpha \nabla \xi \\
\psi &= 0 \\
w &= 0 \\
\frac{\partial w}{\partial t} + (\lambda + \mu) \text{div}(w) v + \alpha m \cdot \nu w + m \cdot \nu v' &= 0 \\
w(0) &= v(0), \\
v(t) &= v^0, \\
v'(t) &= v^1
\end{align}
\end{align}

in $Q$, on $\Sigma_1$, on $\Sigma_2$, on $\Omega$, (3.15)

which has a unique solution with $(v(t), v'(t)) \in C([0, T]; W)$.

Moreover, by Theorem 2.2 (note that we are assuming that $a(x)$ satisfies (2.19) or (2.20), and in this case, Theorem 2.2 on the stabilization of the Lamé system holds), there exists a positive constant $\omega$ such that

\[ E[v, t] \leq E[v, T] e^{-\omega(T-t)}, \quad \forall t \in [0, T] \]

(3.36)

where

\[
E(v, t) = \frac{1}{2} \int_{\Omega} \left[ \rho \left| \frac{\partial v}{\partial t} \right|^2 + (\lambda + \mu) \left| \text{div}(v) v \right|^2 + \left| \alpha \nabla \psi \right|^2 \right] dx
\]

Using the solution $v$ of (3.15), we then consider

\begin{align}
\begin{cases}
\xi' - \Delta \xi = \beta \text{div}(v') \\
\xi = 0 \\
\xi(0) = 0
\end{cases}
\end{align}

in $Q$, on $\Sigma$, in $\Omega$, (3.17)

and

\begin{align}
\begin{cases}
w'' - \mu \Delta w - (\lambda + \mu) \nabla \text{div} w + \alpha \nabla \psi = -\alpha \nabla \xi \\
v' - \Delta \psi + \beta \text{div} w' = 0 \\
w = 0 \\
\frac{\partial w}{\partial t} + (\lambda + \mu) \text{div}(w) v + \alpha m \cdot \nu w + m \cdot \nu v' &= 0 \\
w(0) &= v(0), \\
v'(0) &= v^0(0), \\
\psi(0) &= 0
\end{cases}
\end{align}

in $Q$, on $\Sigma$, on $\Sigma_1$, on $\Sigma_2$, on $\Omega$. (3.18)

Since

\[ \text{div}(v') \in (L^2(0, T; H^{-1}(\Omega)))^n, \]

it follows that (3.17) has a unique solution $\xi$ with

\[ \xi \in C([0, T]; L^2(\Omega) \cap L^2(0, T; H^1_0(\Omega))). \]
In addition, multiplying (3.17) by \( \xi \) and integrating over \( Q \), we obtain

\[
\frac{1}{2} \| \xi(T) \|_2^2 + \int_0^T \| \nabla \xi(t) \|_2^2 \, dt = \beta \int_0^T \int_{\Omega} \text{div}(\xi') \xi \, dx \, dt \\
\leq \beta \int_0^T \| \text{div}(\eta'(t)) \|_\infty \| \nabla \xi(t) \|_2 \, dt \quad \text{(use (2.22))} \\
\leq \frac{1}{2} \int_0^T \| \nabla \xi(t) \|_2^2 \, dt + \frac{\beta \sigma^2}{2} \int_0^T \| \eta'(t) \|_2^2 \, dt \quad \text{(use (3.36))} \\
\leq \frac{1}{2} \int_0^T \| \nabla \xi(t) \|_2^2 \, dt + \frac{\beta \sigma^2 \varepsilon}{\omega} E(v,T) \\
\leq \frac{1}{2} \int_0^T \| \nabla \xi(t) \|_2^2 \, dt + \frac{\beta \sigma^2 \varepsilon}{\omega} E(v,T),
\]

which implies

\[
\| \xi(T) \|_2^2 + \int_0^T \| \nabla \xi(t) \|_2^2 \, dt \leq \frac{2 \beta \sigma^2 \varepsilon}{\omega} E(v,T). \tag{3.29}
\]

On the other hand, since \( \xi \in L^2(0,T;H^1_0(\Omega)) \) and

\[
\int_{\Omega} \nabla \xi \xi = \int_T \xi \xi' \, dt = 0,
\]

then \( (0, -\alpha \nabla \xi, 0) \in L^2(0,T;\mathcal{H}) \). Thus, by the classical theory of semigroups, the nonhomogeneous problem (3.18) has a unique solution with \( \{w, w', \psi\} \in C([0,T];\mathcal{H}) \).

Moreover, the solution can be expressed as

\[
(w, w', \psi) = S(t) \{w(0), w'(0), \psi(0)\} + \int_0^t S(t - \tau) \{0, -\alpha \nabla \xi, 0\} \, d\tau,
\]

where \( S(t) \) denotes the strongly continuous semigroup of contractions generated by the thermoelastic system. By Theorem 2.1, we have

\[
\| S(t) \| \leq e^{(1 - \omega_0^2)/2}, \quad \forall t \geq 0,
\]

then we deduce from (3.19)

\[
E(w,v,t) \leq 2 \| S(t) \|^2 E(w,v,0) + 2 \left( \int_0^t \| S(t - \tau) \| \| \alpha \nabla \xi(\tau) \|_2 \, d\tau \right)^2 \\
\leq 2e^{1 - \omega_0^2} E(w,v,0) + 2e^{1 - \omega_0^2} \int_0^t e^{1 - \omega_0^2 \tau} \int_0^\tau \| \nabla \xi(\tau) \|_2^2 \, d\tau \, d\tau \\
\leq 2e^{1 - \omega_0^2} E(w,v,0) + \frac{4e^{1 - \omega_0^2} \beta \sigma^2 \varepsilon^2}{\omega^2} E(v,T).
\]

Set
\[ u = w - v, \quad \theta = \psi + \xi, \]
and
\[ \phi = -\mathbf{m} \cdot \mathbf{v}(u') + v'. \]
Then \( u, \theta \) satisfies
\[
\begin{align*}
  u'' &- \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \alpha \nabla \theta = 0 & \text{in } Q, \\
  \theta' &- \Delta \theta + j \text{div} w' = 0 & \text{in } Q, \\
  u &- 0 & \text{on } \Sigma, \\
  \frac{\partial u}{\partial n} &- (\lambda + \mu) \text{div}(u) v + \alpha m \cdot n u = \phi & \text{on } \Sigma_2, \\
  u(0) = u'(0) = 0, \quad \theta(0) = 0 & \text{in } \Omega, \\
  (u(T) - v^0, u'(T) - v') &- 0 & \text{in } \Omega.
\end{align*}
\]
We define an operator \( \Lambda \) by
\[
\Lambda(u^0, v^1) = (w(T), u'(T)).
\]
Then it is clear that \( \Lambda \) is a linear operator from \( W \) into \( W \). Moreover, by (3.16) and (3.20), we have
\[
\| \Lambda(u^0, v^1) \|_W \leq E(w, \psi, T) \leq 2 \kappa^{1/2} E(w, \psi, T) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3} \| (v, T) \|
\]
(note definition (2.1) of the norm of \( W \))
\[
\leq 2 \kappa^{1/2} E(w, \psi, 0) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3} \| (v, T) \|
\]
(since \( w(0) = v(0), \ u'(0) = v'(0), \ \psi(0) = 0 \))
\[
= 2 \kappa^{1/2} E(v, 0) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3} \| (v, T) \|
\]
\[
\leq \left[ 2 \kappa^{1/2} (w) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3} \right] \| (v, T) \|
\]
\[
= \left[ 2 \kappa^{1/2} (w) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3} \right] \| (v, T) \|
\]
Therefore,
\[
\| \Lambda \| \leq 2 \kappa^{1/2} (w) + \frac{4 \lambda^2 \beta^2 \sigma^2 e^2}{\omega^3}.
\]
Let \( T_0 \) be large enough so that (2.33) holds if \( T \geq T_0 \). Then \( \Lambda - I \) is an isomorphism from \( W \) onto \( W \). Thus, for any \( (u^0, u^1) \in W \), there exists a unique \( (v^0, v^1) \in W \) such that
\[
(u^0, u^1) = \Lambda(v^0, v^1) - (v^0, v^1)
\]
\[
= (w(T), u'(T)) - (v^0, v^1)
\]
\[
= (u(T), u'(T)).
\]
Consequently, we have constructed a control function $\phi = -m \cdot v(w' + v')$ solving the partial controllability problem (2.30).

On the other hand, multiplying the first equation of (3.15) by $v'$ and integrating over $Q$, we obtain

$$\int_{Q_0} m \cdot v' \psi d\Sigma = E(v, T) - E(v, 0).$$

(3.23)

Multiplying the first equation and second equation of (3.18) by $w'$ and $\frac{\alpha}{T} \psi$, respectively, and integrating over $Q$, we deduce from (3.19) and (3.20) that (the following $c$’s denoting various constants that may depend on $T$)

$$E(w, \psi, T) + \frac{\alpha}{T} \int Q \psi^2 d\sigma + \int_{Q_0} m \cdot v \cdot w' \psi d\Sigma$$

$$= E(w, \psi, 0) - \alpha \int Q w' \cdot \nabla \psi d\sigma$$

$$\leq E(w, \psi, 0) + E(v, T) + \int_0^T E(w, \psi, \tau) d\tau$$

$$\leq cE(w, \psi, 0) + cE(v, T).$$

(3.24)

Noting $E(v, 0) = E(w, \psi, 0)$, we deduce from (3.16), (3.23) and (3.24) that

$$\int_{Q_0} \frac{\partial F}{\partial \psi} d\Sigma = \int_{Q_0} m \cdot v' \psi + w' \psi d\Sigma \leq cE(v, T),$$

which, combined with (3.22), implies (2.34) since $\Lambda - I$ is an isomorphism from $W$ onto itself.

References