

APPROXIMATION OF VISCOSITY SOLUTION BY MORPHOLOGICAL FILTERS

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Abstract. We consider in \mathbb{R}^2 all curvature equation $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$ where G is a nondecreasing function and $\text{curv}(u)$ is the curvature of the level line passing by \mathbf{x} . These equations are invariant with respect to any contrast change $u \rightarrow g(u)$, with g nondecreasing. Consider the contrast invariant operator $T_t : u_o \rightarrow u(t)$. A Matheron theorem asserts that all contrast invariant operator T can be put in a form $(Tu)(\mathbf{x}) = \inf_{B \in \mathcal{B}} \sup_{\mathbf{y} \in B} u(\mathbf{x} + \mathbf{y})$. We show the asymptotic equivalence of both formulations. More precisely, we show that all curvature equations can be obtained as the iteration of Matheron operators T_h^n where $h \rightarrow 0$ and $n \rightarrow \infty$ with $nh = t$.

Résumé. Nous considérons dans \mathbb{R}^2 les équations de courbure $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$ où G est une fonction croissante et $\text{curv}(u)$ représente la courbure de la ligne de niveau passant par le point \mathbf{x} . Ces équations sont invariantes pour tout changement de contraste $u \rightarrow g(u)$, avec g croissante. D'autre part, Matheron a prouvé que tout opérateur invariant par changement de contraste $T_t : u_o \rightarrow u(t)$ peut s'exprimer comme un schéma inf-sup $(Tu)(\mathbf{x}) = \inf_{B \in \mathcal{B}} \sup_{\mathbf{y} \in B} u(\mathbf{x} + \mathbf{y})$. Nous démontrons l'équivalence asymptotique de ces deux approches. Plus précisément, nous prouvons que la solution de viscosité de toute équation de courbure est la limite d'opérateurs de Matheron itérés T_h^n lorsque $h \rightarrow 0$ et $n \rightarrow \infty$ avec $nh = t$.

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1. INTRODUCTION

In image analysis, one of the most basic tasks is to smooth an image $u_0(\mathbf{x})$ for noise removal and shape simplification. Such a smoothing should preserve as much as possible the essential features of an image. This requirement is most easily formalized in terms of invariance. Two invariances requirements are basic in this context: given a smoothing operator T , it should commute with contrast changes, that is, increasing functions. Indeed, for physical and technological reasons, most digital images are known up to a contrast change. The second obvious requirement is geometric invariance: since the position of the camera is in general arbitrary or unknown, the operator T should commute with translations, rotations, and, when possible, with affine and even projective transforms of the image plane.

There are some limitations, however, to the extent of invariance compatible with smoothing. A classical theorem of Matheron [12] asserts that if T is a monotonous operator commuting with translations and contrast

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changes, there exists a family of structuring elements \mathcal{B} such that

$$(Tu)(\mathbf{x}) = \inf_{B \in \mathcal{B}} \sup_{\mathbf{y} \in B} u(\mathbf{x} + \mathbf{y}).$$

In the following, given a family \mathcal{B} , we note

$$(IS_h u)(\mathbf{x}) = \inf_{B \in h\mathcal{B}} \sup_{\mathbf{y} \in \mathbf{x} + B} u(\mathbf{y}),$$

$$(SI_h u)(\mathbf{x}) = \sup_{B \in h\mathcal{B}} \inf_{\mathbf{y} \in \mathbf{x} + B} u(\mathbf{y}),$$

$$(A_h u)(\mathbf{x}) = (IS_h \circ SI_h u)(\mathbf{x}).$$

On the other hand, several contrast invariant image operators based on PDE's have been proposed in recent years [1, 3, 4], basically mean curvature motion $\left(\frac{\partial u}{\partial t} = |Du| \text{curv}(u)\right)$ which commutes with all isometries and contrast changes and the so-called Affine Morphological Scale Space (AMSS), that is $\left(\frac{\partial u}{\partial t} = |Du|(\text{curv}(u))^{\frac{1}{3}}\right)$.

In [11], a multiscale analysis of shapes based on $\frac{\partial u}{\partial t} = |Du|(-\alpha + \text{curv}(u))$, $\alpha > 0$, was proposed for shape encoding. In [4], Catta *et al.* established a link between curvature equations and Matheron operators by proving that if we choose adequately \mathcal{B} , then suitably rescaled iterated Matheron filters converge to the viscosity solution of the mean curvature equation. This result was extended to the AMSS curvature equation by [8] in 2D and more recently by Cao [3] in any dimension. These partial results and propositions have opened a range of mathematical questions, one of which we attempt to solve here. The general questions in dimension 2 we address are:

1) do all adequately rescaled and iterated Matheron filters converge to a curvature equation?

2) Conversely, can all curvature equations $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$ with G a nondecreasing function, be obtained as the limit of iterated Matheron filters, as soon as they have a unique viscosity solution?

In [9], Ishii and Souganidis proved that such curvature equations with an arbitrary nondecreasing function G have a viscosity solution theory yielding existence and uniqueness. Guichard and Morel give a partially positive answer to question 1). More precisely, they give a consistency result, which implies the convergence thanks to a theorem of Barles and Souganidis [2] about the convergence of approximation schemes.

In this paper, we answer positively question 2). More precisely, we prove the two following theorems.

Theorem 1.1. *Given a function G continuous, nondecreasing, there exists a family $\mathcal{B}_{\epsilon, M}$, $\epsilon, M > 0$, such that the function defined with the alternate operator associated with $\mathcal{B}_{\epsilon, M}$*

$$u_{h, \epsilon, M}(\mathbf{x}, nh) = (A_h^n u)(\mathbf{x})$$

tends uniformly on any compact set to the viscosity solution u of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)),$$

when $\epsilon \rightarrow 0$, $M \rightarrow \infty$ and $nh \rightarrow t$:

$$\lim_{\epsilon \rightarrow 0} \lim_{M \rightarrow \infty} \lim_{nh \rightarrow t} u_{h, \epsilon, M}(\mathbf{x}, nh) = u(t, x).$$

Theorem 1.2. *Let $\gamma \in [0; +\infty[$. There exists a family \mathcal{B} such that the function*

$$u_h(\mathbf{x}, nh) = (A_h^n u)(\mathbf{x})$$

tends uniformly when $nh \rightarrow t$ on any compact set to the viscosity solution $u(\mathbf{x}, t)$ of

$$\frac{\partial u}{\partial t} = |Du|(\text{curv}(u))^\gamma.$$

The proof of Theorem 1.1 is easy thanks to a result of Ishii and Souganidis [9] but the family $\mathcal{B}_{\epsilon, M}$ depends on two parameters. Whereas with Theorem 1.2, the case of a power function $G(s) = |s^{\gamma-1}|s$ can be treated with families which do not depend on parameters. The case $\gamma < 1$ is difficult because the family \mathcal{B}_h we get is non local.

Our plan is as follows. In Section 2 we review several aspects and results of image processing theory relevant to questions 1) and 2). In Section 3 we formulate the problem, give the definitions and review the main results of [9]. We adapt the result of Barles and Souganidis to the new definition of viscosity solutions. In Section 4 we give a consistency theorem for Matheron filters. In Section 5, we approximate the viscosity solution in the general case. In Section 6, we construct a Matheron filter adapted to curvature equations with a power larger and smaller than 1 (both cases are quite different).

2. IMAGE PROCESSING

In 1983, Witkin and Koenderink [10,15] introduced the concept of *scale space* in image processing. It consists of a family of operators $(T_t)_{t \geq 0}$ which associates with an image modelled as a real valued function u_o in \mathbb{R}^n (in practice $n = 2, 3$) a sequence of smoothed images $T_t(u_o)(\mathbf{x}) = u(\mathbf{x}, t)$. We call t the scale parameter. A complete axiomatization was presented by Alvarez *et al.* [1] who proved that if a scale space satisfies some stability and invariance properties, then $u(\mathbf{x}, t)$ is viscosity solution of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)) \quad \text{and} \quad u(\mathbf{x}, 0) = u_o(\mathbf{x}),$$

where G is any nondecreasing continuous function with respect to its argument and $\text{curv}(u)$ is the curvature of the level line of the image $u(\mathbf{x}, t)$ passing by \mathbf{x} .

Among curvature equation used in image processing, let us mention:

- $G(s) = 1$, we get an erosion, $G(s) = -1$, we get an dilation (in the Matheron sense).
- $G(s) = s$, we get the mean curvature motion.
- $G(s) = s^\gamma$ (that is $|s|^{\gamma-1}s$), then the associated scale space is scale invariant (commutes with zooms).
- $G(s) = s^{\frac{1}{3}}$, we get the AMSS [1] which is affine invariant.
- $G(s) = 1 + s$, we get a reaction diffusion equation introduced in image processing by Kimia *et al.* [11]. This kind of equation is used to compute the skeleton of a shape provided one adds a stopping test in order to prevent the shape from breaking [13].

3. THE VISCOSITY SOLUTION FRAMEWORK

In this section, we review and adapt several results in viscosity solution theory which we shall use in the following section.

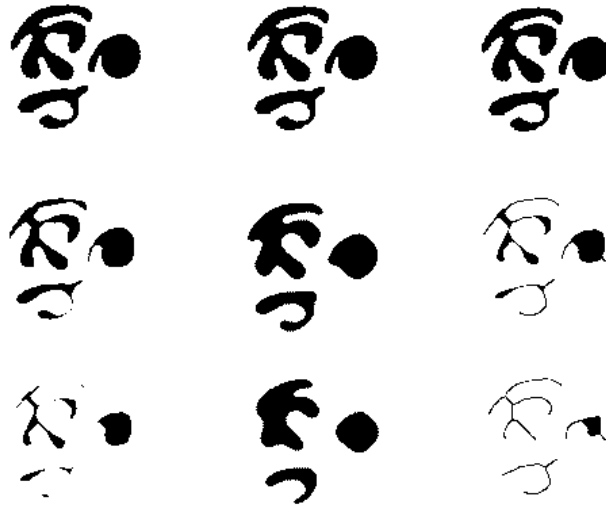


FIGURE 1. On the left column an erosion, on the middle mean curvature motion and on the right the skeletonization of the shapes.

We consider the nonlinear equation

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)) \text{ in } Q_T = \Omega \times (0, T) \quad (1)$$

with $T > 0$. Ω is an open subset of \mathbb{R}^2 , u is the unknown function $u : \Omega \times (0, T) \rightarrow \mathbb{R}$, G is a nondecreasing continuous function, $Du(\mathbf{x})$ is the spatial gradient and $\text{curv}(u)$ is the curvature of the level line passing by \mathbf{x} ,

$$\text{curv}(u) = \text{div} \left(\frac{Du}{|Du|} \right) = \frac{u_{xx}u_y^2 - 2u_xu_yu_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}.$$

Ishii and Souganidis [9] defined a new class of admissible test functions in the definition of viscosity solutions. To this end, let us denote by $\mathcal{F}(G)$ the set of functions $f \in C^2([0, +\infty))$ such that $f(0) = f'(0) = f''(0) = 0$ and $f''(r) > 0$ for $r > 0$ and such that

$$\lim_{p \rightarrow 0} f'(|p|)G\left(\frac{1}{|p|}\right) = \lim_{p \rightarrow 0} f'(|p|)G\left(\frac{-1}{|p|}\right) = 0.$$

Ishii and Souganidis [9] prove that this set is a nonempty cone.

Definition 3.1. Let \mathcal{O} be an open subset of Q_T . A function $\phi \in C^2(\mathcal{O})$ is *admissible* for G if for any $(\mathbf{x}_o, t_o) \in \mathcal{O}$ such that $D\phi(\mathbf{x}_o, t_o) = 0$, there is a constant $\delta > 0$ and functions $f \in \mathcal{F}(G)$ and $\omega \in C([0, +\infty))$ satisfying $\lim_{r \rightarrow 0} \frac{\omega(r)}{r} = 0$, such that, for all $(\mathbf{x}, t) \in B((\mathbf{x}_o, t_o), \delta)$

$$|\phi(\mathbf{x}, t) - \phi(\mathbf{x}_o, t_o) - \phi_t(\mathbf{x}_o, t_o)(t - t_o)| \leq f(|\mathbf{x} - \mathbf{x}_o|) + \omega(|t - t_o|).$$

We note $\mathcal{A}(G)$ the set of admissible functions.

We recall the definition of the upper semicontinuous envelope u^* and the lower semicontinuous envelope u_* of a function u

$$u^*(\mathbf{x}, t) = \limsup_{(\mathbf{y}, s) \rightarrow (\mathbf{x}, t)} (u(\mathbf{y}, s)) \text{ and } u_*(\mathbf{x}, t) = \liminf_{(\mathbf{y}, s) \rightarrow (\mathbf{x}, t)} (u(\mathbf{y}, s)).$$

Definition 3.2. (Ishii-Souganidis) A function $u(\mathbf{x}, t) : \mathcal{O} \rightarrow \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of (1) in \mathcal{O} if $u^* < +\infty$ in \mathcal{O} and for all $\phi \in \mathcal{A}(G)$ and all local finite maximum points (\mathbf{x}_o, t_o) of $u^* - \phi$,

$$\begin{cases} \frac{\partial \phi}{\partial t}(\mathbf{x}_o, t_o) - |D\phi(\mathbf{x}_o, t_o)|G(\text{curv}(\phi)(\mathbf{x}_o, t_o)) \leq 0 & \text{if } D\phi(\mathbf{x}_o, t_o) \neq 0 \\ \frac{\partial \phi}{\partial t}(\mathbf{x}_o, t_o) \leq 0 & \text{if } D\phi(\mathbf{x}_o, t_o) = 0. \end{cases}$$

A function $u(\mathbf{x}, t) : \mathcal{O} \rightarrow \mathbb{R} \cup \{-\infty\}$ is a viscosity supersolution of (1) in \mathcal{O} if $u_* > -\infty$ in \mathcal{O} and for all $\phi \in \mathcal{A}(G)$ and all local finite maximum points (\mathbf{x}_o, t_o) of $u_* - \phi$,

$$\begin{cases} \frac{\partial \phi}{\partial t}(\mathbf{x}_o, t_o) - |D\phi(\mathbf{x}_o, t_o)|G(\text{curv}(\phi)(\mathbf{x}_o, t_o)) \geq 0 & \text{if } D\phi(\mathbf{x}_o, t_o) \neq 0 \\ \frac{\partial \phi}{\partial t}(\mathbf{x}_o, t_o) \geq 0 & \text{if } D\phi(\mathbf{x}_o, t_o) = 0. \end{cases}$$

A function $u(\mathbf{x}, t)$ is a viscosity solution of (1) in \mathcal{O} if u is both a viscosity subsolution and supersolution in \mathcal{O} .

We denote by $BUC(\mathbb{R}^2)$ the set of bounded uniformly continuous functions on \mathbb{R}^2 .

Theorem 3.3. (Ishii-Souganidis) Consider the initial value problem with G a non decreasing continuous function:

$$\begin{cases} \frac{\partial u}{\partial t} - |Du|G(\text{curv}(u)) = 0 \\ u(\mathbf{x}, 0) = u_o(\mathbf{x}). \end{cases}$$

Assume that $u_o \in BUC(\mathbb{R}^2)$. Then there is an unique viscosity solution $u \in BUC(\mathbb{R}^2 \times (0, T))$.

In the following, we need the result of Ishii and Souganidis [9] (Pr. 1.3).

Theorem 3.4. Let G be a nondecreasing continuous function. Let us assume that G_n converge locally uniformly to G , that $\mathcal{F}(G) \subset \mathcal{F}(G_n)$ for any n and that for any $f \in \mathcal{F}(G)$,

$$\liminf_{p \rightarrow 0, n \rightarrow \infty} f'(|p|)G_n\left(\frac{1}{p}\right) \geq 0$$

(resp.

$$\limsup_{p \rightarrow 0, n \rightarrow \infty} f'(|p|)G_n\left(\frac{-1}{p}\right) \leq 0).$$

Let u_n be a subsolution (resp. a supersolution) of

$$\frac{\partial u_n}{\partial t} = |Du_n|G_n(\text{curv}(u_n)) \text{ in } \mathcal{O},$$

and define u^* and u_* : $\mathcal{O} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ by

$$u^*(z) = \limsup_{r \rightarrow 0} \left\{ u_n(y), |y - z| \leq r, n > \frac{1}{r} \right\}$$

$$u_*(z) = \liminf_{r \rightarrow 0} \left\{ u_n(y), |y - z| \leq r, n > \frac{1}{r} \right\}.$$

Assume that $u^*(z) < \infty$ (resp. $u_*(z) > -\infty$) for all $z \in \mathcal{O}$. Then u^* (resp. u_*) is a subsolution (resp. a supersolution) of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)) \text{ in } \mathcal{O}.$$

Our purpose is to approximate the solution u of Theorem 3.3 by an iterated scheme T_h where h is a scale parameter. To prove the convergence, we follow the argumentation developed by Barles and Souganidis [2]. Let u_o be a bounded function on \mathbb{R}^2 . We set

$$u_h(\mathbf{x}, (n+1)h) = T_h u_h(\mathbf{x}, nh) = T_h^{n+1} u_o(\mathbf{x}), \quad u_h(\mathbf{x}, 0) = u_o(\mathbf{x}).$$

Definition 3.5. We note $B(\overline{\Omega})$ the set of bounded functions.

- **Consistency**

A scheme T_h , $h > 0$ is consistent with

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$$

if for any $u \in C^\infty(\overline{\Omega})$ and for any $\mathbf{x} \in \overline{\Omega}$ we have

$$\frac{(T_h u)(\mathbf{x}) - u(\mathbf{x})}{h} = |Du|G(\text{curv}(u)) + o_{\mathbf{x}}(1) \text{ if } Du(\mathbf{x}) \neq 0.$$

If the convergence of $o_{\mathbf{x}}(1)$ is uniform on any compact set on which $Du(\mathbf{x}) \neq 0$, the scheme T_h is said to be uniformly consistent with the PDE.

- **Monotonicity**

A scheme T_h , $h > 0$ is locally monotone if there exists $r > 0$ such that for any functions $u, v \in B(\overline{\Omega})$ with $u(\mathbf{y}) > v(\mathbf{y})$ on $D(\mathbf{x}, r) \setminus \{\mathbf{x}\}$, one has

$$T_h u(\mathbf{x}) \geq T_h v(\mathbf{x}) + o(h).$$

- **Stability**

A scheme T_h , $h > 0$ is stable if for any $u \in B(\overline{\Omega})$ and for any $h > 0$ and n , $T_h^n u \in B(\overline{\Omega})$.

Theorem 3.6. (Barles-Souganidis) We consider a scheme T_h , $h > 0$, uniformly consistent with $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$, monotone, stable and commuting with the addition of constants (i.e. $T_h(u + C) = T_h u + C$). For any \mathbf{x}_o , we set

$$u(\mathbf{x}) = f(|\mathbf{x} - \mathbf{x}_o|).$$

Assume that for any $f \in \mathcal{F}(G)$,

$$\lim_{h \rightarrow 0} \frac{T_h u(\mathbf{x}_o)}{h} = 0.$$

Then when $nh \rightarrow t$, $u_h(\mathbf{x}, nh)$ tends uniformly on any compact set to the viscosity solution $u(\mathbf{x}, t)$ of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)).$$

The proof is an easy adaptation of Barles and Souganidis [2]. For a sake of completeness, we include this adaptation in the Appendix.

4. CONSISTENCY THEOREM FOR MATHERON FILTERS

Theorem 4.1. Asymptotic behaviour. *Let \mathcal{B} be a family of structuring elements in \mathbb{R}^2 which is bounded and isotropic. Assume that each element $B \in \mathcal{B}$ is symmetric with respect to O .*

We define the operators

$$(Tu)(\mathbf{x}) = \sup_{B \in \mathcal{B}} \inf_{\mathbf{y} \in \mathbf{x} + B} u(\mathbf{y})$$

and

$$(T_h u)(\mathbf{x}) = \sup_{B \in h\mathcal{B}} \inf_{\mathbf{y} \in \mathbf{x} + B} u(\mathbf{y}).$$

We set

$$G(b) = T(u)(0),$$

with $u(x, y) = x + by^2$.

Assume that there exists $\gamma \geq 1$

$$G(b) \simeq b^\gamma \text{ when } b \rightarrow 0.$$

If u is a C^n function on \mathbb{R}^2 with $n \geq \gamma + 2$,

- if $Du(\mathbf{x}_o) \neq 0$, then

$$T_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h^{\gamma+1} |Du(\mathbf{x}_o)| \left(\frac{1}{2} \text{curv} u \right)^\gamma + \mathcal{O}(h^{\gamma+2}).$$

The consistency is uniform on a compact set on which $Du(\mathbf{x}) \neq 0$.

- if $Du(\mathbf{x}_o) = 0$, then

$$T_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h^2 \inf_B \sup_{(x,y) \in B} (\lambda_1 x^2 + \lambda_2 y^2) + \mathcal{O}(h^3).$$

λ_1 and λ_2 are the eigenvalues of $D^2u(\mathbf{x})$.

Proof. We only prove the case $Du(\mathbf{x}_o) \neq 0$, the other one is straightforward, see Guichard and Morel [8]. As $T(u - u(\mathbf{x}_o)) = Tu - u(\mathbf{x}_o)$, we can choose $u(\mathbf{x}_o) = 0$. Moreover as T is invariant by any isometry, it is possible to choose the origin O at \mathbf{x}_o , and the orthogonal axes such that $\vec{i} = \frac{Du}{|Du|}$ if $Du(\mathbf{x}_o) \neq 0$. Let \vec{j} be such that (\vec{i}, \vec{j}) is the orthonormal basis. Assume that u is $C^{\gamma+2}$. In what follows, we note (x, y) the coordinates of \mathbf{x} in (O, \vec{i}, \vec{j}) . Then, with these conventions,

$$u(\mathbf{x}) = px + ax^2 + by^2 + cxy + \sum_{2 < k+l \leq \gamma+1} a_{k,l} x^k y^l + \mathcal{O}(|\mathbf{x}|^{\gamma+2})$$

with $p = |Du|(0)$, and $p > 0$. Moreover we have

$$\begin{aligned} a &= \frac{1}{2} D^2u(\vec{i}, \vec{i}), \\ b &= \frac{1}{2} D^2u(\vec{j}, \vec{j}) = \frac{1}{2} |Du| \text{curv} u(0), \end{aligned}$$

and

$$c = D^2u(\vec{i}, \vec{j}).$$

We set

$$Q(x, y) = px + ax^2 + by^2 + cxy + \sum_{2 < k+l \leq \gamma+1} a_{k,l} x^k y^l.$$

So we have

$$(T_h u)(O) = \sup_{B \in \mathcal{B}} \inf_{\mathbf{x}=(x,y) \in hB} (Q(x, y) + \mathcal{O}(|\mathbf{x}|^{\gamma+2})).$$

As \mathcal{B} is bounded, there exists a constant M such that every element B of \mathcal{B} is included in $D(O, M)$. Therefore if $\mathbf{x} \in hB$ then $|\mathbf{x}| < hM$.

As T_h is monotone and commutes with addition of constant, we get

$$T_h u(O) = T_h(Q(x, y)) + \mathcal{O}(|h|^{\gamma+2}).$$

We have the following inequalities for the points (x, y) of hB

$$\begin{aligned} x(p - \epsilon(x)) \left[(|a| + |c|)Mh + \sum_{2 < k+l \leq \gamma+1, 0 < k} |a_{k,l}|M^{k+l-1}h^{k-1}h^l \right] + y^2 \left(b - \sum_{2 < l < \gamma+1} |a_{0,l}|(Mh)^{l-2} \right) \\ \leq Q(x, y) \leq \\ x(p + \epsilon(x)) \left[(|a| + |c|)Mh + \sum_{2 < k+l \leq \gamma+1, 0 < k} |a_{k,l}|M^{k+l-1}h^{k-1}h^l \right] + y^2 \left(b + \sum_{2 < l < \gamma+1} |a_{0,l}|(Mh)^{l-2} \right), \end{aligned}$$

and $\epsilon(x)$ is the sign of x : if $x \leq 0$, $\epsilon(x) = -1$, and if $x > 0$, $\epsilon(x) = 1$.

Since the coefficients a , c and $a_{k,l}$ are bounded and continuous on $D(O, M)$, we note K and K' such that

$$K = \sup_{D(O, M)} \left((|a| + |c|)M + \sum_{2 < k+l \leq \gamma+1, 0 < l} |a_{k,l}|M^{k+l-1} \right),$$

and

$$K' = \sup_{D(O, M)} \left(\sum_{2 < l < \gamma+1} |a_{0,l}|M^{l-3} \right).$$

So we have

$$px - |x|Kh + y^2(b - K'h) \leq Q(x, y) \leq px + |x|Kh + y^2(b + K'h).$$

Thus, for any $B \in \mathcal{B}$,

$$\inf_{(x,y) \in hB} (px - |x|Kh + y^2(b - K'h)) \leq \inf_{(x,y) \in hB} Q(x, y) \leq \inf_{(x,y) \in hB} (px + |x|Kh + y^2(b + K'h)).$$

Since B is symmetric with respect to O , we deduce that it contains points $\mathbf{x} = (x, y)$ such that x is negative. As $p > 0$, we deduce that the infimum is reached for a negative x for a small h . Therefore, we get

$$\inf_{(x,y) \in hB} (px - |x|Kh + y^2(b - K'h)) = \inf_{(x,y) \in hB \cap \{(x,y), x < 0\}} (px + xKh + y^2(b - K'h)).$$

With the same argument,

$$\inf_{(x,y) \in hB \cap \{(x,y), x < 0\}} (px + xKh + y^2(b - K'h)) = \inf_{(x,y) \in hB} (px + xKh + y^2(b - K'h)).$$

Then,

$$\inf_{(x,y) \in hB} (px + xKh + y^2(b - K'h)) \leq \inf_{(x,y) \in hB} Q(x, y) \leq \inf_{(x,y) \in hB} (px - xKh + y^2(b + K'h)).$$

So we deduce

$$SI_h(x(p + Kh) + y^2(b - K'h)) \leq SI_h(Q(x, y)) \leq SI_h(x(p - Kh) + y^2(b + K'h)).$$

As $T_h(x + by^2)(O) = hG(hb)$, we have

$$SI_h(x(p - Kh) + y^2(b + K'h)) = h(p - Kh)G\left(h\frac{b + K'h}{p - Kh}\right),$$

we deduce that

$$SI_h(x(p - Kh) + y^2(b + K'h)) = h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + C_1(b, p, K, K')h^{\gamma+2},$$

and

$$SI_h(x(p + Kh) + y^2(b - K'h)) = h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + C_2(b, p, K, K')h^{\gamma+2},$$

C_1 and C_2 are continuous functions with respect to all their arguments. Then

$$SI_h(Q(x, y)) = h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + \mathcal{O}(h^{\gamma+2}),$$

thus

$$SI_h u = h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + \mathcal{O}(h^{\gamma+2}).$$

From this result, we deduce the consistency if $Du(\mathbf{x}_o) \neq 0$:

$$T_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h^{\gamma+1}|Du(\mathbf{x}_o)|G\left(\frac{1}{2}\text{curv}u\right) + \mathcal{O}(h^{\gamma+2}).$$

In addition, consider Ω a neighborhood of \mathbf{x}_o such that for any $\mathbf{x} \in \Omega$ $|Du(\mathbf{x})| \neq 0$. We have for any $\mathbf{x} \in \Omega$

$$h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + C_1(b, p, K, K')h^{\gamma+2} \leq T_h(u)(\mathbf{x}) - u(x_o) \leq h^{\gamma+1}p\left(\frac{b}{p}\right)^\gamma + C_2(b, p, K, K')h^{\gamma+2},$$

and b, p, K, K' are continuous functions of \mathbf{x} , therefore C_1 and C_2 are bounded functions on Ω . So we deduce that the consistency of T_h is uniform in Ω . □

This theorem enables us to build in Section 6 a family \mathcal{B} with which we define an inf-sup scheme converging to the viscosity solution when G is a power function. But in the general case, we get the following equation if $Du(\mathbf{x}_o) \neq 0$:

$$T_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h|Du(\mathbf{x}_o)|G\left(\frac{1}{2}h\text{curv}u\right) + o(h).$$

Since the scale parameter h is linked to the curvature term $\text{curv}(u)$, $G\left(\frac{1}{2}h\text{curv}u\right)$, it is impossible to separate these two terms when G is not a power function. Instead of considering \mathcal{B}_h as $h\mathcal{B}$, we give an other definition for the scaling of a family \mathcal{B}_h .

Definition 4.2. Let \mathcal{B} be a family of structuring elements. $B \in \mathcal{B}_h$ if and only if there exists $B' \in \mathcal{B}$ and $\theta \in [0, 2\pi]$ such that

$$B = P_\theta A_h B'$$

where

$$P_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

and

$$A_h = \begin{pmatrix} h & 0 \\ 0 & h^{\frac{1}{2}} \end{pmatrix}.$$

Theorem 4.3. (Asymptotic behaviour 2). *Let \mathcal{B} a bounded and isotropic family of structuring elements of \mathbb{R}^2 . Assume that each element $B \in \mathcal{B}$ is symmetric with respect to O .*

Let $(Tu)(\mathbf{x}) = \inf_{B \in \mathcal{B}} \sup_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y})$, (resp. $(Tu)(\mathbf{x}) = \sup_{B \in \mathcal{B}} \inf_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y})$). Define the scheme

$$(T_h u)(\mathbf{x}) = \inf_{B \in \mathcal{B}_h} \sup_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y})$$

(resp. $(T_h u)(\mathbf{x}) = \sup_{B \in \mathcal{B}_h} \inf_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y})$), and set

$$G(b) = T(x + by^2)(0).$$

Then if $u \in C^\infty(\mathbb{R}^2)$, if $Du(\mathbf{x}_o) \neq 0$, we have

$$T_h u(\mathbf{x}) = u(\mathbf{x}_o) + h |Du(\mathbf{x}_o)| G\left(\frac{1}{2} \text{curv}(u)(\mathbf{x}_o)\right) + \mathcal{O}\left(h^{\frac{3}{2}}\right).$$

The consistency is uniform on a compact set on which $Du(\mathbf{x}) \neq 0$.

Proof. As $T(u - u(\mathbf{x}_o)) = Tu - u(\mathbf{x}_o)$, we can choose $u(\mathbf{x}_o) = 0$. Moreover as T is invariant by any isometry, it is possible to take the origin O at \mathbf{x}_o , and the orthogonal axes such that $\vec{i} = \frac{Du}{|Du|}$ if $Du(\mathbf{x}_o) \neq 0$. Let \vec{j} be such that (\vec{i}, \vec{j}) is an orthonormal basis. So we have with $\mathbf{x} = (x, y)$

$$u(\mathbf{x}) = px + ax^2 + by^2 + cxy + \mathcal{O}(|\mathbf{x}|^3)$$

with $p = |Du|(0) > 0$, and

$$\begin{aligned} a &= \frac{1}{2} D^2 u(\vec{i}, \vec{i}), \\ b &= \frac{1}{2} D^2 u(\vec{j}, \vec{j}) = \frac{1}{2} |Du| \text{curv} u(0), \end{aligned}$$

and

$$c = D^2 u(\vec{i}, \vec{j}).$$

We set

$$Q(x, y) = px + ax^2 + by^2 + cxy.$$

We have for $B' \in \mathcal{B}_h$

$$(x', y') \in B' \Leftrightarrow \exists B \in \mathcal{B} \exists (x, y) \in B \exists \theta \in [0, 2\pi] \text{ such that}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = P_\theta A_h \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{cases} x' = h \cos(\theta)x - \sqrt{h} \sin(\theta)y \\ y' = h \sin(\theta)x + \sqrt{h} \cos(\theta)y. \end{cases}$$

Then

$$T_h(Q)(O) = \inf_{B' \in \mathcal{B}_h} \sup_{(x', y') \in B'} (Q(x', y'))$$

therefore

$$T_h(Q)(O) = \inf_{B \in \mathcal{B}} \inf_{0 \leq \theta \leq \pi} \sup_{(x, y) \in B} (Q(h \cos(\theta)x - \sqrt{h} \sin(\theta)y, h \sin(\theta)x + \sqrt{h} \cos(\theta)y)).$$

With the definition of Q

$$T_h(Q)(O) = \inf_{B \in \mathcal{B}} \inf_{0 \leq \theta \leq \pi} \sup_{(x, y) \in B} \left(-p\sqrt{h} \sin(\theta)y + h(p \cos(\theta)x + a \sin^2(\theta)y^2 + b \cos^2(\theta)y^2) + \mathcal{O}(|h|^{\frac{3}{2}}) \right).$$

It is clear that for h small enough the inf-sup is reached when $\theta = 0$. Then we set $\theta = 0$:

$$T_h(Q)(O) = hp \inf_{B \in \mathcal{B}} \sup_{(x, y) \in B} \left(x + \frac{b}{p}y^2 \right) + \mathcal{O}(h^{\frac{3}{2}}).$$

Hence

$$T_h(Q)(O) = hpG\left(\frac{b}{p}\right) + \mathcal{O}(h^{\frac{3}{2}}).$$

Since \mathcal{B} is bounded, there exists a constant M such that for any $B \in \mathcal{B}_h$, $B \subset D(O, h^{\frac{1}{2}}M)$. By monotonicity, we get

$$T_h(u)(O) = T_h(Q)(O) + \mathcal{O}(h^{\frac{3}{2}}).$$

$$T_h(u)(O) = hpG\left(\frac{b}{p}\right) + \mathcal{O}(h^{\frac{3}{2}}).$$

In addition, as in Theorem 4.1, the consistency is uniform on a neighborhood of \mathbf{x}_o on which $Du(\mathbf{x}) \neq 0$. □

5. THE GENERAL CASE OF A NONDECREASING FUNCTION G

In this section, we prove Theorem 1.1 given in the introduction. First we recall this theorem:

Theorem 1.1. *Given a function G continuous, nondecreasing, there exists a family $\mathcal{B}_{\epsilon, M}$, $\epsilon, M > 0$, such that the function defined with the alternate operator associated with $\mathcal{B}_{\epsilon, M}$*

$$u_{h, \epsilon, M}(\mathbf{x}, nh) = (A_h^n u)(\mathbf{x})$$

tends uniformly on any compact set to the viscosity solution u of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)),$$

when $\epsilon \rightarrow 0$, $M \rightarrow \infty$ and $nh \rightarrow t$:

$$\lim_{\epsilon \rightarrow 0} \lim_{M \rightarrow \infty} \lim_{nh \rightarrow t} u_{h, \epsilon, M}(\mathbf{x}, nh) = u(t, x).$$

Proof. 1. We shall first consider the nondecreasing function G such that

$$\begin{cases} i) & \exists A > 0 \quad \forall x \in \mathbb{R}^+ \quad G'(x) \leq A \\ ii) & \exists M > 0 \quad \forall x > M \quad G(x) = G(M) + A(x - M) \\ iii) & \forall x \leq 0 \quad G(x) = 0. \end{cases} \tag{2}$$

Given $N > 0$, we consider the piece of parabola $P_s : (x, y) \in P_s$ if $x = G(s) - y^2s$ and $-N \leq x \leq G(s)$. We call \mathcal{B}_N the smallest isotropic set containing all the sets P_s when $s \in [0; M]$. Some examples of set P_s are given in Figure 2. In addition, \mathcal{B}_N is a bounded family.

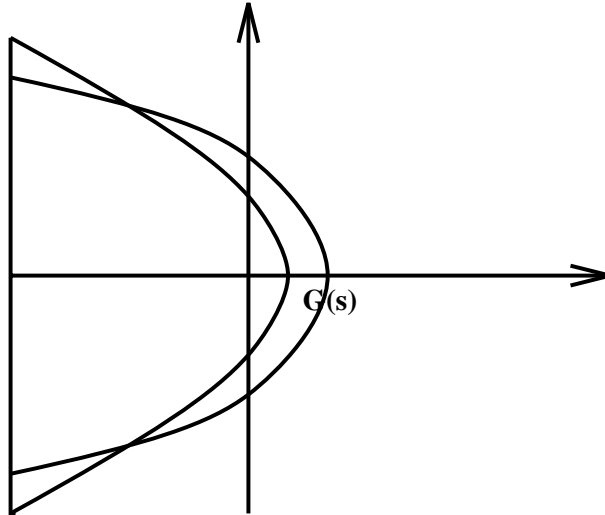


FIGURE 2. Examples of set P_s .

Lemma 5.1. *Let G defined by (2) and the family \mathcal{B}_N as above, for any $b \in \mathbb{R}$ and for N large enough*

$$G(b) = \inf_{B \in \mathcal{B}_N} \sup_{(x,y) \in B} (x + by^2).$$

Proof. Given $s > 0$, we define f_s on \mathbb{R} by

$$\begin{cases} \text{If } x \leq s & f_s(x) = G(s) \\ \text{If } x \geq s & f_s(x) = G(s) + A(x - s). \end{cases}$$

As shown by Figure 3, we have for any $x \in \mathbb{R}$

$$G(x) = \inf_{0 \leq s \leq M} (f_s(x)).$$

We can write

$$f_s(x) = \sup_{0 \leq p \leq A} (G(s) + p(x - s)).$$

So we have

$$G(b) = \inf_{0 \leq s \leq M} \left(\sup_{0 \leq p \leq A} (G(s) - ps + pb) \right).$$

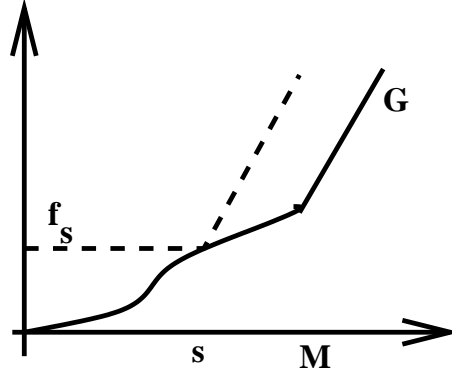


FIGURE 3

We set

$$\begin{cases} x(p) = G(s) - ps \\ y(p) = \sqrt{p}. \end{cases} \tag{3}$$

The point $(x(p), y(p))$ belongs to the parabola P_s . So we deduce

$$G(b) = \inf_{0 \leq s \leq M} \left(\sup_{(x,y) \in P_s} (x + by^2) \right).$$

For N large enough, we have

$$\inf_{B \in \mathcal{B}_N} \sup_{(x,y) \in B} (x + by^2) = \inf_{0 \leq s \leq M} \sup_{(x,y) \in P_s} (x + by^2),$$

so we get the Lemma. □

2. Lemma 5.2. *Let G defined by (2). The functions $u_h(x, nh)$, defined with the inf-sup scheme associated with \mathcal{B}_N , tend uniformly when $nh \rightarrow t$ on any compact to the viscosity solution $u(x, t)$ of*

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)).$$

Proof. With Theorem 4.3, we deduce that the inf-sup scheme is consistent with

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)).$$

Moreover, for any $f \in \mathcal{F}(G)$, we have $f \in C^2([0, +\infty))$ such that $f(0) = f'(0) = f''(0) = 0$ and $f''(r) > 0$ for $r > 0$. With a Taylor expansion, we get

$$f(|x - x_o|) = \mathcal{O}(|x - x_o|^3).$$

With the definition of \mathcal{B}_h , we have

$$T_h f(|\cdot - x_o|)(x_o) = \mathcal{O}\left(h^{\frac{3}{2}}\right).$$

Thus

$$\lim_{h \rightarrow 0} \frac{T_h f(|\cdot - x_o|)(x_o)}{h} = 0.$$

Thus, with Theorem 3.6, we deduce the Lemma. \square

3. Thus, given a nondecreasing continuous function G , let be $\epsilon > 0$, we note f_s^ϵ the function defined on \mathbb{R} by

$$\begin{cases} \text{If } x \leq s & f_s^\epsilon(x) = G(s) \\ \text{If } x \geq s & f_s^\epsilon(x) = G(s) + \frac{1}{\epsilon}(x - s). \end{cases}$$

We set

$$G_\epsilon(x) = \inf_{s \geq 0} (f_s^\epsilon(x)).$$

Lemma 5.3. *Let u_ϵ be the viscosity solution of*

$$\frac{\partial u_\epsilon}{\partial t} = |Du_\epsilon| G_\epsilon(\text{curv}(u_\epsilon)),$$

then $u_\epsilon \rightarrow u$ when $\epsilon \rightarrow 0$ and u is the viscosity solution of

$$\frac{\partial u}{\partial t} = |Du| G(\text{curv}(u)).$$

Proof. We only have to prove that the functions G_ϵ satisfy the assumptions of Theorem 3.4.

The functions G_ϵ converge uniformly locally to G . Let be $f \in \mathcal{F}(G)$, then

$$\lim_{p \rightarrow 0} f'(|p|) G\left(\frac{-1}{|p|}\right) = \lim_{p \rightarrow 0} f'(|p|) G\left(\frac{1}{|p|}\right) = 0.$$

When p is small enough, we have

$$0 \leq f'(|p|) G_\epsilon\left(\frac{1}{|p|}\right) \leq f'(|p|) G\left(\frac{1}{|p|}\right)$$

hence

$$\lim_{p \rightarrow 0} f'(|p|) G_\epsilon\left(\frac{1}{|p|}\right) = 0.$$

In addition, $G_\epsilon\left(\frac{-1}{|p|}\right) = 0$. Finally, let be $f \in \mathcal{F}(G)$, as $f'(|p|) G_\epsilon\left(\frac{1}{|p|}\right) \geq 0$ and $f'(|p|) G_\epsilon\left(\frac{-1}{|p|}\right) = 0$, we deduce the last needed condition to apply the above theorem. \square

4. It remains to approach u_ϵ . Again, given $\epsilon > 0$, we approximate G_ϵ by G_ϵ^M with $M > 0$ and

$$\begin{cases} \text{If } x \leq M & G_\epsilon^M(x) = G_\epsilon(x) \\ \text{If } x \geq M & G_\epsilon^M(x) = G(M) + \frac{1}{\epsilon}(x - M). \end{cases}$$

It is easy to check the assumptions of Theorem 3.4, then

Lemma 5.4. *Let u_M be the viscosity solution of*

$$\frac{\partial u_M}{\partial t} = |Du_M| G_\epsilon^M(\text{curv}(u_M)),$$

then $u_M \rightarrow u$ when $M \rightarrow \infty$ and u is the viscosity solution of

$$\frac{\partial u}{\partial t} = |Du|G_\epsilon(\text{curv}(u)).$$

5. Let $\epsilon, M > 0$ be two parameters. We consider the function G_ϵ^M defined as in step 4. Since G_ϵ^M satisfies the assumptions of Lemma 5.2, step 2, we can construct a family \mathcal{B}_ϵ^M such that the associated functions $u_{\epsilon,h}^M(x, nh)$ tends uniformly on any compact set to the viscosity solution u_ϵ^M of

$$\frac{\partial u}{\partial t} = |Du|G_\epsilon^M(\text{curv}(u)).$$

With Lemma 5.4, we let M tend to ∞ , then u_ϵ^M tends uniformly on any compact set to the viscosity solution u_ϵ of

$$\frac{\partial u}{\partial t} = |Du|G_\epsilon(\text{curv}(u)).$$

With Lemma 5.3, we let ϵ tend to 0, then u_ϵ tend uniformly on any compact set to the viscosity solution u of

$$\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u)).$$

□

6. CASE WHERE $G(s) = s^\gamma$

6.1. Notations

The approach is simpler than in the preceding case because of the concavity or convexity of G . We use the first consistency theorem about the asymptotic behaviour of the scheme T_h when $\mathcal{B}_h = h\mathcal{B}$. We adopt the following notations. Let be $a = \left(\frac{\gamma^{1/2}}{1-\gamma}\right)^{\frac{2}{\gamma+1}}$. We set $I = [0; a]$ and consider the curve $C = (x(t), y(t))$ when $t \in I$ defined by

$$\begin{cases} x(t) = (1-\gamma)t^\gamma \\ y(t) = (\gamma)^{\frac{1}{2}}t^{\frac{\gamma-1}{2}}. \end{cases}$$

The equation of C is

$$y = \gamma \left(\frac{-1}{1-\gamma}\right)^{\frac{\gamma-1}{2\gamma}} (-x)^{\frac{\gamma-1}{2\gamma}}.$$

The exponent of x belongs to the interval $\left[0; \frac{1}{2}\right]$. Let B_t be the set formed by the four points

$$B_t = \{(x(t), y(t)); (x(t), -y(t)); (-x(t), y(t)); (-x(t), -y(t))\}.$$

We call \mathcal{B}_1 the smallest isotropic set containing all the sets B_t when $t \in I$. We notice that the element containing only the origin belongs to \mathcal{B}_1 . Let R_t be the rectangle the vertices of which are the points of B_t . We call \mathcal{B}_2 the smallest isotropic set containing all the sets R_t when $t \in I$.

Theorem 6.1. *Consider the family $\mathcal{B} = \mathcal{B}_1$ or \mathcal{B}_2 just defined above. Set IS_h, SI_h and A_h the operators defined by*

$$\begin{aligned} (IS_h u)(\mathbf{x}) &= \inf_{B \in h\mathcal{B}} \sup_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y}), \\ (SI_h u)(\mathbf{x}) &= \sup_{B \in h\mathcal{B}} \inf_{\mathbf{y} \in \mathbf{x}+B} u(\mathbf{y}), \end{aligned}$$

$$(A_h u)(\mathbf{x}) = (IS_h \circ SI_h u)(\mathbf{x}).$$

If $\gamma \in [0; 1]$ and $\mathcal{B} = \mathcal{B}_2$, the functions $u_h(\mathbf{x}, nh)$, defined with the inf-sup scheme A_h , tend uniformly when $nh \rightarrow t$ on any compact set to the viscosity solution $u(\mathbf{x}, t)$ of

$$\frac{\partial u}{\partial t} = |Du|(\text{curv}(u))^\gamma.$$

If $\gamma \in [1; +\infty]$ and $\mathcal{B} = \mathcal{B}_1$, the functions $u_h(\mathbf{x}, nh)$, defined with the inf-sup scheme A_h , tend uniformly when $nh \rightarrow t$ on any compact set to the viscosity solution $u(\mathbf{x}, t)$ of

$$\frac{\partial u}{\partial t} = |Du|(\text{curv}(u))^\gamma.$$

6.2. γ is larger than 1

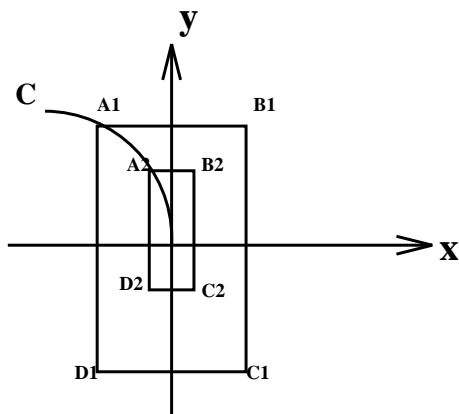


FIGURE 4. The four corners of the rectangle $(A1, B1, C1, D1)$ constitute an element of \mathcal{B}_1 . In the same way the four corners of the rectangle $(A2, B2, C2, D2)$.

Let us first prove Theorem 6.1 when γ is larger than 1.

1. **Lemma 6.2.**

- For all b in I , $\sup_{B_1} \inf_B (x + by^2) = b^\gamma$,
- for all $b \leq 0$, $\sup_{B_1} \inf_B (x + by^2) = 0$.

Proof. First we assume $b \in I$. We call B_t^θ the image of B_t by the rotation centered in O with angle θ . Because of the symmetry of B_t , we restrict θ to $[0; \frac{\pi}{2}]$. So we have

$$\sup_{B_1} \inf_B (x + by^2) = \sup_{t \in I} \sup_{\theta} \inf_{B_t^\theta} (x + by^2).$$

We consider the point $A_t = (x(t), y(t))$. As $t \in I$, we have $|x(t)| \leq y(t)$, we deduce that for all $\theta \in [0; \frac{\pi}{2}]$

$$\inf_{B_t^\theta} (x + by^2) \leq \inf_{B_t^0} (x + by^2),$$

therefore

$$\sup_{\theta} \inf_{B_t^\theta} (x + by^2) = \inf_{B_t^0} (x + by^2).$$

Finally we get

$$\sup_{\mathcal{B}_1} \inf_B (x + by^2) = \sup_{t \in I} \inf_{B_t^\theta} (x + by^2).$$

It remains to prove that

$$\sup_{t \in I} \inf_{B_t^\theta} (x + by^2) = b^\gamma. \tag{4}$$

By construction we have

$$\sup_{t \in I} \inf_{B_t} (x + by^2) = \sup_{t \in I} (x(t) + by(t)^2).$$

On the other hand, since $G(s) = s^\gamma$ is a convex function on I , G is the supremum of its tangent lines. Thus, for any s in I

$$G(s) = \sup_{t \in I} (G(t) + G'(t)(s - t)),$$

or

$$G(s) = \sup_{t \in I} (x(t) + sy(t)^2).$$

From this and the definition of B_t , we get (4).

We now assume $b \leq 0$. For all θ , it is possible to find a point $(x, y) \in B_t^\theta$ with $x \leq 0$. Thus,

$$\inf_{B_t^\theta} (x + by^2) \leq 0.$$

In addition, there is only one point in B_0 : the origin, so

$$\inf_{B_0^\theta} (x + by^2) = 0.$$

Finally,

$$\sup_{t \in I} \inf_{B_t} (x + by^2) = 0.$$

□

2. We have a dual lemma with the inf-sup scheme, since

$$\inf_{\mathcal{B}_1} \sup_B (-u) = - \sup_{\mathcal{B}_1} \inf_B (u).$$

So we can also deduce

Lemma 6.3.

- for all $b \geq 0$, $\inf_{\mathcal{B}_1} \sup_B (x + by^2) = 0$,
- for all b of I , $\inf_{\mathcal{B}_1} \sup_B (x + by^2) = (-b)^\gamma$.

3. Since the family \mathcal{B}_1 is bounded, isotropic and each element is symmetric with respect to O , we deduce with Theorem 4.1 that for any function $u \in C^{\gamma+2}(\mathbb{R}^2)$ in a neighborhood of $\mathbf{x}_o \in \mathbb{R}^2$, if $|Du(\mathbf{x}_o)| \neq 0$

$$\lim_{h \rightarrow 0} \frac{IS_h u(\mathbf{x}_o) - u(\mathbf{x}_o)}{h^{\gamma+1}} = |Du(\mathbf{x}_o)| G(\text{curv}(u)^+(\mathbf{x}_o)),$$

$$\lim_{h \rightarrow 0} \frac{SI_h u(\mathbf{x}_o) - u(\mathbf{x}_o)}{h^{\gamma+1}} = |Du(\mathbf{x}_o)| G(\text{curv}(u)^-(\mathbf{x}_o)),$$

$$\lim_{h \rightarrow 0} \frac{A_h u(\mathbf{x}_o) - u(\mathbf{x}_o)}{h^{\gamma+1}} = |Du(\mathbf{x}_o)|G(\text{curv}(u)(\mathbf{x}_o)),$$

with $(\text{curv}u)^+ = \max(\text{curv}u, 0)$, $(\text{curv}u)^- = \max(-\text{curv}u, 0)$, and

$$G(r) = r^\gamma.$$

In addition, the consistency is uniform.

Then the scheme A_h is uniformly consistent with $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$.

4. It is easy to check that for any $f \in \mathcal{F}(G)$, for any \mathbf{x}_o ,

$$\lim_{h \rightarrow 0} \frac{A_h f(|\cdot - \mathbf{x}_o|)(\mathbf{x}_o)}{h} = 0.$$

In addition, A_h is monotone, stable and commuting with the addition of constants (i.e. $T_h(u + C) = T_h u + C$). Thus, we can apply Theorem 3.6 and complete the proof. □

6.3. γ is smaller than 1

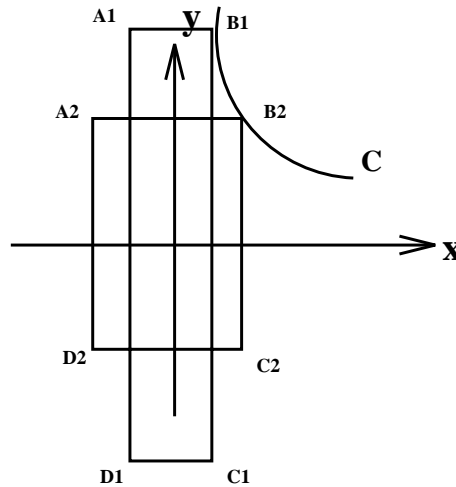


FIGURE 5. The rectangle $(A1, B1, C1, D1)$ constitutes an element of \mathcal{B}_2 . In the same way the rectangle $(A2, B2, C2, D2)$.

We prove Theorem 6.1 if γ is smaller than 1.

1. We prove the consistency result for $IS_h(u)$, and thanks to the relation

$$\sup_{\mathcal{B}_2} \inf_B (u(\mathbf{y})) = - \inf_{\mathcal{B}_2} \sup_B (-u(\mathbf{y})),$$

we deduce the result for SI_h , and then for $C_h(u)$ and A_h . Since the family \mathcal{B}_2 is not bounded, that is to say there is no disk containing every element of \mathcal{B}_2 , the value of $IS_h u(\mathbf{x})$ depends on values of $u(\mathbf{y})$ at points \mathbf{y} arbitrarily far from \mathbf{x} . Therefore, this operator looks like an *a priori* non local operator no matter how small h is. Nevertheless, we shall show that the behaviour of IS_h is very close to an operator defined with a bounded family. We defined this new family \mathcal{B}_ϵ by restricting the domain of t to $[\epsilon; a]$ instead of $[0; a]$ with $\epsilon > 0$. This new family is bounded, isotropic and each element $B \in \mathcal{B}_\epsilon$ is symmetric

with respect to O . Therefore, we can apply Theorem 4.1 about the asymptotic behaviour to IS_ϵ deduced from the bounded family \mathcal{B}_ϵ

$$IS_{h,\epsilon}u(\mathbf{x}_o) = u(\mathbf{x}_o) + h|Du|G_\epsilon(h\text{curv}(u)) + K_{\epsilon,\mathbf{x}_o}h^{2+1/3}$$

and

$$G_\epsilon(b) = \inf_{\mathcal{B}_\epsilon} \sup_B(x + by^2) = \inf_{t \in [\epsilon; a]} \sup_{B_t}(x + by^2).$$

So the idea is to let ϵ tend to 0. First, we compute $G(b) = \inf_{\mathcal{B}_2} \sup_B(x + by^2)$ and we deduce $G_\epsilon(b)$. Then, by using the locality lemma, we compare IS_h and $IS_{h\epsilon}$.

2. **Lemma 6.4.**

- If $b \in I$, $\inf_{\mathcal{B}_2} \sup_B(x + by^2) = b^\gamma$,
- if $b \leq 0$ $\inf_{\mathcal{B}_2} \sup_B(x + by^2) = 0$.

Proof. First, we assume $b \in I$. We call B_t^θ the image of B_t by a rotation of θ . Because of the symmetry of B_t , we restrict θ to $[0; \frac{\pi}{2}]$. So we have

$$\inf_{\mathcal{B}_2} \sup_B(x + by^2) = \inf_{t \in I} \inf_{\theta} \sup_{B_t^\theta}(x + by^2).$$

We consider the point $A_t = (x(t), y(t))$. Since we have $|x(t)| \leq y(t)$, we deduce that for all $\theta \in [0; \frac{\pi}{2}]$

$$\sup_{B_t^\theta}(x + by^2) \geq \sup_{B_t^0}(x + by^2),$$

therefore

$$\inf_{\theta} \sup_{B_t^\theta}(x + by^2) = \sup_{B_t^0}(x + by^2),$$

and finally we get

$$\inf_{\mathcal{B}_2} \sup_B(x + by^2) = \inf_{t \in I} \sup_{B_t^0}(x + by^2).$$

It remains to prove

$$\inf_{t \in I} \sup_{B_t^0}(x + by^2) = b^\gamma. \tag{5}$$

Now, since $G(s) = s^\gamma$ is a concave function on I , G is the infimum of its tangent lines. Thus, for any s in I

$$G(s) = \inf_{t \in I} (G(t) + G'(t)(s - t)),$$

or

$$G(s) = \inf_{t \in I} (x(t) + sy(t)^2).$$

From this we get (5).

We now assume $b \leq 0$. So we have $x + by^2 \leq x$. Then, since the origin belongs to any set B of \mathcal{B}_2 , we have

$$0 \leq \sup_B(x + by^2) \leq \sup_B(x),$$

therefore,

$$0 \leq \inf_{B \in \mathcal{B}_2} \sup_B(x + by^2) \leq \inf_{B \in \mathcal{B}_2} \sup_B(x) = 0.$$

Finally,

$$\inf_{B \in \mathcal{B}_2} \sup_B (x + by^2) = 0.$$

□

3. We have a dual lemma with sup-inf scheme since

$$\inf_{\mathcal{B}_2} \sup_B (-u) = - \sup_{\mathcal{B}_2} \inf_B (u).$$

Lemma 6.5.

- For all $b \geq 0$, $\sup_{\mathcal{B}_2} \inf_B (x + by^2) = 0$,
- for all b of I , $\sup_{\mathcal{B}_2} \inf_B (x + by^2) = (-b)^\gamma$.

4. Then, it is easy to compute G_ϵ .

$$\begin{cases} \text{If } x \geq \epsilon & G_\epsilon(x) = G(x) \\ \text{If } \epsilon \geq x \geq 0 & G_\epsilon(x) = G(\epsilon) + G'(\epsilon)(x - \epsilon) \\ \text{If } 0 \geq x & G_\epsilon(x) = G(\epsilon) - \epsilon G'(\epsilon). \end{cases}$$

We get the following graphics for the two functions G and G_ϵ :

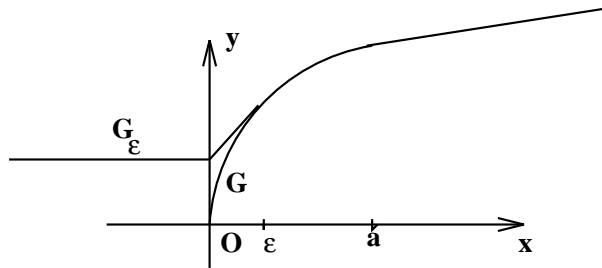


FIGURE 6. G and G_ϵ .

So

$$|G(b) - G_\epsilon(b)| \leq x(\epsilon) = (1 - \gamma)\epsilon^\gamma.$$

5. The next lemma expresses that the families \mathcal{B}_2 and \mathcal{B}_ϵ are very close for the semi-Hausdorff distance. We introduce the Hausdorff semi-distance for two bounded sets of \mathbb{R}^2

$$\delta(B', B) = \sup_{x \in B'} (\inf_{y \in B} |x - y|).$$

Lemma 6.6. Locality

Let $\epsilon > 0$, and $r = h(x^2(\epsilon) + y^2(\epsilon))^{\frac{1}{2}}$. Then for every $B \in h\mathcal{B}_2$, there exists $B' \in \mathcal{B}_{\epsilon h}$, such that

$$\delta(B', B \cap D(0, r)) \leq (1 - \gamma)h\epsilon^\gamma.$$

Proof. If $B \in \mathcal{B}_{\epsilon h}$, we choose for B' the set B . Assume now that B doesn't belong to the family $\mathcal{B}_{\epsilon h}$. We choose \vec{i} so that this vector is parallel to the smaller side of the rectangle B , and \vec{j} is orthogonal to \vec{i} . We consider the element B' of $\mathcal{B}_{\epsilon h}$ like in Figure 7. Then we have

$$\delta(B', B \cap D(0, r)) \leq hx_\epsilon = (1 - \gamma)\epsilon^\gamma h.$$

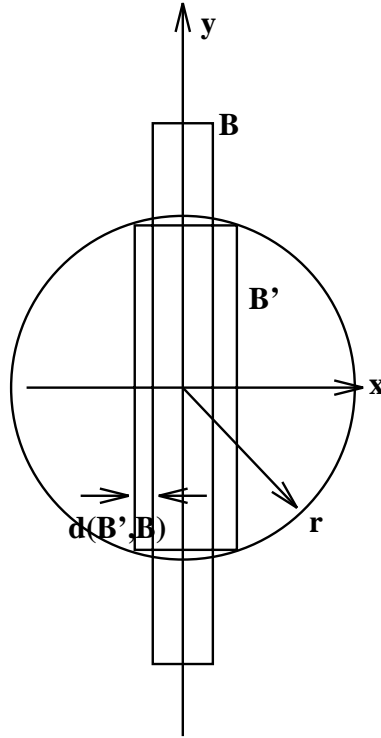


FIGURE 7. B and B' are very close for the semi Hausdorff distance.

□

6. Lemma 6.7. Local Monotonicity

Let $\epsilon > 0$ and $r = h(x^2(\epsilon) + y^2(\epsilon))^{\frac{1}{2}}$. Let u and v be two K -Lipschitz functions such that for every $\mathbf{x} \in D(\mathbf{x}_o, r)$

$$u(\mathbf{x}) \geq v(\mathbf{x}).$$

Then we have

$$IS_h u \geq IS_h v - K(1 - \gamma)h\epsilon^\gamma$$

and

$$IS_{h\epsilon} u(\mathbf{y}) \geq IS_h u(\mathbf{y}) \geq IS_{h\epsilon} u(\mathbf{y}) - K(1 - \gamma)h\epsilon^\gamma.$$

Proof. We have

$$IS_h u(\mathbf{x}_o) \geq \inf_{B \in h\mathcal{B}_2 + \mathbf{x}_o} \sup_{\mathbf{y} \in B \cap D(\mathbf{x}_o, r)} u(\mathbf{y}) \geq \inf_{B \in h\mathcal{B}_2 + \mathbf{x}_o} \sup_{\mathbf{y} \in B \cap D(\mathbf{x}_o, r)} v(\mathbf{y}).$$

Let $B \in h\mathcal{B}_2$, we consider the B' obtained with the preceding lemma. As v is K -Lipschitz, we have for $\mathbf{y} \in B'$ and $\mathbf{x} \in B \cap D$

$$v(\mathbf{y}) \leq v(\mathbf{x}) + K|\mathbf{y} - \mathbf{x}|,$$

therefore,

$$v(\mathbf{y}) \leq \sup_{\mathbf{x} \in B \cap D} v(\mathbf{x}) + K \inf_{\mathbf{x} \in B \cap D} |\mathbf{y} - \mathbf{x}|,$$

so,

$$\sup_{B'} v(\mathbf{y}) \leq \sup_{\mathbf{x} \in B \cap D} v(\mathbf{x}) + K\delta(B', B \cap D),$$

thus,

$$\sup_{B'} v(\mathbf{y}) \leq \sup_{\mathbf{x} \in B \cap D} v(\mathbf{x}) + K(1 - \gamma)h\epsilon^\gamma,$$

we deduce

$$IS_h u(\mathbf{x}_o) \geq \inf_{B \in h\mathcal{B}_2 + \mathbf{x}_o} \sup_{\mathbf{y} \in B'} v(\mathbf{y}) - K(1 - \gamma)h\epsilon^\gamma IS_{h\epsilon} v(\mathbf{y}) - K(1 - \gamma)h\epsilon^\gamma.$$

As $\mathcal{B}_{\epsilon h} \subset h\mathcal{B}_2$, we have

$$IS_{h\epsilon} v(\mathbf{y}) \geq IS_h v(\mathbf{y}).$$

□

7. So we have the relation

$$IS_{h,\epsilon} u(\mathbf{x}_o) = u(\mathbf{x}_o) + h|Du|G_\epsilon(h\text{curv}u) + K_{\epsilon,\mathbf{x}_o} h^{2+1/3}.$$

We set

$$\epsilon = h^\omega.$$

With the preceding results, we have

$$IS_{h,\epsilon} u(\mathbf{x}_o) = IS_h u(\mathbf{x}_o) + K_{1,\epsilon,\mathbf{x}_o} h^{1+\omega\gamma},$$

and

$$G_\epsilon(h\text{curv}u) = G(h\text{curv}u) + K_{2,\epsilon,\mathbf{x}_o} h^{\omega\gamma}.$$

We deduce

$$IS_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h|Du|G(h\text{curv}u) + K_{3,\epsilon,\mathbf{x}_o} h^{1+\omega\gamma} + K_{\epsilon,\mathbf{x}_o} h^{2+1/3}.$$

If $\text{curv}(u)(\mathbf{x}_o) < 0$, then

$$IS_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + K_{3,\epsilon,\mathbf{x}_o} h^{1+\omega\gamma} + K_{\epsilon,\mathbf{x}_o} h^{2+1/3}.$$

If $\text{curv}(u)(\mathbf{x}_o) > 0$, then

$$IS_h u(\mathbf{x}_o) = u(\mathbf{x}_o) + h^{\gamma+1}|Du|(\text{curv}u)^\gamma + K_{3,\epsilon,\mathbf{x}_o} h^{1+\omega\gamma} + K_{\epsilon,\mathbf{x}_o} h^{2+1/3}.$$

In order to get the consistency result, it is enough to choose $\omega > 1$.

Moreover, we can prove that the consistency is uniform on any compact set on which $|Du|(\mathbf{x}) \neq 0$ if $\omega < 2$. To get this result, we only have to prove that the family $\mathcal{B}_{h\epsilon}$ remains included in a ball whose radius tends to 0 with h . For all $B \in \mathcal{B}_{h\epsilon}$, B is included in a ball with radius $r = h(x^2(\epsilon) + y^2(\epsilon))^{1/2}$. An equivalent of r in $h = 0$ is $h^{1+\omega\frac{\gamma-1}{2}}$. Then, with the condition $\omega < 2$, r tends to 0 with h . Therefore the constants $K_{\epsilon,\mathbf{x}_o}$ and $K_{3,\epsilon,\mathbf{x}_o}$ are bounded, so we get the uniform consistency.

Then the scheme A_h is uniformly consistent with $\frac{\partial u}{\partial t} = |Du|G(\text{curv}(u))$ and

$$G(r) = r^\gamma.$$

8. It is easy to check that for any $f \in \mathcal{F}(G)$, for any \mathbf{x}_o ,

$$\lim_{h \rightarrow 0} \frac{A_h f(|\cdot - \mathbf{x}_o|)(\mathbf{x}_o)}{h} = 0.$$

In addition, A_h is monotone, stable and commuting with the addition of constants (*i.e.* $T_h(u + C) = T_h u + C$). Thus, we can apply Theorem 3.6 and complete the proof.

□

7. APPENDIX

Proof of Theorem 3.6. 1. We define

$$u^*(\mathbf{x}, t) = \limsup_{nh \rightarrow t, h \rightarrow 0} u_h(\mathbf{x}, nh)$$

and

$$u_*(\mathbf{x}, t) = \liminf_{nh \rightarrow t, h \rightarrow 0} u_h(\mathbf{x}, nh).$$

Next we prove that u^* is a usc subsolution and u_* is a lsc supersolution. As $u^*(\mathbf{x}, 0) = u_*(\mathbf{x}, 0)$, we deduce with the comparison principle [9] $u^*(\mathbf{x}, t) \leq u_*(\mathbf{x}, t)$. But the opposite inequality is obvious, so we get the equality which means the convergence of the scheme.

2. We only prove that the function u^* is a subsolution since the argument for u_* is identical. Let ϕ be an admissible function and (\mathbf{x}_o, t_o) be a local maximum of $u - \phi$ on $\bar{\Omega}$. We set $D = D((\mathbf{x}_o, t_o), r)$ such that (\mathbf{x}_o, t_o) is a strict local maximum of $u - \phi$ on D . With standard arguments, it is possible to extract a sequence $(\mathbf{x}_h, n_h h) \rightarrow (\mathbf{x}_o, t_o)$ when $h \rightarrow 0$ and $(\mathbf{x}_h, n_h h)$ is a local maximum of $u_h - \phi$ on D . Then, for h small enough and for any $\mathbf{y} \in D(\mathbf{x}, \frac{r}{2})$, we get $(\mathbf{y}, (n_h - 1)h) \in D$. Hence

$$u_h(\mathbf{y}, (n_h - 1)h) \leq u_h(\mathbf{x}_h, n_h h) - \phi(\mathbf{x}_h, n_h h) + \phi(\mathbf{y}, (n_h - 1)h).$$

With the local comparison principle, and the commutation with the addition of constant we deduce

$$T_h u_h(\cdot, (n_h - 1)h)(\mathbf{x}_h) \leq u_h(\mathbf{x}_h, n_h h) - \phi(\mathbf{x}_h, n_h h) + T_h \phi(\cdot, (n_h - 1)h)(\mathbf{x}_h) + o(h). \tag{6}$$

With standard arguments, we get the following lemma.

Lemma 7.1. *If u is a subsolution (resp. supersolution) for any function ϕ admissible C^∞ with $\phi(\mathbf{x}, t) = f(\mathbf{x}) + g(t)$, then u is a subsolution (resp. a supersolution).*

Then we set $\phi(\mathbf{x}, t) = f(\mathbf{x}) + g(t)$ in (6), we get

$$u_h(\mathbf{x}_h, n_h h) \leq u_h(\mathbf{x}_h, n_h h) - f(\mathbf{x}_h) - g(n_h h) + T_h f(\mathbf{x}_h) + g((n_h - 1)h) + o(h),$$

therefore,

$$g(n_h h) - g((n_h - 1)h) + f(\mathbf{x}_h) - T_h f(\mathbf{x}_h) + o(h) \leq 0. \tag{7}$$

3. First, we consider the case $|D\phi|(\mathbf{x}_o, t_o) = |Df|(\mathbf{x}_o) \neq 0$. Then, for h small enough, $|Df|(\mathbf{x}_h) \neq 0$. We conclude with the consistency property

$$hg'(c) - h|Df(\mathbf{x}_h)|G(\text{curv}(f(\mathbf{x}_h))) + o(h) \leq 0$$

with $c \in [(n_h - 1)h; n_h h]$. We divide by h and we let h tend to 0, as G is continuous, we get

$$g'(t_o) - |Df(\mathbf{x}_o)|G(\text{curv}(f(\mathbf{x}_o))) \leq 0.$$

4. We consider now the case $|D\phi|(\mathbf{x}_o, t_o) = 0$. Since ϕ is admissible, there exists a constant $\delta > 0$ and functions $f \in \mathcal{F}(G)$ and $\omega \in C((0, +\infty))$ with $\lim_{r \rightarrow 0} \frac{\omega(r)}{r} = 0$ such that

$$|\phi(\mathbf{x}, t) - \phi(\mathbf{x}_o, t_o) - \phi_t(\mathbf{x}_o, t_o)(t - t_o)| \leq f(|\mathbf{x} - \mathbf{x}_o|) + w(|t - t_o|)$$

for any $(\mathbf{x}, t) \in B((\mathbf{x}_o, t_o), \delta)$.

We assume, as [9], that $\omega \in C^1((0, +\infty))$ and $\omega(0) = \omega'(0) = 0$ and $\omega(r) > 0$ for $r \neq 0$. We set

$$\psi(\mathbf{x}, t) = \phi_t(\mathbf{x}_o, t_o)(t - t_o) + 2f(|\mathbf{x} - \mathbf{x}_o|) + 2w(|t - t_o|).$$

We deduce that $u - \psi$ has a local strict maximum in (\mathbf{x}_o, t_o) . If $\mathbf{x} \neq \mathbf{x}_o$,

$$D\psi(\mathbf{x}, t) = 2f'(|\mathbf{x} - \mathbf{x}_o|) \frac{\mathbf{x} - \mathbf{x}_o}{|\mathbf{x} - \mathbf{x}_o|},$$

and

$$D^2\psi(\mathbf{x}, t) = 2f'(|\mathbf{x} - \mathbf{x}_o|) \frac{I}{|\mathbf{x} - \mathbf{x}_o|} + 2f''(|\mathbf{x} - \mathbf{x}_o|) \frac{(\mathbf{x} - \mathbf{x}_o) \otimes (\mathbf{x} - \mathbf{x}_o)}{|\mathbf{x} - \mathbf{x}_o|^2} - 2f'(|\mathbf{x} - \mathbf{x}_o|) \frac{(\mathbf{x} - \mathbf{x}_o) \otimes (\mathbf{x} - \mathbf{x}_o)}{|\mathbf{x} - \mathbf{x}_o|^3}.$$

- If $\mathbf{x}_h \neq \mathbf{x}_o$, then $D\psi(\mathbf{x}_h, t) \neq 0$, so we have the preceding result

$$\psi_t(\mathbf{x}_h, n_h h) - |D\psi(\mathbf{x}_h, n_h h)|G(\text{curv}(\psi(\mathbf{x}_h, n_h h))) + o(h) \leq 0,$$

therefore,

$$\phi_t(\mathbf{x}_o, t_o) + 2w'(|n_h h - t_o|) - 2f'(|\mathbf{x}_h - \mathbf{x}_o|)G\left(\frac{1}{|\mathbf{x}_h - \mathbf{x}_o|}\right) + o(h) \leq 0.$$

We let h tend to 0, by definition of $\mathcal{F}(G)$, it remains

$$\phi_t(\mathbf{x}_o, t_o) \leq 0.$$

- If $\mathbf{x}_h = \mathbf{x}_o$ for any h , we apply 7 with the function ψ defined in step 4, so we get

$$g(n_h h) - g((n_h - 1)h) + f(\mathbf{x}_h) - 2T_h f(|\cdot - \mathbf{x}_o|)(\mathbf{x}_h) + o(h) \leq 0.$$

We replace \mathbf{x}_h by \mathbf{x}_o

$$hg'(c) - 2T_h f(|\cdot - \mathbf{x}_o|)(\mathbf{x}_o) + o(h) \leq 0$$

with $c \in [(n_h - 1)h; n_h h]$. As we assume

$$\lim_{h \rightarrow 0} \frac{T_h f(|\cdot - \mathbf{x}_o|)(\mathbf{x}_o)}{h} = 0$$

hence

$$\phi_t(\mathbf{x}_o, t_o) \leq 0.$$

□

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