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Correction to "PARTIAL EXACT CONTROLLABILITY AND EXPONENTIAL STABILITY IN HIGHER-DIMENSIONAL LINEAR THERMOELASTICITY"

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In [1], as a consequence of (3.16), we were assuming that

$$\|v'(t)\|_0^2 \leq e^{1-wT}E(v,T),$$

in lines 5 and 6 on page 41. But, the correct inequality is

$$\|v'(t)\|_0^2 \leq 2e^{1-w(T-t)}E(v,T).$$

Consequently, the proof of Theorem 2.3 is not correct. We give here a corrected proof that needs however an additional smallness condition on $\alpha\beta$.

We acknowledge Aissa Guesmia for pointing out this mistake to us.

We now give a corrected and weaker version of Theorem 2.3 with a complete proof. We adopt the notation and numbering of formulas, Theorems and Definitions from [1].

Theorem 2.3. *Let Γ_1 and Γ_2 be given by (1.4) and (1.5), respectively, satisfying (2.12). Assume that the function $a(x)$ satisfies (2.19) or (2.20). Suppose that*

$$\alpha\beta < \frac{\omega}{2\sigma e}.$$

Let T_0 be large enough so that

$$2e^{2(1-\omega T_0)} < 1 - \frac{4\alpha^2\beta^2\sigma^2 e^2}{\omega^2}, \tag{2.33}$$

and $T \geq T_0$, where ω is the constant in Theorem 2.1. Then for any $(u^0, u^1) \in W$, there exists a boundary control function $\phi(x,t)$ with

$$\frac{\phi}{\sqrt{m\cdot\nu}} \in (L^2(\Sigma_2))^n$$

such that the solution of (2.30) satisfies (2.31). Moreover, there exists a positive constant c, independent of (u^0, u^1), such that

$$\| \frac{\phi}{\sqrt{m\cdot\nu}} \|_{(L^2(\Sigma_2))^n} \leq c\|(u^0,u^1)\|_W. \tag{2.34}$$

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Proof of Theorem 2.3. The main idea of the proof is first to construct a linear operator Λ by using the Lamé system and the system of thermoelasticity, and then to show that $I-\Lambda$ is an isomorphism by using the uniform stabilization of these systems.

Given $(v^0, v^1) \in W$, we consider the Lamé system

$$\begin{cases} v'' - \mu\Delta v - (\lambda+\mu)\nabla \mathrm{div}\ v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma_1, \\ \mu\dfrac{\partial v}{\partial \nu} + (\lambda+\mu)\mathrm{div}(v)\nu + am\cdot\nu v - m\cdot\nu v' = 0 & \text{on } \Sigma_2, \\ v(T) = v^0, \quad v'(T) = v^1 & \text{in } \Omega, \end{cases} \tag{3.15}$$

which has a unique solution with

$$(v(t), v'(t)) \in C([0,T]; W).$$

Moreover, by Theorem 2.2 (note that we are assuming that $a(x)$ satisfies (2.19) or (2.20), and in this case, Theorem 2.2 on the stabilization of the Lamé system holds), there exists a positive constant ω such that

$$E(v,t) \le E(v,T)e^{1-\omega(T-t)}, \qquad \forall t \in [0,T] \tag{3.16}$$

where

$$E(v,t) = \frac{1}{2}\int_\Omega \Big[|v'(x,t)|^2 + \mu|\nabla v(x,t)|^2 + (\lambda+\mu)|\mathrm{div} v(x,t)|^2\Big]dx + \frac{1}{2}\int_{\Gamma_2} am\cdot\nu|v(t)|^2 d\Gamma.$$

Using the solution v of (3.15), we then consider

$$\begin{cases} \xi' - \Delta\xi = \beta\mathrm{div}(v') & \text{in } Q, \\ \xi = 0 & \text{on } \Sigma, \\ \xi(0) = 0 & \text{in } \Omega, \end{cases} \tag{3.17}$$

and

$$\begin{cases} w'' - \mu\Delta w - (\lambda+\mu)\nabla\mathrm{div}\ w + \alpha\nabla\psi = -\alpha\nabla\xi & \text{in } Q, \\ \psi' - \Delta\psi + \beta\mathrm{div}\ w' = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ w = 0 & \text{on } \Sigma_1, \\ \mu\dfrac{\partial w}{\partial\nu} + (\lambda+\mu)\mathrm{div}(w)\nu + am\cdot\nu w + m\cdot\nu w' = 0 & \text{on } \Sigma_2, \\ w(0) = v(0), \quad w'(0) = v'(0), \quad \psi(0) = 0 & \text{in } \Omega. \end{cases} \tag{3.18}$$

Since

$$\mathrm{div}(v') \in (L^2(0,T; H^{-1}(\Omega)))^n,$$

it follows that (3.17) has a unique solution ξ with

$$\xi \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

In addition, multiplying (3.17) by ξ and integrating over Q, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\xi(T)\|_0^2 + \int_0^T \|\nabla\xi(t)\|_0^2 dt \\
&= \beta \int_0^T \int_\Omega \operatorname{div}(v')\xi dx dt \\
&\leq \beta \int_0^T \|\operatorname{div}(v'(t))\|_{-1} \|\nabla\xi(t)\|_0 dt \quad \text{(use (2.32))} \\
&\leq \frac{1}{2}\int_0^T \|\nabla\xi(t)\|_0^2 dt + \frac{\beta^2\sigma^2}{2}\int_0^T \|v'(t)\|_0^2 dt \quad \text{(use (3.16))} \\
&\leq \frac{1}{2}\int_0^T \|\nabla\xi(t)\|_0^2 dt + \beta^2\sigma^2 E(v,T)\int_0^T e^{1-\omega(T-t)} dt \\
&\leq \frac{1}{2}\int_0^T \|\nabla\xi(t)\|_0^2 dt + \frac{\beta^2\sigma^2 e}{\omega} E(v,T),
\end{aligned}$$

which implies

$$\|\xi(T)\|_0^2 + \int_0^T \|\nabla\xi(t)\|_0^2 dt \leq \frac{2\beta^2\sigma^2 e}{\omega} E(v,T). \tag{3.19}$$

On the other hand, since $\xi \in L^2(0,T;H_0^1(\Omega))$ and

$$\int_\Omega \nabla\xi dx = \int_\Gamma \nu\xi d\Gamma = 0,$$

then $(0,-\alpha\nabla\xi,0) \in L^1(0,T;\mathcal{H})$. Thus, by the classical theory of semigroups, the nonhomogeneous problem (3.18) has a unique solution with

$$(w,w',\psi) \in C([0,T];\mathcal{H}).$$

Moreover, the solution can be expressed as

$$(w,w',\psi) = S(t)(w(0),w'(0),\psi(0)) + \int_0^t S(t-\tau)(0,-\alpha\nabla\xi,0)d\tau,$$

where $S(t)$ denotes the strongly continuous semigroup of contractions generated by the thermoelastic system. By Theorem 2.1, we have

$$\|S(t)\| \leq e^{(1-\omega t)/2}, \qquad \forall t \geq 0,$$

then we deduce from (3.19)

$$\begin{aligned}
E(w,\psi,t) &\leq 2\|S(t)\|^2 E(w,\psi,0) + 2\Big(\int_0^t \|S(t-\tau)\| \|\alpha\nabla\xi(\tau)\|_0 d\tau\Big)^2 \\
&\leq 2e^{1-\omega t}E(w,\psi,0) + 2\alpha^2 \int_0^t e^{1-\omega(t-\tau)}d\tau \int_0^t \|\nabla\xi(\tau)\|_0^2 d\tau \\
&\leq 2e^{1-\omega t}E(w,\psi,0) + \frac{4\alpha^2\beta^2\sigma^2 e^2}{\omega^2}E(v,T).
\end{aligned} \tag{3.20}$$

Set

$$u = w - v, \quad \theta = \psi + \xi,$$

and

$$\phi = -m \cdot \nu(w' + v'), \tag{3.21}$$

then u, θ satisfies

$$\begin{cases} u'' - \mu\Delta u - (\lambda+\mu)\nabla \text{div } u + \alpha\nabla\theta = 0 & \text{in } Q, \\ \theta' - \Delta\theta + \beta \text{div } u' = 0 & \text{in } Q, \\ \theta = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_1, \\ \mu\dfrac{\partial u}{\partial \nu} + (\lambda+\mu)\text{div}(u)\nu + am\cdot\nu u = \phi & \text{on } \Sigma_2, \\ u(0) = u'(0) = 0, \quad \theta(0) = 0 & \text{in } \Omega, \\ u(T) = w(T) - v^0, \quad u'(T) = w'(T) - v^1 & \text{in } \Omega. \end{cases}$$

We define an operator Λ by

$$\Lambda(v^0, v^1) = (w(T), w'(T)).$$

Then it is clear that Λ is a linear operator from W into W. Moreover, by (3.16) and (3.20), we have

$$\begin{aligned} &\| \Lambda(v^0, v^1) \|_W^2 \\ &\text{(note definition (2.1) of the norm of } W) \\ &\leq E(w,\psi,T) \\ &\leq 2e^{1-\omega T}E(w,\psi,0) + \frac{4\alpha^2\beta^2\sigma^2e^2}{\omega^2}E(v,T) \\ &(\text{since } w(0) = v(0),\ w'(0) = v'(0),\ \psi(0) = 0) \\ &= 2e^{1-\omega T}E(v,0) + \frac{4\alpha^2\beta^2\sigma^2e^2}{\omega^2}E(v,T) \\ &\leq \left[2e^{2(1-\omega T)} + \frac{4\alpha^2\beta^2\sigma^2e^2}{\omega^2}\right]E(v,T) \qquad (\text{use } (3.16)) \\ &= \left[2e^{2(1-\omega T)} + \frac{4\alpha^2\beta^2\sigma^2e^2}{\omega^2}\right] \| (v^0, v^1) \|_W^2 . \end{aligned}$$

Therefore,

$$\| \Lambda \|^2 \leq 2e^{2(1-\omega T)} + \frac{4\alpha^2\beta^2\sigma^2e^2}{\omega^2}.$$

Let T_0 be large enough so that (2.33) holds if $T \geq T_0$. Then $\Lambda - I$ is an isomorphism from W onto W. Thus, for any $(u^0, u^1) \in W$, there exists a unique $(v^0, v^1) \in W$ such that

$$\begin{aligned} (u^0, u^1) &= \Lambda(v^0, v^1) - (v^0, v^1) \\ &= (w(T), w'(T)) - (v^0, v^1) \\ &= (u(T), u'(T)). \end{aligned} \tag{3.22}$$

Consequently, we have constructed a control function $\phi = -m \cdot \nu(w' + v')$ solving the partial controllability problem (2.30).

On the other hand, multiplying the first equation of (3.15) by v_i' and integrating over Q, we obtain

$$\int_{\Sigma_2} m \cdot \nu |v'|^2 d\Sigma = E(v,T) - E(v,0). \tag{3.23}$$

Multiplying the first equation and second equation of (3.18) by w_i' and $\frac{\alpha}{\beta}\psi$, respectively, and integrating over Q, we deduce from (3.19) and (3.20) that (the following c's denoting various constants that may depend on T)

$$\begin{aligned} & E(w,\psi,T) + \frac{\alpha}{\beta}\int_Q |\nabla\psi|^2 dxdt + \int_{\Sigma_2} m \cdot \nu \mid w' \mid^2 d\Sigma \\ & = E(w,\psi,0) - \alpha\int_Q w' \cdot \nabla\xi dxdt \\ & \leq E(w,\psi,0) + cE(v,T) + \int_0^T E(w,\psi,\tau)d\tau \\ & \leq cE(w,\psi,0) + cE(v,T). \end{aligned} \tag{3.24}$$

Noting $E(v,0) = E(w,\psi,0)$, we deduce from (3.16), (3.23) and (3.24) that

$$\int_{\Sigma_2} \frac{|\phi|^2}{m \cdot \nu} d\Sigma = \int_{\Sigma_2} m \cdot \nu \mid v' + w' \mid^2 d\Sigma \leq cE(v,T),$$

which, combined with (3.22), implies (2.34) since $\Lambda - I$ is an isomorphism from W onto itself. $\square$

References

[1] W.J. Liu: Partial exact controllability and exponential stability in higher dimensional linear thermoelasticity, *ESAIM: Control Optim. Calc. Var.,* **3**, 1998, 23–48.