ESAIM: Control, Optimisation and Calculus of Variations December 1998, Vol. 3, 381–405
URL: http://www.emath.fr/cocv/

INFINITE TIME REGULAR SYNTHESIS

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Abstract. In this paper we provide a new sufficiency theorem for regular syntheses. The concept of regular synthesis is discussed in [12], where a sufficiency theorem for finite time syntheses is proved.

There are interesting examples of optimal syntheses that are very regular, but whose trajectories have time domains not necessarily bounded. This research is motivated by the fact that one of the main tools toward the construction of optimal syntheses is the proof of a strong sufficiency theorem.

The regularity assumptions of the main theorem in [12] are verified by every piecewise smooth feedback control generating extremal trajectories that reach the target in finite time with a finite number of switchings (indeed even by more complicate syntheses like the Fuller one presenting trajectories with an infinite number of switchings). In the case of this paper the situation is even more complicate, since we admit both trajectories with finite and infinite time. It is important to notice that, in spite of its complexity, this situation is encountered in many simple cases like linear-quadratic problems (see the example of the last section and [13]).

We use weak differentiability assumptions on the synthesis and weak continuity assumptions on the associated value function. However, in this paper we need the value function to be continuous at the origin (see Remark 3.3 for more details). The general case of synthesis generated by general piecewise smooth feedback deserves a further careful investigation.

1. Introduction

Given an optimization problem for a control system with fixed initial data, there are various ways of giving a solution. One can find an open loop control or try to solve all the problems obtained by varying the initial data. Beside the classical concept of feedback, one can construct a trajectory for every initial data in such a way that the resulting collection has nice regularity properties. This gives raise to what is called a regular synthesis. Firstly introduced by Boltianskii, the concept of regular synthesis has been developed and used by many authors in connection with theoretical problems as well as for special classes of systems [2–8], [10–16], [18]. A synthesis is a mathematical object with more "structure" with respect to a feedback, and there are examples showing that a feedback alone is not sufficient to describe the solution to an optimization problem, see [12].

In this paper we provide a new sufficiency theorem for regular synthesis. The concept of regular synthesis is discussed in [12], where it is proved a

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Received by the journal August 21, 1997. Revised June 17, 1988. Accepted for publication October 8, 1998.

sufficiency theorem for syntheses formed by finite time trajectories. We refer to [12] and the references therein for synthesis theory.

There are interesting examples of optimal syntheses that are very regular, but whose trajectories have time domains not necessarily bounded, see [13]. One of the main tools toward the construction of optimal syntheses is the proof of a strong sufficiency theorem. If an optimization problem is sufficiently regular then every optimal trajectory must satisfy the Pontryagin Maximum Principle (in this case we say that the trajectory is extremal). Hence a first necessary condition for optimality is extremality. However, in general extremality is not a sufficient condition for a single trajectory (see [7], [8], [15]). Moreover, no regularity property can ensure the optimality of an extremal trajectory(see [15]). On the other hand, the sufficiency theorems guarantees that every extremal synthesis that is sufficiently regular is indeed optimal.

Usually one builds a candidate optimal synthesis using regularity properties of extremal trajectories. Thus extremality is a condition granted by construction. The regularity assumptions of the main theorem in [12] are verified by every piecewise smooth feedback control generating extremal trajectories that reach the target in finite time with a finite number of switchings. Moreover, the same theorem applies also to the system considered by Fuller, that presents trajectories with an infinite number of switchings (however it is not clear under which conditions it is possible to apply the theorem to a generic piecewise smooth feedback). In the case of this paper the situation is even more complicate since we admit trajectories defined on unbounded domains. We need weak differentiability assumptions on the synthesis and weak continuity assumptions on the value function associated to the synthesis. However, to treat infinite time trajectories, we have to assume the continuity of the value function at the origin, while only weak upper semicontinuity was assumed in [12] (see Remark 1 for more details). Moreover, we give two different definitions of weak differentiability, see Remark 3.3 of Section 3. The first relies on strong integrability properties (of the Jacobians of the dynamics and the cost) along the trajectories, the other, verified by the linear-quadratic example of Section 4, requires fast in time convergence of trajectories to the target.

We state a result valid for a point target problem. The same result can be used for Bolza problems without final condition when the form of the Lagrangian naturally forces the trajectories to tend asymptotically to a given point. This is the case of some examples, see [13] and Section 4.

We remark that in our problem we admit both trajectories with compact domain and trajectories with unbounded domains at the same time. As remarked in [12], our definition of regular synthesis is quite mild compared to the other definitions given in previous papers [2], [5], [6]. Moreover, these conditions can be checked easily in some interesting examples. The main result is stated for presynthesis even if we do not know interesting examples in which the concept of presynthesis plays a crucial role.

Finally, let us mention that another well known approach ensuring sufficiency theorems is the theory of viscosity solutions for the Hamilton–Jacobi–Bellmann equations, see [1],[9].

In Section 2 we give basic definitions, in Section 3 we state and prove the main result and in section 4 we give examples.

2. Basic definitions

Consider a control system:

$$\dot{x} = f(x, u) \qquad x \in \Omega, \qquad u \in U, \tag{2.1}$$

and the minimization problem:

$$\text{minimize} \quad \int L(\gamma(t), \eta(t))dt, \tag{2.2}$$

where (γ, η) is a trajectory–control pair satisfying some admissibility conditions that will be stated later.

Our basic assumption is

(H). Ω is an open subset of $\mathbb{R}^n$, U is a set, $f : \Omega \times U \to \mathbb{R}^n$ and $L : \Omega \times U \to \mathbb{R}$ are maps such that $f(\cdot, u)$ and $L(\cdot, u)$ are of class C^1 for every fixed $u \in U$.

We use the same notation of [12]. In particular, we use $\tilde{f}$ to denote the $\mathbb{R}^{n+1}$ valued map $(x, u) \to (f(x, u), L(x, u))$.

A *control* is a map $\eta : I \to U$, whose domain is a subinterval I of $\mathbb{R}$ (not necessarily bounded).

If $\mu : A \to B$ is a map, we use $\mathrm{Dom}(\mu)$ to indicate the domain of μ, i.e. the set A. In particular, if $\eta : I \to U$ is a control, then $\mathrm{Dom}(\eta) = I$.

A *trajectory* γ for a control η is an absolutely continuous map $\gamma : \mathrm{Dom}(\eta) \to \Omega$ that satisfies $\dot{\gamma}(t) = f(\gamma(t), \eta(t))$ for almost every $t \in \mathrm{Dom}(\eta)$. If $\mathrm{Dom}(\eta) = [a, b]$ (so that $\mathrm{Dom}(\gamma) = [a, b]$ as well), and $x = \gamma(a)$, $y = \gamma(b)$, we say that γ (or the pair (γ, η)) *goes from x to y*, and that η *steers x to y*, and we write $\gamma^- = x$, $\gamma^+ = y$. In the same way if $\mathrm{Dom}(\eta) = [a, +\infty)$ then we simply write $\gamma^- = x$. Moreover, if $\lim_{t\to+\infty} \gamma(t) = y$ then we write $\gamma^+ = y$.

A control $\eta : I \to U$ is *admissible* if the map $\Omega \times I \ni (x, t) \to \tilde{f}_\eta(x, t) = \tilde{f}(x, \eta(t)) \in \mathbb{R}^{n+1}$ satisfies the following C^1 Carathéodory conditions:

(A) $\tilde{f}_\eta$ is measurable as function of (t, x)

(B) for every compact $K \subset \Omega$ and every compact interval $J \subset I$, there exists an integrable function $\varphi_{K,J}$ such that for every $x \in K$ and $t \in J$:

$$\|\tilde{f}_\eta(x, t)\| + \|D_x \tilde{f}_\eta(x, t)\| \leq \varphi_{K,J}(t). \tag{2.3}$$

If η is an admissible control, γ a trajectory corresponding to η, such that the integral in (2.2) is defined (possibly equal to $+\infty$), then we say that (γ, η) is an *admissible pair* for $\tilde{f}$. We use $\mathrm{Adm}(\tilde{f})$ to denote the set of all admissible pairs for $\tilde{f}$. For an admissible pair (γ, η), we use $J(\gamma, \eta)$ to denote the *cost* of (γ, η), i.e. the value of the integral of (2.2) over $\mathrm{Dom}(\eta)$.

Given $\lambda \in \mathbb{R}_n$ (the space of row n vectors), $\lambda_0 \in \mathbb{R}$, $x \in \Omega$ and $u \in U$, we define:

$$\mathcal{H}(x, \lambda, \lambda_0, u) = \langle \lambda, f(x, u) \rangle + \lambda_0 L(x, u) \tag{2.4}$$

$$H(x, \lambda, \lambda_0) = \inf\{\mathcal{H}(x, \lambda, \lambda_0, u) : u \in U\}. \tag{2.5}$$

The functions $\mathcal{H} : \Omega \times \mathbb{R}_n \times \mathbb{R} \times U \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}_n \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ are known as the *Hamiltonian* and the *minimized Hamiltonian* of $\tilde{f}$.

We say that the pair $(\gamma, \eta) \in \mathrm{Adm}(\tilde{f})$ is *extremal* if there exist an absolutely continuous map $\lambda : \mathrm{Dom}(\eta) \to \mathbb{R}_n$, called the adjoint vector, and a constant $\lambda_0 \geq 0$, not both zero, such that the *adjoint equation*

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \lambda_0, \gamma(t), \eta(t)) \tag{2.6}$$

and the *minimization condition*:

$$H(\gamma(t), \lambda(t), \lambda_0) = \mathcal{H}(\gamma(t), \lambda(t), \lambda_0, \eta(t)) = 0 \tag{2.7}$$

hold for almost every $t \in \mathrm{Dom}(\eta)$. For a discussion of the validity of PMP under our assumptions see [18].

3. The main result

We consider the problem (2.1), (2.2) with a point target that, without loss of generality, will be assumed to be the origin. We consider only trajectories γ whose domain is bounded below, that is either $\mathrm{Dom}(\gamma) = [a, b]$ or $\mathrm{Dom}(\gamma) = [a, +\infty)$ for some $a \in \mathbb{R}$. Hence a trajectory has to start at some time a from a point x and either reach the origin in finite time or converge to it as t tends to $+\infty$. We call $\mathrm{Adm}^0(\tilde{f})$ the set of such trajectories (0 indicates the fact that they end at the origin).

If $\Gamma \subseteq \mathrm{Adm}^0(\tilde{f})$ is an arbitrary set of admissible pairs, we define a function $V_\Gamma : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, called the *value function of* Γ, by letting $V_\Gamma(x)$ be, for $x \in \Omega$, the infimum of the costs $J(\gamma, \eta)$, taken over the set of all $(\gamma, \eta) \in \Gamma$ such that γ goes from x to 0. (In particular, if there is no $(\gamma, \eta) \in \Gamma$ going from x to 0 then $V_\Gamma(x) = +\infty$.) If $\Gamma = \mathrm{Adm}^0(\tilde{f})$, then the function V_Γ is the *value function* of our problem, and in that case we will write $V_{\tilde{f}}$ rather than $V_{\mathrm{Adm}^0(\tilde{f})}$. A pair $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f})$ is optimal if $V_{\tilde{f}}(\gamma^-) = J(\gamma, \eta)$.

A *presynthesis* for our problem on a set $S \subset \Omega$ is a set $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ of admissible pairs $(\gamma_x, \eta_x) \in \mathrm{Adm}^0(\tilde{f})$ such that $\gamma_x^- = x$. The set S is called the *domain* of Γ. If the domain S of Γ consists of all points that can be steered to the origin by an admissible pair, then we say that Γ is *total*. Given a presynthesis Γ we will always renormalize the time along every (γ_x, η_x) in such a way that the domain of η_x (and of γ_x) either is of the form $[0, T_x]$ for some non negative number T_x or it is $[0, +\infty)$.

A presynthesis is *memoryless* if whenever $x \in S$ and $t \in \mathrm{Dom}(\eta_x)$ it follows that $y = \gamma_x(t)$ belongs to S and η_y is the renormalization of the restriction of η_x to the interval $[t, T_x]$ or $[t, +\infty)$. A *synthesis* is a memoryless presynthesis.

If each pair of a presynthesis Γ is optimal (resp. extremal) then we say that Γ is *optimal* (resp. *extremal*). In particular, a presynthesis Γ is a total optimal presynthesis if and only if $V_\Gamma \equiv V_{\tilde{f}}$.

Our goal is to prove that, under suitable regularity assumptions on Γ and V_Γ, a total extremal presynthesis is optimal. We first describe these regularity assumptions in detail.

Given a locally Lipschitz vector field X on Ω, we say that V has the *NDJ ("no downward jumps") property along* X if the following holds:

(NDJ). For every γ integral curve of X, $t \in \mathrm{Dom}(\gamma)$, if $a = \inf \mathrm{Dom}(\gamma)$ then for every $t \in \mathrm{Dom}(\gamma) \setminus \{a\}$ we have

$$\limsup_{h \to 0+} V(\gamma(t)) - V(\gamma(t-h)) \geq 0.$$

We say that V satisfies the *weak continuity conditions* for the control problem (2.1), (2.2) if V is lower semicontinuous, continuous at the origin, and has the NDJ property along the vector field $x \to f(x,u)$ for every $u \in U$.

We now define the "weak differentiability conditions" for Γ. In this definition it is understood that $\tilde{f}(y, \eta_x(t)) = 0$ for every $t \notin \mathrm{Dom}(\eta_x)$. A presynthesis $\Gamma = \{(\gamma_x, \eta_x) : x \in S\}$ is (f,L)*–differentiable* at a point $\bar{x} \in \Omega$ if the following assumption is satisfied:

(A) there exist an open set W, containing $\{\gamma_{\bar{x}}(t) : t \in \mathrm{Dom}(\eta_{\bar{x}})\}$, a neighborhood N of $\bar{x}$ (in Ω) such that $N \subseteq \mathrm{Dom}(\Gamma)$, with the property that

(A1) for every compact $K \subset \Omega$, there exist an integrable function $\varphi_K : [0, +\infty] \to \mathbb{R}$ such that $|\varphi_{J,K}(t)| \leq \varphi_K(t)$ for every compact $J \subset I$, $t \in \mathrm{Dom}(\varphi_{J,K}(t))$ (here $\varphi_{J,K}$ are defined as in (2.3) for the control $\eta_{\bar{x}}$), and, for sufficiently small $\varepsilon > 0$, integrable functions $\psi_\varepsilon : [0, +\infty] \to \mathbb{R}$ such that $\lim_{\varepsilon \to 0} \int_0^{+\infty} \psi_\varepsilon(t)\, dt = 0$, and the inequalities

$$\|\tilde{f}(y, \eta_x(t)) - \tilde{f}(y, \eta_{\bar{x}}(t))\| \leq \psi_\varepsilon(t)$$

$$\|D_y \tilde{f}(y, \eta_x(t)) - D_y \tilde{f}(y, \eta_{\bar{x}}(t))\| \leq \psi_\varepsilon(t)$$

hold for every $y \in W$, $x \in N$ such that $\|x - \bar{x}\| \leq \varepsilon$, $t \in [0, +\infty)$.

(A2) the map $v \to \rho_v$, where ρ_v is the integrable $\mathbb{R}^{n+1}$ valued function on $[0, +\infty)$ given by

$$\rho_v(t) = \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}+v}(t)) - \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))$$

is vaguely differentiable at $v = 0$ or it is weak*-differentiable at $v = 0$, regarded as a map into the space of $\mathbb{R}^{n+1}$–valued Borel measures. That is: for every continuous function $\alpha : [0, +\infty) \to \mathbb{R}$, that vanishes at $+\infty$ ($\lim_{t \to +\infty} \alpha(t) = 0$), the map $\mathbb{R}^n \ni v \to \int_0^{+\infty} \rho_v(t)\alpha(t)\, dt \in \mathbb{R}^{n+1}$ is differentiable at $v = 0$.

Moreover the map $\mathbb{R}^n \ni v \to \int_0^{+\infty} \rho_v''(t)\, dt \in \mathbb{R}$, where ρ_v'' is the $(n+1)$-th component of ρ_v, is differentiable at $v = 0$.

A set $A \subset \Omega$ is $(n-1)$ *dimensional rectifiable* if there exist A_1, A_2, such that $A = A_1 \cup A_2$, A_1 is a finite or countable union of connected embedded

C^1 submanifolds of positive codimension and A_2 verifies $\mathcal{H}^{n-1}(A_2) = 0$ where $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure.

We can now state the main theorem (cf. [12], [18]):

THEOREM 3.1. *Let Ω, U, f, L be such that $0 \in \Omega$ and assumption (H) hold. Let Γ be a total extremal presynthesis. Assume that*

(i) the associated value function V_Γ satisfies the weak continuity conditions,
(ii) $V_\Gamma(0) = 0$,
(iii) Γ is (f, L)–differentiable at all points in the complement of a $(n-1)$ dimensional rectifiable set A.
Then Γ is optimal.

Proof. We first prove the relation between the differential of the value function V_Γ and the adjoint covector fields along the extremal trajectories of the presynthesis. This is of interest in itself so we state it as a separate theorem. □

THEOREM 3.2. *Let Ω, U, f, L be such that $0 \in \Omega$ and assumption (H) hold. Let Γ be an extremal presynthesis. If Γ is (f, L)–differentiable at $\bar{x}$ and if λ denote the adjoint covector associated to the pair $(\gamma_{\bar{x}}, \eta_{\bar{x}}) \in \Gamma$ then $\lambda(0) = D_y V_\Gamma(\bar{x})$.*

Proof of Theorem 3.2. Fix a point $\bar{x}$ of (f, L)–differentiability for Γ. Let W, N, ψ_ε, be as given by Condition *(A)*. Pick $\delta > 0$ such that the compact set

$$W' = \{x : d(x, \{\gamma_{\bar{x}}(t) : t \in \mathrm{Dom}(\eta_{\bar{x}})\}) \leq \delta\}$$

satisfies $W' \subseteq W$. □

Let X be the Banach space $C_0([0, +\infty), \mathbb{R})$ of continuous real-valued functions on $[0, +\infty)$, that vanishes at $+\infty$, endowed with the sup norm. We recall that usually X is defined as the completion for the sup norm of the space of continuous functions with compact support.

With $\rho_v : [0, +\infty) \to \mathbb{R}^{n+1}$ defined as in *(A2)* for $v \in \mathbb{R}^n$, $||v||$ small, let $\rho_v^1, \ldots, \rho_v^{n+1}$ be the components of ρ_v, so each ρ_v^j is an integrable real-valued function on $[0, +\infty)$. Then for each $\alpha \in X$ and each $j \in \{1, \ldots, n+1\}$ the function $v \to \int_0^{+\infty} \alpha(t)\rho_v^j(t)dt$ is differentiable at $v = 0$, so there exists, for each j, a unique α-dependent vector $w^j(\alpha) \in \mathbb{R}^n$ such that

$$\int_0^{+\infty} \alpha(t)\rho_v^j(t)dt = \langle w^j(\alpha), v\rangle + o(||v||) \text{ as } v \to 0. \tag{3.1}$$

It is clear that $w^j(\alpha)$ depends linearly on α, so each component $w_i^j(\alpha)$ $(i = 1, \ldots, n)$ of $w^j(\alpha)$ depends linearly on α as well. For every $v \in \mathbb{R}^n$ sufficiently small, $\rho_v^j\, dt$ is a finite Borel measure, hence an element of X^* (the dual of X). Let us indicate with by $\|\cdot\|_{X^*}$ the norm of X^*. For every fixed $\alpha \in X$, from the differentiability at $v = 0$, it follows that there exists $C_1 > 0$ such that for every $v \neq 0$ sufficiently small

$$\frac{1}{\|v\|}\int \alpha\rho_v^j\, dt \leq C_1.$$

Hence the Uniform Boundedness Theorem implies that there exists a constant C_2 such that

$$\frac{\|\rho_v^j\|_{L^1}}{\|v\|} = \left\|\frac{\rho_v^j}{\|v\|}\right\|_{X^*} \leq C_2, \tag{3.2}$$

for v small and $j \in \{1, \ldots, n+1\}$. (Otherwise there would exist a j and a sequence $\{v_\ell\}$ such that $v_\ell \to 0$, $v_\ell \neq 0$, and $\int_0^{+\infty} ||\rho_{v_\ell}^j(t)||dt > K_\ell ||v_\ell||$, with $K_\ell \to +\infty$. By passing to a subsequence, we may assume that $\frac{v_\ell}{||v_\ell||}$ converges to a unit vector v. Then by (3.1) the continuous linear functionals $\xi_\ell : X \to \mathbb{R}$ given by

$$\xi_\ell(\alpha) = \frac{1}{||v_\ell||}\int_0^{+\infty} \alpha(t)\rho_{v_\ell}^j(t)dt$$

converge pointwise on X to the map $\alpha \to \langle w^j(\alpha), v\rangle$. So the sequence $\{\xi_\ell(\alpha)\}_{\ell=1}^\infty$ is bounded for each α, and then there exists —by Uniform Boundedness— a $C > 0$ such that $||\xi_\ell||_{X^*} \leq C$ for all ℓ. But

$$||\xi_\ell||_{X^*} = \frac{1}{||v_\ell||}\int_0^{+\infty} ||\rho_{v_\ell}^j(t)||dt > K_\ell,$$

and we have derived a contradiction.) From (3.2), it follows in particular that

$$|\langle w^j(\alpha), v\rangle| \leq C||\alpha|| \cdot ||v|| \text{ for } \alpha \in X\,, v \in \mathbb{R}^n,$$

so each w_i^j is a bounded linear functional on X, and $||w_i^j||_{X^*} \leq C$. Therefore each w_i^j is given by a (signed) Borel measure on $[0, +\infty)$, that will also be denoted by w_i^j. Let $\hat{w}_i^j$ denote the indefinite integral of w_i^j, i.e.

$$\hat{w}_i^j(t) = w_i^j([0,t]) \text{ for } t \in (0, +\infty), \;\; \hat{w}_i^j(0) = 0.$$

Let $BV(0,\infty)$ denote the space of all functions $w : [0, +\infty) \to \mathbb{R}$ of bounded variation such that $w(0) = 0$ and w is right-continuous at every $t \in (0, +\infty)$. Let $\mathcal{M}(0,\infty)$ denote the space of all monotonically nondecreasing functions $w : [0, +\infty) \to \mathbb{R}$ such that $w(0) = 0$ and w is right-continuous at every $t \in (0, +\infty)$. Then there is a canonical correspondence between members of $BV(0,\infty)$ and signed Borel measures on $[0, +\infty)$, that assigns to a function $w \in BV(0,\infty)$ the unique Borel measure μ_w such that $\mu_w([0,t]) = w(t)$ for $0 < t < +\infty$. (The measure of a single point set $\{t\}$ is then $\mu_w(t+) - \mu_w(t-)$, which is equal to $\mu_w(t) - \mu_w(t-)$ if $t > 0$ and to $\mu_w(t+)$ if $t = 0$. In particular, $\mu_w(\{t\}) = 0$ iff μ_w is continuous at t.) From now on we identify a function $w \in BV(0,\infty)$ with its corresponding measure μ_w, and write $\int \alpha\, dw$ for $\int \alpha\, d\mu_w$. We emphasize that, if w is not continuous, then for an integral such as $\int_s^t dw$ to be well defined we have to be careful to specify whether the integral is to be interpreted as $\int \chi_{[s,t]} dw$ or $\int \chi_{(s,t]} dw$ or $\int \chi_{[s,t)} dw$ or $\int \chi_{(s,t)} dw$, so we will never write $\int_s^t dw$ unless w is continuous.

Clearly, each $\hat{w}_i^j$ belongs to $BV(0,\infty)$, and (3.1) can be rewritten as

$$\int_0^{+\infty} \alpha(t)\rho_v^j(t)\, dt = \sum_{i=1}^n v^i \int \alpha\, d\hat{w}_i^j + o(||v||), \tag{3.3}$$

where $v = (v^1, \ldots, v^n)$.

We now let $\hat{\rho}_v^j(t) = \int_0^t \rho_v^j(s)ds$, so each $\hat{\rho}_v^j(t)$ is an absolutely continuous function $[0, +\infty) \to \mathbb{R}$ and a member of $BV(0, \infty)$, that satisfies

$$|\hat{\rho}_v^j(t)| \le C_2 ||v|| \text{ for small } v\,,\ t \in [0, +\infty)\,,\ j \in \{1, \ldots, n+1\}.$$

We let Y be the space $L^\infty([0, +\infty), \mathbb{R})$ of all real-valued bounded measurable functions on $[0, +\infty)$, endowed with the weak* topology of Y regarded as the dual of $L^1([0, +\infty), \mathbb{R})$, so a net $\{y_d : d \in D\}$ of members of Y —where D is a directed set— converges to a $y \in Y$ iff $\int \alpha(t) y_d(t)dt \to \int \alpha(t) y(t)dt$ for all $\alpha \in L^1([0, +\infty), \mathbb{R})$. We remark that the topology of Y is metrizable on subsets of Y that are bounded in the L^∞ norm. Therefore, as long as we are dealing with bounded sets the topology is entirely characterized by the convergence of sequences.

We show that

$$\hat{\rho}_v^j = \sum_{i=1}^n v^i \hat{w}_i^j + o(||v||), \tag{3.4}$$

in the sense that

$$\lim_{v \to 0} \frac{1}{||v||} \left| \hat{\rho}_v^j - \sum_{i=1}^n v^i \hat{w}_i^j \right| = 0 \ \text{ in } Y\,, \quad \text{for } j = 1, \ldots, n+1. \tag{3.5}$$

To see this, fix j, and write

$$Q^j(v)(t) = \frac{1}{||v||} \left| \hat{\rho}_v^j(t) - \sum_{i=1}^n v^i \hat{w}_i^j(t) \right|. \tag{3.6}$$

Let $\rho_v^j(t) = \rho_v^{j,+}(t) - \rho_v^{j,-}(t)$, where $\rho_v^{j,+}(t) = \max(\rho_v^j(t), 0)$, and define

$$\hat{\rho}_v^{j,+}(t) = \int_0^t \rho_v^{j,+}(s)ds, \quad \hat{\rho}_v^{j,-}(t) = \int_0^t \rho_v^{j,-}(s)ds,$$

so each $\hat{\rho}_v^{j,+}$, $\hat{\rho}_v^{j,-}$ is a monotonically nondecreasing continuous function on $[0, +\infty)$ with the property that $\hat{\rho}_v^{j,+}(0) = \hat{\rho}_v^{j,-}(0) = 0$, and $\lim_{t \to +\infty} \hat{\rho}_v^{j,+}(0) + \hat{\rho}_v^{j,-}(0) \le C||v||$. If $\{v_k\}_{k=1}^\infty$ is a sequence of nonzero vectors in $\mathbb{R}^n$ converging to 0, then it follows from Helly's Theorem that there exists a subsequence $\{v_{k(\ell)}\}$ and functions $\omega^{j,+}$, $\omega^{j,-}$ belonging to $\mathcal{M}(0, \infty)$, such that

$$\frac{1}{||v_{k(\ell)}||} \hat{\rho}_{v_{k(\ell)}}^{j,+}(t) \to \omega^{j,+}(t) \ \text{ and } \ \frac{1}{||v_{k(\ell)}||} \hat{\rho}_{v_{k(\ell)}}^{j,-}(t) \to \omega^{j,-}(t) \tag{3.7}$$

for all $t \in [0, +\infty)$ that are points of continuity of $\omega^{j,+}$ and $\omega^{j,-}$. By passing to a subsequence, if necessary, we may assume that there is a vector v such that $||v|| = 1$ for which

$$\lim_{\ell \to \infty} \frac{v_{k(\ell)}}{||v_{k(\ell)}||} = v. \tag{3.8}$$

It then follows that

$$\frac{1}{||v_{k(\ell)}||}\int_0^{+\infty}\alpha(t)\rho^{j,+}_{v_{k(\ell)}}(t)dt \to \int \alpha d\omega^{j,+}$$

and

$$\frac{1}{||v_{k(\ell)}||}\int_0^{+\infty}\alpha(t)\rho^{j,-}_{v_{k(\ell)}}(t)dt \to \int \alpha d\omega^{j,-}$$

for all $\alpha \in X$. Indeed, if $\alpha(t) = \sum_{k=1}^{n} a_k \chi_{[t_k,t_{k+1}]}(t)$, for some $a_k \in \mathbb{R}$ and some points $t_k \in \mathbb{R}$ of continuity for $\omega^{j,\pm}$, the convergence easily follows from (3.7). By approximation, we obtain the conclusion for every α. Therefore, if we let $\omega^j = \omega^{j,+} - \omega^{j,-}$, we see that $\omega^j \in BV(0,\infty)$ and

$$\frac{1}{||v_{k(\ell)}||}\int_0^{+\infty}\alpha(t)\rho^{j}_{v_{k(\ell)}}(t)dt \to \int \alpha d\omega^{j}$$

for all $\alpha \in X$.

On the other hand, (3.3) and (3.8) imply that

$$\frac{1}{||v_{k(\ell)}||}\int_0^{+\infty}\alpha(t)\rho^{j}_{v_{k(\ell)}}(t)dt \to \sum_{i=1}^{n} v^i \int \alpha d\hat{w}^j_i$$

for all $\alpha \in X$. So $\omega^j \equiv \sum_{i=1}^{n} v^i \hat{w}^j_i$, and then (3.7) implies that

$$\frac{1}{||v_{k(\ell)}||}\hat{\rho}^j_{v_{k(\ell)}}(t) \to \sum_{i=1}^{n} v^i \hat{w}^j_i(t)$$

for all $t \in [0,+\infty)$ except possibly on a countable set. But

$$\frac{1}{||v_{k(\ell)}||}\sum_{i=1}^{n} v^i_{k(\ell)} \hat{w}^j_i(t) \to \sum_{i=1}^{n} v^i \hat{w}^j_i(t)$$

for all t. So

$$Q^j(v_{k(\ell)})(t) = \frac{1}{||v_{k(\ell)}||}\left|\hat{\rho}^j_{v_{k(\ell)}}(t) - \sum_{i=1}^{n} v^i_{k(\ell)} \hat{w}^j_i(t)\right| \to 0 \tag{3.9}$$

for almost all t. Therefore, since $Q^j(v_{k(\ell)})(t)$ is bounded by a fixed constant, independent of t and ℓ, the Dominated Convergence Theorem implies that $Q^j(v_{k(\ell)}) \to 0$ in Y. So we have shown that every sequence $\{v_k\}$ converging to 0 in $\mathbb{R}^n$ has a subsequence $\{v_{k(\ell)}\}$ for which $Q^j(v_{k(\ell)}) \to 0$ in Y. This implies that (3.6) holds.

We can rewrite (3.6) in vector form by introducing the column-vector-valued functions $\hat{\rho}_v : [a,b] \to \mathbb{R}^{n+1}$ given by $\hat{\rho}_v(t) = \int_0^t \rho_v(s)ds$, and the matrix-valued function $\hat{W} : [0,+\infty) \to \mathbb{R}^{n+1\times n}$ (with $n+1$ rows and n

columns) whose entry in column i, row j, is $\hat{w}_i^j$. Then (3.6) clearly implies that the scalar function

$$t \to \frac{1}{||v||} \left\| \hat{\rho}_v(t) - \hat{W}(t).v \right\|$$

converges to 0 in Y. It will also be convenient to split ρ_v into a n-dimensional part ρ'_v corresponding to the first n components and a scalar part ρ''_v corresponding to the $n+1$-th component. With the obvious definition of $\hat{\rho}'_v$, $\hat{\rho}''_v$, $\hat{W}'$, $\hat{W}''$, we can conclude that the scalar functions

$$t \to \frac{1}{||v||} \left\| \hat{\rho}'_v(t) - \hat{W}'(t).v \right\| \quad \text{and} \quad t \to \frac{1}{||v||} \left| \hat{\rho}''_v(t) - \hat{W}''(t).v \right|$$

converge to 0 in Y.

Now consider $x = \bar{x} + \varepsilon v$, $v \in \mathbb{R}^n$, $\|v\| = 1$ and $\varepsilon \in \mathbb{R}$ small.

Let φ be such that $\|f(x, \nu_{\bar{x}}(t)) - f(y, \nu_{\bar{x}}(t)\| \leq \varphi(t)\|x - y\|$ for every $x, y \in W'$ and $t \in [a, b]$. There exists $\bar{\varepsilon}$ such that:

$$\int_0^{+\infty} \psi_\varepsilon(t)\, dt < \frac{\delta}{2\, e^{\|\varphi\|_{L^1}}}$$

for every $\varepsilon \in [0, \bar{\varepsilon}]$. From now on we consider only those x corresponding to $\varepsilon \leq \bar{\varepsilon}$. Fix now such an x. There exists $T(x)$ such that:

$$\|\gamma_x(T(x))\| < \frac{\delta}{4e^{\|\varphi\|_{L^1}}}, \qquad \|\gamma_{\bar{x}}(T(x))\| < \frac{\delta}{4e^{\|\varphi\|_{L^1}}}, \tag{3.10a}$$

and for $t \geq T(x)$

$$\|\gamma_x(t) - \gamma_{\bar{x}}(t)\| \leq \|x - \bar{x}\|. \tag{3.10b}$$

Notice that we may have that the trajectory γ_x (or $\gamma_{\bar{x}}$) reaches the origin in finite time. But in this case we consider it as prolonged on $[0, +\infty)$ by $\gamma_x(t) = 0$ for $t \geq \sup(\mathrm{Dom}(\gamma_x))$. This is compatible with the definition $\hat{f}(\gamma_x(t), \eta_x(t)) = 0$ for $t \notin Dom(\eta_x)$.

Define $\mu_x(t) = \gamma_x(-t)$, $\nu_x(t) = \eta_x(-t)$ so that $\mu_x : [-T(x), 0] \to \mathbb{R}^n$ is γ_x run backward in time and it satisfies the equation:

$$\dot{\mu}_x(t) = -f(\mu_x(x), \nu_x(t)).$$

Now

$$\begin{aligned}
\|\mu_x(t) - \mu_{\bar{x}}(t)\| \leq\ & \|\mu_x(-T(x)) - \mu_{\bar{x}}(-T(x))\| \\
& + \int_{-T(x)}^{t} \|f(\mu_x(s), \nu_x(s)) - f(\mu_{\bar{x}}(s), \nu_{\bar{x}}(s))\| ds \\
\leq\ & \|\mu_x(-T(x)) - \mu_{\bar{x}}(-T(x))\| \\
& + \int_{-T(x)}^{t} \|f(\mu_x(s), \nu_x(s)) - f(\mu_x(s), \nu_{\bar{x}}(s))\| ds \\
& + \int_{-T(x)}^{t} \|f(\mu_x(s), \nu_{\bar{x}}(s)) - f(\mu_{\bar{x}}(s), \nu_{\bar{x}}(s))\| ds \\
\leq\ & \|\mu_x(-T(x)) - \mu_{\bar{x}}(-T(x))\| \\
& + \int_{-T(x)}^{t} \psi_\varepsilon(-s) ds + \int_0^t \varphi(-s)\|\mu_x(s) - \mu_{\bar{x}}(s)\| ds
\end{aligned}$$

as long as $\mu_x(s), \mu_{\bar{x}}(s) \in W'$ for every $s \in [-T(x), 0]$. Then

$$\begin{aligned}\|\mu_x(t)-\mu_{\bar{x}}(t)\| &\leq e^{\|\varphi\|_{L^1}}\left(\|\mu_x(-T(x))-\mu_{\bar{x}}(-T(x))\|+\int_{-T(x)}^{t}\psi_\varepsilon(-s)ds\right)\\ &\leq e^{\|\varphi\|_{L^1}}\left(\|\gamma_x(T(x))\|+\|\gamma_{\bar{x}}(T(x))\|+\int_0^{+\infty}\psi_\varepsilon(s)ds\right)\\ &\leq \frac{\delta}{4}+\frac{\delta}{4}+\frac{\delta}{2}=\delta.\end{aligned} \tag{3.11}$$

Assume by contradiction that $\mu_x(t)$ is not in W' for some $t \in [-T(x), 0]$ and let $\bar{t}$ the first time in which this happens. Using the estimate (3.11) at time $\bar{t}$ we obtain

$$\|\mu_x(\bar{t})-\mu_{\bar{x}}(\bar{t})\| < \delta \tag{3.12}$$

that gives us the contradiction. Hence we can conclude that for $\varepsilon \in [0, \bar{\varepsilon}]$, $\mu_{\bar{x}+\varepsilon v}(t) \in W'$ for $t \in [-T(x), 0]$ and from now on we consider only those $x = \bar{x} + \varepsilon v$ for which $\varepsilon \leq \bar{\varepsilon}$.

For simplicity, denote by φ the integrable function $\varphi_{W'}$ of *(A1)*. From *(A1)* it follows

$$\begin{aligned}\sup_{y\in W'}\|D_y\tilde{f}(y,\nu_x(t))\| &\leq \sup_{y\in W'}\|D_y\tilde{f}(y,\nu_x(t))-D_y\tilde{f}(y,\nu_{\bar{x}}(t))\|\\ &\quad+\sup_{y\in W'}\|D_y\tilde{f}(y,\nu_{\bar{x}}(t))\|\\ &\leq \psi_\varepsilon(t)+\varphi(t).\end{aligned} \tag{3.13}$$

Notice that from (3.12) we know that the segment joining $\mu_x(t)$ to $\mu_{\bar{x}}(t)$ is inside W' thus we can apply the mean value inequality and use (3.13). Hence

$$\begin{aligned}\|\mu_x(t)-\mu_{\bar{x}}(t)\| &\leq \|\mu_x(-T(x))-\mu_{\bar{x}}(-T(x))\|\\ &\quad+\int_{-T(x)}^{t}\|f(\mu_x(s),\nu_x(s))-f(\mu_{\bar{x}}(s),\nu_x(s))\|ds\\ &\quad+\int_{-T(x)}^{t}\|f(\mu_{\bar{x}}(s),\nu_x(s))-f(\mu_{\bar{x}}(s),\nu_{\bar{x}}(s))\|ds\\ &\leq \|\mu_x(-T(x))-\mu_{\bar{x}}(-T(x))\|\\ &\quad+\int_{-T(x)}^{t}(\psi_\varepsilon(s)+\varphi(s))\|\mu_x(s)-\mu_{\bar{x}}(s)\|ds\\ &\quad+\int_{-T(x)}^{t}\|\rho_{\varepsilon v}(-s)\|ds\end{aligned}$$

and applying Gronwall Lemma, from *(A2)*

$$\begin{aligned}\|\mu_x(t)-\mu_{\bar{x}}(t)\| &\leq e^{\|\varphi\|_{L^1}+\|\psi_\varepsilon\|_{L^1}}\|\mu_x\Big(-T(x))-\mu_{\bar{x}}(-T(x))\|\\ &\qquad+\int_{-T(x)}^{t}\|\rho_{\varepsilon v}(-s)\|ds\Big)\\ &\leq e^{\|\varphi\|_{L^1}+\|\psi_\varepsilon\|_{L^1}}(\|x-\bar{x}\|+C\|\varepsilon v\|)\\ &= e^{\|\varphi\|_{L^1}+\|\psi_\varepsilon\|_{L^1}}(1+C)\,\|x-\bar{x}\|.\end{aligned} \tag{3.14}$$

Now, from (3.2), (3.10) and (3.14), we have that there exists a constant C_3 such that

$$\|\gamma_x(t) - \gamma_{\bar{x}}(t)\| \leq C_3 \|x - \bar{x}\|. \tag{3.15}$$

We now prove the differentiability of V_Γ at $\bar{x}$. We have:

$$\begin{aligned} \frac{d}{dt}(\gamma_x(t) - \gamma_{\bar{x}}(t)) &= f(\gamma_x(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \\ &= f(\gamma_x(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_x(t)) \\ &\qquad + f(\gamma_{\bar{x}}(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \\ &= \int_0^1 D_y f(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) \\ &\qquad - \gamma_{\bar{x}}(t)), \eta_x(t)) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))\, d\theta + \rho_v'(t) \\ &= A(t) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) + \rho_v'(t) + Z_x(t), \end{aligned}$$

where

$$A(t) = D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)),$$

$$\begin{aligned} Z_x(t) &= Z_x^1(t) + Z_x^2(t), \\ Z_x^1(t) &= \int_0^1 B_1(x, t, \theta) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) d\theta, \\ Z_x^2(t) &= \int_0^1 B_2(x, t, \theta) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) d\theta, \end{aligned} \tag{3.16}$$

$$\begin{aligned} B_1(x,t,\theta) &= D_y f(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_x(t)) \\ &\qquad - D_y f(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t)), \\ B_2(x,t,\theta) &= D_y f(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t)) \\ &\qquad - D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)), \end{aligned}$$

and, as explained before, we write $\rho = (\rho', \rho'')$ with $\rho'(t) \in \mathbb{R}^n$, $\rho''(t) \in \mathbb{R}$. We use $M(t,s)$ to denote the fundamental matrix solution of the linear system:

$$\dot{w} = A(t) \cdot w(t)\,, \;\; 0 \leq t < +\infty,$$

and observe that the matrix-valued map $t \to A(t)$ is Lebesgue integrable, because of the integral bound $\|A(t)\| \leq \psi(t)$, which follows from *(A1)*. Then $M : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^{n \times n}$ is continuous, and

$$\gamma_x(t) - \gamma_{\bar{x}}(t) = M(t,0) \cdot v + \int_0^t M(t,s) \cdot \rho_v'(s)\, ds + \int_0^t M(t,s) \cdot Z_x(s)\, ds. \tag{3.17}$$

Next, we claim that

$$\lim_{v \to 0} \frac{1}{||v||} \sup \left\{ \left\| \int_0^t M(t,s) \cdot Z_x(s)\, ds \right\| : 0 \leq t < +\infty \right\} = 0. \tag{3.18}$$

Indeed, (3.15) implies that

$$\sup\{\|\gamma_x(t)-\gamma_{\bar{x}}(t)\| : 0 \le t < +\infty\} = O(\|v\|).$$

Moreover, $\gamma_{\bar{x}}(t)+\theta(\gamma_x(t)-\gamma_{\bar{x}}(t)) \in W' \subset W$ if $t \in [0,+\infty)$, $\theta \in [0,1]$. Then from the definition of Z_x^1 and *(A1)* we get the bound $\|Z_x^1(t)\| \le O(\varepsilon)\psi_\varepsilon(t)$ (because $\|B_1(x,t,\theta)\| \le \psi_\varepsilon(t)$), so

$$\|Z_x^1\|_{L^1} = O(\varepsilon)\|\psi_\varepsilon\|_{L^1} = o(\varepsilon). \tag{3.19}$$

To estimate Z_x^2, suppose first that $x_j \to \bar{x}$ as $j \to \infty$. Then it is clear that $\|B_2(x_j,t,\theta)\| \le 2\varphi(t)$ for each fixed t,θ, and $B_2(x_j,t,\theta) \to 0$ as $j \to \infty$. Then $\int_0^1 \|B_2(x_j,t,\theta)\|d\theta \to 0$ as $j \to \infty$ for each fixed t by the Dominated Convergence Theorem, and $\int_0^1 \|B_2(x_j,t,\theta)\|d\theta \le \varphi(t)$. Using Dominated Convergence again we conclude that $\int_0^{+\infty}\int_0^1 \|B_2(x_j,t,\theta)\|d\theta\,dt \to 0$. So, if we let

$$\beta(v) = \int_0^{+\infty}\int_0^1 \|B_2(x,t,\theta)\|d\theta\,dt,$$

we see that $\beta(v) \to 0$ as $v \to 0$. Then

$$\begin{aligned}\int_0^{+\infty} \|Z_x^2(t)\|dt &\le \int_0^{+\infty}\Big(\int_0^1 \|B_2(x,t,\theta)\|d\theta\Big)\|\gamma_x(t)-\gamma_{\bar{x}}(t)\|dt \\ &\le \beta(v)\sup\{\|\gamma_x(t)-\gamma_{\bar{x}}(t)\| : 0 \le t < +\infty\} \\ &= o(\varepsilon).\end{aligned}$$

Therefore $\|Z_x^2\|_{L^1} = o(\varepsilon)$. This fact, together with (3.16) and (3.19) imply that

$$\|Z_x\|_{L^1} = o(\varepsilon).$$

Since $t \mapsto A(t)$ is Lebesgue integrable, M is bounded on $[0,+\infty)\times[0,+\infty)$, hence

$$\int_0^t \|M(t,s)\|\cdot\|Z_x(s)\|\,ds \le o(\varepsilon), \tag{3.20}$$

so (3.18) follows.

We now analyze the second term of the right-hand side of (3.17). Using integration by parts, we write

$$\begin{aligned}\int_0^t M(t,s)\cdot\rho_v'(s)\,ds &= M(t,t)\hat{\rho}_v'(t) - \int_0^t \frac{\partial}{\partial s}M(t,s)\cdot\hat{\rho}_v'(s)\,ds \\ &= \hat{\rho}_v'(t) + \int_0^t M(t,s)A(s)\hat{\rho}_v'(s)\,ds,\end{aligned}$$

since $M(t,t) = \text{identity}$ and $\frac{\partial}{\partial s}M(t,s) = -M(t,s)A(s)$. Define

$$\Lambda(t) = \hat{W}'(t) + \int_0^t M(t,s)\hat{W}'(s)\,ds.$$

Then Λ is an $n \times n$-matrix-valued function on $[0, +\infty)$ whose components belong to $BV(0, \infty)$. We then have

$$\int_0^t M(t,s) \cdot \rho'_v(s)\, ds - \Lambda(t) \cdot v$$
$$= \hat{\rho}'_v(t) - \hat{W}'(t) \cdot v + \int_0^t M(t,s) A(s) \Big(\hat{\rho}'_v(s) - \hat{W}'(s) \cdot v \Big) ds.$$

In view of (3.9), the scalar function

$$s \to \frac{1}{||v||} \Big\| \hat{\rho}'_v(s) - \hat{W}'(s) \cdot v \Big\|$$

converges to 0 in Y. Therefore for any $\bar{t}$

$$\lim_{v \to 0} \frac{1}{||v||} \int_0^{+\infty} ||M(\bar{t}, s)|| \cdot ||A(s)|| \cdot \Big\| \hat{\rho}'_v(s) - \hat{W}'(s) \cdot v \Big\| \, ds = 0.$$

Since $M(t,s) = M(t, \bar{t}) M(\bar{t}, s)$, and M is bounded, we can conclude that

$$\lim_{v \to 0} \frac{1}{||v||} \sup_{0 \le t < +\infty} \left\| \int_0^t M(t,s) A(s) \Big(\hat{\rho}'_v(s) - \hat{W}'(s) \cdot v \Big) ds \right\| = 0.$$

Therefore (3.9) implies that the scalar functions

$$t \to \frac{1}{||v||} \Big\| \int_0^t M(t,s) \cdot \rho'_v(s)\, ds - \Lambda(t) \cdot v \Big\|$$

converge to 0 in Y. In view of (3.17), (3.20), it follows that the scalar functions

$$t \to \frac{1}{||v||} \Big\| \gamma_x(t) - \gamma_{\bar{x}}(t) - M(t, T) \cdot v - \Lambda(t) v \Big\|$$

converge to 0 in Y. This says, in particular, that the map $x \to \gamma_x$ is "differentiable at $x = \bar{x}$ and its differential is the linear map $v \to M(\cdot, T) \cdot v - \Lambda(\cdot) v$," where differentiability is understood in the precise sense spelled out above, namely, the difference $\gamma_x(t) - \gamma_{\bar{x}}(t) - M(t,T).v - \Lambda(t).v$ is $o(||v||)$ in the sense that

$$\lim_{v \to 0} \frac{1}{||v||} \int_0^{+\infty} \sigma(t) \Big\| \gamma_x(t) - \gamma_{\bar{x}}(t) - M(t, 0).v - \Lambda(t).v \Big\| \, dt = 0 \qquad (3.21)$$

for every integrable function $\sigma : [0, +\infty) \to \mathbb{R}$.

Now

$$V_\Gamma(x) - V_\Gamma(\bar{x}) = \int_0^{T_x} L(\gamma_x(t), \eta_x(t)) dt - \int_0^{T_{\bar{x}}} L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) dt.$$

By definition, if $T_x < +\infty$ we prolong (γ_x, η_x) on $[T_x, +\infty)$ by setting $\tilde{f}(\gamma_x(t), \eta_x(t)) = 0$ for $t \notin \mathrm{Dom}(\eta_x)$. Hence, we can write both integrals with 0 and $+\infty$ as extrema of integration, so that

$$\begin{aligned} V(x) - V(\bar{x}) &= \int_0^{+\infty} (L(\gamma_x(t), \eta_x(t)) - L(\gamma_{\bar{x}}(t), \eta_x(t)))dt \\ &\quad + \int_0^{+\infty} (L(\gamma_{\bar{x}}(t), \eta_x(t)) - L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)))dt \\ &= \int_0^{+\infty} \int_0^1 D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) \\ &\quad - \gamma_{\bar{x}}(t)), \eta_x(t))\, d\theta \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))\, dt \\ &\quad + \int_0^{+\infty} \rho''_{\varepsilon v}(t)\, dt \\ &= \int_0^{+\infty} a(t) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) dt \\ &\quad + \int_0^{+\infty} \rho''_v(t) dt + \int_0^{+\infty} z_x(t) dt, \end{aligned}$$

where

$$a(t) = D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)),$$

$$\begin{aligned} z_x(t) &= z_x^1(t) + z_x^2(t), \\ z_x^1(t) &= \int_0^1 b_1(x,t,\theta) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) d\theta, \\ z_x^2(t) &= \int_0^1 b_2(x,t,\theta) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t)) d\theta, \end{aligned} \tag{3.22}$$

$$\begin{aligned} b_1(x,t,\theta) &= D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_x(t)) \\ &\quad - D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t)), \\ b_2(x,t,\theta) &= D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t)) - D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)). \end{aligned}$$

An argument identical to that used above to estimate Z_x proves that the L^1 norm of z_x is $o(||v||)$ as $v \to 0$. Also, since $a(\cdot)$ is integrable and the scalar functions $t \to \frac{1}{||v||}||\gamma_x(t) - \gamma_{\bar{x}}(t) - M(t,T)\cdot v - \Lambda(t)\cdot v||$ go to 0 in Y, we can conclude that the integral $\int_0^{+\infty} a(t)\cdot(\gamma_x(t) - \gamma_{\bar{x}}(t) - M(t,T)\cdot v - \Lambda(t)\cdot v)dt$ is $o(||v||)$. Finally, the integral $\int_0^{+\infty} \rho''_v(t)dt$ is differentiable as function of v at $v = 0$. If we use Δ to denote its differential, it follows that

$$V_\Gamma(x) - V_\Gamma(\bar{x}) = \left(\int_0^{+\infty} a(t) \Big(M(t,0) + \Lambda(t) \Big)\, dt + \Delta \right) \cdot v + o(||v||). \tag{3.23}$$

This shows that V is differentiable at $\bar{x}$ and gives us an explicit expression for its differential.

Let denote by (λ, λ_0) the adjoint covector along $(\gamma_{\bar{x}}, \eta_{\bar{x}})$. We have:

$$\begin{aligned}&\lambda_0(V_\Gamma(\bar{x}+\varepsilon v) - V_\Gamma(\bar{x}))\\&= \int_0^{+\infty} \lambda_0 L(\gamma_x(t), \eta_x(t))dt - \int_0^{+\infty} \lambda_0 L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))dt,\end{aligned}$$

and again we can use 0 and $+\infty$ as extrema of integration for both integrals. Notice that we have $\gamma_x(t) = \gamma_x(T_x)$ for every $t \in [T_x, +\infty)$. Moreover the equation:

$$\dot{\lambda}(t) = -\lambda(t) \cdot D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(x)) - \lambda_0 D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \tag{3.24}$$

is valid on $[0, +\infty)$ defining $\lambda(t) = \lambda(T_{\bar{x}})$ for $t \in [T_{\bar{x}}, +\infty)$. Hence

$$\begin{aligned}\lambda_0(V(\bar{x}+\varepsilon v) - V(\bar{x})) &= \int_0^{+\infty} \lambda_0(L(\gamma_x(t), \eta_x(t)) - L(\gamma_{\bar{x}}(t), \eta_x(t)))dt\\&\quad + \int_0^{+\infty} \lambda_0(L(\gamma_{\bar{x}}(t), \eta_x(t)) - L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)))dt\\&= I_1 + I_2.\end{aligned}$$

We start estimating I_1:

$$\begin{aligned}I_1 &= \lambda_0 \int_0^{+\infty}\int_0^1 D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_x(t)) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))d\theta\, dt\\&= \lambda_0 \int_0^{+\infty}\int_0^1 \big(D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_x(t))\\&\qquad - D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t))\big)\\&\qquad \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))d\theta\, dt\\&\quad + \lambda_0 \int_0^{+\infty}\int_0^1 \big(D_y L(\gamma_{\bar{x}}(t)+\theta(\gamma_x(t)-\gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t)) - D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))\big)\\&\qquad \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))d\theta\, dt\\&\qquad + \int_0^{+\infty} \lambda_0 D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \cdot (\gamma_x(t) - \gamma_{\bar{x}}(t))dt\end{aligned} \tag{3.25}$$

and the first two terms can be written as $o(\varepsilon)$. Indeed

$$\|\gamma_x(t) - \gamma_{\bar{x}}(t)\| = O(\varepsilon),$$

and $\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)) \in W'$, then the first integrand is bounded by $\psi_\varepsilon(t)O(\varepsilon)$. For the second, notice that for every fixed t the first factor is bounded by $2\varphi(s)$ and tends to zero as ε tends to zero, therefore by dominate convergence we obtain the conclusion. Now using (3.24)

$$
\begin{aligned}
I_1 &= \int_0^{+\infty} \langle -\dot{\lambda}(t) - \lambda(t) \cdot D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle dt \, + o(\varepsilon) \\
&= - \int_0^{+\infty} \frac{d}{dt} \langle \lambda(t), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle dt \\
&\qquad + \int_0^{+\infty} \langle \lambda(t), f(\gamma_x(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \rangle dt \\
&\qquad - \int_0^{+\infty} \langle \lambda \cdot D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle dt \, + o(\varepsilon) \\
&= \langle \lambda(0), \gamma_x(0) - \gamma_{\bar{x}}(0) \rangle - \lim_{t \to +\infty} \langle \lambda(t), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle \\
&\qquad + \int_0^{+\infty} \langle \lambda(t), f(\gamma_{\bar{x}}(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \rangle dt \\
&\qquad + \int_0^{+\infty} \langle \lambda(t), f(\gamma_x(t), \eta_x(t)) - f(\gamma_{\bar{x}}(t), \eta_x(t)) \rangle dt \\
&\qquad + \int_0^{+\infty} \langle \lambda \cdot D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle dt \, + o(\varepsilon).
\end{aligned}
$$

Since $D_x \tilde{f}(\gamma_{\bar{x}}, \eta_{\bar{x}}) \in L^1(0, +\infty)$, we have that $\|\lambda(t)\|$ is bounded and then from $\lim_{t \to +\infty} \gamma_x(t) = \lim_{t \to +\infty} \gamma_{\bar{x}}(t) = 0$, it follows that the second addendum is zero. The sum of the last two integrals is $o(\varepsilon)$, indeed we can reason as for (3.25) replacing L with f. From the minimization condition of the Maximum Principle:

$$
\begin{aligned}
&\langle \lambda(t), f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \rangle + \lambda_0 L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \\
&\leq \langle \lambda(t), f(\gamma_{\bar{x}}(t), \eta_x(t)) \rangle + \lambda_0 L(\gamma_{\bar{x}}(t), \eta_x(t))
\end{aligned} \tag{3.26}
$$

for every x and almost every $t \in \mathrm{Dom}(\eta_{\bar{x}}) \cap \mathrm{Dom}(\eta_x)$. Moreover, with the above definitions, the minimization condition holds also for $t \notin \mathrm{Dom}(\eta_{\bar{x}})$ in a trivial way. Indeed the first term of (3.26) is always zero, while the second is either zero (if $t \notin \mathrm{Dom}(\eta_x)$) or equal to the Hamiltonian, evaluated at $\gamma_{\bar{x}}(t) = 0$ and $\eta_x(t)$, that is positive. From these facts we have:

$$
I_1 \geq \langle \lambda(0), x - \bar{x} \rangle - \lambda_0 \int_0^{+\infty} \lambda_0 (L(\gamma_{\bar{x}}(t), \eta_x(t)) - L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))) dt + o(\varepsilon)
$$

and then:

$$
I_1 + I_2 \geq \langle \lambda(0), \varepsilon v \rangle + o(\varepsilon).
$$

We can divide by ε and pass to the limit as ε goes to zero, obtaining:

$$
\lambda_0 \langle D_y V_\Gamma(\bar{x}), v \rangle \geq \langle \lambda(0), v \rangle.
$$

Since both terms are linear in v, we obtain:

$$
\lambda_0 \langle D_y V_\Gamma(\bar{x}), v \rangle = \langle \lambda(0), v \rangle.
$$

We have $\lambda_0 \neq 0$ otherwise $\lambda = 0$, but this contradicts the non triviality of adjoint covector. Hence, it is possible to normalize the adjoint covector setting $\lambda_0 = 1$. We finally obtain:

$$D_y V_\Gamma(\bar{x}) = \lambda(0). \tag{3.27}$$

This proves, in particular, the uniqueness of $(\lambda(\cdot), \lambda_0)$ up to multiplication by a positive constant, since the Cauchy problem for the adjoint equation has unique solutions. This concludes the proof of Theorem 3.2.

We now complete the proof of Theorem 3.1. From the minimization condition of Pontryagin Maximum Principle (2.7), we have:

$$\langle \lambda(t), f(\gamma_x(t), \omega) \rangle + L(\gamma_x(t), \omega) \geq 0 \tag{3.28}$$

for every $x \in S$, every $\omega \in U$ and almost every $t \in \mathrm{Dom}(\gamma_x)$. Using a sequence of times tending to 0, at which the inequality is true, from the continuity of the quantities involved in (3.28), we obtain that the inequality is true at time 0. Hence, from (3.27), we obtain:

$$\langle D_y V_\Gamma(x), f(x, \omega) \rangle + L(x, \omega) \geq 0, \tag{3.29}$$

for every $x \in S$ at which Γ is $(f.L)$–differentiable and every $\omega \in U$.

At this stage, we are ready to prove optimality. Fix $x \in S$ and consider an admissible pair $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f})$ verifying $\gamma^- = x$ and $J(\gamma, \eta) < +\infty$. If $\mathrm{Dom}(\gamma)$ is bounded then we can reason exactly as in [12]. Thus we assume that $\mathrm{Dom}(\gamma) = [0, +\infty)$. Now, take a sequence T_j that tend to $+\infty$. For each fixed T_j, there exists a sequence (γ_l^j, η_l^j) of admissible pairs, with $\mathrm{Dom}(\gamma_l^j) = [0, T_j]$, η_l^j piecewise constant, $\gamma_l^j(T_j) = \gamma(T_j)$, such that γ_l^j tend to γ uniformly on $[0, T_j]$, and

$$\left| \int_0^{T_j} L(\gamma_l^j(t), \eta_l^j(t))\, dt - \int_0^{T_j} L(\gamma, \eta)\, dt \right| \to 0,$$

as l tends to $+\infty$. This argument is used for the finite time case [12], for the proof see [17].

By a transversality argument, identical to that one used in [12], we can find a sequence of points y_i^j converging to $\gamma(T_j)$ such that for every l the trajectory $\gamma_i^{j,l}$, corresponding to control η_l^j and verifying $\gamma_i^{j,l}(T_j) = y_i^j$, intersects the (n-1) dimensional rectifiable set A a finite or countable number of times. Hence using (3.29), the lower semicontinuity of V_Γ and the property *(NDJ)* (exactly as in the finite time case, see [12]), we obtain:

$$V_\Gamma(\gamma_i^{j,l}(0)) \leq \int_0^{T_j} L(\gamma_i^{j,l}(t), \eta_l^j(t))\, dt + V_\Gamma(y_i^j). \tag{3.30}$$

We now let i tend to infinity. Since trajectories corresponding to control η_l^j depend continuously on initial data, we obtain that $\gamma_i^{j,l}$ tends uniformly to γ_l^j. Moreover, using (2.3) for η_l^j we have that

$$\lim_{i \to +\infty} \int_0^{T_j} L(\gamma_i^{j,l}(t), \eta_l^j(t))\, dt = \int_0^{T_j} L(\gamma_l^j(t), \eta_l^j(t))\, dt.$$

Hence, passing to the limit in the inequality (3.30) and using the lower semicontinuity of V_Γ we obtain:

$$V_\Gamma(\gamma_l^j(0)) \leq \int_0^{T_j} L(\gamma_l^j(t), \eta_l^j(t))\, dt + \liminf_{i\to+\infty} V_\Gamma(y_i^j).$$

We now pass to the limit in l. Using again the lower semicontinuity of V_Γ and the uniform convergence of γ_l^j to γ on $[0, T_j]$, it follows

$$V_\Gamma(x) = V_\Gamma(\gamma(0)) \leq \int_0^{T_j} L(\gamma(t), \eta(t))\, dt + \liminf_{i\to+\infty} V_\Gamma(y_i^j). \tag{3.31}$$

Finally, we pass to the limit in j. From the continuity of V_Γ at the origin and $V_\Gamma(0) = 0$, we have

$$\lim_{j\to+\infty} \liminf_{i\to+\infty} V_\Gamma(y_i^j) = 0.$$

Hence, since (γ, η) is an admissible pair, from (3.31) letting j tend to $+\infty$, we obtain

$$V_\Gamma(x) \leq J(\gamma, \eta).$$

Since (γ, η) was an arbitrary admissible pair steering x to the origin

$$V_\Gamma(x) \leq V_{\tilde{J}}(x),$$

therefore Γ is optimal.

Remark 3.3. The main theorem admits various generalizations.

Target.
This proof is valid under the hypothesis that the target $\mathcal{T}$ is the origin. The same proof can be extended to cover the case of for more general smooth targets under additional assumptions in the definition of (f, L)-differentiability. More precisely we ask:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \|\gamma^+_{\bar{x}+v\varepsilon} - \gamma^+_{\bar{x}}\| = 0.$$

However, this condition seems to be too stringent to fit with the interesting application. While in the finite time case, the result is valid for general targets, see [12]. It is possible to consider a final cost $\psi : \mathcal{T} \to \mathbb{R}$, substituting the condition $V_\Gamma(0) = 0$ with the condition $V_\Gamma(x) \leq \psi(x)$ for every $x \in \mathcal{T}$.

Totality.
The hypothesis requiring that the presynthesis is total can be replaced by the following. It is sufficient to ask S to be open and $\gamma_x(t) \in S$ for every $x \in S$ and every $t \in \mathrm{Dom}(\gamma_x)$. Another interesting sufficient condition stated in term of the cost is the following. There exists a real number $\bar{J} > 0$ such that:

(a) For every $x \in S$ we have $J(\gamma_x, \eta_x) \leq \bar{J}$

(b) For every $x \in S$ and every $(\gamma, \eta) \in \mathrm{Adm}^0(\tilde{f})$ verifying $\gamma^- = x$, we have that $J(\gamma, \eta) \leq \bar{J}$ implies that, for every $t \in \mathrm{Dom}(\gamma)$, $\gamma(t) \in S$.

CONTINUITY AT THE ORIGIN.
It is clear that a sufficient assumptions to carry out the proof in the same way is that

$$\limsup_{j \to +\infty} \liminf_{i \to +\infty} V_\Gamma(y_i^j) \leq 0.$$

This can be guaranteed for example by asking that for every admissible pair (γ, η) that steers a point $x \in S$ to the origin in infinite time, there exists a sequence of time T_j such that $T_j \to +\infty$ and

$$\limsup_{j \to +\infty} V_\Gamma(\gamma(T_j)) \leq V_\Gamma(0).$$

We can always choose the sequene y_i^j in such a way that $\|y_i^j - \gamma(T_j) - f(\gamma(T_j), u)\| \to 0$ as $j \to +\infty$ for some $u \in U$. Thus from the *(NDJ)* assumption we obtain

$$\liminf_{j \to +\infty} V_\Gamma(y_i^j) \leq \limsup_{j \to +\infty} V_\Gamma(y_i^j) \leq V_\Gamma(\gamma(T_j)),$$

concluding. This condition is veryfied by Example 1 of [12].

We may also ask the existence of a set C such that every trajectory γ reaching the origin in infinite time is definitely in C (that is there exist $\bar{T}$ such that $\gamma(t) \in C$ for every $t \geq \bar{T}$) and V_Γ is upper semicontinuous at 0 along C, that is $\limsup_{y \to 0, y \in C} V_\Gamma(y) \leq V_\Gamma(0)$. Again, this more geometric condition is satisfied by Example 1 of [12]. Notice that both conditions for the linear quadratic example of next section are equivalent to continuity at the origin for V_Γ.

DIFFERENTIABILITY CONDITIONS.
We give an alternative definition of differentiability of the synthesis that guarantees the same conclusion of the theorem. We briefly indicate the modifications of the proof needed to reach the conclusion in this case.

This new conditions are important since the example of a linear quadratic problem we give in the next section, satisfies these assumptions (not the assumption *(A)*).

Given $\bar{x}$, define, as in the proof of Theorem 3.1,

$$A(t) = D_y f(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)),$$

and let $M(t, s)$ be the fundamental matrix solution of the associated linear system. A presynthesis Γ is (f, L)-differentiable in $\bar{x}$ if the following holds.

(A)' there exist an open set W, containing $\{\gamma_{\bar{x}}(t) : t \in \mathrm{Dom}(\eta_{\bar{x}})\}$, a neighborhood N of $\bar{x}$ (in Ω) such that $N \subseteq \mathrm{Dom}(\Gamma)$, with the property that

(Aa) there exists $\delta > 0$ such that the compact set $W' = \{x : d(x, \{\gamma_{\bar{x}}(t) : t \in \mathrm{Dom}(\eta_{\bar{x}})\}) \leq \delta\}$ satisfies $W' \subseteq W$ and

$$\limsup_{x \to \bar{x}} \liminf_{T \to +\infty} \|\gamma_x(T) - \gamma_{\bar{x}}(T)\| \exp\left(\|\varphi_{W',[0,T]}\|_{L^1}\right) \leq \frac{\delta}{2}$$

(Ab) there exist an integrable function $\psi : [0, +\infty] \to \mathbb{R}$ and, for sufficiently small $\varepsilon > 0$, integrable functions $\psi_\varepsilon : [0, +\infty] \to \mathbb{R}$ such that $\lim_{\varepsilon \to 0} \int_0^{+\infty} \psi_\varepsilon(t)\, dt = 0$, and the inequalities

$$\|\tilde{f}(y, \eta_x(t)) - \tilde{f}(y, \eta_{\bar{x}}(t))\| \leq \psi_\varepsilon(t)$$

$$\|D_y \tilde{f}(y, \eta_x(t)) - D_y \tilde{f}(y, \eta_{\bar{x}}(t))\| \leq \psi_\varepsilon(t)$$

$$\|D_y L(\gamma_{\bar{x}}(t) + \theta(\gamma_x(t) - \gamma_{\bar{x}}(t)), \eta_{\bar{x}}(t))\| \leq \psi(t)$$

hold for every $y \in W$, $x \in N$ such that $\|x - \bar{x}\| \leq \varepsilon$, $t \in [0, +\infty)$ and $\theta \in [0, 1]$.

(Ac) the map $v \to \rho_v$, where ρ_v is the integrable $\mathbb{R}^{n+1}$ valued function on $[0, +\infty)$ given by

$$\rho_v(t) = \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}+v}(t)) - \tilde{f}(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t))$$

is vaguely differentiable at $v = 0$ or it is weak*-differentiable at $v = 0$, regarded as a map into the space of $\mathbb{R}^{n+1}$–valued Borel measures. That is: for every continuous function $\alpha : [0, +\infty) \to \mathbb{R}$, that vanishes at $+\infty$ ($\lim_{t \to +\infty} \alpha(t) = 0$), the map $\mathbb{R}^n \ni v \to \int_0^{+\infty} \rho_v(t)\alpha(t)\, dt \in \mathbb{R}^{n+1}$ is differentiable at $v = 0$.

Moreover the map $\mathbb{R}^n \ni v \to \int_0^{+\infty} \rho_v''(t)\, dt \in \mathbb{R}$, where ρ_v'' is the (n+1)-th component of ρ_v, is differentiable at $v = 0$.

(Ad) define C_2 as in (3.2), then the function

$$t \to \left\| D_y L(\gamma_{\bar{x}}(t), \eta_{\bar{x}}(t)) \cdot \left(M(t, 0) + C_2 \int_0^t M(t, s)\, ds \right) \right\|$$

is integrable.

(Ae) let $\lambda_{\bar{x}}$ denote the covector associated to $\gamma_{\bar{x}}$, then for every $x \in N$

$$\lim_{t \to +\infty} \langle \lambda_{\bar{x}}(t), \gamma_x(t) - \gamma_{\bar{x}}(t) \rangle = 0.$$

Notice that the condition *(Ac)* is the same as *(A2)*.

The proof now can be modified in the following way.

From *(Ac)*, we have that, as before, the scalar functions

$$t \to \frac{1}{||v||} \left\| \int_0^t M(t, s) \cdot \rho_v'(s)\, ds - \Lambda(t) \cdot v \right\|$$

converge to 0 in Y.

Now condition *(Aa)* ensures that trajectories γ_x are near to $\gamma_{\bar{x}}$ for x near to $\bar{x}$. More precisely, we can prove the estimate (3.12) for ε sufficiently small. The estimate (3.15) is then obtained defining $T(x)$ in a similar way and using the function $\varphi_{W',[0,T(x)]}$ instead of $\varphi_{W'}$.

The differentiability in Y of the map $t \to \gamma_x(t) - \gamma_{\bar{x}}(t)$ can now be proved only on compact intervals using the same proof and *(Ab)*. The differentiability of V_Γ at $\bar{x}$ is now obtained using the assumption *(Ad)* to pass to the limit for t tending to $+\infty$.

The link of $D_y V_\Gamma$ with $\lambda_{\bar{x}}(0)$ is obtained in the same way, making use of the estimate *(Ae)*.

The rest of the proof does not change.

4. Examples

The aim of this section is to give examples of syntheses that satisfy the assumption *(A)'* and present both finite time and infinite time optimal trajectories.

Example 4.1. Consider the system (cf. [13]):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad |u| \le 1 \tag{4.1}$$

and the minimization problem:

$$\min \int_0^{+\infty} \left(x_1^2(t) + x_2^2(t)\right) \, dt.$$

Let us describe the optimal synthesis. There are two optimal singular trajectories that lie on the line $x_2 = -x_1$, respectively for $-1 \le x_1 < 0$ and $0 < x_1 \le 1$. These trajectories reach the origin in infinite time using the feedback control $u(x) = -x_2$. The other trajectories start as bang–bang trajectories ad either reach one of the two singular trajectories or reach the origin in finite time (the latter happens only for trajectories reaching the origin with control ± 1, see Fig. 1).

This minimization problem is not written as the optimization problem we considered in section 2. However, it is clear that any admissible trajectory must tend to the origin at infinity. Even a stronger property is true: namely if $x(\cdot)$ is optimal and $x(t) = 0$ for some t, then necessarily $x(s) = 0$ for $t \ge s$. Otherwise the trajectory defined by $\tilde{x}(s) = x(s)\chi_{[0,t]}(s)$ would achieve a better performance. Hence, we can introduce the final constraint assigning the origin as point target.

For convenience, we will verify the assumptions of our theorem in a neighborhood of the origin. This clearly is enough if we want to prove optimality on some open neighborhood of the origin covered by trajectory of the synthesis (choosing the open set in such a way that the trajectories do not exit from it). For the global synthesis, one can easily check the assumptions estimating the effects of the switchings of the optimal trajectories. This is entirely similar to the computations exploited in [12] for Fuller phenomenon (now with only a finite number of switchings and hence no problem of convergence).

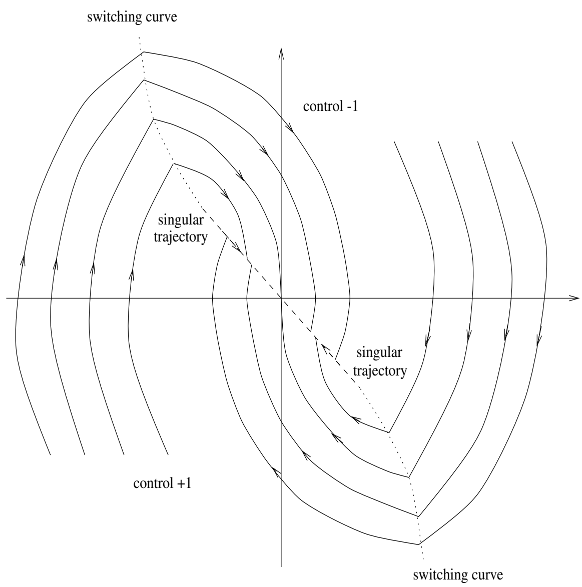

Fig.1

Consider a point x in a neighborhood of the origin, not on the two singular trajectories or on the trajectories reaching the origin in finite time. The time $t(x)$, in which the optimal trajectory starting from x reaches one of the two singular trajectories, depends smoothly on x. Consider the curve $\rho \mapsto x_\rho$ such that $t(x_\rho) = t(x)$ and $x_0 = x$. Let us check the assumption *(Ac)*. We first prove the differentiability for a fixed $\alpha \in \mathcal{C}_0$ along $x'_\rho|_{\rho=0}$ (the tangent vector to the curve $\rho \mapsto x_\rho$ at $\rho = 0$), and then along $f(\gamma_x(0), \eta_x(0))$. We have

$$\tilde{f}(\gamma_x(t), \eta_{x_\rho}(t)) - \tilde{f}(\gamma_x(t), \eta_x(t)) = \begin{pmatrix} 0 \\ \eta_{x_\rho}(t) - \eta_x(t) \\ 0 \end{pmatrix}. \tag{4.2}$$

The only nonzero component of (4.2) is the second one. Hence we have to estimate only the integral:

$$\begin{aligned} \int_0^{+\infty} \left(\eta_{x_\rho}(t) - \eta_x(t)\right) \alpha(t)\, dt &= \int_{t(x)}^{+\infty} e^{t(x)-t} \left(\gamma_{x_\rho}(t(x)) - \gamma_x(t(x))\right) \alpha(t)\, dt \\ &= C(\alpha) \left(\gamma_{x_\rho}(t(x)) - \gamma_x(t(x))\right), \end{aligned}$$

for some constant $C(\alpha) > 0$. We obtain immediately the differentiability using the regularity of $t(x)$ and of the singular trajectory. The computations for the differentiability in the other direction are straightforward. Now, the vectors x'_ρ and $f(\gamma_y(0), \eta_y(0))$ form a basis for every y in a neighborhood of x, and are smooth functions of y. Moreover, the derivatives along these directions depend continuously on y. Hence, we obtain the weak* differentiability for every $\alpha \in \mathcal{C}_0$. The other assumptions of Theorem 3.1 are easily verified, therefore we can apply the theorem obtaining the optimality of the synthesis.

EXAMPLE 4.2. Consider the system

$$\begin{cases} \dot{x}_1 = \varphi(x_2) \\ \dot{x}_2 = u \end{cases} \qquad |u| \leq 1 \tag{4.3}$$

where $\varphi \in \mathcal{C}^\infty$ verifies $\varphi(x_2) = x_2 + o(x_2)$, and the minimization problem:

$$\min \int_0^{+\infty} \left(x_1^2(t) + x_2^2(t)\right) \, dt.$$

The optimal synthesis near the origin is entirily similar to the one described in the previous example. The singular trajectories now lay on the set of zeroes of the function

$$\frac{\partial \varphi}{\partial x_2}(x_1^2 + x_2^2) - 2x_2 \, \varphi.$$

Call S this set and let S' be its intersections with the union of the second and the forth orthant. Then S' can be characterized giving x_2 as a function of x_1, say $x_2 = \phi(x_1)$, with $\phi'(0) = 1$. The function φ can be highly nonlinear, e.g. $\varphi(x_2) = \text{arctg}(x_2)$, showing that syntheses with the above features are not necessarily linked to linear–quadratic problems.

REMARK 4.3. Consider the same problem of Example 4.1 but now with a cost function Q having a single zero, say at $\bar{x}$, and a quadratic behaviour near $\bar{x}$ (e.g. $Q(x - \bar{x})$ is quadratic), and a perturbed dynamics admitting the zero velocity at $\bar{x}$ (the perturbation being small in the $\mathcal{C}^3$ norm). Then the optimal synthesis has the same structure with $\bar{x}$ playing the same role of the origin in Example 4.1. (This follows from the analysis of the structural stability of two dimensional syntheses discussed in [3].) This proves structural stability of the synthesis of Example 4.1 and hence illustrates the applicability of Theorem 3.1.

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