

## OPTIMAL CONTROL OF AN ILL-POSED ELLIPTIC SEMILINEAR EQUATION WITH AN EXPONENTIAL NON LINEARITY

E. CASAS, O. KAVIAN, AND J.-P. PUEL

ABSTRACT. We study here an optimal control problem for a semilinear elliptic equation with an exponential nonlinearity, such that we cannot expect to have a solution of the state equation for any given control. We then have to speak of pairs (control, state). After having defined a suitable functional class in which we look for solutions, we prove existence of an optimal pair for a large class of cost functions using a non standard compactness argument. Then, we derive a first order optimality system assuming the optimal pair is slightly more regular.

### 1. INTRODUCTION

In this paper we are concerned with the optimal control of the following semilinear elliptic equation

$$\begin{cases} -\Delta y = e^y + u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n > 2$ , is a bounded open set,  $\Gamma$  being the boundary, which is assumed to be Lipschitz. The function  $u$  is the control, that will be taken in some space  $L^p(\Omega)$ , and  $y$  denotes the state in our control problem.

The equation (1.1) appears in several contexts: we refer for instance to D.A. Franck-Kamenetskii [5] for combustion theory in chemical reactors, S. Chandrasekhar [3] in the study of stellar structures. The equation (1.1) is ill-posed in the sense that there is no solution for some controls  $u$  and many solutions can be found for some others (see for instance I.M. Gelfand [7], M.G. Crandall and P.H. Rabinowitz [4], F. Mignot and J.P. Puel [8, 9], Th. Gallouët, F. Mignot and J.P. Puel [6]). Because of the term  $e^y$ , we need

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E. Casas: Dpt. Matemática Aplicada Y Ciencias de la Computación, E.T.S.I.I y T., Universidad de Cantabria, Av. Los Castros s/n 39005 Santander, Spain. E-mail: [casas@etsiso.macc.unican.es](mailto:casas@etsiso.macc.unican.es).

O. Kavian: Université de Versailles Saint-Quentin et Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France. E-mail: [kavian@math.uvsq.fr](mailto:kavian@math.uvsq.fr).

J.-P. Puel: Université de Versailles Saint-Quentin et Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France. E-mail: [jppuel@cmplx.polytechnique.fr](mailto:jppuel@cmplx.polytechnique.fr).

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to explain what we mean by a solution of (1.1). We will say that  $y$  is a solution of (1.1) if it belongs to the class of functions

$$Y = \{y \in H_0^1(\Omega) : e^y \in L^1(\Omega)\} \tag{1.2}$$

and it satisfies the equation in the distribution sense. Then the optimal control problem will be formulated in the following terms

$$(P) \begin{cases} \text{Minimize } J(y, u) := \int_{\Omega} L(x, y(x))dx + \frac{N}{p} \int_{\Omega} |u(x)|^p dx \\ (y, u) \in Y \times K \text{ satisfies (1.1),} \end{cases}$$

where  $K$  is a nonempty closed convex subset of  $L^p(\Omega)$ ,  $2 \leq p < +\infty$ ,  $N \geq 0$  and  $L : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function of class  $C^1$  with respect to the second variable and satisfying appropriate growth conditions which will be shown to be the following

$$\left| \frac{\partial L}{\partial y}(x, y) \right| \leq a_1(x) + \alpha_1(|y^-|^{\theta_1} + e^y), \tag{1.3}$$

$$L(x, y) \geq a_2(x) - \alpha_2(|y^-|^{\theta_2} + e^y), \tag{1.4}$$

for some  $a_1, a_2 \in L^1(\Omega)$ ,  $\alpha_1, \alpha_2 \geq 0$ ,  $\theta_1 = np/(n - 2p)$  if  $p < n/2$ , and  $0 \leq \theta_1 < +\infty$  if  $p \geq n/2$ , and  $1 \leq \theta_2 < p$ . For instance,  $y_d \in Y$  being given, a typical functional  $J$  would be

$$J(y, u) := \frac{1}{2} \int_{\Omega} |y(x) - y_d(x)|^2 dx + \frac{N}{p} \int_{\Omega} |u(x)|^p dx. \tag{1.5}$$

We should emphasize on the fact that one of the main difficulties of the problem is to choose an appropriate class of solutions such that (P) has a solution in that class.

The plan of the paper is as follows. In Section 2 we will analyze the state equation and we will establish the necessary background to study the control problem. The existence of a solution for (P) is studied in Sections 3 and 4 for the cases  $p > 2$  and  $p = 2$ , respectively. The case  $p = 2$  presents some difficulties and we will be able to prove the existence of an optimal control under some additional assumption on the function  $L$ . We should note that, as it seems to us, the case  $1 \leq p < 2$  cannot be treated with the techniques we use in this paper. Finally in Section 5 the optimality conditions will be investigated.

## 2. ANALYSIS OF THE STATE EQUATION

We start this section by establishing that any solution of (1.1) in the sense defined in Section 1 is a solution in the variational sense in  $H_0^1(\Omega)$ ; this requires to prove some regularity of the term  $e^y$ .

LEMMA 2.1. *Let  $y \in Y$  be a solution of (1.1), then  $e^y \in H^{-1}(\Omega)$ ,  $e^y z \in L^1(\Omega)$  for every  $z \in H_0^1(\Omega)$  and*

$$\int_{\Omega} \nabla y(x) \nabla z(x) dx = \int_{\Omega} [e^{y(x)} + u(x)] z(x) dx. \tag{2.1}$$

*Proof.* By the definition of a solution of (1.1), for all  $z \in C_c^\infty(\Omega)$  we have

$$\int_{\Omega} \nabla y(x) \nabla z(x) dx = \int_{\Omega} [e^{y(x)} + u(x)] z(x) dx. \tag{2.2}$$

Given  $z \in L^\infty(\Omega) \cap H_0^1(\Omega)$ , we can take a sequence  $\{z_k\}_{k=1}^\infty \subset C_c^\infty(\Omega)$ , with  $\|z_k\|_\infty \leq \|z\|_\infty + 1$ , for every  $k \in \mathbb{N}$  and  $z_k \rightarrow z$  in  $H_0^1(\Omega)$  and  $z_k \rightharpoonup z$  in  $L^\infty(\Omega)$ -w\*. Then for all  $k \geq 1$  we can replace  $z$  by  $z_k$  in (2.2) and pass to the limit when  $k \rightarrow \infty$  to obtain that the identity in (2.2) is also true for any  $z \in L^\infty(\Omega) \cap H_0^1(\Omega)$ .

Let us take now  $z \in H_0^1(\Omega)$  such that  $z \geq 0$  and set

$$T_k(z)(x) := \begin{cases} k & \text{if } z(x) > k, \\ z(x) & \text{if } 0 \leq z(x) \leq k. \end{cases}$$

Then  $T_k(z) \in L^\infty(\Omega) \cap H_0^1(\Omega)$  and

$$\int_{\Omega} \nabla y(x) \nabla T_k(z)(x) dx = \int_{\Omega} [e^{y(x)} + u(x)] T_k(z)(x) dx \quad \forall k \in \mathbb{N}.$$

Since  $T_k(z) \rightarrow z$  in  $H_0^1(\Omega)$ , the only trouble to pass to the limit in this identity comes from the term  $e^y z_k$ . As  $T_k(z) \geq 0$  and  $e^y > 0$ , then from the monotone convergence theorem, taking into account that  $T_k(z)(x) \uparrow z(x)$  for almost all  $x \in \Omega$ , we deduce

$$\begin{aligned} \int_{\Omega} e^{y(x)} z(x) dx &= \lim_{k \rightarrow +\infty} \int_{\Omega} e^{y(x)} T_k(z)(x) dx \\ &= \lim_{k \rightarrow +\infty} \left\{ \int_{\Omega} \nabla y(x) \nabla T_k(z)(x) dx - \int_{\Omega} u(x) T_k(z)(x) dx \right\} \\ &= \int_{\Omega} \nabla y(x) \nabla z(x) dx - \int_{\Omega} u(x) z(x) dx < +\infty. \end{aligned}$$

Next, for a general  $z \in H_0^1(\Omega)$ , we notice that  $z = z^+ - z^-$  with  $z^+, z^- \in H_0^1(\Omega)$ . This proves that (2.1) is satisfied. Moreover  $e^y \in H^{-1}(\Omega)$  and (2.1) shows that for all  $z \in H_0^1(\Omega)$

$$\langle e^y, z \rangle = \int_{\Omega} e^{y(x)} z(x) dx.$$

□

Now we state a very important identity to derive some estimates, in terms of  $u$ , on the solution of (1.1).

**THEOREM 2.2.** *Let  $u \in L^2(\Omega)$  and assume that  $y \in H^2(\Omega)$  is a solution of (1.1). Then for any  $x_0 \in \mathbb{R}^n$  we have*

$$\begin{aligned} &\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla y(x)|^2 dx + \frac{1}{2} \int_{\Gamma} |\nabla y(\sigma)|^2 [\nu(\sigma) \cdot (\sigma - x_0)] d\sigma \\ &= n \int_{\Omega} (e^{y(x)} - 1) dx - \int_{\Omega} u(x) [(x - x_0) \cdot \nabla y(x)] dx, \end{aligned} \tag{2.3}$$

where  $\nu(\sigma)$  denotes the outward unit normal vector to  $\Gamma$  at the point  $\sigma$ .

*Proof.* Since  $y \in H^2(\Omega)$ , then  $e^y = -\Delta y - u \in L^2(\Omega)$ . Therefore we can multiply the equation (1.1) by any function of  $L^2(\Omega)$  and make the integration in  $\Omega$ . In particular we can take  $(x - x_0) \cdot \nabla y(x) \in L^2(\Omega)$  as this function:

$$\int_{\Omega} [-\Delta y][(x - x_0) \cdot \nabla y] = \int_{\Omega} e^y [(x - x_0) \cdot \nabla y] dx + \int_{\Omega} u [(x - x_0) \cdot \nabla y] dx. \tag{2.4}$$

Let us make an integration by parts in the first integral

$$\begin{aligned} & \int_{\Omega} [-\Delta y][(x - x_0) \cdot \nabla y] \\ &= \int_{\Omega} \nabla y \nabla [(x - x_0) \cdot \nabla y] - \int_{\Gamma} \partial_{\nu} y(\sigma) [(\sigma - x_0) \cdot \nabla y(\sigma)] d\sigma \\ &= \sum_{i,j=1}^n \int_{\Omega} \partial_{x_i} y \partial_{x_i} [(x_j - x_{0j}) \partial_{x_j} y] dx - \int_{\Gamma} \partial_{\nu} y(\sigma) [(\sigma - x_0) \cdot \nabla y(\sigma)] d\sigma \\ &= \sum_{i=1}^n \int_{\Omega} |\partial_{x_i} y|^2 dx + \sum_{i,j=1}^n \int_{\Omega} \partial_{x_i} y (x_j - x_{0j}) \partial_{x_i x_j}^2 y dx \\ &\quad - \int_{\Gamma} \partial_{\nu} y(\sigma) [(\sigma - x_0) \cdot \nabla y(\sigma)] d\sigma \\ &= \sum_{i=1}^n \int_{\Omega} |\partial_{x_i} y|^2 dx + \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} (x_j - x_{0j}) \partial_{x_j} (\partial_{x_i} y)^2 dx \\ &\quad - \int_{\Gamma} \partial_{\nu} y(\sigma) [(\sigma - x_0) \cdot \nabla y(\sigma)] d\sigma. \end{aligned}$$

Now we can integrate by parts in the last relation, taking into account that for  $\sigma \in \Gamma$  we have  $\nabla y(\sigma) = \pm |\nabla y(\sigma)| \nu(\sigma)$ , then  $(\sigma - x_0) \cdot \nabla y(\sigma) = \pm |\nabla y(\sigma)| \nu(\sigma) \cdot (\sigma - x_0)$  and  $\partial_{\nu} y(\sigma) = \nabla y(\sigma) \cdot \nu(\sigma) = \pm |\nabla y(\sigma)|$ , we get

$$\begin{aligned} & \int_{\Omega} [-\Delta y][(x - x_0) \cdot \nabla y] = \sum_{i=1}^n \int_{\Omega} |\partial_{x_i} y|^2 dx \\ &+ \frac{1}{2} \sum_{i,j=1}^n \left\{ - \int_{\Omega} (\partial_{x_i} y)^2 + \int_{\Gamma} (x_j - x_{0j}) \nu_j(x) (\partial_{x_i} y)^2 d\sigma \right\} \\ &\quad - \int_{\Gamma} \partial_{\nu} y(\sigma) [(\sigma - x_0) \cdot \nabla y(\sigma)] d\sigma \\ &= \left(1 - \frac{n}{2}\right) \sum_{i=1}^n \int_{\Omega} |\nabla y|^2 dx - \frac{1}{2} \int_{\Gamma} [\nu(\sigma) \cdot (\sigma - x_0)] |\nabla y(\sigma)|^2 d\sigma. \end{aligned} \tag{2.5}$$

In order to handle the first term in the right hand side of (2.4), for  $k \geq 1$  set

$$T_k(y)(x) := \begin{cases} +k & \text{if } y(x) > +k, \\ y(x) & \text{if } -k \leq y(x) \leq k, \\ -k & \text{if } y(x) < -k. \end{cases}$$

One checks easily that  $T_k(y) \rightarrow y$  in  $H_0^1(\Omega)$  and a.e.,  $e^{T_k(y)} \rightarrow e^y$  in  $L^2(\Omega)$  and a.e. On the other hand  $e^{T_k(y)} \nabla T_k(y) \rightarrow e^y \nabla y$  in  $L^1(\Omega)$ ; but as  $\nabla(e^{T_k(y)}) = e^{T_k(y)} \nabla T_k(y)$ , we conclude that  $e^y \nabla y = \nabla(e^y - 1)$  in  $L^1(\Omega)$ . Therefore

$$\int_{\Omega} e^y \nabla y \cdot \Phi dx = - \int_{\Omega} (e^y - 1) \nabla \cdot \Phi dx$$

for all  $\Phi \in (C^1(\mathbb{R}^n))^n$ , and we may write:

$$\begin{aligned} \int_{\Omega} e^y [(x - x_0) \cdot \nabla y] dx &= \sum_{j=1}^n \int_{\Omega} (x_j - x_{0j}) e^y \partial_{x_j} y dx \\ &= \sum_{j=1}^n \int_{\Omega} (x_j - x_{0j}) \partial_{x_j} [e^y - 1] dx - \sum_{j=1}^n \int_{\Omega} [e^y - 1] dx = -n \int_{\Omega} [e^y - 1] dx. \end{aligned} \tag{2.6}$$

Putting together (2.5) and (2.6) in (2.4), we get (2.3). □

LEMMA 2.3. *Let us assume that  $u \in L^2(\Omega)$ ,  $\Omega$  is star-shaped with respect to some interior point  $x_0$  (i.e.  $(\sigma - x_0) \cdot \nu(\sigma) \geq 0$  for  $\sigma \in \Gamma$ ) and  $y \in H_0^1(\Omega)$  satisfies  $\Delta y \in L^2(\Omega)$ . Then we have*

$$\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla y(x)|^2 dx \leq \int_{\Omega} \Delta y [(x - x_0) \cdot \nabla y] dx. \tag{2.7}$$

*Proof.* Let us consider a sequence of bounded open sets  $\Omega_1 \supset \Omega_2 \supset \dots \supset \bar{\Omega}$ , with  $\bigcap_{j=1}^{\infty} \Omega_j = \bar{\Omega}$ , with  $\Gamma_j = \partial\Omega_j$  of class  $C^{1,1}$  and such that all of them are star-shaped with respect to  $x_0$ . For each  $j$  we consider the problem

$$\begin{cases} -\Delta z = (e^y + u)\chi_{\Omega} & \text{in } \Omega_j, \\ z = 0 & \text{on } \Gamma_j, \end{cases}$$

where  $\chi_{\Omega}$  is the characteristic function of  $\Omega$ . This linear problem has a unique solution  $y_j \in H_0^1(\Omega_j) \cap H^2(\Omega_j)$ . We extend each  $y_j$  and  $y$  to  $\Omega_1$  by zero, thus  $y_j, y \in H_0^1(\Omega_1)$ . From the above equation we get

$$\int_{\Omega_1} |\nabla y_j|^2 dx = \int_{\Omega_j} |\nabla y_j|^2 dx = \int_{\Omega_j} [e^y + u] y_j dx \leq \|e^y + u\|_{L^2(\Omega)} \|y_j\|_{L^2(\Omega)},$$

which proves the boundedness of  $\{y_j\}_{j=1}^{\infty}$  in  $H_0^1(\Omega_1)$ . By taking a subsequence we can assume that  $y_j \rightarrow \tilde{y}$  weakly in  $H_0^1(\Omega_1)$  and  $y_j(x) \rightarrow \tilde{y}(x)$  for almost all  $x \in \Omega_1$ . Obviously we have that  $\tilde{y}(x) = 0$  for  $x \in \Omega_1 \setminus \bar{\Omega}$ . Then we have that  $\tilde{y} = 0$  on  $\Gamma$  and  $-\Delta \tilde{y} = \lim_{j \rightarrow \infty} -\Delta y_j = e^y + u$  in  $\Omega$ . This leads to the equality  $\tilde{y} = y$ . Moreover

$$\begin{aligned} \int_{\Omega_1} |\nabla y|^2 dx &\leq \liminf_{j \rightarrow \infty} \int_{\Omega_1} |\nabla y_j|^2 dx \leq \limsup_{j \rightarrow \infty} \int_{\Omega_1} |\nabla y_j|^2 dx \\ &= \limsup_{j \rightarrow \infty} \int_{\Omega_j} |\nabla y_j|^2 dx = \lim_{j \rightarrow \infty} \int_{\Omega_j} [e^y + u] y_j dx \\ &= \int_{\Omega} [e^y + u] y dx = \int_{\Omega} |\nabla y|^2 dx = \int_{\Omega_1} |\nabla y|^2 dx, \end{aligned}$$

hence  $\lim_{j \rightarrow \infty} \|y_j\|_{H_0^1(\Omega_1)} = \|y\|_{H_0^1(\Omega_1)}$ , which proves the strong convergence  $y_j \rightarrow y$  in  $H_0^1(\Omega_1)$ .

Now arguing again as in the proof of Theorem 2.2 and using the fact that  $(\sigma - x_0) \cdot \nu_j(\sigma) \geq 0$  for every  $\sigma \in \Gamma_j$  (here  $\nu_j$  is the outward normal to  $\Gamma_j$ ), we deduce

$$\begin{aligned} \int_{\Omega} (-\Delta y)[(x - x_0) \cdot \nabla y] dx &= \lim_{j \rightarrow \infty} \int_{\Omega} (-\Delta y_j)[(x - x_0) \cdot \nabla y_j] dx \\ &= \lim_{j \rightarrow \infty} \int_{\Omega_j} (-\Delta y_j)[(x - x_0) \cdot \nabla y_j] dx \\ &= \lim_{j \rightarrow \infty} \left\{ \left(1 - \frac{n}{2}\right) \int_{\Omega_j} |\nabla y_j|^2 dx - \frac{1}{2} \int_{\Gamma_j} [(\sigma - x_0) \cdot \nu_j(\sigma)] |\nabla y_j|^2 d\sigma \right\} \\ &\leq \lim_{j \rightarrow \infty} \left(1 - \frac{n}{2}\right) \int_{\Omega} |\nabla y_j|^2 dx = \left(1 - \frac{n}{2}\right) \int_{\Omega} |\nabla y|^2 dx. \end{aligned}$$

□

**Remark.** Inequality (2.7) makes sense when we assume only that  $y \in Y$  and  $u \in L^2(\Omega)$  but we do not know whether it holds under these assumptions. Actually (2.7) is true for solutions  $(y, u) \in Y \times K$  such that there is a sequence of solutions  $(y_k, u_k) \in Y \times K$  with  $y_k \in H^2(\Omega)$  satisfying (1.1),  $u_k \rightarrow u$  strongly in  $L^2(\Omega)$  (indeed this implies that  $y_k \rightarrow y$  in  $H_0^1(\Omega)$ -weak and  $e^{y_k} \rightarrow e^y$  in  $L^1(\Omega)$ ). □

**COROLLARY 2.4.** *Let us assume that  $u \in L^2(\Omega)$ ,  $\Omega$  is star-shaped with respect to some interior point  $x_0$  and  $y \in Y$  is a solution of (1.1) such that  $e^y \in L^2(\Omega)$ . Then we have*

$$\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla y(x)|^2 dx \leq n \int_{\Omega} (e^{y(x)} - 1) dx - \int_{\Omega} u(x)[(x - x_0) \cdot \nabla y(x)] dx. \tag{2.8}$$

*Proof.* If  $y \in H^2(\Omega)$  this inequality is an immediate consequence of (2.3). Indeed it is enough to note that  $\nu(\sigma) \cdot (\sigma - x_0) \geq 0$  for almost every  $\sigma \in \Gamma$  because  $\Omega$  is assumed to be star-shaped with respect to  $x_0$ . Since we have not assumed  $\Gamma$  to be of class  $C^{1,1}$  or  $\Omega$  to be convex, we cannot deduce the  $H^2(\Omega)$ -regularity of  $y$  from the fact that  $e^y + u \in L^2(\Omega)$ . However (2.6) is still valid, therefore the result follows from the previous lemma. □

As a consequence of this corollary, we deduce some estimates for  $y$  in terms of  $u$ .

**THEOREM 2.5.** *Let us assume that  $y$  is a solution of (1.1) that satisfies the inequality (2.8). Then there exist positive constants  $C_i$ ,  $1 \leq i \leq 4$ , independent of  $y$  and  $u$  such that*

$$\|e^y y\|_{L^1(\Omega)} \leq C_1 \|u\|_{L^2(\Omega)}^2 + C_2, \tag{2.9}$$

$$\|y\|_{H_0^1(\Omega)} \leq C_3 \|u\|_{L^2(\Omega)} + C_4. \tag{2.10}$$

*Proof.* Thanks to Theorem 2.1 we know that

$$\int_{\Omega} |\nabla y(x)|^2 dx = \int_{\Omega} e^{y(x)} y(x) dx + \int_{\Omega} u(x) y(x) dx. \tag{2.11}$$

Combining this identity with (2.8) we deduce

$$\begin{aligned} & \left(\frac{n}{2} - 1\right) \int_{\Omega} e^y y dx + \left(\frac{n}{2} - 1\right) \int_{\Omega} u y dx \\ & \leq n \int_{\Omega} [e^y - 1] dx - \int_{\Omega} u [(x - x_0) \cdot \nabla y] dx. \end{aligned}$$

Which yields

$$\int_{\Omega} e^y y dx \leq \frac{2n}{n - 2} \int_{\Omega} [e^y - 1] dx + c_1 \|u\|_{L^2(\Omega)} \|y\|_{H_0^1(\Omega)}. \tag{2.12}$$

Let us set

$$\Omega_n = \left\{ x \in \Omega : y(x) > \frac{4n}{n - 2} \right\}.$$

Then from (2.12) we get

$$\int_{\Omega} e^y y dx \leq \frac{1}{2} \int_{\Omega_n} e^y y dx + \int_{\Omega \setminus \Omega_n} e^{\frac{4n}{n-2}} dx + c_1 \|u\|_{L^2(\Omega)} \|y\|_{H_0^1(\Omega)},$$

therefore

$$\frac{1}{2} \int_{\Omega} e^y y dx \leq c_2 - \frac{1}{2} \int_{\Omega \setminus \Omega_n} e^y y dx + c_1 \|u\|_{L^2(\Omega)} \|y\|_{H_0^1(\Omega)}.$$

Taking into account that  $e^t |t| \leq c(n) < +\infty$  for any real number  $t \leq \frac{4n}{n-2}$ , we get from the previous inequality

$$\int_{\Omega} e^y |y| dx \leq c_3 + c_1 \|u\|_{L^2(\Omega)} \|y\|_{H_0^1(\Omega)}. \tag{2.13}$$

Using this inequality in (2.11), we obtain (2.10). Finally using (2.10) in (2.13) we get the estimate (2.9). □

We finish this section by proving two propositions that will be very useful in the next sections. Here we will use the notation  $y^+ = \max\{y, 0\}$  and  $y^- = \max\{-y, 0\}$ .

**PROPOSITION 2.6.** *Let  $y$  be a solution of (1.1) corresponding to a function  $u \in L^p(\Omega)$ . Then  $y^+ \in L^r(\Omega)$  for all  $1 \leq r < +\infty$ ;  $y^- \in L^q(\Omega)$  with*

$$q = \begin{cases} +\infty & \text{if } p > n/2, \\ < +\infty & \text{if } p = n/2, \\ np/(n - 2p) & \text{if } p < n/2. \end{cases}$$

*Moreover  $\|y^-\|_{L^q(\Omega)} \leq C_{p,q} \|u^-\|_{L^p(\Omega)}$  for some constant  $C_{p,q} > 0$  independent of  $u$  and  $y$ ; if we denote by  $[r]$  the integer part of  $r$  and  $k := [r] + 1$  then*

$$\|y^+\|_{L^r(\Omega)} \leq (k!)^{1/r} (C_1 \|u\|_{L^2(\Omega)} + C_2)^{1/r}.$$

*Proof.* Let us take  $k = [r] + 1$ . Then  $|y^+|^r < k! e^y \in L^1(\Omega)$ , which proves that  $y^+ \in L^r(\Omega)$ . Using the estimate  $\int_{\Omega} e^y dx \leq C_1 \|u\|_{L^2(\Omega)}^2 + C_2$  one gets the last estimate of the proposition.

Now assume that  $p < n/2$  (the case  $p \geq n/2$  may be treated in the same way). Consider  $\psi \in H_0^1(\Omega)$  satisfying  $-\Delta \psi = -u^-$ . We have

$$-\Delta y \geq -u^- = -\Delta \psi$$

and by the maximum principle we conclude that  $y \geq \psi$  in  $\Omega$ . As  $\psi < 0$ , we conclude that  $0 \leq y^- \leq \psi^- = |\psi|$ . Now recall that by a classical result of G. Stampacchia [12] (theorems 4.1 and 4.2), if for some  $s \geq 2$  (and  $s < n$ ) one has  $-\Delta z = T \in W^{-1,s}(\Omega)$  and  $z \in H_0^1(\Omega)$ , then

$$\|z\|_{L^{s^*}(\Omega)} \leq C(n, s)(\|T\|_{W^{-1,s}(\Omega)} + \|z\|_{L^2(\Omega)}), \quad \frac{1}{s^*} = \frac{1}{s} - \frac{1}{n}$$

Here we have  $-\Delta \psi = -u^- \in L^p(\Omega)$  and  $L^p(\Omega) \subset W^{-1,s}(\Omega)$  if

$$\frac{1}{s} = \frac{1}{p} - \frac{1}{n}.$$

Consequently one sees that the estimate of the proposition on  $\|y^-\|_{L^q(\Omega)}$  holds if  $q = s^*$ , which means

$$\frac{1}{q} = \frac{1}{p} - \frac{2}{n}.$$

□

PROPOSITION 2.7. *Let  $u$  and  $y$  be as in Proposition 2.6 and satisfying the inequality (2.8). Then  $y \in L^p(\Omega)$  and*

$$\|y\|_{L^p(\Omega)} \leq C_{1,p}\|u\|_{L^p(\Omega)} + C_{2,p},$$

for some constants  $C_{i,p} > 0$  independent of  $u$  and  $y$ .

*Proof.* Let us take  $k = [p] + 1$ . From (2.9), the inequality  $e^t \leq |t|e^t + 1$  for every  $t \in \mathbb{R}$  and taking into account that  $p \geq 2$ , we deduce

$$\|y^+\|_{L^p(\Omega)} \leq \left(k! \int_{\Omega} e^y dx\right)^{\frac{1}{p}} \leq \left(c_1\|u\|_{L^2(\Omega)}^2 + c_2\right)^{1/p}$$

$$\leq c_3\|u\|_{L^2(\Omega)}^{2/p} + c_4 \leq c_3(\|u\|_{L^2(\Omega)} + 1) + c_4 \leq c_5\|u\|_{L^p(\Omega)} + c_6.$$

On the other hand, from Proposition 2.6 it follows that  $y^- \in L^p(\Omega)$ . Indeed it is enough to note that  $np/(n - 2p) > p$  if  $p < n/2$ . Moreover

$$\|y^-\|_{L^p(\Omega)} \leq C_p\|u^-\|_{L^p(\Omega)}.$$

Now it is enough to write  $y = y^+ - y^-$  and to use the two inequalities obtained above to achieve the desired result. □

### 3. EXISTENCE OF AN OPTIMAL CONTROL. CASE $p > 2$

The aim of this section is to study the existence of a solution for the optimal control problem. As usual, to prove the existence of such a solution, we take a minimizing sequence  $\{(y_k, u_k)\}_{k=1}^{\infty}$  of feasible elements. Assuming that either  $K$  is bounded in  $L^p(\Omega)$  or  $N > 0$ , we can deduce that  $\{u_k\}_{k=1}^{\infty}$  is bounded. The difficult part is to deduce that  $\{y_k\}_{k=1}^{\infty}$  is bounded in  $H_0^1(\Omega)$ . If the elements  $(y_k, u_k)$  satisfy the inequality (2.8), then Theorem 2.5 provides the necessary inequalities to deduce the boundedness of  $\{y_k\}_{k=1}^{\infty}$ . Unfortunately (2.8) has been proved to hold only for solutions of the state equation with  $e^y \in L^2(\Omega)$ . This leads us to consider the following class of states: we denote by  $\mathcal{Y}$  the subset of  $Y$  formed by the functions which satisfy (1.1) and (2.8) for some control  $u$  (note that for  $q > n/2$  one has



$W^{2,q}(\Omega) \cap H_0^1(\Omega) \subset \mathcal{Y}$ ). Not every element of  $\mathcal{Y}$  needs to be a solution of (1.1) with  $e^y \in L^2(\Omega)$ . In fact we have the following result.

**THEOREM 3.1.** *Let us assume that  $\{(y_k, u_k)\}_{k=1}^\infty \subset \mathcal{Y} \times L^p(\Omega)$  is a sequence of functions satisfying (1.1) and converging weakly to some element  $(y, u)$  in  $H_0^1(\Omega) \times L^p(\Omega)$ , with  $p > 2$ . Then  $e^{y_k} \rightarrow e^y$  in  $L^1(\Omega)$ ,  $y \in \mathcal{Y}$  and  $(y, u)$  satisfies also (1.1). The same result holds if  $y_k \rightharpoonup y$  in  $H_0^1(\Omega)$  and  $u_k \rightarrow u$  strongly in  $L^2(\Omega)$ .*

*Proof.* The weak convergence  $y_k \rightharpoonup y$  in  $H_0^1(\Omega)$  implies the strong convergence  $y_k \rightarrow y$  in  $L^2(\Omega)$ . Then taking a subsequence if necessary, we can assume that  $y_k(x) \rightarrow y(x)$  for almost every point  $x \in \Omega$ . In particular  $e^{y_k(x)} \rightarrow e^{y(x)}$  almost everywhere in  $\Omega$ . Let us use Vitali’s theorem to prove the convergence  $e^{y_k} \rightarrow e^y$  in  $L^1(\Omega)$ . First let us note that the boundedness of  $\{u_k\}_{k=1}^\infty$  in  $L^2(\Omega)$  along with (2.9) imply that  $\|e^{y_k} y_k\|_{L^1(\Omega)} \leq C$  for some constant  $C < +\infty$  and all  $k \in \mathbb{N}$ . Now given  $\epsilon > 0$ , let us take  $m > 0$  such that  $C/m < \epsilon/2$  and  $\delta = \epsilon/(2e^m)$ . Then for every measurable set  $E \subset \Omega$ , with  $\text{meas}(E) < \delta$ , we have

$$\begin{aligned} \int_E e^{y_k(x)} dx &\leq \frac{1}{m} \int_{\{x \in E: y_k(x) > m\}} e^{y_k(x)} y_k(x) dx + \int_{\{x \in E: y_k(x) \leq m\}} e^m dx \\ &\leq \frac{1}{m} \int_\Omega e^{y_k(x)} |y_k(x)| dx + e^m \text{meas}(E) \leq \frac{C}{m} + e^m \delta < \epsilon \quad \forall k \in \mathbb{N}, \end{aligned}$$

which allows to conclude the desired convergence. Now it is easy to pass to the limit in the state equation satisfied by  $(y_k, u_k)$  and to conclude that  $(u, y)$  satisfies (1.1).

Let us prove that  $(y, u)$  satisfies (2.8). First of all we will prove that there exists a subsequence, that we will denote in the same way, such that  $\nabla y_k(x) \rightarrow \nabla y(x)$  for almost all point  $x \in \Omega$ . To achieve this aim, we remark that  $e^{y_k} \rightarrow e^y$  in  $L^1(\Omega)$  while  $(u_k)_k$  is bounded in  $L^p(\Omega)$ : therefore  $(\Delta y_k)_k$  is bounded in  $L^1(\Omega)$ . Now by a result due to L. Boccardo and F. Murat [2] (theorem 2.1) one may conclude that there exists a subsequence (still denoted by)  $(y_k)_k$  such that  $y_k \rightharpoonup y$  in  $H^1(\Omega)$  weakly and  $\nabla y_k \rightarrow \nabla y$  almost everywhere.

Now the weak convergence  $\nabla y_k \rightharpoonup \nabla y$  in  $(L^2(\Omega))^n$  along with the pointwise convergence implies the strong convergence in  $(L^r(\Omega))^n$  for all  $r < 2$  (we use here the fact that the weak convergence in  $L^2(\Omega)$  implies that the sequence  $|\nabla y_k - \nabla y|^r$  is equi-integrable and we apply Vitali’s theorem). In particular the strong convergence of  $\nabla y_k \rightarrow \nabla y$  holds in  $L^{p'}(\Omega)$ , with  $(1/p) + (1/p') = 1$  (this is the only place where we need the assumption  $p > 2$ ). Then we can pass to the limit in the inequality (2.8) to obtain:

$$\begin{aligned} \left(\frac{n}{2} - 1\right) \int_\Omega |\nabla y|^2 dx &\leq \liminf_{k \rightarrow \infty} \left(\frac{n}{2} - 1\right) \int_\Omega |\nabla y_k|^2 dx \\ &\leq \liminf_{k \rightarrow \infty} \left\{ n \int_\Omega [e^{y_k} - 1] dx - \int_\Omega u_k [(x - x_0) \cdot \nabla y_k] dx \right\} \\ &= n \int_\Omega [e^y - 1] dx - \int_\Omega u [(x - x_0) \cdot \nabla y] dx, \end{aligned}$$

which concludes the proof when  $(y_k, u_k) \rightharpoonup (y, u)$  in  $H_0^1(\Omega) \times L^p(\Omega)$  and  $p > 2$ . On the other hand it is clear that when  $p = 2$  and  $u_k \rightarrow u$  strongly in  $L^2(\Omega)$ , in the above inequalities one has

$$\int_{\Omega} u_k(x - x_0) \cdot \nabla y_k dx \rightarrow \int_{\Omega} u(x - x_0) \cdot \nabla y dx$$

as  $k \rightarrow \infty$ . □

Now we reformulate the control problem as follows

$$(\mathcal{P}) \begin{cases} \text{Minimize } J(y, u) := \int_{\Omega} L(x, y(x)) dx + \frac{N}{p} \int_{\Omega} |u(x)|^p dx \\ (y, u) \in \mathcal{Y} \times K \text{ satisfies (1.1).} \end{cases}$$

The fact of taking  $(y, u) \in \mathcal{Y} \times K$  imposes a restriction on the class of solutions of (1.1), but it is not restrictive with respect to the controls. More precisely, we have the following result

**PROPOSITION 3.2.** *Let us assume that  $\Omega$  is star-shaped with respect to some point  $x_0 \in \Omega$  and that (1.1) has a solution for some control  $u \in L^p(\Omega)$ ,  $p \geq 2$ . Then there exists a solution  $z$  of (1.1) corresponding to the same control  $u$  and belonging to the class  $\mathcal{Y}$ .*

Before proving this proposition, we state and prove the following lemma:

**LEMMA 3.3.** *For  $k \geq 1$  and  $t \in \mathbb{R}$  let us denote  $f_k(t) := \min\{e^k, e^t\}$ . Then there is  $z_k \in H_0^1(\Omega)$  such that*

$$\begin{cases} -\Delta z = f_k(z) + u & \text{in } \Omega, \\ z_k = 0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Moreover  $z_k \leq z_{k+1}$  and the sequence  $\{z_k\}_{k=1}^{\infty}$  is bounded in  $H_0^1(\Omega)$ .

*Proof.* We denote with  $y$  a solution of (1.1) associated to the control  $u$ . Let us take  $y_0 \in H_0^1(\Omega)$  such that  $-\Delta y_0 = u$ . Then we have the following three inequalities:

$$-\Delta y_0 \leq f_k(y_0) + u, \quad -\Delta y \geq f_k(y) + u, \quad -\Delta(y - y_0) = e^y > 0.$$

From the last relation we deduce that  $y_0 \leq y$  in  $\Omega$ . From the two first relations it follows that  $y_0$  is a subsolution and  $y$  is a supersolution of (3.1). Combining all these, by the now classical techniques introduced by D.H. Sattinger [10] (see also H. Amann [1]) we get the existence of a solution  $z_k$  of (3.1) such that  $y_0 \leq z_k \leq y$ . Moreover we have  $z_k \leq z_{k+1}$  and

$$\begin{aligned} \int_{\Omega} |\nabla z_k|^2 dx &= \int_{\Omega} [f_k(z_k) + u] z_k dx \leq \int_{\Omega} [e^{z_k} + |u|] |z_k| dx \\ &\leq \int_{\Omega} [e^y + |u|] |z_k| dx \leq \|e^y + |u|\|_{H^{-1}(\Omega)} \|z_k\|_{H_0^1(\Omega)}, \end{aligned}$$

which implies that  $\{z_k\}_{k=1}^{\infty}$  is bounded in  $H_0^1(\Omega)$ . □

*Proof of Proposition 3.2.* The sequence  $\{z_k\}_{k=1}^{\infty}$  being given by Lemma 3.3, taking a subsequence, denoted in the same way, we infer that there exists an element  $z \in H_0^1(\Omega)$  such that

$$z_k \rightarrow z \text{ weakly in } H_0^1(\Omega), \quad z_k(x) \rightarrow z(x) \text{ a.e. } x \in \Omega. \tag{3.2}$$

Therefore  $f_k(z_k(x)) \rightarrow e^{z(x)}$  for almost all  $x \in \Omega$  and  $f_k(z_k) \leq e^{z_k} \leq e^y$ . Then we can apply the dominated convergence theorem to obtain that  $e^{z_k} \rightarrow e^z$  in  $L^1(\Omega)$ . Now it is easy to pass to the limit and to deduce that  $z$  satisfies (1.1). We are going to prove that  $z \in \mathcal{Y}$ . First of all let us remark that  $f_k(z_k) + u \in L^2(\Omega)$ , hence  $\Delta z_k \in L^2(\Omega)$ . Therefore we can multiply (3.1) by  $(x - x_0) \cdot \nabla z_k$  and integrate over  $\Omega$  and get:

$$\begin{aligned}
 & - \int_{\Omega} \Delta z_k [(x - x_0) \cdot \nabla z_k] dx \\
 &= \int_{\Omega} f_k(z_k) [(x - x_0) \cdot \nabla z_k] dx + \int_{\Omega} u [(x - x_0) \cdot \nabla z_k] dx. \tag{3.3}
 \end{aligned}$$

Let us define the function  $F_k : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F_k(t) = \begin{cases} e^t - 1 & \text{if } t \leq k, \\ e^k(t - k + 1) - 1 & \text{if } t > k. \end{cases}$$

Then  $F_k$  is the primitive of  $f_k$  (which is defined in Lemma 3.3) satisfying  $F_k(0) = 0$  and  $F_k(t) \leq e^t - 1$ . Arguing as in the proof of Theorem 2.2 we obtain

$$\int_{\Omega} f_k(z_k) [(x - x_0) \cdot \nabla z_k] dx = -n \int_{\Omega} F_k(z_k) dx \geq -n \int_{\Omega} [e^{z_k} - 1] dx. \tag{3.4}$$

Now from Lemma 2.3 we get

$$\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla z_k|^2 dx \leq \int_{\Omega} (\Delta z_k) [(x - x_0) \cdot \nabla z_k] dx. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), the following inequality is obtained

$$\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla z_k|^2 dx \leq n \int_{\Omega} [e^{z_k} - 1] dx - \int_{\Omega} u [(x - x_0) \cdot \nabla z_k] dx.$$

Using (3.2) and the convergence  $e^{z_k} \rightarrow e^z$  in  $L^1(\Omega)$ , it comes

$$\left(\frac{n}{2} - 1\right) \int_{\Omega} |\nabla z|^2 dx \leq n \int_{\Omega} [e^z - 1] dx - \int_{\Omega} u [(x - x_0) \cdot \nabla z] dx,$$

which proves that  $z \in \mathcal{Y}$  as desired. □

In the next proposition we state some interesting properties of the set of feasible controls.

**PROPOSITION 3.4.** *Let  $p \geq 2$  and let  $\Omega$  be star-shaped with respect to one of its interior points. Then the set of controls  $u \in L^p(\Omega)$  for which there exists a solution  $y \in \mathcal{Y}$  is non empty, convex and closed in  $L^p(\Omega)$ .*

*Proof.* It is enough to take any function  $y \in C_c^\infty(\Omega)$  and to put  $u = -\Delta y - e^y$  to deduce that the set of feasible controls is non empty. Let us take a sequence of feasible controls  $\{u_k\}_{k=1}^\infty$  converging in  $L^p(\Omega)$  to some function  $u$ . Thanks to Theorem 2.5, we know that the corresponding states  $\{y_k\}_{k=1}^\infty \subset \mathcal{Y}$  are bounded in  $H_0^1(\Omega)$ . Then Theorem 3.1 claims that any weak limit of  $\{y_k\}_{k=1}^\infty$  is an element of  $\mathcal{Y}$  satisfying (1.1) along with the control  $u$ , which proves that the set of feasible controls is closed.

Now let us prove the convexity. If  $u_1$  and  $u_2$  are two feasible controls, with associated states  $y_1$  and  $y_2$ , respectively, and  $\lambda \in (0, 1)$ , we set  $u =$

$\lambda u_1 + (1 - \lambda)u_2$  and  $y = \lambda y_1 + (1 - \lambda)y_2$ . Let us take  $\psi \in H_0^1(\Omega)$  such that  $-\Delta\psi = u$ . Then

$$-\Delta\psi \leq e^\psi + u, \quad \text{and} \quad -\Delta y = \lambda e^{y_1} + (1 - \lambda)e^{y_2} + u \geq e^y + u.$$

Therefore  $\psi$  is a subsolution of (1.1) for the control  $u$  and  $y$  is a supersolution. On the other hand,  $-\Delta(y - \psi) \geq e^y > 0$ , then  $\psi \leq y$ . Therefore we deduce the existence of a solution  $z$  of (1.1) associated to the control  $u$ , with  $\psi \leq z \leq y$ . Finally, the conclusion follows from Proposition 3.2.  $\square$

So far we have studied the properties of the feasible pairs  $(y, u) \in \mathcal{Y} \times L^p(\Omega)$  satisfying (1.1). Let us say something about the action of functional  $J$  on these pairs. For each one of these pairs  $(y, u)$ ,  $J$  is well defined and  $-\infty < J(y, u) \leq +\infty$ . Indeed the only trouble can come from the integral of  $L(x, y(x))$ . With the notation of (1.4), let us set

$$f(x) = L(x, y(x)) + \alpha_2 \left( |y^-(x)|^{\theta_2} + e^{y(x)} \right) - a_2(x).$$

Then  $f$  is a nonnegative measurable function and consequently its integral is well defined as a number in  $[0, +\infty]$ . On the other hand, it is enough to use Theorem 2.5 and Proposition 2.6 and the assumptions on  $\theta_2$  to deduce that  $|y|^{\theta_2}$  and  $e^y$  are integrable functions. Therefore the integral of  $L(x, y(x))$  is well defined, though it could be  $+\infty$  in some cases.

Finally we establish our result of existence of a solution to  $(\mathcal{P})$ .

**THEOREM 3.5.** *Let us assume that  $p > 2$  and*

- (i) *There exists a pair  $(y, u) \in \mathcal{Y} \times K$  satisfying (1.1).*
- (ii) *Either  $K$  is bounded or  $N > 0$ .*

*Then problem  $(\mathcal{P})$  has at least one solution.*

*Proof.* Let us assume that  $\mu := \inf(\mathcal{P}) < +\infty$  (otherwise the theorem is obvious). Let  $\{(y_k, u_k)\}_{k=1}^\infty$  be a minimizing sequence for problem  $(\mathcal{P})$ :  $J(y_k, u_k) \downarrow \mu$ . Thus we can suppose that  $J(y_k, u_k) \leq \mu + 1$ . The main point in the proof is to establish the boundedness of  $\{u_k\}_{k=1}^\infty$  in  $L^p(\Omega)$ . This is obvious if  $K$  is bounded. Let us assume that  $K$  is not bounded. We have

$$\begin{aligned} \int_\Omega a_2(x)dx + \frac{N}{p} \int_\Omega |u_k(x)|^p dx - \alpha_2 \int_\Omega |y_k^-(x)|^{\theta_2} dx - \alpha_2 \int_\Omega e^{y_k(x)} dx \\ \leq J(y_k, u_k) \leq \mu + 1. \end{aligned}$$

Using the fact that the exponent  $q$  defined in proposition 2.6 satisfies  $q > p$  we conclude that  $\|y_k^-\|_{\theta_2}^{\theta_2} \leq C\|u_k\|_p^{\theta_2}$ . In the same way we observe that

$$\int_\Omega e^{y_k(x)} dx \leq C(\|u_k\|_2^2 + 1),$$

and finally we obtain:

$$\|u_k\|_p^p \leq C + C\|u_k\|_p^2 + C\|u_k\|_p^{\theta_2}.$$

As  $\theta_2 < p$  and  $p > 2$ , this implies that  $(u_k)_k$  is bounded in  $L^p(\Omega)$ .

The boundedness of  $\{y_k\}_{k=1}^\infty$  in  $H_0^1(\Omega)$  is a consequence of (2.10). Therefore, taking a subsequence if necessary, we can assume that  $(y_k, u_k) \rightharpoonup (y, u)$  weakly in  $H_0^1(\Omega) \times L^p(\Omega)$ . Theorem 3.1 asserts that  $y \in \mathcal{Y}$  and  $(y, u)$  satisfies (1.1). Moreover the convexity and closedness of  $K$  in  $L^p(\Omega)$  implies that  $u \in K$ . Thus  $(y, u)$  is a feasible pair for problem  $(\mathcal{P})$ . Let us prove that it

is a solution. Since  $y_k \rightarrow y$  strongly in  $L^2(\Omega)$  we can take a subsequence in such a way that  $y_k(x) \rightarrow y(x)$  for almost all  $x \in \Omega$ . Let us set

$$f_k(x) = L(x, y_k(x)) + \alpha_2 \left( |y_k^-(x)|^{\theta_2} + e^{y_k(x)} \right) - a_2(x)$$

and

$$f(x) = L(x, y(x)) + \alpha_2 \left( |y(x)|^{\theta_2} + e^{y(x)} \right) - a_2(x).$$

Then  $f_k(x) \rightarrow f(x)$  almost everywhere and  $f_k \geq 0$ . Therefore we can apply Fatou's Lemma and the convergences  $e^{y_k} \rightarrow e^y$  (Theorem 3.1) and  $|y_k|^{\theta_2} \rightarrow |y|^{\theta_2}$  (Proposition 2.7) in  $L^1(\Omega)$  to derive

$$\begin{aligned} J(y, u) &= \int_{\Omega} f(x) dx - \int_{\Omega} \left\{ \alpha_2 \left( |y^-(x)|^{\theta_2} + e^{y(x)} \right) - a_2(x) \right\} dx \\ &+ \frac{N}{p} \int_{\Omega} |u(x)|^p dx \leq \liminf_{k \rightarrow \infty} \left\{ \int_{\Omega} f_k(x) dx \right. \\ &\quad \left. - \int_{\Omega} \left[ \alpha_2 \left( |y_k^-(x)|^{\theta_2} + e^{y_k(x)} \right) - a_2(x) \right] dx + \frac{N}{p} \int_{\Omega} |u_k(x)|^p dx \right\} \\ &= \liminf_{k \rightarrow \infty} \left\{ \int_{\Omega} L(x, y_k(x)) dx + \frac{N}{p} \int_{\Omega} |u_k(x)|^p dx \right\} \\ &= \liminf_{k \rightarrow \infty} J(y_k, u_k) = \inf(\mathcal{P}). \end{aligned}$$

This concludes the proof. □

We conclude this section by studying the uniqueness of the solution.

**THEOREM 3.6.** *Let  $p \geq 2$  and assume that  $\Omega$  is star-shaped with respect to some  $x_0 \in \Omega$  and that the set of admissible pairs  $(y, u) \in \mathcal{Y} \times K$  satisfying (1.1) is not empty. Assume also that the function  $t \mapsto L(x, t)$  is monotone on  $\mathbb{R}$ , non decreasing and convex for almost all  $x \in \Omega$ . Then problem (P) has at most one solution if one of the following conditions holds:*

- (i)  $N > 0$ ;
- (ii)  $L(x, \cdot)$  is strictly increasing;
- (iii)  $L(x, \cdot)$  is strictly convex.

*Proof.* Let us assume that  $(y_1, u_1)$  and  $(y_2, u_2)$  are two different solutions of (P). We note in particular that  $y_1 \neq y_2$  and we set  $u = (u_1 + u_2)/2$ . Looking at the proof of Proposition 3.4, we remark that one may prove the existence of a solution  $y \in \mathcal{Y}$  of (1.1) corresponding to the control  $u$ , with  $y \leq (y_1 + y_2)/2$ . In fact we have that this inequality is strict in  $\Omega$ . Indeed

$$-\Delta[(y_1 + y_2)/2 - y] \geq \frac{1}{2}(e^{y_1} + e^{y_2}) - e^y \geq e^{(y_1+y_2)/2} - e^y \geq 0 \quad \text{in } \Omega$$

and due to the fact that  $y_1 \neq y_2$ , we have  $\frac{1}{2}(e^{y_1} + e^{y_2}) - e^y \not\equiv 0$  on  $\Omega$ . Now by the strong maximum principle we conclude that  $(y_1 + y_2)/2 > y$  in  $\Omega$ . Therefore

$$\begin{aligned} \int_{\Omega} L(x, y(x)) dx &\leq \int_{\Omega} L(x, (y_1(x) + y_2(x))/2) dx \\ &\leq \frac{1}{2} \left( \int_{\Omega} L(x, y_1(x)) dx + \int_{\Omega} L(x, y_2(x)) dx \right), \end{aligned}$$

and the inequality

$$\int_{\Omega} L(x, y(x)) dx \leq \frac{1}{2} \left( \int_{\Omega} L(x, y_1(x)) dx + \int_{\Omega} L(x, y_2(x)) dx \right)$$

is strict if the strict convexity or monotonicity of  $L(x, \cdot)$  is assumed. If the previous inequality is strict or if  $N > 0$  we deduce

$$J(y, u) < \frac{1}{2}[J(y_1, u_1) + J(y_2, u_2)] = \inf(\mathcal{P}),$$

which is a contradiction with the fact that  $(u, y)$  is feasible for problem  $(\mathcal{P})$ . □

#### 4. EXISTENCE OF AN OPTIMAL CONTROL. CASE $p = 2$

As we noticed before, when  $p = 2$  the conclusions of Theorem 3.1 holds only when we know that  $u_k \rightarrow u$  strongly in  $L^2(\Omega)$ . The difficulty comes from the fact that the strong convergence  $\nabla y_k \rightarrow \nabla y$  can only be proved in  $L^r(\Omega)$ , with  $r < 2$ . Hence we cannot pass to the limit in the inequality (2.8) satisfied for every  $(y_k, u_k)$ . Therefore we cannot deduce that  $(y, u)$  satisfies (2.8), and consequently we are not able to prove the existence of a solution in  $\mathcal{Y} \times K$ . Since we have estimates on the state (see Theorem 2.5) only for elements of  $\mathcal{Y}$ , we cannot deduce, in general, the boundedness in  $Y \times K$  of a minimizing sequence of problem  $(\mathcal{P})$ . In this section we will show that, under some additional assumptions on the function  $L$ , it is possible to have a minimizing sequence  $\{(y_k, u_k)\}_{k=1}^{\infty}$  of  $(\mathcal{P})$  with  $\{y_k\}_{k=1}^{\infty} \subset \mathcal{Y}$ . In this way, we can deduce the boundedness of the states and prove the existence of an optimal solution.

**THEOREM 4.1.** *Let us assume that  $\Omega$  is star-shaped with respect to one of its interior points. We also make the hypotheses*

(i) *The function  $t \mapsto L(x, t)$  defined on  $\mathbb{R}$  is monotone non decreasing for almost every  $x \in \Omega$ .*

(ii) *There exists a pair  $(y, u) \in Y \times K$  satisfying (1.1).*

(iii) *Either  $K$  is bounded or  $N > 0$  and  $\theta_2 < 2$  in (1.4).*

*Then problems  $(\mathcal{P})$  and  $(\mathcal{P})$  have at least one solution. For each solution  $(\bar{y}, \bar{u})$  of  $(\mathcal{P})$ , we can find  $\tilde{y} \in \mathcal{Y}$  such that  $(\tilde{y}, \bar{u})$  is a solution of  $(\mathcal{P})$  and  $(\mathcal{P})$ . Moreover, if  $L(x, \cdot)$  is strictly increasing for almost all  $x \in \Omega$ , then any optimal solution of  $(\mathcal{P})$  is also a solution of  $(\mathcal{P})$ .*

*Proof.* Let us take a minimizing sequence  $\{(y_k, u_k)\}_{k=1}^{\infty} \subset Y \times K$ . From Proposition 3.2 we deduce the existence of elements  $z_k \in \mathcal{Y}$  such that  $(z_k, u_k)$  satisfies (1.1). Moreover, by looking at the proof of the mentioned proposition, we know that  $z_k \leq y_k$  in  $\Omega$ . Now using the monotonicity of  $L(x, \cdot)$ , we get that  $J(z_k, u_k) \leq J(y_k, u_k)$ . Therefore  $\{(z_k, u_k)\}_{k=1}^{\infty}$  is also a minimizing sequence of  $(\mathcal{P})$ . Arguing as in the proof of Theorem 3.5, we can obtain a subsequence, denoted in the same way, converging to an element  $(\bar{y}, \bar{u}) \in Y \times K$  solution of Problem  $(\mathcal{P})$ .

If  $\bar{y} \notin \mathcal{Y}$ , we can apply again Proposition 3.2 to deduce the existence of an element  $\tilde{y} \in \mathcal{Y}$ , with  $\tilde{y} \leq \bar{y}$ , such that  $(\tilde{y}, \bar{u})$  satisfies (1.1). Again the monotonicity of  $L(x, \cdot)$  leads to  $J(\tilde{y}, \bar{u}) \leq J(\bar{y}, \bar{u})$ . So  $(\tilde{y}, \bar{u})$  is a solution of  $(\mathcal{P})$ , and consequently of  $(\mathcal{P})$  too.

Finally, if  $L(x, \cdot)$  is strictly increasing and  $\bar{y} \notin \mathcal{Y}$ , then  $J(\bar{y}, \bar{u}) < J(\bar{y}, \bar{u})$ , which contradicts the optimality of  $(\bar{y}, \bar{u})$ .  $\square$

### 5. THE OPTIMALITY CONDITIONS

The aim of this section is to derive some optimality conditions for the control problem. We will prove two theorems corresponding to the cases  $K = L^p(\Omega)$  and  $K \subset L^p(\Omega)$  with  $K \neq L^p(\Omega)$ . Let us start with the first case.

**THEOREM 5.1.** *Assume that  $\Omega$  is star-shaped with respect to one of its interior points,  $p \geq 2$  and  $K = L^p(\Omega)$ . If  $(\bar{y}, \bar{u})$  is a solution of problem (P) (resp. (P)) with  $e^{\bar{y}} \in L^p(\Omega)$ , then*

$$\begin{cases} -\Delta \bar{y} = e^{\bar{y}} + \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

and setting  $\bar{\phi} := -N|\bar{u}|^{p-2}\bar{u}$ , one has  $\bar{\varphi} \in W_0^{1,s}(\Omega)$  for every  $s < n/(n-1)$  and

$$\begin{cases} -\Delta \bar{\varphi} = e^{\bar{y}}\bar{\varphi} + \frac{\partial L}{\partial y}(x, \bar{y}) & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma. \end{cases} \tag{5.2}$$

Moreover, if  $p > n/2$ ,  $n \leq 5$  and  $a_1 \in L^{2n/(n+2)}(\Omega)$  in (1.3), then  $\bar{\varphi} \in H_0^1(\Omega)$ .

*Proof.* Let us take  $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$  such that  $\Delta z \in L^p(\Omega)$ . For every  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ , we set

$$y_\lambda = \bar{y} + \lambda z \quad \text{and} \quad u_\lambda = \bar{u} - \lambda \Delta z + e^{\bar{y}} - e^{\bar{y} + \lambda z}.$$

Then Corollary 2.4 implies that  $y_\lambda \in \mathcal{Y}$ , with  $-\Delta y_\lambda = e^{y_\lambda} + u_\lambda$ . On the other hand  $u_\lambda \in L^p(\Omega)$ . Then  $(y_\lambda, u_\lambda)$  is a feasible point for (P) (resp. (P)), consequently, using Lebesgue's convergence theorem along with assumption (1.3), we get

$$\begin{aligned} 0 &\leq \lim_{\lambda \rightarrow 0} \frac{J(y_\lambda, u_\lambda) - J(\bar{y}, \bar{u})}{\lambda} \\ &= \lim_{\lambda \rightarrow 0} \int_\Omega \frac{L(x, y_\lambda(x)) - L(x, \bar{y}(x))}{\lambda} dx + \lim_{\lambda \rightarrow 0} \frac{N}{p} \int_\Omega \frac{|u_\lambda(x)|^p - |\bar{u}(x)|^p}{\lambda} dx \\ &= \int_\Omega \frac{\partial L}{\partial y}(x, \bar{y}(x))z(x) dx + N \int_\Omega |\bar{u}|^{p-2}\bar{u} (-\Delta z - e^{\bar{y}}z) dx. \end{aligned}$$

From the linearity of the previous relation with respect to  $z$  we deduce that

$$\int_\Omega \frac{\partial L}{\partial y}(x, \bar{y}(x))z(x) dx + N \int_\Omega |\bar{u}|^{p-2}\bar{u} (-\Delta z - e^{\bar{y}}z) dx = 0$$

for every  $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$  such that  $\Delta z \in L^p(\Omega)$ . Let us set  $\bar{\varphi} := -N|\bar{u}|^{p-2}\bar{u}$ , then

$$\int_\Omega \bar{\varphi} (-\Delta z - e^{\bar{y}}z) dx = \int_\Omega \frac{\partial L}{\partial y}(x, \bar{y}(x))z(x) dx \tag{5.3}$$

Given  $f \in C_0^\infty(\Omega)$ , let  $z \in H_0^1(\Omega)$  be the solution of Dirichlet problem

$$\begin{cases} -\Delta z = f & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{5.4}$$

Then  $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$  and for  $s < n/(n - 1)$  and  $(1/s) + (1/s') = 1$  we have (see G. Stampacchia [12], Theorems 4.1 and 4.2)

$$\|z\|_{L^\infty(\Omega)} \leq C_s \|f\|_{W^{-1,s'}(\Omega)}. \tag{5.5}$$

Combining (5.3), (5.4) and (5.5) we get

$$\begin{aligned} \int_\Omega \bar{\varphi} f dx &= \int_\Omega \bar{\varphi} (-\Delta z) dx = \int_\Omega \left\{ \frac{\partial L}{\partial y}(x, \bar{y}(x)) z(x) + e^{\bar{y}} z \right\} dx \\ &\leq \left( \left\| \frac{\partial L}{\partial y}(x, \bar{y}) \right\|_{L^1(\Omega)} + \|e^{\bar{y}}\|_{L^1(\Omega)} \right) \|z\|_{L^\infty(\Omega)} \\ &\leq C_s \left( \left\| \frac{\partial L}{\partial y}(x, \bar{y}) \right\|_{L^1(\Omega)} + \|e^{\bar{y}}\|_{L^1(\Omega)} \right) \|f\|_{W^{-1,s'}(\Omega)}. \end{aligned}$$

Taking into account that  $C_0^\infty(\Omega)$  is dense in  $W^{-1,s'}(\Omega)$ , we deduce from the above inequality that  $\bar{\varphi} \in \left( W^{-1,s'}(\Omega) \right)' = W_0^{1,s}(\Omega)$  for every  $s < n/(n - 1)$ . The fact that (5.2) follows from (5.3) is a straightforward consequence of the definition of  $\bar{\varphi}$ .

Finally, if  $p > n/2$ , from the fact that  $-\Delta \bar{y} = e^{\bar{y}} + \bar{u} \in L^p(\Omega)$  and using again the above mentioned results of G. Stampacchia [12], it follows that  $\bar{y} \in L^\infty(\Omega)$ . On the other hand,  $W_0^{1,s}(\Omega) \subset L^{ns/(n-s)}(\Omega) \subset H^{-1}(\Omega)$  if  $s$  is close enough to  $n/(n - 1)$  and  $n \leq 5$ . Therefore the right hand side of (5.2) belongs to  $H^{-1}(\Omega)$ , which allows to conclude that  $\bar{\varphi} \in H_0^1(\Omega)$ .  $\square$

In case of a problem with control constraints, we have the following result.

**THEOREM 5.2.** *Let us assume that  $\Omega$  is star-shaped with respect to some  $x_0 \in \Omega$ ,  $p \geq 2$  and  $p > n/2$ . If  $(\bar{y}, \bar{u})$  is a solution of problem (P) (resp. (P)) with  $e^{\bar{y}} \in L^p(\Omega)$ , then there exist a real number  $\bar{\alpha} \geq 0$  and a function  $\bar{\varphi} \in W_0^{1,s}(\Omega)$  for every  $s < n/(n - 1)$  such that*

$$\bar{\alpha} + \|\bar{\varphi}\|_{W_0^{1,s}(\Omega)} > 0; \tag{5.6}$$

$$\begin{cases} -\Delta \bar{y} = e^{\bar{y}} + \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 & \text{on } \Gamma; \end{cases} \tag{5.7}$$

$$\begin{cases} -\Delta \bar{\varphi} = e^{\bar{y}} \bar{\varphi} + \bar{\alpha} \frac{\partial L}{\partial y}(x, \bar{y}) & \text{in } \Omega, \\ \bar{\varphi} = 0 & \text{on } \Gamma; \end{cases} \tag{5.8}$$

$$\int_\Omega (\bar{\varphi} + \bar{\alpha} N |\bar{u}|^{p-2} \bar{u}) (u - \bar{u}) dx \geq 0 \quad \forall u \in K. \tag{5.9}$$

Moreover, if  $n \leq 5$  and  $a_1 \in L^{2n/(n+2)}(\Omega)$  in (1.3), then  $\bar{\varphi} \in H_0^1(\Omega)$ .



The proof of this theorem requires some previous lemmas. First of all, let us remark that  $\bar{y} \in L^\infty(\Omega)$ . Indeed  $-\Delta\bar{y} = e^{\bar{y}} + \bar{u} \in L^p(\Omega)$  with  $p > n/2$ , thus it is enough to use again the mentioned results of G. Stampacchia [12] to deduce the boundedness of  $\bar{y}$ . In particular we have that if  $(\bar{y}, \bar{u})$  is a solution of (P), then it is also a solution of (P) because  $\bar{y} \in \mathcal{Y}$ ; see Corollary 2.4.

Given  $\epsilon > 0$  we define  $J_\epsilon : L^p(\Omega) \times L^p(\Omega) \longrightarrow \mathbb{R}$  by

$$J_\epsilon(u, w) := J(y_{(u,w)}, u) + \frac{1}{p\epsilon} \int_\Omega |e^{y_{(u,w)}} - w|^p dx + \frac{1}{p} \int_\Omega |u - \bar{u}|^p dx + \frac{1}{p} \int_\Omega |e^{\bar{y}} - w|^p dx,$$

where  $y_{(u,w)}$  is the unique element of  $H_0^1(\Omega)$  solving the boundary value problem

$$\begin{cases} -\Delta y = u + w & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{5.10}$$

Once again using the mentioned results of G. Stampacchia [12] we obtain that  $y_{(u,w)} \in L^\infty(\Omega)$  and therefore  $J_\epsilon$  is well defined. Now we consider the following control problem

$$(P_\epsilon) \begin{cases} \text{Minimize } J_\epsilon(u, w) \\ (u, w) \in K \cap \bar{B}_1(\bar{u}) \times \bar{B}_1(e^{\bar{y}}), \end{cases}$$

where  $\bar{B}_1(\bar{u})$  (resp.  $\bar{B}_1(e^{\bar{y}})$ ) denotes the closed unit ball of  $L^p(\Omega)$  with center at  $\bar{u}$  (resp.  $e^{\bar{y}}$ ). We have the following result.

LEMMA 5.3. *Problem (P $_\epsilon$ ) has at least one solution  $(u_\epsilon, w_\epsilon)$ . Moreover we have*

$$\lim_{\epsilon \rightarrow 0} \|u_\epsilon - \bar{u}\|_{L^p(\Omega)} = \lim_{\epsilon \rightarrow 0} \|w_\epsilon - e^{\bar{y}}\|_{L^p(\Omega)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \|e^{y_\epsilon} - w_\epsilon\|_{L^p(\Omega)}^p = 0, \tag{5.11}$$

$$\lim_{\epsilon \rightarrow 0} \|y_\epsilon - \bar{y}\|_{L^\infty(\Omega)} = \lim_{\epsilon \rightarrow 0} \|y_\epsilon - \bar{y}\|_{H_0^1(\Omega)} = 0, \tag{5.12}$$

where  $y_\epsilon$  is the solution of (5.10) corresponding to  $(u_\epsilon, w_\epsilon)$ .

*Proof.* The existence of a solution is obvious because of the convexity, boundedness and closedness of the set of feasible controls as well as the weak lower semicontinuity of  $J_\epsilon$ . Furthermore  $\{(u_\epsilon, w_\epsilon)\}_{0 < \epsilon < 1}$  is bounded in  $L^p(\Omega) \times L^p(\Omega)$ , consequently  $\{y_\epsilon\}_{0 < \epsilon < 1}$  is bounded in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ . Then we can take subsequences such  $\epsilon_k \searrow 0$  as  $k \rightarrow \infty$  and  $(u_{\epsilon_k}, w_{\epsilon_k}) \rightarrow (\tilde{u}, \tilde{w})$  weakly in  $L^p(\Omega) \times L^p(\Omega)$ . Hence we get from (5.10) that  $y_{\epsilon_k} \rightarrow \tilde{y} = y_{(\tilde{u}, \tilde{w})}$  strongly in  $H_0^1(\Omega) \cap L^\infty(\Omega)$ .

Since  $(\bar{u}, e^{\bar{y}})$  is a feasible control for  $(P_\epsilon)$  and  $\bar{y} = y_{(\bar{u}, e^{\bar{y}})}$ , we have that  $J_\epsilon(u_\epsilon, w_\epsilon) \leq J_\epsilon(\bar{u}, e^{\bar{y}}) = J(\bar{y}, \bar{u})$ . Then we have

$$\lim_{\epsilon \rightarrow 0} \|e^{y_\epsilon} - w_\epsilon\|_{L^p(\Omega)} = 0,$$

which implies that  $e^{\tilde{y}} = \tilde{w}$ . Therefore  $\tilde{y} \in \mathcal{Y}$  is a solution of (1.1) corresponding to  $\tilde{u}$ . So  $(\tilde{y}, \tilde{u})$  is a feasible point for problem (P) (and also for

( $\mathcal{P}$ )), hence

$$\begin{aligned} J(\bar{y}, \bar{u}) &\leq J(\tilde{y}, \tilde{u}) \leq J(\tilde{y}, \tilde{u}) + \frac{1}{p} \|\tilde{u} - \bar{u}\|_{L^p(\Omega)}^p + \frac{1}{p} \|e^{\tilde{y}} - e^{\bar{y}}\|_{L^p(\Omega)}^p \\ &\leq \liminf_{k \rightarrow \infty} J_{\epsilon_k}(u_{\epsilon_k}, w_{\epsilon_k}) \leq J(\bar{y}, \bar{u}), \end{aligned}$$

which leads to the equalities  $\tilde{u} = \bar{u}$  and  $\tilde{y} = \bar{y}$ . Now the convergences (5.11) and (5.12) follow from the above inequality and the state equation (5.10).  $\square$

LEMMA 5.4. *There exist  $\epsilon_0 > 0$  and a sequence  $\{(u_\epsilon, w_\epsilon)\}_{0 < \epsilon < \epsilon_0}$  such that  $(u_\epsilon, w_\epsilon)$  is a solution of  $(P_\epsilon)$  and for all  $\epsilon > 0$  such that  $\epsilon < \epsilon_0$ :*

$$\|\bar{u} - u_\epsilon\|_{L^p(\Omega)} < 1 \quad \text{and} \quad \|\bar{u} - u_\epsilon\|_{L^p(\Omega)} < 1. \tag{5.13}$$

Furthermore, there exists  $\varphi_\epsilon \in W_0^{1,s}(\Omega)$  for every  $s < n/(n - 1)$  such that

$$\begin{cases} -\Delta y_\epsilon = u_\epsilon + w_\epsilon & \text{in } \Omega, \\ y_\epsilon = 0 & \text{on } \Gamma; \end{cases} \tag{5.14}$$

$$\begin{cases} -\Delta \varphi_\epsilon = e^{y_\epsilon} \varphi_\epsilon + \frac{\partial L}{\partial y}(x, y_\epsilon) + g_\epsilon & \text{in } \Omega, \\ \varphi_\epsilon = 0 & \text{on } \Gamma; \end{cases} \tag{5.15}$$

$$\forall u \in K, \quad \int_\Omega \{ \varphi_\epsilon + N|u_\epsilon|^{p-2}u_\epsilon + |u_\epsilon - \bar{u}|^{p-2}(u_\epsilon - \bar{u}) \} (u - u_\epsilon) dx \geq 0, \tag{5.16}$$

where  $g_\epsilon \rightarrow 0$  in  $L^{p'}(\Omega)$  as  $\epsilon \searrow 0$ .

*Proof.* The existence of a sequence  $\{(u_\epsilon, w_\epsilon)\}_{0 < \epsilon < \epsilon_0}$  of solutions satisfying (5.13) is a consequence of Lemma 5.3. Given  $h \in L^p(\Omega)$  we denote by  $z_h \in H_0^1(\Omega) \cap L^\infty(\Omega)$  the solution of

$$\begin{cases} -\Delta z = h & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

Let us set  $y_\epsilon = y_{(u_\epsilon, w_\epsilon)}$  and

$$\varphi_\epsilon = \frac{1}{\epsilon} |e^{y_\epsilon} - w_\epsilon|^{p-2} (e^{y_\epsilon} - w_\epsilon) + |e^{\bar{y}} - w_\epsilon|^{p-2} (e^{\bar{y}} - w_\epsilon).$$

From the optimality of  $(u_\epsilon, w_\epsilon)$  we deduce that for all  $u \in K$  and  $w \in L^p(\Omega)$ :

$$\frac{\partial J_\epsilon}{\partial u}(u_\epsilon, w_\epsilon)(u - u_\epsilon) \geq 0 \quad \text{and} \quad \frac{\partial J_\epsilon}{\partial w}(u_\epsilon, w_\epsilon)w = 0. \tag{5.17}$$

We compute the second derivative for each  $w \in L^p(\Omega)$

$$\frac{\partial J_\epsilon}{\partial w}(u_\epsilon, w_\epsilon)w = \int_\Omega \left\{ \frac{\partial L}{\partial y}(x, y_\epsilon) + e^{y_\epsilon} \varphi_\epsilon + g_\epsilon \right\} z_w dx - \int_\Omega \varphi_\epsilon w dx = 0, \tag{5.18}$$

where

$$g_\epsilon = -e^{y_\epsilon} |e^{\bar{y}} - w_\epsilon|^{p-2} (e^{\bar{y}} - w_\epsilon).$$

From (5.11) and (5.12) we get that  $g_\epsilon \rightarrow 0$  strongly in  $L^{p'}(\Omega)$ .

Computing now the first derivative of (5.17)

$$\begin{aligned} \frac{\partial J_\epsilon}{\partial u}(u_\epsilon, w_\epsilon)(u - u_\epsilon) &= \int_\Omega \left\{ \frac{\partial L}{\partial y}(x, y_\epsilon) + e^{y_\epsilon} \varphi_\epsilon + g_\epsilon \right\} z_{u-u_\epsilon} dx \\ &+ \int_\Omega \{N|u_\epsilon|^{p-2}u_\epsilon + |u_\epsilon - \bar{u}|^{p-2}(u_\epsilon - \bar{u})\} (u - u_\epsilon) dx \geq 0. \end{aligned}$$

Taking  $w = u - u_\epsilon$  in (5.18), one sees that (5.16) follows from this inequality.

Finally, from (5.18) we get that

$$\int_\Omega \varphi_\epsilon(-\Delta z) dx = \int_\Omega \left\{ \frac{\partial L}{\partial y}(x, y_\epsilon) + e^{y_\epsilon} \varphi_\epsilon + g_\epsilon \right\} z dx$$

for every  $z \in H_0^1(\Omega)$  with  $-\Delta z \in L^p(\Omega)$ . Arguing as in the proof of Theorem 5.1, we deduce from here that  $\varphi_\epsilon \in W_0^{1,s}(\Omega)$  for every  $s < n/(n - 1)$  and (5.15) holds. □

Now we are ready to conclude the proof of Theorem 5.2.

*Proof of Theorem 5.2.* If  $\{\varphi_\epsilon\}_{0 < \epsilon < \epsilon_0}$  is bounded in  $L^1(\Omega)$ , then  $\{\varphi_\epsilon\}_{0 < \epsilon < \epsilon_0}$  is also bounded in  $W_0^{1,s}(\Omega)$  and it is easy to pass to the limit in (5.14)–(5.16) with the aid of Lemma 5.3 and to deduce (5.7)–(5.9) with  $\bar{\alpha} = 1$ . Otherwise we take

$$\alpha_\epsilon = \frac{1}{\|\varphi_\epsilon\|_{L^1(\Omega)}}$$

and we redefine  $\varphi_\epsilon$  as  $\alpha_\epsilon \varphi_\epsilon$ . Then (5.15) and (5.16) can be written

$$\begin{cases} -\Delta \varphi_\epsilon = e^{y_\epsilon} \varphi_\epsilon + \alpha_\epsilon \frac{\partial L}{\partial y}(x, y_\epsilon) + \alpha_\epsilon g_\epsilon & \text{in } \Omega, \\ \varphi_\epsilon = 0 & \text{on } \Gamma \end{cases}$$

and

$$\forall u \in K, \quad \int_\Omega \{ \varphi_\epsilon + \alpha_\epsilon N|u_\epsilon|^{p-2}u_\epsilon + \alpha_\epsilon |u_\epsilon - \bar{u}|^{p-2}(u_\epsilon - \bar{u}) \} (u - u_\epsilon) dx \geq 0,$$

respectively. Now  $\{\varphi_\epsilon\}_{0 < \epsilon < \epsilon_0}$  is bounded in  $L^1(\Omega)$  and  $\alpha_\epsilon \rightarrow 0$ , then we can pass to the limit in the previous relations and to obtain (5.8) and (5.9) with  $\bar{\alpha} = 0$ . It remains to prove that (5.6) holds, or equivalently that  $\bar{\varphi} \neq 0$ . From the equation satisfied by  $\varphi_\epsilon$  we deduce that  $\{\varphi_\epsilon\}_{0 < \epsilon < \epsilon_0}$  is bounded in  $W_0^{1,s}(\Omega)$  for every  $s < n/(n - 1)$ . Then we can take a subsequence, denoted in the same way, such that  $\varphi_\epsilon \rightarrow \bar{\varphi}$  weakly in  $W_0^{1,s}(\Omega)$ , hence also strongly in  $L^1(\Omega)$ . Now the equality  $\|\varphi_\epsilon\|_{L^1(\Omega)} = 1$  leads to  $\|\bar{\varphi}\|_{L^1(\Omega)} = 1$ .

The  $H_0^1(\Omega)$ -regularity of  $\bar{\varphi}$  claimed in the theorem follows as in the proof of Theorem 5.1. □

In some cases we can prove that  $\bar{\alpha}$  can be chosen equal to one in the system (5.7)–(5.9).

**COROLLARY 5.5.** *Under the assumptions of Theorem 5.2, if furthermore there exists an open set  $\omega \subset \Omega$  and a neighborhood  $\mathcal{W}$  of zero in  $C_0^\infty(\omega)$  such that  $\bar{u} + w \in K$  for every  $w \in \mathcal{W}$ , then (5.7)–(5.9) holds with  $\bar{\alpha} = 1$ .*

*Proof.* If  $\bar{\alpha} = 0$  in (5.7)-(5.9), then from the hypothesis of the corollary and (5.9) we deduce that  $\bar{\varphi}(x) = 0$  for almost all  $x \in \omega$ . Now (5.8) is

$$\begin{cases} -\Delta \bar{\varphi} = e^{\bar{y}} \bar{\varphi} & \text{in } \Omega. \\ \bar{\varphi} = 0 & \text{on } \Gamma. \end{cases}$$

Then we deduce that  $\bar{\varphi} = 0$  in  $\Omega$  (see J.C. Saut and B. Scheurer [11]), which contradicts (5.6). □

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