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ON THE OPTIMAL CONTROL OF IMPLICIT SYSTEMS

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Abstract. In this paper we consider the following problem, known as implicit Lagrange problem: find the trajectory x argument of

$$\min \int_0^1 L(x, \dot{x})\, dt$$

where the constraints are defined by an implicit differential equation

$$F(x, \dot{x}) = 0$$

with $\dim F = n - q < \dim x = n$. We define the geometric framework of a q-π-submanifold in the tangent bundle of a surrounding manifold X, which is an extension of the π-submanifold geometric framework defined by Rabier and Rheinboldt for control systems. With this geometric framework, we define a class of well-posed implicit differential equations for which we obtain locally a controlled vector field on a submanifold W of the surrounding manifold X by means of a reduction procedure. We then show that the implicit Lagrange problem leads locally to an explicit optimal control problem on the submanifold W, for which the Pontryagin maximum principle is naturally apply.

1. Introduction

We consider for the state x of $\mathbb{R}^n$ the implicit differential equation

$$F(x, \dot{x}) = 0. \tag{1.1}$$

In this equation the control u does not appear explicitly, but only because there are less equations than unknowns, namely $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-q}$, where $q < n$ (see [6].) Here, the control variable u belongs to $\mathbb{R}^q$. The cost function is the Lagrangian $L(x, \dot{x})$ of $T\mathbb{R}^n$. A process is a trajectory $x(\cdot)$ belonging to $C^1([0,1], \mathbb{R}^n)$ the set of continously differentiable functions (resp. $KC^1([0,1], \mathbb{R}^n)$ the set of continuous and piecewise differentiable functions, $AC([0,1], \mathbb{R}^n)$ the set of absolutely continuous functions, see the footnotes of the subsection 5.2.) A trajectory $x(\cdot)$ is admissible if $x(0) = a$, $x(1) = b$ and

$$F(x(t), \dot{x}(t)) = 0, \forall t \in [0,1] \text{ (resp. a.e. on } [0,1]).$$

For any admissible trajectory $x(\cdot)$ the cost is

$$J(x(\cdot)) = \int_0^1 L(x(t), \dot{x}(t))dt.$$

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An admissible trajectory $\bar{x}(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ (resp. $KC^1([0,1],\mathbb{R}^n)$, $AC([0,1],\mathbb{R}^n)$) is a weak minimum (resp. strong minimum) of J if

$$J(\bar{x}(\cdot)) \leq J(x(\cdot))$$

for any admissible trajectory $x(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ (resp. $KC^1([0,1],\mathbb{R}^n)$, $AC([0,1],\mathbb{R}^n)$). An admissible trajectory $\bar{x}(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ (resp. $KC^1([0,1],\mathbb{R}^n)$, $AC([0,1],\mathbb{R}^n)$) is a weak local minimum (resp. strong local minimum) of J if there exist an $\varepsilon > 0$ such that for any trajectory $x(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ (resp. $KC^1([0,1],\mathbb{R}^n)$, $AC([0,1],\mathbb{R}^n)$) such that $\| x(\cdot) - \bar{x}(\cdot) \|_1 < \varepsilon$ (resp. $\| x(\cdot) - \bar{x}(\cdot) \|_0 < \varepsilon$) where

$$\| x(\cdot) \|_1 = \max_{t\in[0,1]} \max\{| x(t) |, | \dot{x}(t) |\}$$

$$(\text{resp. } \| x(\cdot) \|_0 = \max_{t\in[0,1]} | x(t) |)$$

then

$$J(\bar{x}(\cdot)) \leq J(x(\cdot)).$$

REMARK 1.1. a) An admissible trajectory $x(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ which is a strong (local) minimum is also a weak (local) minimum, meanwhile a trajectory $x(\cdot)$ belonging to $C^1([0,1],\mathbb{R}^n)$ can be a weak (local) minimum without to be a strong (local) minimum.
b) The necessary conditions for the weak local minimum are also necessary conditions for the strong local minimum, and the sufficiency conditions for the strong local minimum are also the sufficiency conditions for the weak local minimum.

We will subsequently turn our attention to the geometry of the implicit differential equation (1.1). More precisely, we will extend the definitions of π-submanifold, reducible and completely reducible π-submanifold in [16] to our situation (see also [15].) Let us consider the manifold $X = \mathbb{R}^n$ and its tangent bundle $TX = T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Let us assume that the subset $M = F^{-1}(0)$ is a submanifold of TX (it is the case when F is a submersion). A trajectory $x(\cdot)$ is admissible if $(x(t), \dot{x}(t))$ belongs to M for any $t \in [0,1]$. The implicit Lagrange problem[1] is

$$\mathcal{P}_0 \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in M \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt.$$

If $x(\cdot)$ is an admissible trajectory then $x(t)$ has to belong to the set $W = \pi(M)$ for any $t \in [0,1]$. Let us assume that W is a submanifold of X, then $\dot{x}(t)$ has to belong to the subspace $T_{x(t)}W$ of $T_{x(t)}X$ for any t in $[0,1]$ and thus $(x(t), \dot{x}(t))$ belongs to the set $M_1 = TW \cap M$ for any t in $[0,1]$. In other

[1]When $q = n$, the constraint $F(x,\dot{x}) = 0$ being absent, this is the simple problem of the calculus of variations (see [3, 4, 5, 10, 11])

$$\min_{\substack{x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt.$$

words, any admissible trajectory for the implicit Lagrange problem $(\mathcal{P}_0)$ is an admissible trajectory for the following implicit Lagrange problem

$$\mathcal{P}_1 \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in M_1 \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt.$$

Thus, if $\bar{x}(\cdot)$ is a solution of $\mathcal{P}_0$ then it is a solution of $\mathcal{P}_1$ and conversely. Moreover, the startpoint a and the endpoint b have to belong to W. This replacement of the submanifold M of TX by the submanifold M_1 of TX is the reduction procedure and M_1 is called the reduction of M. Let us assume that we are able to do with M_1, what we have done with M, then we construct a submanifold $W_1 = \pi(M_1)$ of X and a submanifold M_2 of TX. If $x(\cdot)$ is an admissible trajectory for the problem $\mathcal{P}_1$ then it is an admissible trajectory for the problem

$$\mathcal{P}_2 \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in M_2 \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt.$$

Let us assume that we construct by induction a sequence of implicit Lagrange problem

$$\mathcal{P}_k \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in M_k \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt$$

such that $W_k = \pi(M_k)$ is a submanifold of X and $M_{k+1} = M_k \cap TW_k$ is a submanifold of TX, then any admissible trajectory $x(\cdot)$ of $\mathcal{P}_k$ is an admissible trajectory of $\mathcal{P}_{k+1}$. Therefore any admissible trajectory of $\mathcal{P}_0$ is an admissible trajectory of $\mathcal{P}_k$ for any k. Thus any admissible trajectory of $\mathcal{P}_0$ is an admissible trajectory of the following implicit Lagrange problem

$$\mathcal{P}_c \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in C(M) \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt$$

with $C(M) = \cap_{k\geq 0} M_k$. Clearly, the strong (resp. weak) minimum of $\mathcal{P}_0$ are the strong (resp. weak) minimum of $\mathcal{P}_c$ and the points a et b have to belong to $\pi(C(M))$. Furthermore, if the sequence $\{M_k\}_{k\geq 0}$ is stationary then $C(M)$ is a submanifold of TX and the smallest integer α such that $M_k = M_\alpha$, $\forall i \geq \alpha$ will be called the index. Then we can wonder if $\mathcal{P}_c$ is equivalent to an explicit control problem (see the subsection 5.2 of the Appendix). This will be the case for the class of well-posed implicit differential equations.

2. Definitions and Main results

Using the geometric framework of q-π-submanifold of the section 3, we are able to define a well-posed implicit differential equation.

Definition 2.1. An implicit differential equation (1.1) is *well-posed* if the set

$$M = F^{-1}(0)$$

is a completely reducible q-π-submanifold of $T\mathbb{R}^n$ such that the core $C(M)$ is not empty.

Definition 2.2. The *index* of a well-posed implicit differential equation (1.1) is the maximum over the index of the non-empty connected component of the core $C(M)$.

Now, we consider only well-posed implicit differential equations. As in the introduction, for each q-π-submanifold M_k of $T\mathbb{R}^n$ of the chain of reduction we consider the implicit Lagrange problem

$$\mathcal{P}_k \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in M_k \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt$$

and for $C(M)$ the core of M we consider the implicit Lagrange problem

$$\mathcal{P}_c \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in C(M) \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t),\dot{x}(t))dt.$$

The sequence $\{\mathcal{P}_k\}_{k\geq 0}$ is called the chain of reduced implicit Lagrange problems of the well-posed implicit differential equation (1.1) and $\mathcal{P}_c$ is called the central implicit Lagrange problem. Evidently, the points a and b have to belong to $\pi(M_k)$ for any k and thus to belong to $\pi(C(M))$.

Definition 2.3. Any point x of $\mathbb{R}^n$ is *consistent* with a well-posed implicit differential equation F if it belongs to the projection of the core $C(M)$.

According to the definitions we can formulate the following theorems (proofs are given in the subsection 5.1 of the Appendix).

Theorem 2.4. *Let F be a well-posed implicit differential equation ($q < n$) of $T\mathbb{R}^n$, $\{P_k\}_{k\geq 0}$ its chain of reduced implicit Lagrange problems and $\mathcal{P}_c$ its central implicit Lagange's problem. Then, any admissible trajectory $x(\cdot)$ of $\mathcal{P}_0$ is an admissible trajectory of $\mathcal{P}_k$ for any k. In particular, any admissible trajectory $x(\cdot)$ of $\mathcal{P}_0$ is an admissible trajectory of $\mathcal{P}_c$ and any strong (resp. weak) minimum $\bar{x}(\cdot)$ of $\mathcal{P}_c$ is a strong (resp. weak) minimum of $\mathcal{P}_0$.*

Theorem 2.4 shows that the strong (resp. weak) minimum are living in the core $C(M)$. According to the theorem 3.31, the q-π-submanifold $C(M)$ is locally the image of a controlled vector field. Thus, we are able to show that locally the trajectories of $C(M)$ are in bijection with the trajectories of the controlled vector field χ.

Theorem 2.5. *(Local equivalence) Let F be a well-posed implicit differential equation, $C(M)$ its core, (x_0, p_0) a point belonging to $C(M)$ and $W = \pi(V)$ a local projection of $C(M)$ at (x_0, p_0), O an open set of $\mathbb{R}^q$, χ a controlled vector field given by the theorem 3.31. If $x(\cdot)$ is a local trajectory of $C(M)$ such that $(x(t), \dot{x}(t))$ belongs to V for any t, then there* *exists an unique continuous (resp. piecewise continuous) control $u(\cdot)$ taking its value in O such that $(x(t), \dot{x}(t)) = \chi(x(t), u(t))$[2] for any t. Conversely, for any initial condition x_0 belonging to W and for any continuous (resp. piecewise continuous) control $u(\cdot)$ taking its value in O there exists a unique local trajectory of $C(M)$ such that $(x(t), \dot{x}(\cdot))$ belongs to V for any t.*

Then, on the one hand, we have shown that the strong (resp. weak) minimum of the implicit Lagrange problem $\mathcal{P}_0$ are the strong (resp. weak) minimum of the central implicit Lagrange problem $\mathcal{P}_c$ (theorem 2.4), and on the other hand that (locally) the admissible trajectories of $\mathcal{P}_c$ are in bijection with the admissible trajectories of the controlled vector field χ (theorem 2.5). Now, let us consider a strong minimum $\bar{x}(\cdot)$ of the central implicit Lagrange problem $\mathcal{P}_c$. Let τ be a point in the set T and $W = \pi(V)$ a local projection of $C(M)$ at $(\bar{x}(\tau), \dot{\bar{x}}(\tau))$, O an open set of $\mathbb{R}^q$, χ a controlled vector field given by the theorem 3.31. There exists $\varepsilon > 0$ such that for any point t belonging to the interval $I_\varepsilon = [\tau - \varepsilon, \tau + \varepsilon]$ then $(\bar{x}(t), \dot{\bar{x}}(t))$ belongs to V. We naturally consider the following local implicit Lagrange problem

$$\mathcal{P}_{c,\varepsilon} \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in V \\ x(\tau-\varepsilon)=\bar{x}(\tau-\varepsilon) \\ x(\tau+\varepsilon)=\bar{x}(\tau+\varepsilon)}} \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(x(t), \dot{x}(t))dt.$$

THEOREM 2.6. *(Local optimality) Let F be a well-posed implicit differential equation. If $\bar{x}(\cdot)$ is a strong minimum of the central implicit Lagrange problem $\mathcal{P}_c$ then for any τ belonging to T there exists $\varepsilon > 0$ such that the trajectory $\dot{\bar{x}}_{|I_\varepsilon}(\cdot)$ is a strong minimum of the implicit Lagrange problem $\mathcal{P}_{c,\varepsilon}$*

Let us also consider the following local explicit optimal control problem

$$\mathcal{P}_{e,\varepsilon} \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))=\chi(x(\cdot),u(\cdot)) \\ u(\cdot)\in O \\ x(\tau-\varepsilon)=\bar{x}(\tau-\varepsilon) \\ x(\tau+\varepsilon)=\bar{x}(\tau+\varepsilon)}} \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(x(t), u(t))dt.$$

THEOREM 2.7. *$\bar{x}(\cdot)$ is strong (local) minimum of the implicit Lagrange problem $\mathcal{P}_{c,\varepsilon}$ if, and only if, the corresponding admissible process $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a strong (local) minimum for the explicit control problem $\mathcal{P}_{e,\varepsilon}$.*

This leads to consider the following local implicit Lagrange problem. Let $W = \pi(V)$ be a local projection such that there exists an open set O in $\mathbb{R}^q$ and a controlled vector field $\chi : W \times O \to TW$ given by the theorem 3.31

$$\mathcal{P}_V \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))\in V \\ x(0)=a \\ x(1)=b}} \int_0^1 L(x(t), \dot{x}(t))dt$$

where the points a and b belong to W. The admissible trajectories (resp. strong minimum) of $\mathcal{P}_V$ are in bijection with the admissible processes (resp.

[2]In a local coordinate system x of W, $\chi(x, u)$ takes the form $(x, f(x, u))$.

strong minimum) of the explicit optimal control problem

$$\mathcal{P}_e \qquad \min_{\substack{(x(\cdot),\dot{x}(\cdot))=\chi(x(\cdot),u(\cdot))\\ u(\cdot)\in O\\ x(0)=a\\ x(1)=b}} \int_0^1 L(\chi(x(t),u(t))dt.$$

Finally, we choose a local coordinate system $x = (x^1, \cdots, x^{r_\alpha})$ of W and then apply the Maximum Principle to the problem $\mathcal{P}_e$ with the pseudo-Hamiltonian

$$H^{\psi_0} : T^\star W \times O \to \mathbb{R}$$

$$(x, \psi, u) \mapsto \sum_{i=1}^{r_\alpha} \psi_i f^i(x, u) + \psi_0 L(x, f(x, u))$$

and the controlled vector field

$$\overrightarrow{H}^{\psi_0} : T^\star W \times O \to T(T^\star W)$$

$$(x, \psi, u) \mapsto \overrightarrow{H}^{\psi_0}(x, \psi, u) = \sum_{i=1}^{r_\alpha} \frac{\partial H^{\psi_0}}{\partial \psi_i}(x, \psi, u)\frac{\partial}{\partial x^i} - \sum_{i=1}^{r_\alpha} \frac{\partial H^{\psi_0}}{\partial x^i}(x, \psi, u)\frac{\partial}{\partial \psi_i}$$

where $\psi_0 = 0, 1$.

Remark 2.8. Obviously, the necessary conditions of optimality are invariant by bundle isomorphism h. Let us consider a bundle isomorphism h

$$\tilde{x} = \tilde{X}(x), \ \tilde{u} = \tilde{U}(x, u)$$

with inverse

$$x = X(\tilde{x}), \ u = U(\tilde{x}, \tilde{u}).$$

In the new coordinates $(\tilde{x}, \tilde{u})$ the controlled vector field is

$$\tilde{f}(\tilde{x}, \tilde{u}) = \frac{\partial \tilde{X}}{\partial x}(X(\tilde{x}))f(X(\tilde{x}), U(\tilde{x}, \tilde{u})),$$

the Lagrangian $\tilde{L}(\tilde{x}, \tilde{u}) = L(X(\tilde{x}), U(\tilde{x}, \tilde{u}))$ and the pseudo-Hamiltonian

$$\tilde{H}(\tilde{x}, \tilde{\psi}, \tilde{u}) = \textstyle\sum_{i=1}^{r_\alpha} \tilde{\psi}_i \tilde{f}^i(\tilde{x}, \tilde{u}) - \tilde{\psi}_0 \tilde{L}(\tilde{x}, \tilde{u})).$$

The extremals $(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot))$ are the projection of a triplet $(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot), \bar{\tilde{u}}(\cdot))$ such that

$(\tilde{a})$: $(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot), \bar{\tilde{u}}(\cdot))$ is a trajectory of the controlled vector field ($\tilde{\psi}_0 = 0, 1$)

$$\overrightarrow{\tilde{H}}^{\tilde{\psi}_0} : T^\star \tilde{W} \times \tilde{O} \to T(T^\star \tilde{W})$$

$$(\tilde{x}, \tilde{\psi}, u) \mapsto \overrightarrow{\tilde{H}}^{\tilde{\psi}_0}(\tilde{x}, \tilde{\psi}, \tilde{u}) = \sum_{i=1}^{r_\alpha} \frac{\partial \tilde{H}^{\tilde{\psi}_0}}{\partial \tilde{\psi}_i}(\tilde{x}, \tilde{\psi}, \tilde{u})\frac{\partial}{\partial \tilde{x}^i} - \sum_{i=1}^{n} \frac{\partial \tilde{H}^{\tilde{\psi}_0}}{\partial \tilde{x}^i}(\tilde{x}, \tilde{\psi}, \tilde{u})\frac{\partial}{\partial \tilde{\psi}_i}$$

$(\tilde{b})$: for any t belonging to $[0, 1]$ (resp. a.e. on $[0, 1]$)

$$\tilde{H}^{\tilde{\psi}_0}(\bar{\tilde{x}}(t), \bar{\tilde{u}}(t), \bar{\tilde{\psi}}(t)) = \max_{\tilde{u} \in \tilde{O}} \tilde{H}^{\tilde{\psi}_0}(\bar{\tilde{x}}(t), u, \bar{\tilde{\psi}}(t)).$$

Clearly, the extremals $(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot))$ are in bijection with the extremals $(\bar{x}(\cdot), \bar{\psi}(\cdot))$ via the relationship

$$(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot)) = (\tilde{X}(\bar{x}(\cdot), \frac{{}^t\partial X}{\partial \tilde{x}}(\tilde{X}(\bar{x}(\cdot)))\bar{\psi}(\cdot)).$$

For any triplet $(\bar{x}(\cdot), \bar{\psi}(\cdot), \bar{u}(\cdot))$ such that (a) and (b) are satisfied, the triplet

$$(\bar{\tilde{x}}(\cdot), \bar{\tilde{\psi}}(\cdot), \bar{\tilde{u}}(\cdot)) = (\tilde{X}(\bar{x}(\cdot)), \frac{{}^t\partial X}{\partial \tilde{x}}(\tilde{X}(\bar{x}(\cdot)))\bar{\psi}(\cdot), \tilde{U}(\bar{x}(\cdot), \bar{u}(\cdot)))$$

satisfies $(\tilde{a})$ and $(\tilde{b})$.

Example 2.9. *The controlled rigid pendulum.* A mass m is attached at the extremity of a rigid massless wire of length l and fixed at the origin. τ is the tension of the wire, g the gravity constant and the control $u = (u_1, u_2)$ acts on the mass. The equations of the system are

$$\begin{array}{rcl} m\ddot{x}_1 & = & -\frac{\tau}{l}x_1 + u_1 \\ m\ddot{x}_2 & = & -\frac{\tau}{l}x_2 + mg + u_2 \\ 0 & = & x_1^2 + x_2^2 - l^2 \end{array} .$$

In order to return to an implicit differential equation and to use the reduction procedure we consider the following mapping $F_0 : T\mathbb{R}^7 \to \mathbb{R}^5$

$$F_0(x,p) = \begin{pmatrix} p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1p_7 - p_5 \\ p_4 + x_2p_7 - p_6 - g \\ x_1^2 + x_2^2 - l^2 \end{pmatrix}, \quad p = \dot{x}$$

where $\dot{x}_5 = u_1/m$, $\dot{x}_6 = u_2/m$ and $\dot{x}_7 = \tau/ml$ and the submanifold, of $T\mathbb{R}^7$, $M_0 = F_0^{-1}(0)$. M_0 has dimension 9. The equation of the set $W_0 = \pi(M_0)$ is

$$x_1^2 + x_2^2 - l^2 = 0$$

it is a submanifold of $\mathbb{R}^7$ of dimension 6. $TW_0 = G_0^{-1}(0)$ where G_0 is the mapping

$$G_0(x,p) = \begin{pmatrix} x_1^2 + x_2^2 - l^2 \\ p_1x_1 + p_2x_2 \end{pmatrix}.$$

Thus the reduction of M_0 is $M_1 = TW_0 \cap M_0 = F_1^{-1}(0)$, where F_1 is the mapping

$$F_1(x,p) = \begin{pmatrix} p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1p_7 - p_5 \\ p_4 + x_2p_7 - p_6 - g \\ x_1^2 + x_2^2 - l^2 \\ x_1x_3 + x_2x_4 \end{pmatrix},$$

it is a submanifold of dimension 8. The equations of the set $W_1 = \pi(M_1)$ are

$$\begin{array}{rcl} x_1^2 + x_2^2 - l^2 & = & 0 \\ x_1x_3 + x_2x_4 & = & 0\,; \end{array}$$

and it is a submanifold of $\mathbb{R}^7$ of dimension 5. $TW_1 = G_1^{-1}(0)$, where G_1 is the mapping

$$G_1(x,p) = \begin{pmatrix} x_1^2 + x_2^2 - l^2 \\ x_1x_3 + x_2x_4 \\ p_1x_1 + p_2x_2 \\ p_1x_3 + x_1p_3 + p_2x_4 + x_2p_4 \end{pmatrix}.$$

Thus $M_2 = TW_1 \cap M_1 = F_2^{-1}(0)$, where F_2 is the mapping

$$F_2(x,p) = \begin{pmatrix} p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1p_7 - p_5 \\ p_4 + x_2p_7 - p_6 - g \\ x_1^2 + x_2^2 - l^2 \\ x_1x_3 + x_2x_4 \\ x_3^2 + x_4^2 + x_1p_5 + x_2p_6 + x_2g - l^2p_7 \end{pmatrix},$$

is a submanifold of dimension 7. Finally the set $W_2 = \pi(M_2)$ is in fact W_1 and thus $M_3 = TW_2 \cap M_2 = TW_1 \cap M_2 = M_2$ (since $M_2 \subset TW_1$), then $C(M_0)$ is the submanifold M_2 and $W = \pi(C(M_0))$ is the submanifold W_2. Moreover $C(M) = \chi(W \times \mathbb{R}^2)$ where $\chi(x,v) = (x, f(x,v))$ is the vector field of the state $x \in W$ depending on $v = (v_1, v_2) \in \mathbb{R}^2$ such that

$$f(x,v) = \begin{pmatrix} x_3 \\ x_4 \\ -\frac{x_1}{l^2}(x_3^2 + x_4^2) + \frac{x_2}{l^2}(x_2v_1 - x_1(v_2+g)) \\ -\frac{x_2}{l^2}(x_3^2 + x_4^2) - \frac{x_1}{l^2}(x_2v_1 - x_1(v_2+g)) \\ v_1 \\ v_2 \\ \frac{1}{l^2}(x_3^2 + x_4^2 + x_1v_1 + x_2(v_2+g)) \end{pmatrix}$$

and $v_1 = u_1/m$ et $v_2 = u_2/m$. On the other hand, from the relation $x_1 = l\sin\theta$ and $x_2 = l\cos\theta$ ($\theta \in]-\pi,\pi[$) we obtain

$$\begin{cases} x_3 &= l\dot\theta\cos\theta \\ x_4 &= -l\dot\theta\sin\theta \end{cases} \quad \begin{cases} \dot x_3 &= -l\dot\theta^2\sin\theta + l\ddot\theta\cos\theta \\ \dot x_4 &= -l\dot\theta^2\cos\theta - l\ddot\theta\sin\theta \\ \dot x_7 &= \dot\theta^2 + v_1\frac{\sin\theta}{l} + v_2\frac{\cos\theta}{l}. \end{cases}$$

Therefore, θ satisfy the following second order implicit differential equation

$$l\ddot\theta = v_1\cos\theta - (v_2+g)\sin\theta \tag{2.1}$$

Thus, we take for W the parameterization

$$x = X(z) = (l\sin\theta, l\cos\theta, l\vartheta\cos\theta, -l\vartheta\sin\theta, y_5, y_6, y_7),$$

where $z = (\theta, \vartheta, y_5, y_6, y_7) \in]-\pi,\pi[\times\mathbb{R}^4$, the controlled vector field

$$\begin{aligned}\chi(z,v) = (z, g(z,v)) \;&=\; \vartheta\frac{\partial}{\partial\theta} + (v_1\frac{\cos\theta}{l} - (v_2+g)\frac{\sin\theta}{l})\frac{\partial}{\partial\vartheta} + v_1\frac{\partial}{\partial y_5} \\ &\quad + v_2\frac{\partial}{\partial y_6} + (\vartheta^2 + v_1\frac{\sin\theta}{l} + (v_2+g)\frac{\cos\theta}{l})\frac{\partial}{\partial y_7}\end{aligned}$$

and the Lagrangian $\tilde{L}(z,v) = L(X(z), \frac{\partial X}{\partial z}(z)g(z,v))$. Then the problem $\mathcal{P}_c$ is equivalent to the explicit optimal control problem

$$\min_{\substack{(z(t),\dot{z}(t))=\chi(z(t),v(t))\ \forall t\in[0,1] \\ v(t)\in\mathbb{R}^2 \\ z(0)\in\{a\}\times\mathbb{R}^3 \\ z(1)\in\{b\}\times\mathbb{R}^3}} \int_0^1 \tilde{L}(z(t),v(t))dt,$$

for which we obtain the necessary conditions of optimality with the pseudo-Hamiltonian

$$\begin{aligned} H^{\psi_0}(z,\psi,v) &= \psi_1\vartheta + \psi_2(v_1\frac{\cos\theta}{l} - (v_2+g)\frac{\sin\theta}{l}) + \psi_3 v_1 + \psi_4 v_2 \\ &+ \psi_5(\vartheta^2 + v_1\frac{\sin\theta}{l} + (v_2+g)\frac{\cos\theta}{l}) - \psi_0\tilde{L}(z,v). \end{aligned}$$

REMARK 2.10. For this system, the kinetic energy is $T(\theta,\dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2$, the potential energy is $V(\theta) = -mgl\cos\theta$ and the Lagrangian is

$$\mathtt{L} = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

The virtual work of the control u is $\delta W_u = Q\delta\theta = (u_1 l\cos\theta - u_2 l\sin\theta)\delta\theta$ and for the tension it is zero. The Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathtt{L}}{\partial\dot{\theta}} - \frac{\partial\mathtt{L}}{\partial\theta} = Q,$$

gives the second order differential equation 2.1.

3. Geometry of Implicit Differential Equations

For the problem $\mathcal{P}$, M is a submanifold of $T\mathbb{R}^n$; it is obvious that the reduction procedure that we present in the introduction is not applicable to any submanifold M of $T\mathbb{R}^n$, especially the submanifolds M for which $\pi_{|M}$ admits singularities. In this section we will define the class of submanifolds of $T\mathbb{R}^n$ that will be allowed for the problem $\mathcal{P}$. For this class of submanifolds we will be able to apply locally the reduction procedure. First of all, let us make some comparisons with the definition of π-submanifolds given in [16]. The authors' concern is to answer to the problem of the existence and uniqueness of solutions, namely to put M in the form

$$M = \phi(Y) \tag{3.1}$$

for a section $\phi : Y \to TY$ of a connected submanifold Y of $\mathbb{R}^n$ with a dimension equal to that of M. In this situation, M is equivalent to an ordinary differential equation and, thus, the problem of existence and uniqueness is solved. Here, it is not our purpose to obtain the existence and uniqueness of the solutions (since in this case the optimal control problem admits an obvious solution, namely the trajectory which (possibly) goes from a to b) but to have the existence and uniqueness of a family of solutions, in other words to find a submanifold Y of $\mathbb{R}^n$, an open set U of $\mathbb{R}^q$ and a mapping

$\chi : Y \times U \to TY$ such that the diagram

$$\begin{array}{cccc} & \chi & & \\ Y \times U & \to & TY & \subset T\mathbb{R}^n \\ Pr & \searrow & \downarrow & \pi \\ & & Y & \subset \mathbb{R}^n \end{array}$$

switches and such that

$$M = \chi(Y \times U). \tag{3.2}$$

This occurs for the submanifold $C(M)$ in the example of the controlled rigid pendulum. Even though in [16] the equality (3.1) suggest that the submanifold M is locally embedded in a tangent bundle TY of a submanifold Y of $\mathbb{R}^n$ with the same dimension as that M, in our situation the equality (3.2) suggests that the submanifold M is locally embedded in a tangent bundle TY of a submanifold Y of $\mathbb{R}^n$ of dimension less than or equal to the dimension of M.

3.1. q-π-SUBMANIFOLD

For our geometric framework we will consider separable, Haussdorf manifold X with finite dimension and, for reasons of convenience, they are assumed to be smooth (although they could be of class C^k, $k \geq 2$). Let us recall some elements of differential geometry. The dimension of a manifold M is the maximal dimension among the dimension of the connected components Ξ of M. A pure manifold M is a manifold such that all the connected components Ξ have the same dimension. For any manifold X, the points belonging to the tangent bundle TX are denoted by (x, p) with x belonging to X and p belonging to T_xX. The canonical projection $\pi : TX \to X$ is the mapping such that $\pi(x, p) = x$. For the manifold $\mathbb{R}^n$, the tangent bundle is identified with $\mathbb{R}^n \times \mathbb{R}^n$ and the projection π is identified with the projection onto the first factor. Moreover, for any submanifold Y of X and any point belonging to Y, the subspace T_xY is identified with a subspace of T_xX and, thus, TY is identified with a submanifold of TX. Subsequently, the following notation $f : (X, a) \to (Y, b)$ means that the mapping f is defined in an open neighborhood U of a in X and $b = f(a)$. As in the case of manifolds, all the mappings are assumed to be smooth (once again they could be of class C^k, $k \geq 2$). For any mapping $f : X \to Y$ and any point x belonging to X the linear tangent mapping is denoted by T_xf. Now let us give the definition of subimmersion and the subimmersion theorem

DEFINITION 3.1. *(subimmersion)* Let X, Y be manifolds and a mapping $f : X \to Y$.

- (a): f is a *subimmersion* at $x \in X$ if $r = \text{rank } T_xf$ is constant in an open neighborhood of x in X.
- (b): f is a subimmersion on X if it is a subimmersion at x for all points x of X. In particular, for each connected component Ξ of X the rank r has a constant value on Ξ, we shall call it the rank of f on Ξ.

THEOREM 3.2. *(subimmersion theorem) Let X be connected manifolds and $f : X \to Y$ a subimmersion with rank r. Then, the following statements hold* (a): *for any* y *belonging to* $f(X)$ *the set* $M = f^{-1}(y)$ *is a submanifold of dimension* $m - r$ *and* $T_x M = Ker T_x f$.

(b): *for any point* x *belonging to* X *there exists an open neighborhood* V *of* x *in* X *such that the set* $W = f(V)$ *is a submanifold of dimension* r *and* $T_y W = Im T_x f$ *for any point* y *belonging to* W. *Moreover, if* N *is any submanifold of* X *of dimension* r *such that* x *belongs to* N *and* $T_x N \cap \ker T_x f = \{0\}$ *then the mapping* $f_{|N}$ *is a local diffeomorphism of some open neighborhood of* x *in* N *onto an open neighborhood of* y *in* $f(V)$.

Proof. see [9] □

For an implicit differential equation (1.1) the following proposition gives a criterion for the projection $\pi_{|M}$ to be a subimmersion.

PROPOSITION 3.3. *Let* $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-q}$ *be a mapping with* $0 \leq q \leq n$ *such that* $DG(x,p)$ *has full rank* $n - q$ *in an open neighborhood of a point* (x_0, p_0) *belonging to* $G^{-1}(0)$ *and* U *an open neighborhood of this point in* $\mathbb{R}^n \times \mathbb{R}^n$ *such that the set* $M = U \cap G^{-1}(0)$ *is a submanifold of* $\mathbb{R}^n \times \mathbb{R}^n$ *of dimension* $n + q$. *Then, the mapping* $\pi_{|M}$ *where* π *is the projection onto the first factor is a subimmersion at* (x_0, p_0) *of rank* $r = \rho + q$ *if, and only if,* $\text{rank } D_p G(x,p) = \rho \leq n - q$ *is constant in an open neighborhood of* (x_0, p_0) *in* M.

Proof. On the one hand, for any point (x,p) belonging to M, the tangent space $T_{(x,p)}M$ is equal to $\ker DG(x,p)$ and his dimension is equal to $n+q$. On the other hand, the linear tangent mapping $T_{(x,p)}(\pi_{|M}) : T_{(x,p)}M \to T_x\mathbb{R}^n$ is the restriction of the canonical projection to the subspace $T_{(x,p)}M$. Namely, the mapping

$$(\delta x, \delta p) \in T_{(x,p)}M \mapsto \delta x.$$

Then, for any point (x,p) belonging to M

$$\ker T_{(x,p)}(\pi_{|M}) = \{(\delta x, \delta p) \in T_{(x,p)}M / \delta x = 0\} = \{0\} \times \ker D_p G(x,p). \tag{3.3}$$

Clearly, the mapping $\pi_{|M}$ has constant rank r in an open neighborhood of (x_0, p_0) in M if, and only if, $\dim \text{Im } T_{(x,p)}(\pi_{|M}) = r$ in an open neighborhood of (x_0, p_0) in M; therefore if, and only if,

$$\dim \ker T_{(x,p)}(\pi_{|M}) = \dim T_{(x,p)}M - \dim \text{Im } T_{(x,p)}(\pi_{|M}) = n + q - r \tag{3.4}$$

in an open neighborhood of (x_0, p_0) in M. According to (3.3), (3.4) holds if, and only if, $\dim \ker D_p G(x,p) = n + q - r = n - \rho$ in an open neighborhood of (x_0, p_0) in M; that is if, and only if, $\dim \text{Im } D_p G(x,p) = \rho$ in an open neighborhood of (x_0, p_0) in M. □

Now we give the definition of a q-π-submanifold M of TX.

DEFINITION 3.4. *(q-π-submanifold)* Let X be a manifold, q a fixed integer less than or equal to the dimension of X, M a submanifold of TX and (x,p) a point of M. M is a *q-π-submanifold* of TX at (x,p) (in an neighborhood of (x,p) in M) if the following conditions hold

- (a): there exists a connected open neighborhood U of (x,p) in M and a submanifold Y of X such that $\dim Y + q = \dim U$ and U is a subset of TY.
- (b): the mapping $\pi_{|U} : U \to X$ is a subimmersion in the neighborhood of (x,p).

M is a q-π-submanifold of TX if for any point (x,p) belonging to M, M is a q-π-submanifold at (x,p).

Remark 3.5. a) If M is a q-π-submanifold at a point (x,p) of M, then we can assume that the mapping $\pi_{|U} : U \to X$ is a subimmersion on U, even if this means shrinking U. Moreover, for any point (x,p) belonging to U, the first condition of the definition holds (It is enough to take U and Y). Thus, for any point (x,p) belonging to U, U is a q-π-submanifold at (x,p); in other words U is a q-π-submanifold of TX.
b) When M is not a q-π-submanifold of TX we can consider the set, possibly empty, of points (x,p) of M such that M is q-π-submanifold of TX at (x,p). If it is a non-empty set, according to a), it is an open set of M and a q-π-submanifold of TX.

The definition of a q-π-submanifold can be formulated in the following way

Definition 3.6. *(bis)* Let X be a manifold and q an integer less than or equal to the dimension of X. A submanifold M of TX is a *q-π-submanifold* of TX if for any connected component Ξ of M the following conditions hold.

- (a): for any point (x,p) of Ξ there exists an open neighborhood U in Ξ of (x,p) and a submanifold Y of X such that $\dim Y + q = \dim \Xi$ and U is a subset of TY.
- (b): the mapping $\pi_{|\Xi} : \Xi \to X$ is a subimmersion in a neighborhood of any point (x,p) of Ξ.

Remark 3.7. This definition extends the definition of a π-submanifold in [16], which is the case $q = 0$ of our definition. The first condition means exactly that M is locally embedded in the tangent bundle TY of a submanifold Y of X of dimension less than or equal to the dimension of M. For the second, according to the subimmersion theorem, for any point (x,p) of M there exists an open neighborhood V of (x,p) in M such that $W = \pi(V)$ is a submanifold of X of dimension the rank of the mapping $\pi_{|\Xi}$ at (x,p). This is the local analogous of the condition, $W = \pi(M)$ is a submanifold of X, supposed in the global reduction procedure. For any connected component Ξ of M the inequality $2q \leq \dim \Xi$ is satisfied ($\dim \Xi \leq 2 \dim Y$, $\dim Y + q = \dim \Xi$). We shall use this inequality to prove a property of the index.

Example 3.8. Let $\chi : X \times U \to TX$ be a smooth controlled vector field with $\dim U = q$. Let us assume that

- (a): the mapping $\frac{\partial \chi}{\partial u}(x,u)$ has full rank q.
- (b): for any (x,p) belonging to TX either the equation $\chi(x,u) = (x,p)$ has a unique solution or it does not have any solution.
- (c): the mapping $(x,u) \mapsto \chi(x,u)$ is proper.

Clearly, under this mild assumption the set $M = \chi(X \times U)$ is a submanifold of TX[3]. Obviously, $\dim M = \dim X + \dim U$ and $\pi_{|M}$ is a subimmersion. Thus, M is a q-π-submanifold.

In the example of the controlled rigid pendulum, the submanifold M_0 of $T\mathbb{R}^7$ is connected and its dimension is equal to 7. Therefore, for any point (x,p) belonging to M_0 the first condition holds with $U = M_0$, $Y = \mathbb{R}^7$ and $q = 2$. Moreover, for any point (x,p) belonging to M_0, rank $D_pG(x,p) = 4$; then, according to the proposition 3.3, the mapping $\pi_{|M_0}$ is a subimmersion of rank $4+2=6$. Therefore, the submanifold M_0 is a 2-π-submanifold of $T\mathbb{R}^7$.

Now we shall give some definitions: the order of point (x,p) of M is the rank of the mapping $\pi_{|\Xi}$ at this point, we shall denote it by ord $_M(x,p)$. Since, the mapping $\pi_{|\Xi}$ has, locally, constant rank, ord $_M(x,p)$ is constant for each point of any connected component of M, then we may define ord $_M\Xi$ as the order of one of its points and it is less than or equal to the dimension of the submanifold Y. A submanifold W as in the remark 3.7 is called a local projection of M at (x,p).

REMARK 3.9. With the notations of the definition of a q-π-submanifold, U is a submanifold of TY and since the mapping $\pi_{|\Xi} : \Xi \to X$ is a subimmersion, the mapping $\pi_{|U} : U \to X$ is a subimmersion. This is satisfied if, and only if, the mapping $\pi_{|U} : U \to Y$ is a subimmersion. The order of a point belonging to Ξ is also the rank of the mapping $\pi_{|U} : U \to Y$ at this point. Then, we can see U as a submanifold of TY and as a submanifold of TX.

The following theorem ensures that a q-π-submanifold M is, locally, the image of a unique controlled vector field.

THEOREM 3.10. *(Existence and uniqueness) Given X a manifold and M a q-π-submanifold of TX such that*

$$\dim \Xi = \text{ord }_M\Xi + q \tag{3.5}$$

for each connected component Ξ of M, then for each point (x,p) of M there exists a local projection $W = \pi(V)$ of M at (x,p), an open set O of $\mathbb{R}^q$ and a unique smooth mapping $\chi : W \times O \to TW$ such that

$$V = \chi(W \times O),\ p = \chi(x,0),\ rank\frac{\partial \chi}{\partial u}(x,0) = q$$

and such that $Pr = \pi \circ \chi$ where Pr is the canonical projection from $W \times O$ onto W. Moreover, if $W' = \pi(V')$ is another local projection of M at (x,p) such that there exists an open set O' of $\mathbb{R}^q$ and a unique smooth mapping $\chi' : W' \times O' \to TW'$ such that

$$V' = \chi(W' \times O'),\ p = \chi'(x,0),\ rank\frac{\partial \chi'}{\partial u'}(x,0) = q$$

and such that $Pr' = \pi \circ \chi'$ where Pr' is the canonical projection from $W' \times O'$ onto W', then there exists a diffeomorphism $h : (W \times O, (x,0)) \to (W' \times O', (x,0))$ such that $\chi = \chi' \circ h$ and $Pr = Pr' \circ h$.

Proof. Let (x_0,p_0) be a point of M. According to (3.5) and the remark 3.9 the mapping $\pi_{|U} : U \to Y$ is a subimmersion with rank equal to Ord $_M\Xi =$

[3]Since the mapping χ is an injective proper immersion.

$\dim \Xi - q = \dim Y$, therefore it is a submersion. Then, there exists an open neighborhood V of (x_0, p_0) in U such that the local projection $W = \pi(V)$ is an open set of Y. Since W is an open set of Y, the tangent bundle TW of W is equal to $(\pi_{|TY})^{-1}(W) = \pi^{-1}(W) \cap TY$. Moreover, $V \subset TY$ and $V \subset \pi^{-1}(W)$ then $V \subset \pi^{-1}(W) \cap TY = TW$. Then, V is a submanifold of TW of dimension $\dim Y + q$. This last property is also satisfied when we shrink V or W (for any open set V' of V, $W' = \pi(V')$ is also a local projection and V' is a submanifold of TW' by the same arguments; for any open set W' of W setting $V' = (\pi_{|TY})^{-1}(W') \cap V$ then $W' = \pi(V')$ and V' is also a submanifold of TW'). Thus, even if this means shrinking V, we can assume that there exists a chart (W, ψ) of Y such that W is identified with an open set of $\mathbb{R}^m$, also denoted by W, then TW is identified with $W \times \mathbb{R}^m$, π is identified with the canonical projection and V is identified with a submanifold of $W \times \mathbb{R}^m$ projected onto W. We can also assume that there exists an open set Ω of (x_0, p_0) in $W \times \mathbb{R}^m$, an open set Ω_v of 0 in $\mathbb{R}^{m-q}$ and a submersion $G : \Omega \to \Omega_v$ such that $V \cap \Omega = G^{-1}(0) \cap \Omega$. Since V is projected onto W then $D_pG(x_0, p_0)$ has full rank; ; even if this means shrinking Ω, there exists an open set Ω_u of 0 in $\mathbb{R}^q$ and a mapping $H : \Omega \to \Omega_u$ such that for the mapping $\Phi : \Omega \to W \times \Omega_u \times \Omega_v$ defined by $(y, u, v) = \Phi(x, p) = (x, H(x, p), G(x, p))$, $D\Phi(x_0, p_0)$ is an isomorphism. According to the local inverse functions theorem there exists an open set $\Omega' \subset \Omega$ of (x_0, p_0) in $W \times \mathbb{R}^m$ and an open set Ω'' of $(x_0, 0, 0)$ in $W \times \Omega_u \times \Omega_v$ such that $\Phi_{|\Omega'} : \Omega' \to \Omega''$ is a diffeomorphism. We can assume that Ω'' has the form $W' \times \Omega'_u \times \Omega'_v$ where W' is an open set of W, Ω'_u is an open set of Ω_u and Ω'_v is an open set of Ω_v. Moreover there exists an open set Ω'_p of p in $\mathbb{R}^m$ such that $\Omega' = \Phi^{-1}_{|\Omega'}(\Omega'')$ has the form $W' \times \Omega'_p$. Let $V_0 = V \cap \Omega'$, then $W_0 = \pi(V_0)$ is a local projection of M at (x_0, p_0). Given $\Phi^{-1}_{|\Omega'}(y, u, v) = (y, \phi(y, u, v))$ the inverse mapping of $\Phi_{|\Omega'}$, then $V_0 = \{(x, \phi(x, u, 0),\ x \in W_0,\ u \in \Omega'_u\} = \Phi^{-1}_{|\Omega'}(W_0 \times \Omega'_u \times \{0\})$. Therefore, we define $O = \Omega'_u \times \{0\}$ and $\chi = \Phi^{-1}_{|W_0 \times O}$. Then $V_0 = \chi(W_0 \times O)$, $\chi(x_0, 0) = p_0$ and $\text{rank}\dfrac{\partial \chi}{\partial u}(x_0, 0) = \text{rank } \dfrac{\partial \phi}{\partial u}(x_0, 0, 0) = q$. Given $W' = \pi(V')$ another local projection of M at (x, p) such that there exists an open set O' of $\mathbb{R}^q$ and a unique smooth mapping $\chi' : W' \times O' \to TW'$ such that $V' = \chi(W' \times O')$, $p = \chi'(x, 0)$, $\text{rank}\dfrac{\partial \chi'}{\partial u'}(x, 0) = q$ and such that $Pr' = \pi \circ \chi'$ where Pr' is the canonical projection from $W' \times O'$ onto W', then the implicit equation $\chi'(x, u') = \chi(x, u)$ is locally invertible relative to u and u' and the existence of h follows. □

REMARK 3.11. The diagrams

$$\begin{array}{ccccc} W' \times O' & \xrightarrow{\chi'} & TW' & \subset TX \\ & {\scriptstyle Pr'} \searrow & \downarrow \pi & \\ & & W' & \subset X \end{array} \quad \text{and} \quad \begin{array}{ccccc} W \times O & \xrightarrow{\chi} & TW & \subset TX \\ & {\scriptstyle Pr} \searrow & \downarrow \pi & \\ & & W & \subset X \end{array}$$

are equivalent in a neighborhood of $(x, 0)$ by the bundle isomorphism h. Such bundle isomorphisms define in control theory a feedback.

3.2. Reduction of a q-π-submanifold

Since the condition of the theorem 3.10 is generally not fulfilled, then we have to define the reduction procedure. In the case of the global reduction procedure, we have seen that any admissible trajectory $x(\cdot)$ for M_0 is necessarily an admissible trajectory for $M_1 = M_0 \cap TW_0$ and M_1 is the reduction of M_0. In order to have, locally, the same reduction procedure, we use local projection. Let $x(\cdot)$ be a local admissible trajectory, in other words $(x(\cdot), \dot{x}(\cdot))$ is a trajectory of M passing through the point $(x_0, p_0) = (x(t_0), \dot{x}(t_0)) \in M$ at time t_0, where t_0 is a point of continuity; since M is a q-π-submanifold , then for a local projection $W = \pi(V)$ at (x_0, p_0) of M and for t in a neighborhood of t_0, $(x(t), \dot{x}(t)) \in V$, whence $x(t) \in W = \pi(V)$ and $(x(t), \dot{x}(t)) \in TW$. In particular $(x_0, p_0) \in TW \cap V \subset TW \cap M$. This leads to the following definition

Definition 3.12. Let X be a manifold and M a q-π-submanifold of TX. A point $(x, p) \in M$ is a *point of reducibility* of M if there exists a local projection $W = \pi(V)$ of M at X such that $p \in T_x W$, in other words (x, p) belongs to $TW \cap M$. The (possibly empty) set of the points of reducibility of M is the reduction of M, we denote it by M'

Remark 3.13. If such local projection $W = \pi(V)$ exists then for any other local projection $W' = \pi(V')$ of M at (x, p), since $T_x W' = T_x W$, (x, p) belongs to $TW' \cap M$. Thus, the definition does not depend on the choice of the local projection W.

Example 3.14. *A q-π-submanifold M such that the reduction M' is the empty set.*

In $T\mathbb{R}^5$ let M be the submanifold given by the implicit differential equation $F_0 = 0$ where

$$F_0(x, p) = \begin{pmatrix} x_1 - x_2 \\ p_1 - 1 \\ p_2 \\ p_3 - p_4 - p_5 \end{pmatrix}.$$

Clearly, M is a 1-π-submanifold of $T\mathbb{R}^5$, $W = \pi(M)$ is the submanifold of dimension 4 of $\mathbb{R}^5$ given by the equation $x_1 - x_2 = 0$ and $M' = TW \cap M = \emptyset$.

Now that we have stated the reduction procedure and a new set, namely M' the reduction of M, we are going to see under which conditions for M, the reduction M' is a submanifold and a q-π-submanifold. First of all, we can establish the following results

Proposition 3.15. *Let X be a manifold and M a q-π-submanifold of TX. Then, the reduction M' of M is a closed subset of M.*

Proof. The cases $M = M'$ and $M = \emptyset$ are obvious. Let us assume that $\emptyset \subsetneq M' \subsetneq M$ and let us consider the subset $N = M \setminus M'$. Let (x, p) be a point of N and $W = \pi(V)$ a local projection of M at this point. W is a submanifold of X of dimension $r = \mathrm{ord}_M \Xi$. Upon shrinking V, we can assume that there exists a chart (Ω, ϕ) of X such that $W \subset \Omega$, $\phi(\Omega)$ is an open set of $\mathbb{R}^n$, $\phi(W)$ is an open set of $\mathbb{R}^r$, $T\Omega$ is identified with $\Omega \times \mathbb{R}^n$, TW is identified with $W \times \mathbb{R}^r$ and V is identified with a submanifold of $\Omega \times \mathbb{R}^n$. Since, the point (x, p) belongs to N and the point x belongs to W

the point p does not belong to T_xW, and in the chart this means that p does not belong to $\mathbb{R}^r$. Thus, there exists an open neighborhood V' of (x,p) in V such that $V' \cap TW = \emptyset$. Therefore, the set N is an open set of M and then M' is a closed subset of M. □

The following proposition is used to establish the conditions such that the reduction M' of a q-π-submanifold M is a submanifold.

Proposition 3.16. *Let X be a manifold and M a q-π-submanifold of TX such that its reduction M' is not empty. Given (x,p) a point of M', then for any local projection $W = \pi(V)$ of M at (x,p)*

$$\dim[T_{(x,p)}TW \cap T_{(x,p)}M] \geq \text{ord}_{\,M}(x,p) + q.$$

Proof. Let (x_0,p_0) be a point of M' and $W = \pi(V)$ a local projection of M at (x_0,p_0). According to the definition of a q-π-submanifold there exists an open neighborhood U of (x_0,p_0) in M and a submanifold Y of X such that $\dim Y + q = \dim \Xi$ and U is a subset of TY. We can choose $W = \pi(V)$ such that $V \subset U$, therefore $W \subset \pi(U) \subset Y$. Let $i : W \to Y$ be the canonical embedding of W in Y and

$$\Sigma = i^\star TY = \bigcup_{x \in W} \{x\} \times T_xY \subset TY.$$

Σ is a submanifold of TY of dimension equal to the sum of the dimension of W and of the dimension of Y. Moreover, TW and V are submanifold of Σ. Indeed, let (x,p) be a point of Σ. Then, according to the construction of Σ, x belongs to W and p belongs to T_xY. Since Y and W are submanifolds of X, there exists in a neighborhood of the point x in X two submersions $g : (\mathbb{R}^n, x) \to (\mathbb{R}^{n-r}, 0)$ and $h : (\mathbb{R}^n, x) \to (\mathbb{R}^{n-m}, 0)$ such that in a neighborhood of x, $W = g^{-1}(0)$ and $Y = h^{-1}(0)$ where $m = \dim Y = \dim \Xi - q$ and $r = \text{ord}_{\,M}(x,p) = \text{ord}_{\,M}(x_0,p_0)$. Thus in a neighborhood of (x,p) in TX, Σ is the zero set of the submersion

$$\begin{aligned} \sigma : (TX, (x,p)) &\to (\mathbb{R}^{n-r} \times \mathbb{R}^{n-m}, (0,0)) \\ (x,p) &\mapsto \sigma(x,p) = (g(x), dh(x).p). \end{aligned}$$

Thus, Σ is a submanifold of TX of dimension $n - r - (n - m) = r + m$ and therefore a submanifold of TY. According to the construction, TW is a subset of Σ and a submanifold of TX, therefore TW is a submanifold of Σ. Finally, since V is subset of Σ (for any point (x,p) of $V \subset U \subset TY$, x belongs to W and p belongs to T_xY) and a submanifold of TY, V is a submanifold of Σ. Since $\dim V = dimY + q$, $\dim TW = 2r$ and $\dim \Sigma = \dim Y + r$ then

$$\begin{aligned} \dim[T_{(x,p)}TW \cap T_{(x,p)}V] &\geq \dim T_{(x,p)}TW + \dim T_{(x,p)}V - \dim T_{(x,p)}\Sigma \\ &= \text{ord}_{\,M}(x,p) + q. \end{aligned}$$

□

Theorem 3.17. *Let X be a manifold and M a q-π-submanifold of TX such that its reduction M' is not empty. If for each point (x,p) of the reduction M' of M there exists a local projection $W = \pi(V)$ of M at (x,p) such that*

$$\dim[T_{(x,p)}TW \cap T_{(x,p)}M] = \text{ord}_{\,M}(x,p) + q, \tag{3.6}$$

then M' is a submanifold and $T_{(x,p)}M' = T_{(x,p)}TW \cap T_{(x,p)}M$. Moreover, if in an open neighborhood of (x,p) in $TW \cap M$

$$\pi_{|T_{(x',p')}M'} \text{ has constant rank,} \tag{3.7}$$

then for any connected component Ξ of M, Ξ' the reduction of Ξ is either empty or a pure q-π-submanifold of dimension $\operatorname{ord}_M \Xi + q$. *In particular M' is a q-π-submanifold.*

Proof. Given (x,p) a point of M' and $W = \pi(V)$ a local projection of M at (x,p) such that (3.6) is satisfied, then the submanifolds TW and V are transversal in the bundle Σ at (x,p), therefore $TW \cap V$ is a submanifold of TW and of M of dimension $\operatorname{ord}_M(x,p) + q$. The mapping $\pi_{|V}$ has constant rank on V, even if this means shrinking V. But, on the one hand,

$$M' \cap V = \{(x,p) \in V / \exists \text{ a local projection } W_{(x,p)} = \pi(V_{(x,p)}) \;/\; p \in T_x W_{(x,p)}\}$$

and, on the other hand, $W = \pi(V)$ is a local projection of M at each points of V, therefore

$$M' \cap V = \{(x,p) \in V \;/\; p \in T_x W\} = TW \cap V.$$

In other words M' is a submanifold of M and $T_{(x,p)}M' = T_{(x,p)}TW \cap T_{(x,p)}V = T_{(x,p)}TW \cap T_{(x,p)}M$.

Therefore the connected component of M' containing the point (x,p) has a dimension equal to $\operatorname{ord}_M(x,p) + q$. Let us prove that the first condition of the definition of a q-π-submanifold hold. Given Ξ' the connected component of M' containing (x,p), there exists an open neighborhood U' of (x,p) in Ξ' such that $U' \subset TW$, in fact $U' = TW \cap V$ that we can assume included in Ξ'. If we set down $Y' = W$ then $\dim Y' + q = \dim \Xi'$ and the condition holds. Moreover, if condition (3.7) holds then $\pi_{|\Xi'}$ has constant rank and therefore according to the previous results, M' is a q-π-submanifold of X. □

The theorem 3.17 justified the following definition

DEFINITION 3.18. Let X be a manifold and M a q-π-submanifold M of TX such that its reduction M' is not empty. M is a *reducible q-π-submanifold* if for any point (x,p) of the reduction M' of M there exists a local projection $W = \pi(V)$ of M at (x,p) such that

$$\dim[T_{(x,p)}TW \cap T_{(x,p)}M] = \operatorname{ord}_M(x,p) + q \tag{3.8}$$

and if in an open neighborhood of (x,p) in $TW \cap M$

$$\pi_{|T_{(x',p')}M'} \text{ has constant rank} \tag{3.9}$$

where $T_{(x',p')}M' = T_{(x',p')}TW \cap T_{(x',p')}M$.

REMARK 3.19. If M is a reducible q-π-submanifold of class c^k, $k \geq 2$, then its reduction is a q-π-submanifold of class C^{k-1}.

EXAMPLE 3.20. *q-π-submanifold M such that the reduction M' is not empty and is not a q-π-submanifold.* Let us consider in $T\mathbb{R}^3$, the 1-π-submanifold given by the implicit differential equation $F_0 = 0$ where

$$F_0(x,p) = \begin{pmatrix} x_1 \\ p_1 + x_2 + p_3^2 \end{pmatrix}.$$

Clearly, $W=\pi(M)$ is the submanifold of dimension 2 of $\mathbb{R}^3$ given by the equation $x_1=0$. Its reduction M_1 is equal to $F_1^{-1}(0)$ where

$$F_1(x,p)=\begin{pmatrix} x_1 \\ p_1 \\ x_2+p_3^2 \end{pmatrix}.$$

is not a 1-π-submanifold.

We can now provide a new formulation of the theorem 3.10

THEOREM 3.21. *(Existence and uniqueness) Given X a manifold and M a q-π-submanifold of TX such that its reduction M' is M, then for each point (x,p) of M there exists a local projection $W=\pi(V)$ of M at (x,p), an open set O of $\mathbb{R}^q$ and a unique smooth mapping $\chi : W\times O\to TW$ such that*

$$V=\chi(W\times O),\;\; p=\chi(x,0),\;\; rank\frac{\partial\chi}{\partial u}(x,0)=q$$

and such that $Pr=\pi\circ\chi$ where Pr is the canonical projection from $W\times O$ onto W. Moreover, if $W'=\pi(V')$ is another local projection of M at (x,p) such that there exists an open set O' of $\mathbb{R}^q$ and a unique smooth mapping $\chi' : W'\times O'\to TW'$ such that

$$V'=\chi(W'\times O'),\;\; p=\chi'(x,0),\;\; rank\frac{\partial\chi'}{\partial u'}(x,0)=q$$

and such that $Pr'=\pi\circ\chi'$ where Pr' is the canonical projection from $W'\times O'$ onto W', then there exists a diffeomorphism $h : (W\times O,(x,0))\to(W'\times O',(x,0))$ such that $\chi=\chi'\circ h$ and $Pr=Pr'\circ h$.

Proof. The connected component Ξ' of M' is exactly the connected component Ξ of M, but according to the theorem 3.17 $\dim\Xi=\dim\Xi'=\mathrm{ord}\,_M\Xi+q$ then assumption (3.5) of the theorem 3.10 holds for any connected component of M. □

REMARK 3.22. For any q-π-submanifold M of TX such that for any connected component Ξ of M the equality (3.5) holds then M is equal to its reduction. Clearly, for any point (x,p) of M there exists a local projection $W=\pi(V)$ of M at (x,p) such that p belongs to $T_xW\cap M$.

This remark leads to the following proposition

PROPOSITION 3.23. *Let X be a manifold, q a fixed integer less than or equal to the dimension of X and M a submanifold of TX. Let us assume that for any point (x,p) in M there exists a connected open neighborhood U of (x,p) in M and a submanifold Y of X such that $\dim Y+q=\dim U$ and U is a subset of TY. Then M is a reducible q-π-submanifold of TX equal to its reduction M' if, and only if, for any connected component Ξ of M there exists a point (x,p) in Ξ, such that the linear mapping $T_{(x,p)}\pi : T_{(x,p)}M\to T_{(x,p)}Y$ is surjective.*

Proof. If M is a reducible q-π-submanifold of TX equal to its reduction then for any connected component Ξ and for any point (x,p) of Ξ the linear mapping $T_{(x,p)}\pi : T_{(x,p)}M\to T_{(x,p)}Y$ is surjective. Clearly, this must hold only for one point (x,p) of any connected component Ξ of M. Conversely, let us assume that for any connected component Ξ of M there exists a point (x,p) in Ξ such that the linear mapping $T_{(x,p)}\pi : T_{(x,p)}M\to T_{(x,p)}Y$

is surjective. Thus, the mapping $\pi_{|U} : U \to Y$ has full rank $\dim Y$ and according to the remark 3.9 the mapping $\pi_{|U} : U \to X$ has constant rank. Then M is a q-π-submanifold of TX. In particular for each connected component Ξ of M the equality (3.5) is satisfied. Thus, according to the remark 3.22, M is equal to its reduction. □

3.3. COMPLETELY REDUCIBLE q-π-SUBMANIFOLD

Given a reducible q-π-submanifold M of a manifold TX, then according to the theorem 3.17 its reduction M_1 is a q-π-submanifold. Clearly, the reduction M_2 of M_1 may be empty (example 3.14) and in the case where M_2 is not empty, M_2 may not be reducible (example 3.20). If M_1 is reducible then M_2 is a q-π-submanifold. Thus, we can, if it is possible, consider the successive reductions of M(example 2.9). For reasons of convenience, we shall say that the empty set is a reducible q-π-submanifold such that its reduction is the empty set. These considerations lead us to consider the definition of a completely reducible q-π-submanifold

Definition 3.24. Let X be a manifold and M a q-π-submanifold of TX. We shall say that M is a *completely reducible q-π-submanifold* if it is reducible and if its reduction M' is a completely reducible q-π-submanifold.

The definition means that it is possible to construct a sequence of reducible q-π-submanifolds M_k, $k \geq 0$ such that $M = M_0$, $M_{k+1} = M'_k$ if $M_k \neq \emptyset$ and $M_{k+1} = M_k$ if $M_k = \emptyset$. This sequence of q-π-submanifold is called the chain of reduction of M. If for an integer α, $M_{\alpha+1} = M_\alpha$ then the sequence M_k becomes stationary at and after the integer α. Since the sequence $\dim M_k$ is decreasing, we can expect the chain of reduction of M to be stationary.

Theorem 3.25. *(Stationarity) Let X be a manifold and M a completely reducible q-π-submanifold of TX, m its dimension and $\{M_k\}_{k\geq 0}$ its chain of reduction.*

- (a): *given Ξ_{m+1-2q} a non-empty connected component of M_{m+1-2q} and for any $k = 0, \cdots, m-2q$, Ξ_k the connected component of M_k containing Ξ_{k+1}, then there exists a smallest integer α, $0 \leq \alpha \leq \dim \Xi_0 - 2q \leq m-2q$, such that $\Xi'_k = \Xi_k$ for any $k \geq \alpha$.*
- (b): *the reduction M_{m+2-2q} of M_{m+1-2q} is M_{m+1-2q}.*

Proof. When $M_{m+1-2q} = \emptyset$ we do not take (a) into account and with the convention, (b) is obvious. Let us assume that M_{m+1-2q} is not empty and let us consider Ξ_{m+1-2q} one of its non empty connected component. For any $k = 0, \cdots, m-2q$ the reduction Ξ'_k of Ξ_k is by definition the set of points of reducibility of Ξ_k, namely the set of points of reducibility of M_k belonging to Ξ_k i.e. $\Xi'_k = \Xi_k \cap M_{k+1}$. Thus from the construction of the sequence Ξ_k, the connected component Ξ_{k+1} of M_{k+1} included in Ξ_k is a connected component of the reduction Ξ'_k of Ξ_k and a closed subset of Ξ_k. Then we have

$$\Xi_{k+1} \subset \Xi'_k \subset \Xi_k, \qquad k = 0, m-2q. \tag{3.10}$$

Therefore, the sequence $\nu_k = \dim \Xi_k$, $k = 0, \cdots, m-2q$ satisfy

$$2q \leq \nu_{m-2q+1} \leq \nu_{m-2q} \leq \cdots \leq \nu_0 \leq m \tag{3.11}$$

and thus there exists a smallest integer α between 0 and $\nu_0 - 2q$ such that $\nu_\alpha = \nu_{\alpha+1}$. Thus $\Xi_{\alpha+1}$ is an open set of Ξ_α. Since $\Xi_{\alpha+1}$ is also a non empty closed subset of Ξ_α then $\Xi_\alpha = \Xi_{\alpha+1}$. According to (3.10) the reduction Ξ'_α of Ξ_α is exactly $\Xi_{\alpha+1}$ and therefore, for any $k = \alpha, \cdots, 2m - q + 1$, the reduction Ξ'_k of Ξ_k is Ξ_{k+1}. Finally, the reduction Ξ'_{m+1-2q} of any connected component Ξ_{m+1-2q} of M_{m+1-2q} is Ξ_{m+2-2q}, thus $M_{m+1-2q} = M_{m+2-2q}$. □

Definition 3.26. *(Index)* Let X be a manifold, M a completely reducible q-π-submanifold of TX, m the dimension of, and $\{M_k\}_{k\geq 0}$ the chain of reduction of M.

(a): the *core* of the completely reducible q-π-submanifold M is the limit of its chain of reduction and we denote it by $C(M)$.
(b): the *index* of any non-empty connected component Ξ of $C(M)$ is the integer α in 3.25(a).
(c): the *index* of any point (x,p) of $C(M)$ is the index of the connected component of $C(M)$ containing (x,p). In particular it is less than or equal to $\dim \Xi_0 - 2q$.

Remark 3.27. a) According to the remark 3.5a) for any point (x,p) belonging to a q-π-submanifold M of TX then there exists a connected open neighborhood U which is a q-π-submanifold of TX. If M is a completely reducible q-π-submanifold of TX then U is a completely reducible q-π-submanifold of TX, and $C(U)$ the core of U is included in the core of M. Obviously, for any point (x,p) of $C(U)$ the index of (x,p) seen as a point of $C(U)$ is equal to the index of (x,p) seen as a point of $C(M)$. Generally speaking, any open set U of a q-π-submanifold M of TX is a q-π-submanifold of TX and if M is a completely reducible q-π-submanifold then U is a completely reducible q-π-submanifold. Thus, for any point (x,p) of M such that there exists an open set U which is a completely reducible q-π-submanifold such that (x,p) belongs to $C(U)$, then the index of (x,p) seen as a point of $C(U)$ does not depend on U; when M is a completely reducible q-π-submanifold, it is equal to the index of (x,p) see as a point of $C(M)$.
b) As in the remark 3.5b), when M is not a completely reducible q-π-submanifold of TX we can consider the set, possibly empty, of points (x,p) of M such that there exists an open set U of M which is a completely reducible q-π-submanifold of TX. Then, this set is an open set in M and it is a completely reducible q-π-submanifold of TX. In the example 3.20, it is the set of points (x,p) of $T\mathbb{R}^3$ such that p_3 is not equal to zero.

Remark 3.28. According to the remark 3.19 for a completely reducible q-π-submanifold M of class $C^l, l \geq m+2-2q$, the q-π-submanifolds M_k of the chain of reduction are of class C^{l-k}. Clearly, the core $C(M)$, which is the q-π-submanifold M_{m+1-2q} is a q-π-submanifold of class $C^{l-m-1+2q}$. When M has class C^l with $l < m+2-2q$ then, the chain of reduction is only defined for $k \leq l$ since the reduction M_l is not defined. Consequently, if M_l the reduction of M_{l-1} is not equal to M_l, it is not possible to construct $C(M)$ the core of M. Obviously, when M_{k+1} the reduction of M_k is equal to M_k with $k \leq l-1$ then, the core $C(M)$ is equal to M_k.

EXAMPLE 3.29. *The controlled rigid pendulum.* The connected submanifold $C(M)$ has an index equal to 2.

EXAMPLE 3.30. Under the mild assumption of the example 3.8 the connected manifold $M = \chi(X \times U)$ has an index equal to 0.

According to the above theorem the core $C(M)$ of a completely reducible q-π-submanifold M is the completely reducible q-π-submanifold M_{m+1-2q} which is reducible and equal to its reduction. Thus we can formulate the following theorem

THEOREM 3.31. *(Existence and uniqueness) Given X a manifold and M a completely reducible q-π-submanifold then its core $C(M)$ is either empty or a reducible q-π-submanifold equal to its reduction and for any point (x,p) of M there exists a local projection $W = \pi(V)$ of M at (x,p), an open set O of $\mathbb{R}^q$ and a unique smooth mapping $\chi : W \times O \to TW$ such that*

$$V = \chi(W \times O), \; p = \chi(x,0), \; rank\frac{\partial \chi}{\partial u}(x,0) = q$$

and such that $Pr = \pi \circ \chi$ where Pr is the canonical projection from $W \times O$ onto W. Moreover, if $W' = \pi(V')$ is another local projection of M at (x,p) such that there exists an open set O' of $\mathbb{R}^q$ and a unique smooth mapping $\chi' : W' \times O' \to TW'$ such that

$$V' = \chi(W' \times O'), \; p = \chi'(x,0), \; rank\frac{\partial \chi'}{\partial u'}(x,0) = q$$

and such that $Pr' = \pi \circ \chi'$ where Pr' is the canonical projection from $W' \times O'$ onto W', then there exists a diffeomorphism $h : (W \times O, (x,0)) \to (W' \times O', (x,0))$ such that $\chi = \chi' \circ h$ and $Pr = Pr' \circ h$.

REMARK 3.32. In our definition of a q-π-submanifold we assume that the integer q is the same for each connected component Ξ of M; in fact we can extend the definition if we assume that the integer q depends on the connected component Ξ. In other words M, is a disjoint reunion of q_i-π-submanifold N_i of TX of dimension n_i, $i \geq 1$, where the q_i are integers less than or equal to $\dim X$. We will say again that M is a q-π-submanifold, where q is an integer n-tuple $(q_1, q_2, \ldots, q_n)$ (n possibly infinite). On the other hand the definition of the reduction is still valid and M is a (completely) reducible q-π-submanifold of TX if each q_i-π-submanifold N_i is a (completely) reducible q_i-π-submanifold of TX. For a completely reducible q-π-submanifold M of TX we can define in the same way the chain of reduction $\{M_k\}_{k \geq 0}$. Clearly, if for each q_i-π-submanifold N_i, $\{N_{ik}\}_{k \geq 0}$ is its chain of reduction, then, $M_k = \bigcup_i N_{ik}$. Moreover, according to the theorem 3.25 for each q_i-π-submanifold N_i of TX, the reduction $N_{n_i+2-2q_i}$ of $N_{n_i+1-2q_i}$ is $N_{n_i+1-2q_i}$; consequently, if we pose $\alpha = \max\{n_i + 2 - 2q, \; i \geq 1\}$ then M_α the reduction of $M_{\alpha-1}$ is exactly $M_{\alpha-1}$. Then, the core is the q-π-submanifold $M_{\alpha-1}$. Clearly, the theorem 3.31 still holds.

4. ALGORITHM OF REDUCTION

For a q-π-submanifold M we can set the following algorithm in a neighborhoods of any point (x_0, p_0) of M, allowing us to know if M is completely reducible, to find $C(M)$ and to obtain the controlled vector field.

Step 0: Assume M_0 is a non-empty q-π-submanifold and (x_0, p_0) belongs to $M_0 = M$. Let Ξ_0 be the connected component of M_0 which contains (x_0, p_0). From the definition of a q-π-submanifold there exists an open set U_0 of (x_0, p_0) in Ξ_0, a submanifold Y_0 of X such that $\dim Y_0 + q = m_0 + q = \dim \Xi_0$ and we have also seen that U_0 is a submanifold of TY_0. We can place ourselves in a chart of Y_0 at x_0, even if this means shrinking U_0. Thus Y_0 is an open set of $\mathbb{R}^{m_0}$, that we denote again by Y_0, $TY_0 = Y_0 \times \mathbb{R}^{m_0}$ and U_0 is a submanifold of $Y_0 \times \mathbb{R}^{m_0}$ of dimension $m_0 + q$. Then there exists in a neighborhood of (x_0, p_0) in $Y_0 \times \mathbb{R}^{m_0}$ a submersion $G_0 : (Y_0 \times \mathbb{R}^{m_0}, (x_0, p_0)) \to (\mathbb{R}^{m_0 - q}, 0)$ such that $U_0 = G_0^{-1}(0)$. In this way $\pi_{|\Xi_0}$ is a subimmersion in a neighborhood of (x_0, p_0) if, and only if, $D_p G_0(x, p)$ has constant rank $\rho_0 = r_0 - q \leq m_0 - q$ in a neighborhood of (x_0, p_0) in U_0. Then any local projection $W_0 = \pi(V_0)$ of U_0 (or M_0) in (x_0, p_0) is a submanifold of Y_0 of dimension r_0. Thus there exists in a neighborhood of x_0 in Y_0 a submersion $g_0 : (Y_0, x_0) \to (\mathbb{R}^{m_0 - r_0}, 0)$ such that $W_0 = g_0^{-1}(0)$ in a neighborhood of x_0 in Y_0. The tangent bundle TW_0 is the subset of points (x, p) of TY_0 for which the following equations are satisfied

$$g_0(x) = 0, \quad Dg_0(x)p = 0.$$

According to the definition 3.12 the reduction M_1 of M_0 is in a neighborhood of (x_0, p_0) the (possibly empty) subset of points (x, p) such that

$$Dg_0(x)p = 0, \quad G_0(x, p) = 0.$$

More particularly here, (x_0, p_0) is a point of reducibility of M_0 if, and only if, $Dg_0(x_0)p_0 = 0$. If M_1 is not a q-π-submanifold of TX then the algorithm is stopped. If M_1 is empty then $C(M)$ is empty.

Step k: Assume M_k is a a non-empty q-π-submanifold and (x_0, p_0) belongs to M_k. Let Ξ_k be the connected component of M_k which contains (x_0, p_0). As before there exists an open set U_k of (x_0, p_0) in Ξ_k, a submanifold Y_k of X such that $\dim Y_k + q = m_k + q = \dim \Xi_k$ and U_k is a submanifold of TY_k. Even if this means shrinking U_k, let us place ourselves once more in a chart of Y_k at x_0. Then Y_k is an open set of $\mathbb{R}^{m_k}$ that we denote by Y_k, $TY_k = Y_k \times \mathbb{R}^{m_k}$ and U_k is a submanifold of $Y_k \times \mathbb{R}^{m_k}$ of dimension $m_k + q$. Then there exists in a neighborhood of (x_0, p_0) in $Y_k \times \mathbb{R}^{m_k}$ a submersion $G_k : (Y_k \times \mathbb{R}^{m_k}, (x_0, p_0)) \to (\mathbb{R}^{m_k - q}, 0)$ such that $U_k = G_k^{-1}(0)$. In this way $\pi_{|\Xi_k}$ is a subimmersion in a neighborhood of (x_0, p_0), if, and only if, $D_p G_k(x, p)$ has constant rank $\rho_k = r_k - q \leq m_k - q$ in a neighborhood of (x_0, p_0) in U_k. Then any local projection $W_k = \pi(V_k)$ of U_k (or M_k) in (x_0, p_0) is a submanifold of Y_k of dimension r_k. There exists in a neighborhood of x_0 in Y_k a submersion $g_k : (Y_0, x_0) \to (\mathbb{R}^{m_k - r_k}, 0)$ such that $W_k = g_k^{-1}(0)$ in a neighborhood of x_0 in Y_k. The tangent bundle TW_k is the subset of points (x, p) of TY_k for which the following equations are satisfied

$$g_k(x) = 0, \quad Dg_k(x)p = 0.$$

The reduction M_{k+1} of M_k is in a neighborhood of (x_0, p_0) the (possibly empty) subset of points such that

$$Dg_k(x)p = 0, \quad G_k(x, p) = 0.$$

More particularly here, (x_0, p_0) is a point of reducibility of M_k if, and only if, $Dg_k(x_0)p_0 = 0$. If M_{k+1} is not a q-π-submanifold then the algorithm is stopped. If M_{k+1} is empty then $C(M)$ is empty.

Step $m+1-2q$: Let us assume that M_{m+1-2q} is a non-empty q-π-submanifold of TX and (x_0, p_0) belongs to M_{m+1-2q}. Then the proof of the theorem 3.10 gives the controlled vector field.

REMARK 4.1. For each step k we define the submersions

$$G_k : (Y_k \times \mathbb{R}^{m_k}, (x_0, p_0)) \to (\mathbb{R}^{m_k - q}, 0)$$

$$g_k : (Y_k, x_0) \to (\mathbb{R}^{m_k - r_k}, 0)$$

with $m_0 = \dim \Xi_0$ and $m_{k+1} = r_k = \rho_k + q \leq m_k$. Since the sequences of integers ρ_k and m_k are decreasing there exists an integer α such that $\rho_{k-1} = \rho_{\alpha-1}$ and $m_k = m_\alpha$ for any integer $k \geq \alpha$. We have the following sequence of inequalities

$$q \leq \cdots = m_{\alpha+1} = m_\alpha < m_{\alpha-1} < \cdots < m_{k+1} < m_k < \cdots < m_1 < m_0 < n.$$

We find once again that the index α is, at most, equal to $m_0 - q = \dim \Xi_0 - 2q$ and it is equal to $m_0 - q$ when $m_{k+1} = m_k - 1$ for all $k < \alpha$.

We shall end this section by showing first how to obtain g_k from G_k, then how to characterize the reducibility of M_k in a neighborhood of (x_0, p_0) and finally how to find G_{k+1} with g_k and G_k.

Construction of g_k: we show the existence of a local coordinates system on a submanifold N_k of U_k such that $\pi_{|N_k} : N_k \to W_k$ is a local diffeomorphism. Since, N_k has to be a submanifold of U_k of dimension r_k, we must obtain a submersion $\tilde{G}_k$ of $Y_k \times \mathbb{R}^{m_k}$ in $\mathbb{R}^{2m_k - r_k}$. Let $A_k : \mathbb{R}^{m_k} \to \mathbb{R}^{m_k - \rho_k}$ be a linear mapping such that

$$\ker A_k \cap \ker D_p G_k(x_0, p_0) = \{0\}, \quad \operatorname{rank} A_k = m_k - \rho_k$$

We define $\tilde{G}_k$ by $\tilde{G}_k(x, p) = (G_k(x, p), A_k(p - p_0))$. $D\tilde{G}_k(x_0, p_0)$ has full rank $2m_k - r_k$, and so $\tilde{G}_k$ is a submersion in an open neighborhood of (x_0, p_0) in TY_k. We take $N_k = \tilde{G}_k^{-1}(0)$ in a neighborhood of (x_0, p_0) in U_k. The tangent space is

$$T_{(x_0, p_0)} N_k = \ker D\tilde{G}_k(x_0, p_0) = \ker A_k \cap \ker DG_k(x_0, p_0).$$

Then, according to the following equality

$$T_{(x_0,p_0)} N_k \cap \ker D(\pi_{|U_k})(x_0, p_0) = \ker D\tilde{G}_k(x_0, p_0) \cap \{0\} \times \ker D_p G_k(x_0, p_0) = 0$$

and the subimmersion theorem, $\pi_{|N_k} : N_k \to W_k$ is a local diffeomorphism.

The local coordinate system of N_k in a neighborhood of (x_0, p_0) is obtained in the following way. Let $E_k = \operatorname{Im} D_p G_k(x_0, p_0)$ and F_k be any complement of E_k in $\mathbb{R}^{m_k - q}$ and P_k the projection of $\mathbb{R}^{m_k - q} = E_k \oplus F_k \simeq E_k \times F_k$ onto F_k.

LEMMA 4.2. *The subspace $K_k = \ker P_k D_x G_k(x_0, p_0)$ of $\mathbb{R}^{m_k}$ has dimension r_k. Moreover, for any complement L_k of K_k in $\mathbb{R}^{m_k}$, the linear mapping*

$$I_k = P_k D_x G_k(x_0, p_0)_{|L_k} : L_k \to F_k$$

is an isomorphism.

Proof. For the subspace K_k we have the first inequality

$$\dim K_k = m_k - \dim \mathrm{Im}\ P_k D_x G_k(x_0, p_0) \geq m_k - (m_k - r_k) = r_k$$

Let us show the other inequality. According to the definition of the projection P_k

$$P_k DG(x_0, p_0) = P_k D_x G_k(x_0, p_0).$$

Consequently, for any $(\delta x, \delta p) \in T_{(x_0,p_0)} N_k$, $\delta x \in K_k$. Conversely, for any $\delta x \in K_k$ there exists a $\delta p \in \mathbb{R}^{m_k}$ such that $(\delta x, \delta p) \in T_{(x_0,p_0)} N_k$. Indeed, if $\delta x \in K_k$, then $D_x G_k(x_0, p_0)\delta x \in \mathrm{Im}\ D_p G_k(x_0, p_0)$, and so there exists $\delta p' \in \mathbb{R}^{m_k}$ such that $DG_k(x_0, p_0)(\delta x, \delta p') = 0$. But, since A_k is an isomorphism from $\ker D_p G_k(x_0, p_0)$ into $\mathbb{R}^{m_k - \rho_k}$ there exists a $\delta p'' \in \ker D_p G_k(x_0, p_0)$ such that

$$A_k(\delta p' + \delta p'') = 0$$

whence $(\delta x, \delta p) \in T_{(x_0,p_0)} N_k$ for $\delta p = \delta p' + \delta p''$. Thus

$$D\pi(x_0, p_0) T_{(x_0,p_0)} N_k = K_k$$

and $\dim K_k \leq \dim T_{(x_0,p_0)} N_k = r_k$. Finally, according to the definition of L_k and I_k, $\ker I_k = L_k \cap K_k = \{0\}$. □

Thus any point x of $\mathbb{R}^{m_k}$ is splitting in a unique way as $x = \bar{x} + \tilde{x}$, where $\bar{x}$ belongs to K_k and $\tilde{x}$ belongs to L_k. Let us consider in a open neighborhood of (x_0, p_0) the equation

$$\tilde{H}_k(\bar{x}, \tilde{x}, p) = \tilde{G}_k(\bar{x} + \tilde{x}, p) = (G_k(\bar{x} + \tilde{x}, p), A_k(p - p_0)) = 0.$$

Since

$$D_{(\tilde{x},p)} \tilde{H}(x_0, p_0) = \begin{pmatrix} D_x G_k(x_0, p_0)_{|L_k} & D_p G(x_0, p_0) \\ 0 & A(p - p_0) \end{pmatrix}$$

is an isomorphism then in an open neighborhood of (x_0, p_0)

$$N_k = \{(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})\}$$

with $(\bar{x}_0, \phi_k(\bar{x}_0), \psi_k(x_0)) = (\bar{x}_0, \tilde{x}_0, p_0)$. Lastly, in a neighborhood of x_0,

$$W_k = \pi(N_k) = \{(\bar{x}, \phi_k(\bar{x})\} = g_k^{-1}(0).$$

We can therefore establish $g_k(x) = \tilde{x} - \phi_k(\bar{x})$.

Criterion of reducibility: in order to characterize the reducibility of M_k in a neighborhood of (x_0, p_0) we give the following result

Proposition 4.3. *Given $g_k : \mathbb{R}^{m_k} = K_k \oplus L_k \to L_k \simeq \mathbb{R}^{m_k - r_k}$ the above mapping, then for any $\delta x = \delta\bar{x} + \delta\tilde{x}$, $\delta y = \delta\bar{y} + \delta\tilde{y}$ of $\mathbb{R}^{m_k}$*

$$Dg(x_0)\delta x = \delta\tilde{x} \text{ and } D^2 g_k(x_0)(\delta x, \delta y) = I_k^{-1} P_k B_k(\delta\bar{x}, \delta\bar{y})$$

where B_k is the bilinear mapping on K_k

$$B_k(\bullet, \blacksquare) = D^2 G_k(x_0, p_0)((\bullet, D\psi_k(\bar{x}_0)\bullet), (\blacksquare, D\psi_k(\bar{x}_0)\blacksquare))$$

Proof. For any $\delta x = \delta\bar{x} + \delta\tilde{x}$, $\delta y = \delta\bar{y} + \delta\tilde{y}$ of $\mathbb{R}^{m_k}$

$$Dg_k(x_0)\delta x = \delta\tilde{x} - D\phi_k(\bar{x}_0)\delta\bar{x}$$

$$D^2 g_k(x_0)(\delta x, \delta y) = -D^2\phi_k(\bar{x}_0)(\delta\bar{x}, \delta\bar{y}).$$

According to the relationship $\tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})) = 0$ we have

$$D_{\bar{x}} \tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})) + D_{\tilde{x}} \tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})) D\phi_k(\bar{x})$$

$$+ D_p\tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x}))D\psi_k(\bar{x}) = 0,$$

and at the point (x_0, p_0) this gives

$$D_{\bar{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0) + D_{\tilde{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)D\phi_k(\bar{x}_0) + D_p\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)D\psi_k(\bar{x}_0) = 0.$$

But

$$D_{\bar{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0) = \begin{pmatrix} D_xG_k(x_0, p_0)_{|K_k} \\ 0 \end{pmatrix}$$

$$D_{\tilde{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0) = \begin{pmatrix} D_xG_k(x_0, p_0)_{|L_k} \\ 0 \end{pmatrix}$$

$$D_p\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0) = \begin{pmatrix} D_pG_k(x_0, p_0) \\ A_k \end{pmatrix}.$$

Thus, we obtain

$$D_xG_k(x_0, p_0)_{|K_k} + D_xG_k(x_0, p_0)_{|L_k}D\phi_k(\bar{x}_0) + D_pG_k(x_0, p_0)D\psi_k(\bar{x}_0) = 0$$
$$A_kD\psi_k(\bar{x}_0) = 0.$$

Let us apply P_k on the second equality, then we find $D\phi_k(\bar{x}_0) = 0$.

According to the relationship

$$D_{\bar{x}}\tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})) + D_{\tilde{x}}\tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x}))D\phi_k(\bar{x}) + D_p\tilde{H}_k(\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x}))D\psi_k(\bar{x}) = 0,$$

we obtain at the point $(\bar{x}, \tilde{x}, p) = (\bar{x}, \phi_k(\bar{x}), \psi_k(\bar{x})$ the following equality

$$\begin{aligned} & D^2_{\bar{x}^2}\tilde{H}_k(\bar{x}, \tilde{x}, p)(\delta\bar{x}, \delta\bar{y}) + D^2_{\tilde{x}\bar{x}}\tilde{H}_k(\bar{x}, \tilde{x}, p)(\delta\bar{x}, D\phi_k(\bar{x})\delta\bar{y}) \\ + \quad & D^2_{p\bar{x}}\tilde{H}_k(\bar{x}, \tilde{x}, p)(\delta\bar{x}, D\psi_k(\bar{x})\delta\bar{y}) + D^2_{\bar{x}\tilde{x}}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\phi_k(\bar{x})\delta\bar{x}, \delta\bar{y}) \\ + \quad & D^2_{\tilde{x}^2}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\phi_k(\bar{x})\delta\bar{x}, D\phi_k(\bar{x})\delta\bar{y}) \\ + \quad & D^2_{p\tilde{x}}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\phi_k(\bar{x})\delta\bar{x}, D\psi_k(\bar{x})\delta\bar{y}) \\ + \quad & D_{\tilde{x}}\tilde{H}_k(\bar{x}, \tilde{x}, p)D^2\phi_k(\bar{x})(\delta\bar{x}, \delta\bar{y}) + D^2_{\bar{x}p}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\psi_k(\bar{x})\delta\bar{x}, \delta\bar{y}) \\ + \quad & D^2_{\tilde{x}p}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\psi_k(\bar{x})\delta\bar{x}, D\phi_k(\bar{x})\delta\bar{y}) \\ + \quad & D^2_{p^2}\tilde{H}_k(\bar{x}, \tilde{x}, p)(D\psi_k(\bar{x})\delta\bar{x}, D\psi_k(\bar{x})\delta\bar{y}) \\ + \quad & D_p\tilde{H}_k(\bar{x}, \tilde{x}, p)D^2\psi_k(\bar{x})(\delta\bar{x}, \delta\bar{y}) = 0 \end{aligned}$$

which thus gives at the point (x_0, p_0) the equality

$$\begin{aligned} & D^2_{\bar{x}^2}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)(\delta\bar{x}, \delta\bar{y}) + D^2_{\bar{x}p}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)(D\psi_k(\bar{x}_0)\delta\bar{x}, \delta\bar{y}) \\ + \quad & D^2_{p\bar{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)(\delta\bar{x}, D\psi_k(\bar{x}_0)\delta\bar{y}) \\ + \quad & D^2_{p^2}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)(D\psi_k(\bar{x}_0)\delta\bar{x}, D\psi_k(\bar{x}_0)\delta\bar{y}) \\ + \quad & D_p\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)D^2\psi_k(\bar{x}_0)(\delta\bar{x}, \delta\bar{y}) \\ + \quad & D_{\tilde{x}}\tilde{H}_k(\bar{x}_0, \tilde{x}_0, p_0)D^2\phi_k(\bar{x}_0)(\delta\bar{x}, \delta\bar{y}) = 0. \end{aligned}$$

According to the definition of $\tilde{H}_k$ we obtain for G_k the relationship

$$\begin{aligned} & D_xG_k(x_0, p_0)_{|L_k}D^2\phi_k(\bar{x}_0)(\delta\bar{x}, \delta\bar{y}) = \\ - \quad & D_pG_k(x_0, p_0)D^2\psi_k(\bar{x}_0)(\delta\bar{x}, \delta\bar{y}) \\ - \quad & D^2G_k(x_0, p_0)((\delta\bar{x}, D\psi_k(\bar{x}_0)\delta\bar{x}), (\delta\bar{y}, D\psi_k(\bar{x}_0)\delta\bar{y})). \end{aligned}$$

Let us apply P_k on the left, then we obtain

$$I_k D^2\phi_k(\bar{x}_0)(\delta\bar{x},\delta\bar{y}) = \\ -P_k D^2 G_k(x_0,p_0)((\delta\bar{x}, D\psi_k(\bar{x}_0)\delta\bar{x}),(\delta\bar{y}, D\psi_k(\bar{x}_0)\delta\bar{y})).$$

□

PROPOSITION 4.4. *Let (x_0,p_0) be a point of reducibility of M_k and $W_k = \pi(V)$ a local projection of M_k in (x_0,p_0). M_k is reducible in a neighborhood of (x_0,p_0) if, and only if,*

$$\operatorname{rank}\begin{pmatrix} P_k B_k(p_0,\bullet) & P_k D_x G_k(x_0,p_0) \\ D_x G_k(x_0,p_0) & D_p G_k(x_0,p_0)\end{pmatrix} = 2m_k - (r_k+q) \tag{4.1}$$

and if in a neighborhood of (x_0,p_0) in $TW_k \cap V_k$

$$\dim[\ker P_k(x,p)D_xG_k(x,p)\cap \ker D_pG_k(x,p)] = \\ \dim[\ker P_k D_x G_k(x_0,p_0)\cap \ker D_p G_k(x_0,p_0)]$$

where $P_k(x,p)$ is the projection onto a complement of

$$E_k(x,p) = \operatorname{Im}\, D_pG_k(x,p).$$

Proof. Clearly, the tangent space of TW_k at (x_0,p_0) is the subset of points $(\delta x,\delta p)$ of TY_k which satisfy the following system

$$Dg_k(x_0)\delta x = 0 \\ D^2g_k(x_0)(p_0,\delta x) + Dg_k(x_0)\delta p = 0$$

and consequently $T_{(x_0,p_0)}TW_k \cap T_{(x_0,p_0)}M_k$ is the subset of points such that

$$Dg_k(x_0)\delta x = 0 \\ D^2g_k(x_0)(p_0,\delta x) + Dg_k(x_0)\delta p = 0 \\ D_xG_k(x_0,p_0)\delta x + D_pG_k(x_0,p_0)\delta p = 0.$$

But, on the one hand, according to proposition 4.3, the first equality means that $\delta x = \delta\bar{x}$, namely δx belongs to K_k. And on the other hand, we obtain the same thing when P_k is applied on the third equation. Thus we can leave aside the first equation. Now, $T_{(x_0,p_0)}TW_k \cap T_{(x_0,p_0)}M_k$ is the kernel of the linear mapping $DR_k(x_0,p_0) : \mathbb{R}^{m_k}\times\mathbb{R}^{m_k} \to L_k \times \mathbb{R}^{m_k-q}$ where

$$DR_k(x,p) = \begin{pmatrix} D^2g_k(x)(p,\bullet) & Dg_k(x) \\ D_xG_k(x,p) & D_pG_k(x,p)\end{pmatrix}.$$

Therefore, the subspace $T_{(x_0,p_0)}TW_k \cap T_{(x_0,p_0)}M$ has dimension r_k+q if, and only if, the linear mapping $DR_k(x_0,p_0)$ has full rank $m_k - r_k + m_k - q = 2m_k - (r_k+q)$, namely if, and only if, $\dim\ker DR_k(x_0,p_0) = r_k+q$. A point $(\delta x,\delta p)$ belongs to $\ker DR_k(x_0,p_0)$ if, and only if, $(\delta x,\delta p)$ satisfy the following system

$$\begin{cases} D^2g_k(x_0)(p_0,\delta x) + Dg_k(x_0)\delta\tilde{p} & = & 0 \\ D_xG_k(x_0,p_0)\delta x + D_pG_k(x_0,p_0)\delta p & = & 0.\end{cases}$$

According to proposition 4.3 this is equivalent to the system

$$\begin{cases} I_k^{-1}P_kB_k(p_0,\delta\bar{x}) + \delta\tilde{p} & = & 0 \\ D_xG_k(x_0,p_0)\delta x + D_pG_k(x_0,p_0)\delta p & = & 0.\end{cases}$$

And according to lemma 4.2 this is equivalent to the system

$$\begin{cases} P_k B_k(p_0, \delta\bar{x}) + P_k D_x G_k(x_0, p_0)\delta\tilde{p} &= 0 \\ D_x G_k(x_0, p_0)\delta x + D_p G_k(x_0, p_0)\delta p &= 0. \end{cases}$$

Lastly, as we have seen above, the second equation implies that $\delta x = \delta\bar{x}$. Thus, we are able to write δx instead of $\delta\bar{x}$ in the first equation of this system. Obviously we are also able to write δp instead of $\delta\tilde{p}$ in the first equation. Then we obtain the system

$$\begin{cases} P_k B_k(p_0, \delta x) + P_k D_x G_k(x_0, p_0)\delta p &= 0 \\ D_x G_k(x_0, p_0)\delta x + D_p G_k(x_0, p_0)\delta p &= 0. \end{cases}$$

Thus, $\dim\ker DR_k(x_0, p_0) = r_k + q_k$ if, and only if, (4.1) holds.

Now, assume that in an open neighborhood of (x_0, p_0) in $TW_k \cap V_k$

$$\text{rank } \pi_{|T_{(x,p)}(TW_k \cap V_k)} \text{ is constant.} \tag{4.2}$$

As we can see, $DR_k(x_0, p_0)$ is precisely the derivative of the mapping

$$\begin{aligned} R_k : TY_k &\to L_k \times \mathbb{R}^{m_k - q} \\ (x, p) &\mapsto (Dg_k(x, p), G_k(x, p)) \end{aligned}$$

where the zero set in a neighborhood of (x_0, p_0) is exactly $TW_k \cap V_k$. Thus, in a neighborhood of (x_0, p_0), $T_{(x,p)}(TW_k \cap V_k)$ is the kernel of $DR_k(x, p)$. The condition (4.2) holds if, and only if, the mapping

$$(\delta x, \delta p) \in \ker DR_k(x, p) \mapsto \delta x \tag{4.3}$$

has constant rank in neighborhood of (x_0, p_0) in $TW_k \cap V_k$. Compute the kernel of this mapping for any point (x, p) in an open neighborhood of (x_0, p_0). This is the set of $(\delta x, \delta p)$ of $\mathbb{R}^{m_k} \times \mathbb{R}^{m_k}$ such that

$$\begin{gathered} \delta x = 0 \\ D^2 g_k(x)(p, \delta x) + Dg_k(x)\delta p = 0 \\ D_x G_k(x, p)\delta x + D_p G_k(x, p)\delta p = 0 \end{gathered}$$

namely, the set $\{0\} \times [\ker Dg_k(x) \cap \ker D_p G_k(x, p)]$. However,

$$\ker Dg_k(x) = \ker P_k(x, p) D_x G_k(x, p).$$

Indeed the equality $Dg_k(x)\delta p = 0$ means that δp belongs to $T_x W_k$ which is the projection of $T_x M_k$. Therefore, there exists δq such that $D_x G_k(x, p)\delta p + D_p G_k(x, p)\delta q = 0$. And if we apply $P_k(x, p)$, we obtain that

$$\ker Dg_k(x) \subset \ker P_k(x, p) D_x G_k(x, p).$$

Since $\dim\ker Dg_k(x) = \dim\ker P_k(x, p) D_x G_k(x, p)$ we have the equality. Thus, the mapping (4.3) has constant rank if, and only if,

$$\begin{aligned} &\dim[\ker P_k(x, p) D_x G_k(x, p) \cap \ker D_p G_k(x, p)] = \\ &\qquad \dim[\ker P_k D_x G_k(x_0, p_0) \cap \ker D_p G_k(x_0, p_0)] \end{aligned}$$

□

Construction of G_{k+1}*:* we construct another submersion for M_k and then we obtain a submersion for M_{k+1}. Let C_k be a linear mapping from $\mathbb{R}^{m_k - (\rho_k + q)}$ to $\mathbb{R}^{m_k - q}$ such that $\text{Im } C_k \oplus E_k = \mathbb{R}^{m_k - q}$[4], $\mathcal{K}_k$ the kernel of $D_p G_k(x_0, p_0)$ and $\mathcal{L}_k$ a complement of $\mathcal{K}_k$. The linear mapping $J_k : \mathcal{L}_k \times \mathbb{R}^{m_k - (\rho_k + q)} \to$

[4]Such mapping is one-to-one since it has rank $m_k - (\rho_k + q)$.

$\mathbb{R}^{m_k-q}$ given by $J_k(u,v) = D_pG_k(x_0,p_0)u + C_kv$ is an isomorphism. Indeed, for any (u,v) such that $D_pG_k(x_0,p_0)u+C_kv=0$ since u belongs to $\mathcal{L}_k$ a complement of $\mathcal{K}_k = \ker D_pG_k(x_0,p_0)$, then C_kv belongs to Im $D_pG_k(x_0,p_0) = E_k$, therefore $C_kv=0$, and $v=0$ since C_k is one-to-one. Consequently, u belongs to $\mathcal{K}_k$ and then $u=0$. Now, for the mapping

$$\begin{gathered} F_k : TY_k \times \mathbb{R}^{m_k-(\rho_k+q)} \to \mathbb{R}^{m_k-q} \\ (x,p,u) \mapsto G_k(x,p) + C_ku \end{gathered}$$

the zero set $\tilde{M}_k$ is a submanifold of dimension $2m_k - \rho_k$. Using the implicit mapping theorem we obtain the following parameterization of $\tilde{M}_k$

$$\check{p} = a_k(x,\hat{p}), \qquad u = b_k(x,\hat{p})$$

where $\check{p}$ belongs to $\mathcal{L}_k$ and $\hat{p}$ to $\mathcal{K}_k$. Then we define in a neighborhood of (x_0,p_0) in TY_k the mapping

$$\begin{gathered} \tilde{F}_k : TY_k \to \mathcal{L}_k \simeq \mathbb{R}^{\rho_k} \\ (x,p) \mapsto \tilde{F}_k(x,p) = \check{p} - a_k(x,\hat{p}) \end{gathered}$$

and the mapping

$$\begin{gathered} \tilde{G}_k : TY_k \to \mathbb{R}^{m_k-(\rho_k+q)} \times \mathcal{L}_k \\ (x,p) \mapsto \tilde{G}_k(x,p) = (g_k(x), \tilde{F}_k(x,p)). \end{gathered}$$

PROPOSITION 4.5. *In an neighborhood of (x_0,p_0) the q-π-submanifold M_k is the zero set of the mapping $\tilde{G}_k$ and the q-π-submanifold M_{k+1} is the zero set of the mapping $\tilde{F}_{k|TW_k}$.*

Proof. According to the construction, the mapping $\tilde{G}_k$ is a submersion in a neighborhood of (x_0,p_0), therefore the zero set of $\tilde{G}_k$ is a submanifold with the same dimension of M. Then we have only to prove that $M_k \subset \tilde{G}_k^{-1}(0)$ in a neighborhood of (x_0,p_0) to set the equality $M_k = \tilde{G}_k^{-1}(0)$. For any point (x,p) of M_k in a neighborhood of (x_0,p_0), $G_k(x,p)=0$, therefore $(x,p,0)$ belongs to $\tilde{M}_k$ which implies that $\tilde{F}_k(x,p)=0$. Lastly, $G_k(x,p)=0$ implies that $g_k(x)=0$. Then the inclusion has been proved. The reducibility assumption of M_k in a neighborhood of (x_0,p_0) gives

$$\operatorname{rank} \begin{pmatrix} D^2g_k(x_0)(p_0,\bullet) & Dg_k(x_0) \\ D_x\tilde{G}_k(x_0,p_0) & D_p\tilde{G}_k(x_0,p_0) \end{pmatrix} = 2m_k - (r_k+q).$$

Therefore, for any δv of $\mathcal{L}_k$ there exists $(\delta x, \delta p)$ such that

$$\begin{aligned} D^2g_k(x_0)(p_0,\delta x) + Dg_k(x_0)\delta p &= 0 \\ Dg_k(x_0)\delta p &= 0 \\ D_x\tilde{F}_k(x_0,p_0)\delta x + D_p\tilde{F}_k(x_0,p_0)\delta p &= \delta v. \end{aligned}$$

In other words for any δv of $\mathcal{L}_k$ there exists $(\delta x, \delta p)$ belonging to $T_{(x_0,p_0)}TW_k$ such that

$$D_x\tilde{F}_k(x_0,p_0)\delta x + D_p\tilde{F}_k(x_0,p_0)\delta p = \delta v.$$

Therefore the mapping $F_{k|TW_k}$ is a submersion in a neighborhood of (x_0,p_0) and the zero set is $TW_k \cap V_k$. □

We now see that, with a chart of W_k, we may define $G_{k+1} = \tilde{F}_{k|TW_k}$.

5. Appendix

5.1. Proof of main results

Proof of Theorem 2.4. Let $x(\cdot)$ be an admissible trajectory of $\mathcal{P}_0$ (supposed continuously differentiable in the interval $[0,1]$ for reasons of convenience[5]). Let t_0 belongs to the open interval $]0,1[$, $(x_0,p_0)=(x(t_0),\dot{x}(t_0))$ and $W_0=\pi(V_0)$ be a local projection of M_0 at (x_0,p_0), then for any t in a open neighborhood of t_0, $(x(t),\dot{x}(t))$ belongs to V_0. Consequently, $x(t)$ belongs to W_0 for any t in an open neighborhood of t_0. Thus $(x(t),\dot{x}(t))$ belongs to $TW_0\cap V_0$ for any t in an open neighborhood of t_0. Since $TW_0\cap V_0$ is equal to M_1 in an open neighborhood of (x_0,p_0), $(x(t),\dot{x}(t))$ belongs to M_1 for any t in open neighborhood of t_0; this is the case for t_0. We have shown that for any t_0 belonging to the open interval $]0,1[$, $(x(t_0),\dot{x}(t_0))$ belongs to M_1. Let us prove that $(x(0),\dot{x}(0))$ (resp. $(x(1),\dot{x}(1))$) belongs to M_1. Let $\{t_n\}_{n\geq 0}$ be a sequence of $]0,1[$ converging to 0 (resp. 1), we then have

$$\lim_{n\to\infty}(x(t_n),\dot{x}(t_n))=(x(0),\dot{x}(0))$$

$$(\text{resp. } \lim_{n\to\infty}(x(t_n),\dot{x}(t_n))=(x(1),\dot{x}(1))).$$

But for any n, $(x(t_n),\dot{x}(t_n))$ belongs to M_1 which is a closed subset of M_0, therefore $(x(0),\dot{x}(0))$ (resp. $(x(1),\dot{x}(1))$) belongs to M_1. Thus we have shown that $x(\cdot)$ is an admissible trajectory of the problem $\mathcal{P}_1$. By induction we show that $x(\cdot)$ is an admissible trajectory of the problem $\mathcal{P}_k$ for any k. From the definition of the core $C(M)$, $x(\cdot)$ is an admissible trajectory of $\mathcal{P}_c$. Let $\bar{x}(\cdot)$ be a strong (resp. weak) trajectory of $\mathcal{P}_c$, clearly it is an admissible trajectory of $\mathcal{P}_0$. Assume that $\bar{x}(\cdot)$ is not an strong (resp. weak) minimum of $\mathcal{P}_0$, then there exists an admissible trajectory $\tilde{x}(\cdot)$ of $\mathcal{P}_0$ such that $J(\tilde{x}(\cdot))<J(\bar{x}(\cdot))$, but any admissible trajectory of $\mathcal{P}_0$ is an admissible trajectory of $\mathcal{P}_c$. Therefore, $\tilde{x}(\cdot)$ is an admissible trajectory of $\mathcal{P}_c$ such that $J(\tilde{x}(\cdot))<J(\bar{x}(\cdot))$; then, we obtain a contradiction with the optimal character of $\bar{x}(\cdot)$. □

Proof of Theorem 2.5. According to the proof of the theorem 3.10 the continuous mapping (resp. piecewise continuous) $u(\cdot)=H(x(\cdot),\dot{x}(\cdot))$ is the solution. The converse is direct. □

Remark 5.1. According to theorem 3.31 if we take another local projection $W'=\pi(V')$ of $C(M)$ at (x_0,p_0), another open set O' and another controlled vector field χ' then the bundle isomorphism h gives a bijection between the trajectories of χ and χ'. Then, the theorem 2.5 is independent of the choice of the triplets (W,O,χ).

[5]In the case of continuous and piecewise differentiable admissible trajectory $x(\cdot)$ we proceed from the same way for any interval $]0,\tau_1[$, $]\tau_k,\tau_{k+1}[$, $]\tau_n,1[$, where $0\leq\tau_1<\cdots<\tau_n\leq 1$ are the points of discontinuity of $\dot{x}$ and for any admissible trajectory $x(\cdot)$ which is absolutely continuous we use the fact that there exists a denumerable sequence $(I_n)_{n\in N}$ of disjoint interval in $[0,1]$ such that the Lebesgue's measure of the set $I-\cup_n I_n$ is zero and the restriction of $(x(\cdot),\dot{x}(\cdot))$ in each interval I_n is extended on the interval $\bar{I}_n$ by a continuous mapping $(x_n(\cdot),\dot{x}_n(\cdot))$.

Proof of Theorem 2.6. If $\bar{x}_{|I_\varepsilon}(\cdot)$ is not a strong minimum of the implicit Lagrange problem $\mathcal{P}_{c,\varepsilon}$ then there exists an admissible trajectory $\tilde{x}(\cdot)$ of $\mathcal{P}_{c,\varepsilon}$ such that

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\tilde{x}(t), \dot{\tilde{x}}(t))dt < \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\bar{x}(t), \dot{\bar{x}}(t))dt.$$

Then, the construction of the admissible trajectory of the implicit Lagrange problem $\mathcal{P}_c$

$$x^\star(t) = \begin{cases} \bar{x}(t) & \text{if } t \in [0, \tau-\varepsilon] \cup]\tau+\varepsilon, 1], \\ \tilde{x}(t) & \text{if } t \in]\tau-\varepsilon, \tau+\varepsilon], \end{cases}$$

gives the inequality

$$J(x^\star(\cdot)) < J(\bar{x}(\cdot))$$

which contradicts the optimality of $\bar{x}(\cdot)$. □

Proof of Theorem 2.7. Let $\bar{x}(\cdot)$ be a strong minimum of $\mathcal{P}_{c,\varepsilon}$ and $\bar{u}(\cdot)$ the corresponding control. Assume that the control $\bar{u}(\cdot)$ is not an optimal control of $\mathcal{P}_e$ then there exists an admissible control $\tilde{u}(\cdot)$ of the explicit optimal control problem $\mathcal{P}_e$ such that for the process $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ we have the inequality

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\tilde{x}(t), \tilde{u}(t)))dt < \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\bar{x}(t), \bar{u}(t)))dt.$$

According to the theorem 2.5 the trajectory $\tilde{x}(\cdot)$ is an admissible trajectory of $\mathcal{P}_{c,\varepsilon}$ such that

$$\begin{aligned} \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\tilde{x}(t), \dot{\tilde{x}}(t))dt &= \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\tilde{x}(t), \tilde{u}(t)))dt \\ &< \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\bar{x}(t), \bar{u}(t)))dt \\ &= \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\bar{x}(t), \dot{\bar{x}}(t))dt. \end{aligned}$$

Which is impossible. Conversely, given $(\bar{x}(\cdot), \bar{u}(\cdot))$ an optimal process of the explicit optimal control problem $\mathcal{P}_e$, then according to theorem 2.5 the trajectory $\bar{x}(\cdot)$ is an admissible trajectory of $\mathcal{P}_{c,\varepsilon}$. If it is not a strong minimum then there exists an admissible trajectory $\tilde{x}(\cdot)$ of $\mathcal{P}_{c,\varepsilon}$ such that

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\tilde{x}(t), \dot{\tilde{x}}(t))dt < \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\bar{x}(t), \dot{\bar{x}}(t))dt.$$

But, according to the theorem 2.5 for the trajectory $\tilde{x}(\cdot)$ there exists a unique control $\tilde{u}(\cdot)$ such that $(\tilde{x}(t), \dot{\tilde{x}}(t)) = \chi(\tilde{x}(t), \tilde{u}(t))$. Thus, $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is an admissible process of $\mathcal{P}_e$ such that

$$\begin{aligned} \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\tilde{x}(t), \tilde{u}(t)))dt &= \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\tilde{x}(t), \dot{\tilde{x}}(t))dt \\ &< \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\bar{x}(t), \dot{\bar{x}}(t))dt \\ &= \int_{\tau-\varepsilon}^{\tau+\varepsilon} L(\chi(\bar{x}(t), \bar{u}(t)))dt \end{aligned}$$

which is impossible. □

5.2. Pontryagin Maximum Principle

For the classic problem of optimal control ([1, 2, 12, 14]) we are given a state variable x in $\mathbb{R}^n$, a control variable u belonging to a closed subset U of $\mathbb{R}^q$, a vector field $f(x,u)$ of the state depending on the control variable, a startsubmanifold X_0 of $\mathbb{R}^n$, an endsubmanifold X_1 of $\mathbb{R}^n$ and a cost function $L(x,u)$. For any control u belonging to $KC([0,1],\mathbb{R}^q)$ the set of piecewise continuous functions[6] (resp. measurable and bounded[7]) the Cauchy's problem

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(t)) \ \forall t \in [0,1] \text{ (resp. a.e. on } [0,1]) \\ x(0) & = & a \end{array} \right. \tag{5.1}$$

admits an unique trajectory belonging to $KC^1([0,1],\mathbb{R}^n)$, the set of continuous and piecewise differentiable functions[8] (resp. $AC([0,1],\mathbb{R}^n)$, the set of absolutely continuous functions[9]). The pairs trajectory/control $(x(\cdot),u(\cdot))$ are called the processes. A process is admissible if $u(t) \in U$ for any t, $x(0)$ belongs to X_0 and $x(1)$ belongs to X_1. For any admissible process (x,u) we associate the cost

$$J(x,u) = \int_0^1 L(x(t),u(t))dt.$$

An admissible process $\bar{u}$ is optimal if $J(\bar{x},\bar{u})$ is the minimum of J on the set of admissible processes, namely the solution of the following problem

$$\mathcal{P} \qquad \min_{\substack{\dot{x}(\cdot)=f(x(\cdot),u(\cdot)) \\ x(0)\in X_0 \\ x(1)\in X_1}} \int_0^1 L(x(t),u(t))dt.$$

The Pontryagin Maximum Principle gives the necessary conditions for optimality.

Theorem 5.2. *(Maximum Principle) If $(\bar{x}(\cdot),\bar{u}(\cdot))$ is an optimal process for the problem $\mathcal{P}$, then there exists a non zero Lagrange multiplier*

$$(\bar{\psi}_0(\cdot),\bar{\psi}_1(\cdot),\cdots,\bar{\psi}_n(\cdot)) = (\psi_0(\cdot),\psi(\cdot))$$

belonging to $KC^1([0,1],\mathbb{R}^{n+1\star})$ (resp. $AC([0,1],\mathbb{R}^{n+1\star})$) and satisfying the following conditions

(a): *for any t belonging to $[0,1]$ (resp. a.e. on $[0,1]$)*

$$\dot{\bar{\psi}}_0(t) = 0, \ \dot{\bar{\psi}}_i(t) = -\frac{\partial H}{\partial x^i}(\bar{x}(t),\bar{u}(t),\bar{\psi}_0(t),\bar{\psi}(t)), \ i = 1,\cdots,n.$$

(b): *for any t belonging to $[0,1]$ (resp. a.e. on $[0,1]$)*

$$H(\bar{x}(t),\bar{u}(t),\bar{\psi}_0(t),\bar{\psi}(t)) = max_{u\,\in\,U} H(\bar{x}(t),u,\bar{\psi}_0(t),\bar{\psi}(t)).$$

(c): $\bar{\psi}_0(0) \leq 0$, $\bar{\psi}(0) \perp T_{\bar{x}(0)}X_0$ *and* $\bar{\psi}(1) \perp T_{\bar{x}(1)}X_1$

[6]For such control u, there exists a finite number of points $0 < \tau_1 < \cdots < \tau_n < 1$ such that u is continuous on any open interval $]0,\tau_1[,]\tau_k,\tau_{k+1}[,]\tau_n,1[$, and such that the right and left limits of u at τ_k exist. We denote the set of points of continuity by T.

[7]For this class of control, T is the set of Lebesgue points.

[8]For such trajectories x the function $\dot{x}(\cdot)$ belongs to $KC([0,1],\mathbb{R}^n)$.

[9]Recall that for any $\varepsilon > 0$ there exist a $\delta > 0$ such that for any finite collection $(]a_k,b_k[)_{k=1,\cdots,n}$ of non overlapping open interval such that $\sum_{k=1}^n \mid b_k - a_k \mid < \delta$ then $\sum_{k=1}^n \parallel x(b_k) - x(a_k) \parallel < \varepsilon$.

where $H(x, u, \psi_0, \psi) = \sum_{i=1}^{n} \psi_i f^i(x, u) + \psi_0 L(x, u)$.

PROOF OF THEOREM 2.7. See [1, 2, 7, 14] and [8, 13] for the nonsmooth case. □

REMARK 5.3. a) As usual, we can only consider the two cases $\bar{\psi}_0(0) = 0$ and $\bar{\psi}_0(0) = -1$ and then consider the following pseudo-Hamiltonian

$$H^{\psi_0} : T^{\star}\mathbb{R}^n \times U \to \mathbb{R}$$

$$H^{\psi_0}(x, \psi, u) = \sum_{i=1}^{n} \psi_i f^i(x, u) + \psi_0 L(x, u)$$

where $\psi_0 = 0, 1$. The necessary conditions (a) and (b) become

(a'): the triplet $(\bar{x}(\cdot), \bar{\psi}(\cdot), \bar{u}(\cdot))$ is a trajectory of the controlled vector field

$$\overrightarrow{H}^{\psi_0} : T^{\star}\mathbb{R}^n \times U \to T(T^{\star}\mathbb{R}^n)$$

$$(x, \psi, u) \mapsto \overrightarrow{H}^{\psi_0}(x, \psi, u) = \sum_{i=1}^{n} \frac{\partial H^{\psi_0}}{\partial \psi_i}(x, \psi, u)\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} \frac{\partial H^{\psi_0}}{\partial x^i}(x, \psi, u)\frac{\partial}{\partial \psi_i}$$

where $\psi_0 = 0, 1$.

(b'): for any t belonging to $[0, 1]$ (resp. a.e. on $[0, 1]$)

$$H^{\psi_0}(\bar{x}(t), \bar{u}(t), \bar{\psi}(t)) = max_{u \in U} H^{\psi_0}(\bar{x}(t), u, \bar{\psi}(t)).$$

b) the trajectories $(\bar{x}(\cdot), \bar{\psi}(\cdot))$ which are the projection of a triplet

$$(\bar{x}(\cdot), \bar{\psi}(\cdot), \bar{u}(\cdot))$$

satisfying the conditions (a'), (b') and (c) are called the extremals of $\mathcal{P}$.

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