

STABILITY RESULTS FOR SOME NONLINEAR ELLIPTIC EQUATIONS INVOLVING THE p -LAPLACIAN WITH CRITICAL SOBOLEV GROWTH

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Abstract. This article is devoted to the study of a perturbation with a viscosity term in an elliptic equation involving the p -Laplacian operator and related to the best constant problem in Sobolev inequalities in the critical case. We prove first that this problem, together with the equation, is stable under this perturbation, assuming some conditions on the data. In the next section, we show that the zero solution is strongly isolated in some sense, among the space of the solutions. Actually, we end the paper by giving some analogous results in the case where the data present symmetries.

Résumé. Dans cet article, l'auteur se propose d'étudier une perturbation de type visqueux dans une équation elliptique contenant l'opérateur du p -Laplacien, et provenant d'un problème de meilleure constante pour les inégalités de Sobolev dans le cas critique. Nous montrons ici que ce problème, ainsi que l'équation associée est stable sous cette perturbation, moyennant quelques hypothèses sur les données. Dans la suite, nous prouvons que la solution identiquement nulle est, dans un certain sens, isolée dans l'espace des solutions. Enfin, nous terminons cette étude par des résultats analogues dans le cas où les données possèdent des symétries.

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1. INTRODUCTION

In this paper, we are interested in the stability under a perturbation with a viscosity term of the following nonlinear elliptic PDE's involving the p -Laplacian with critical Sobolev growth:

$$\begin{cases} -\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + a(x) |u|^{p-2} u = f(x) |u|^{p^*-2} u \\ u \in W_0^{1,p}(\Omega). \end{cases} \quad (1)$$

Here, Ω denotes a bounded open set of \mathbb{R}^N , a and f are smooth on $\overline{\Omega}$, p is a real in $(1, N)$, and $p^* = Np/(N-p)$ is the critical exponent for the Sobolev embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$.

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When a is constant and $f \equiv 1$, equation (1) is the Euler equation of the following minimisation problem:

$$\lambda(N, p, \Omega) = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} (|\nabla u|^p + a|u|^p)}{\left(\int_{\Omega} |u|^{p^*}\right)^{\frac{p}{p^*}}}. \quad (2)$$

In other words, $\lambda(N, p, \Omega)^{-1/p}$ is the first best constant for the embedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ (see Hebey [11]).

In the case where $\Omega = \mathbb{R}^N$, the supremum defined by:

$$K(N, p) = \sup_{u \in C_0^\infty(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |u|^{p^*}\right)^{\frac{1}{p^*}}}{\left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}}$$

has been computed by Aubin [1] and Talenti [16], and has value:

$$K(N, p) = \frac{p-1}{N-p} \left(\frac{N-p}{N(p-1)}\right)^{\frac{1}{p}} \left(\frac{\Gamma(N+1)}{\Gamma\left(\frac{N}{p}\right) \Gamma\left(N+1-\frac{N}{p}\right) \omega_{N-1}}\right)^{\frac{1}{N}}$$

where ω_{N-1} denotes the volume of the unit sphere in \mathbb{R}^N . Furthermore, this supremum is achieved on the following functions:

$$u_\mu(x) = \left(\mu + r^{\frac{p}{p-1}}\right)^{1-\frac{N}{p}}$$

where μ is a parameter and r the euclidean norm of x .

When Ω is arbitrary, Hebey [10], Hebey-Vaugon [8] (in the case where $p = 2$), and Demengel-Hebey (in the case where $p > 1$) study the existence of extremal functions (*i.e.* which realize the extremum) for problem (1) (see also Lions [13, 14]):

Let a and f be C^∞ functions on $\overline{\Omega}$. We assume that Ω is bounded and regular. Then, we define

$$\lambda(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_{\Omega} (|\nabla u|^p + a(x)|u|^p)}{\left(\int_{\Omega} f(x)|u|^{p^*}\right)^{\frac{p}{p^*}}}.$$

Note that, in this kind of problems, though the infimum value is in general not known, the case \mathbb{R}^N acts as reference, since the authors above show that, if $\lambda(\Omega) < K(N, p)^{-p} \|f\|_{L^\infty}^{(p/N)-1}$, then this infimum is realized on a positive solution of the following equation:

$$-\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + a(x) |u|^{p-2} u = \lambda(\Omega) f(x) |u|^{p^*-2} u. \quad (3)$$

By regularity results, such as developed in Guedda-Veron [6, 7], $u \in C^{1,\alpha}(\overline{\Omega})$, and by the Vazquez strict maximum principle [18], u is positive in Ω . They give also symmetry conditions on Ω and invariance conditions on a and f , which imply the existence's condition $\lambda(\Omega) < K(N, p)^{-p} \|f\|_{L^\infty}^{p/N-1}$.

Our aim in this article is to show, for $p < 2$ and under the condition above, that these solutions are stable under some viscosity perturbation. Furthermore, the proof of this result presents the advantage to give another proof of the existence theorem given by Demengel-Hebey.

Remark that, if u is a solution of (3) and if $\int_{\Omega} f(x)|u|^{p^*} = 1$, then u realizes the infimum $\lambda(\Omega)$. Moreover, since the embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is not compact, we can not solve equation (3) by standard variational

arguments. In the perturbed equation, the viscosity term $-\epsilon\Delta u_\epsilon$ compensates the loss of compactness and give us a method in order to find a solution.

2. NOTATIONS AND RESULTS

In this paper, Ω will denote a C^1 domain of \mathbb{R}^N , where $N \geq 3$. Let $p \in (1, N)$ be a real, and let a, f be two C^∞ functions defined on $\overline{\Omega}$. We are interested in the following problem:

$$\begin{cases} -\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + a(x) |u|^{p-2} u = f(x) |u|^{p^*-2} u \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{4}$$

where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent for the embeddings $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

We make the following assumptions:

- the function a is such that the operator

$$L(u) = -\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + a(x) |u|^{p-2} u$$

is coercive, in the sense that there exists a positive constant C such that, for all functions u in $W_0^{1,p}(\Omega)$,

$$J(u) = \int_{\Omega} (|\nabla u|^p + a(x)|u|^p) \geq C \int_{\Omega} |u|^p.$$

- the function f is positive somewhere in $\overline{\Omega}$ (this assumption is necessary, since L is coercive).

Demengel and Hebey proved in [4] existence's results in the case where the data Ω , a , and f present some symmetries (for a Riemannian manifold, this problem is treated by Druet in [3]). Since we are not interested here in finding concrete conditions for the existence of extremal functions, we only consider the general case. (We make at the end of the paper a brief study of the presence of symmetries.) Then, the result of [4] can be written in the following simplified form: let us define the set

$$\mathcal{W}_{p^*} = \left\{ v \in W_0^{1,p}(\Omega); \int_{\Omega} f(x)|v|^{p^*} = 1 \right\}.$$

Theorem 2.1. *Let us suppose that, for all $x \in \overline{\Omega}$ such that $f(x) > 0$, we have:*

$$K(N, p)^p \lambda f(x)^{1-p/N} < 1$$

where

$$\lambda = \inf_{v \in \mathcal{W}_{p^*}} J(v). \tag{5}$$

Then, there exists a solution u of (4) in $W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, which is positive. Moreover, u realizes the infimum in (5).

In the first section, we shall prove, in the case where $p < 2$, a stability result of the positive solutions of equation (4) under some perturbation by a viscous term. More precisely, we consider the variational problem:

$$\lambda_\epsilon = \inf_{\int_{\Omega} f|v|^{p^*}=1} \left(\frac{\epsilon}{2} \int_{\Omega} |\nabla v|^2 + J(v) \right) \tag{6}$$

where ϵ is a positive real, which will tend to 0 later. Since $p < 2$, p^* is subcritical for the embedding of $H_0^1(\Omega)$ in $L^q(\Omega)$, and then, by standard compactness arguments, this problem admits a non zero solution u_ϵ . In addition, u_ϵ solves the following equation:

$$-\frac{\epsilon}{p}\Delta u_\epsilon - \operatorname{div}\left(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\right) + a(x)|u_\epsilon|^{p-2}u_\epsilon = \mu_\epsilon f(x)|u_\epsilon|^{p^*-2}u_\epsilon \tag{7}$$

where μ_ϵ is some Lagrange multiplier.

Then, we prove the following result:

Theorem 2.2. *We suppose that the assumptions in Theorem 2.1 hold. Let u_ϵ be a solution of (6), which is positive. Then, up to a subsequence (u_ϵ) converges strongly in $W_0^{1,p}(\Omega)$ to a solution u of (4) which belongs to $W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, is positive in Ω , and realizes the infimum in (5).*

In the next section, we are interested in the weak continuity of the set of solutions for equation (4) and we prove the following theorem:

Theorem 2.3. *Let (u_n) be a bounded sequence (in $W_0^{1,p}(\Omega)$) of solutions of (4) which are non identically zero. We assume that, at every point x in $\overline{\Omega}$ where $f(x) > 0$,*

$$K(N, p)^p \lambda f(x)^{1-p/N} < 1 \tag{8}$$

and that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x)|u_n|^{p^*} = \lambda^{N/p}.$$

Then, (u_n) converges strongly in $W_0^{1,p}(\Omega)$ to a nonzero solution u of (4).

Finally, in the last section, we present a brief discussion concerning the case where the domain Ω is invariant under the action of a subgroup G of the orthogonal group $O_N(\mathbb{R}^N)$ and give stability results for positive and nodal solutions (we say nodal for a solution wich changes sign).

3. STABILITY OF POSITIVE SOLUTIONS

Let us consider the following problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + a(x)|u|^{p-2}u = f(x)|u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, u > 0 & \text{in } \Omega \end{cases} \tag{9}$$

where $p < 2$.

This has been solved by Demengel-Hebey in [4] for every $p \in (1, N)$. Our goal here is to study its stability when the operator L is perturbed by adding to it $-\epsilon\Delta$ and when ϵ goes to zero. Since $p < 2$, the operator $L - \epsilon\Delta$ is smoothing L . We now introduce some notations:

- $\mathcal{H}_{p^*} = \{v \in H_0^1(\Omega); \int_{\Omega} f(x)|v|^{p^*} = 1\}$.
- For $\epsilon \geq 0$, and $v \in H_0^1(\Omega)$,

$$J_\epsilon(v) = \frac{\epsilon}{2} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} (|\nabla v|^p + a(x)|v|^p) \tag{10}$$

$$\lambda_\epsilon = \inf_{v \in \mathcal{H}_{p^*}} J_\epsilon(v). \tag{11}$$

The functional J_ϵ represents the energy functional for the operator $-\epsilon\Delta + L$ and the real λ_ϵ is the minimal energy under the condition $\int_{\Omega} f|v|^{p^*} = 1$ (we shall see later that the real λ is also the minimal energy of the initial problem).

3.1. The perturbed problem

Before studying the perturbed equation, let us note that:

Lemma 3.1. $\lambda = \lambda_0$

Proof. We obviously have $\lambda \leq \lambda_0$. To prove the reverse inequality, we first state that

$$\overline{\mathcal{H}_{p^*}}^{W_0^{1,p}(\Omega)} = \mathcal{W}_{p^*}.$$

Indeed, let $v \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} f(x)|v|^{p^*} = 1$. Then, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of functions in $H_0^1(\Omega)$ such that v_n converges to v in $W_0^{1,p}(\Omega)$, almost everywhere in Ω and, due to the Sobolev embedding theorem, in $L^{p^*}(\Omega)$. From this, we deduce that $\int_{\Omega} f(x)|v_n|^{p^*} \rightarrow \int_{\Omega} f(x)|v|^{p^*} = 1$. Defining, for n large enough in order to have $(\int_{\Omega} f|u|^{p^*})^{1/p^*} < \frac{1}{2}$,

$$w_n = \frac{v_n}{(\int_{\Omega} f(x)|v_n|^{p^*})^{\frac{1}{p^*}}},$$

one clearly has $J_0(w_n) \rightarrow J_0(v)$, i.e.

$$\forall \eta > 0, \exists N \in \mathbb{N}, n > N \implies |J_0(w_n) - J_0(v)| < \eta,$$

and consequently

$$\lambda_0 \leq J_0(w_n) < J_0(v) + \eta.$$

This inequality being true for arbitrary v and η , the proof is completed. We now give the main result of this section. □

Proposition 3.1. *Let $\epsilon > 0$ be given. Then, there exists $u_{\epsilon} \in \mathcal{H}_{p^*}$, $u_{\epsilon} \geq 0$ a.e. in Ω , which is a solution of the minimization problem (11). Furthermore, u_{ϵ} is a weak solution of the equation:*

$$-\frac{\epsilon}{p} \Delta u_{\epsilon} - \operatorname{div} \left(|\nabla u_{\epsilon}|^{p-2} \nabla u_{\epsilon} \right) + a(x) |u_{\epsilon}|^{p-2} u_{\epsilon} = \mu_{\epsilon} f(x) |u_{\epsilon}|^{p^*-2} u_{\epsilon} \tag{12}$$

where

$$\mu_{\epsilon} = \lambda_{\epsilon} + \epsilon \left(\frac{1}{p} - \frac{1}{2} \right) \int_{\Omega} |\nabla u_{\epsilon}|^2.$$

Proof. To prove the existence of u_{ϵ} , let $(v_n)_{n \in \mathbb{N}}$ be a minimizing sequence for J_{ϵ} . Since $|\nabla|v_n|| = |\nabla v_n|$, one can assume that v_n is nonnegative, for all $n \in \mathbb{N}$. Using the coercivity of L , one gets that (v_n) is bounded in $H_0^1(\Omega)$. From Banach-Alaoglu and Rellich-Kondrakov theorems, there exists a subsequence, still denoted (v_n) , and a function u_{ϵ} in $H_0^1(\Omega)$, such that:

- $v_n \rightharpoonup u_{\epsilon}$ in $H_0^1(\Omega)$ (and then in $W_0^{1,p}(\Omega)$);
- $v_n \rightarrow u_{\epsilon}$ in $L^{p^*}(\Omega)$;
- $v_n \rightarrow u_{\epsilon}$ a.e. in Ω ;

where $u_{\epsilon} \in H_0^1(\Omega)$.

By the last two assertions, one gets that $u_{\epsilon} \geq 0$ a.e. in Ω , and that $\int_{\Omega} f|u_{\epsilon}|^{p^*} = 1$. By the lower semi-continuity of the norms in $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$, one obtains

$$J_{\epsilon}(u_{\epsilon}) \leq \liminf_{n \rightarrow \infty} J_{\epsilon}(v_n) = \lambda_{\epsilon}(G)$$

and then u_ϵ solves (11). We now prove that u_ϵ is a weak solution of equation (12).

Let $v \in \mathcal{D}(\Omega)$ be given. Then, for every real t small enough,

$$J_\epsilon \left(\frac{u_\epsilon + tv}{\left(\int_\Omega f|u_\epsilon + tv|^{p^*}\right)^{\frac{1}{p^*}}} \right) \geq J_\epsilon(u_\epsilon).$$

By expanding the left hand side in powers of t to the first order, one gets:

$$\begin{aligned} t \left(\frac{\epsilon}{p} \int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla v + \int_\Omega a(x) |u_\epsilon|^{p-2} u_\epsilon v \right) &\geq t \left(\int_\Omega f(x) |u_\epsilon|^{p^*-2} u_\epsilon v \right) \\ &\times \left(\frac{\epsilon}{p} \int_\Omega |\nabla u_\epsilon|^2 + \int_\Omega (|\nabla u_\epsilon|^p + a(x) |u_\epsilon|^p) \right) + \mathcal{O}(t^2). \end{aligned}$$

It follows that for every function v in $\mathcal{D}(\Omega)$,

$$\frac{\epsilon}{p} \int_\Omega \nabla u_\epsilon \nabla v + \int_\Omega |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla v + \int_\Omega a(x) |u_\epsilon|^{p-2} u_\epsilon v = \mu_\epsilon \int_\Omega f(x) |u_\epsilon|^{p^*-2} u_\epsilon v \tag{13}$$

with $\mu_\epsilon = \lambda_\epsilon + \epsilon \left(\frac{1}{p} - \frac{1}{2}\right) \int_\Omega |\nabla u_\epsilon|^2$. This completes the proof. □

In order to prove the stability of problem (4) under the perturbation defined in (7), we need some further results on the behaviour of the sequences (u_ϵ) and (μ_ϵ) .

Proposition 3.2. *The sequence (λ_ϵ) tends to λ as ϵ goes to 0.*

Proof. Let $\epsilon > \eta > 0$ be given. Then, $\forall v \in \mathcal{H}_{p^*}, J_\eta(v) < J_\epsilon(v)$. Thus, $\lambda_\eta \leq \lambda_\epsilon$. We derive from this that (λ_ϵ) has a limit as ϵ goes to 0. Moreover, by Lemma 3.1, this limit is greater than λ . Let us prove the reverse inequality.

Let $\delta > 0$ be given. As we remarked in the proof of Lemma 3.1, there exists $v_\delta \in \mathcal{H}_{p^*}$ such that,

$$\lambda \leq J_0(v_\delta) \leq \lambda + \delta.$$

Hence,

$$\lambda_\epsilon \leq J_\epsilon(v_\delta) < \lambda + \delta + \frac{\epsilon}{2} \int_\Omega |\nabla v_\delta|^2.$$

δ being arbitrary, we let ϵ go to 0, and obtain

$$\lim_{\epsilon \rightarrow 0} \lambda_\epsilon \leq \lambda + \delta$$

which ends the proof. □

Now, we give a strong convergence result concerning the perturbation term.

Proposition 3.3. *Let (u_ϵ) be a sequence of solutions given by Proposition 3.1. Then, $(\sqrt{\epsilon}u_\epsilon)$ converges strongly to 0 in $H_0^1(\Omega)$.*

Proof. For $\epsilon > \eta > 0$ being given, one has

$$\lambda_\epsilon - \lambda_\eta \geq J_\epsilon(u_\epsilon) - J_\eta(u_\epsilon) \geq \frac{\epsilon - \eta}{2} \int_\Omega |\nabla u_\epsilon|^2.$$

By two repeated applications of Proposition 3.2 above, letting first η , and next ϵ go to 0, one obtains the result. □

Corollary 3.4. *The sequence (μ_ϵ) tends to λ as ϵ goes to 0.*

3.2. Convergence of the perturbed problem

We shall get a solution of (4) by extracting subsequences from the initial sequence $(u_\epsilon)_{\epsilon>0}$. By the way, the coercivity of the operator L , together with the convergence of (λ_ϵ) , imply that the sequence (u_ϵ) is bounded in $W_0^{1,p}(\Omega)$. Then, up to a subsequence, it converges weakly in this space. The main difficulty here is to prove that the limit is not identically zero. This will be done in the next section.

Proposition 3.5. *Let us suppose that there exists a subsequence of (u_ϵ) which converges weakly in $W_0^{1,p}(\Omega)$ to some function $u \neq 0$. Then, u belongs to $W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in (0, 1)$, and u is a solution of (4). Furthermore, u realizes the infimum (5).*

Proof. Up to a subsequence, one can assume that:

- $u_\epsilon \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$;
- $u_\epsilon \rightarrow u$ in $L^k(\Omega)$, $\forall k < p^*$;
- $u_\epsilon \rightarrow u$ a.e. in Ω .

It turns out that $u \geq 0$ a.e. in Ω .

In addition, the sequence $(|\nabla u_\epsilon|)$ is bounded in $L^p(\Omega)$, so $(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon)$ is bounded in $L^{p'}(\Omega)$, where p' is the Hölder conjugate of p . Hence, there exists $\Sigma \in L^{p'}(\Omega)$, such that, up to a subsequence,

- $|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \rightharpoonup \Sigma$, weakly in $L^{p'}(\Omega)$.

Since $(\sqrt{\epsilon}u_\epsilon)$ converges strongly to 0 in $H_0^1(\Omega)$, passing to limit in (12), one obtains

$$-\operatorname{div}(\Sigma) + a(x)|u|^{p-2}u = \lambda f(x)|u|^{p^*-2}u.$$

Moreover, from (12), $\operatorname{div}(\epsilon \nabla u_\epsilon + |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon)$ is bounded in $L^{\frac{p^*}{p^*-1}}(\Omega)$, and then in $L^1(\Omega)$. Using Lemma 3.2 below, one obtains that $\Sigma = |\nabla u|^{p-2} \nabla u$, and that u is a solution of

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + a(x)|u|^{p-2}u = \lambda f(x)|u|^{p^*-2}u.$$

By regularity results, such as developed in Guedda-Veron [6], Tolksdorf [17], and by the Vazquez strict maximum principle [18], one gets that u belongs to $C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in (0, 1)$ and that $u > 0$ in Ω . Furthermore, multiplying the equation by u and integrating over Ω , one can see that λ and $\int_\Omega f|u|^{p^*}$ are positive. Multiplying u by $\lambda^{\frac{N-p}{p^2}}$, one obtains a solution of (4). □

Lemma 3.2. *Let Ω be a bounded domain of \mathbb{R}^N , p a real strictly larger than 1, and (u_ϵ) a sequence of $H_0^1(\Omega)$. We assume that:*

1. $\sqrt{\epsilon}u_\epsilon \rightarrow 0$ strongly in $H_0^1(\Omega)$.
2. $(u_\epsilon)_{\epsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.
3. $\left(\operatorname{div}\left(\frac{\epsilon}{p}\nabla u_\epsilon + |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon\right)\right)_{\epsilon>0}$ is bounded in $L^1(\Omega)$.

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $(u_\epsilon)_{\epsilon>0}$ converges a.e. to u , $(\nabla u_\epsilon)_{\epsilon>0}$ converges a.e. to ∇u , $(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon)_{\epsilon>0}$ converges a.e. and weakly in $L^{p'}(\Omega)$ to $|\nabla u|^{p-2} \nabla u$ (In fact, (∇u_ϵ) converges strongly to ∇u in $L^q(\Omega)$, for all $q < p$).

Proof. The proof is based on Evans ([9] Th. 3, Chap. 4) and Courilleau-Demengel's arguments ([5] Prop. 3.1). From the boundedness in $L^{p'}(\Omega)$ of $(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon)$, one obtains that, up to a subsequence,

- $\sqrt{\epsilon}\nabla u_\epsilon \rightarrow 0$ a.e. in Ω ;

- $u_\epsilon \rightarrow u$ a.e. in Ω ;
- $\Sigma_\epsilon = |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \rightharpoonup \Sigma$ weakly in $L^{p'}(\Omega)$;
- $u_\epsilon \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$.

Let $\delta > 0$ be given. By Egoroff's theorem, there exists a universally measurable set $E_\delta \subset\subset \Omega$ such that $\text{meas}(\Omega \setminus E_\delta) < \delta$, and u_ϵ (respectively $\sqrt{\epsilon} \nabla u_\epsilon$) tends to u (respectively 0) uniformly in E_δ . This implies in particular that $\sqrt{\epsilon} \nabla u_\epsilon \rightarrow 0$ strongly in $L^q(E_\delta)$, for all $q \leq \infty$.

Now, let $\eta > 0$ be given. By the uniform convergence in E_δ , there exists $\epsilon_0 > 0$ such that

$$\left(\epsilon < \epsilon_0 \implies \forall x \in E_\delta, |u_\epsilon(x) - u(x)| < \frac{\eta}{2} \right).$$

Let us consider the following cut-off function β_η :

$$\beta_\eta(t) = \begin{cases} t & \text{if } |t| \leq \eta \\ \eta \frac{t}{|t|} & \text{if } |t| > \eta \end{cases}$$

and define $\tilde{\Sigma} = |\nabla u|^{p-2} \nabla u$ ($\tilde{\Sigma} \in L^{p'}(\Omega)$). Since β_η is piecewise C^1 and continuous, one has that $\beta_\eta \circ (u_\epsilon - u) \in W^{1,p}(\Omega)$. Furthermore, one can easily see that

$$\left(\Sigma_\epsilon - \tilde{\Sigma} \right) \cdot \nabla (\beta_\eta \circ (u_\epsilon - u)) \geq 0 \quad \text{dans } \Omega,$$

since $\nabla (\beta_\eta \circ (u_\epsilon - u)) = \nabla (u_\epsilon - u)$ for $\epsilon < \epsilon_0$ in E_δ . Now consider the integral

$$\int_{E_\delta} \left(\left(\Sigma_\epsilon - \tilde{\Sigma} \right) \cdot \nabla (u_\epsilon - u) \right) (x) dx = \int_{E_\delta} \left(\left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon - \tilde{\Sigma} \right) \cdot \nabla (u_\epsilon - u) \right) (x) dx - \int_{E_\delta} \frac{\epsilon}{p} (\nabla u_\epsilon \cdot \nabla (u_\epsilon - u)) (x) dx.$$

Writing the second integral on the right hand side as

$$-\frac{\epsilon}{p} \int_{E_\delta} (\nabla u_\epsilon \cdot \nabla (u_\epsilon - u)) (x) dx = \frac{1}{p} \left(-\|\sqrt{\epsilon} u_\epsilon\|_{L^2(E_\delta)}^2 + \int_{E_\delta} \epsilon (\nabla u_\epsilon \cdot \nabla u) (x) dx \right)$$

one sees that it goes to 0 when ϵ goes to 0, since $\sqrt{\epsilon} u_\epsilon$ converges to 0 in $L^{p'}(E_\delta)$ and in $L^2(E_\delta)$. Now, let us treat the first integral as follows:

$$\begin{aligned} \int_{E_\delta} \left(\left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon - \tilde{\Sigma} \right) \cdot \nabla (u_\epsilon - u) \right) (x) dx &= \int_{\Omega} \left(\left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon - \tilde{\Sigma} \right) \cdot \nabla (\beta_\eta \circ (u_\epsilon - u)) \right) (x) dx \\ &= - \int_{\Omega} \left(\left(\text{div} \left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon \right) \right) (\beta_\eta \circ (u_\epsilon - u)) \right) (x) dx \\ &\quad - \int_{\Omega} \left(\tilde{\Sigma} \cdot \nabla (\beta_\eta \circ (u_\epsilon - u)) \right) (x) dx. \end{aligned}$$

On the one hand, $\beta_\eta \circ (u_\epsilon - u)$ converges weakly in $W_0^{1,p}(\Omega)$, hence

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \left(\tilde{\Sigma} \cdot \nabla (\beta_\eta \circ (u_\epsilon - u)) \right) (x) dx = 0.$$

On the other hand, by point 3, there exists $C > 0$ such that, for all $\epsilon > 0$,

$$\left| - \int_{\Omega} \left(\text{div} \left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon \right) (\beta_\eta \circ (u_\epsilon - u)) \right) (x) dx \right| \leq \eta \int_{\Omega} \left| \text{div} \left(\frac{\epsilon}{p} \nabla u_\epsilon + \Sigma_\epsilon \right) (x) \right| dx \leq C\eta.$$

Consequently, one has that

$$\limsup_{\epsilon \rightarrow 0} \int_{E_\delta} \left((\Sigma_\epsilon - \tilde{\Sigma}) \cdot \nabla(u_\epsilon - u) \right) (x) dx \leq C\eta.$$

Since this is true for all η , one gets that $(\Sigma_\epsilon - \tilde{\Sigma}) \cdot \nabla(u_\epsilon - u)$ converges a.e. to 0 on E_δ . Using Lemma 3.3 below, one has that (∇u_ϵ) converges a.e. to ∇u in E_δ , for all $\delta > 0$, and δ being arbitrary, ∇u_ϵ tends to ∇u almost everywhere in Ω . This implies in particular that Σ_ϵ converges to $\tilde{\Sigma}$ a.e., and, since (Σ_ϵ) is bounded in $L^{p'}(\Omega)$, weakly in $L^{p'}(\Omega)$. Finally, $\Sigma = \tilde{\Sigma}$.

It could be easily derived from Egoroff’s theorem that the convergence of (∇u_ϵ) to ∇u holds also in every $L^q(\Omega)$ spaces, for $q < p$.

To complete the proof, we give Lemma 3.3, which may be found in [4]. □

Lemma 3.3. *Let p be in $(1, \infty)$, and let (X_k) be a sequence of \mathbb{R}^N and $X \in \mathbb{R}^N$, such that*

$$\lim_{k \rightarrow \infty} \left(|X_k|^{p-2} X_k - |X|^{p-2} X \right) (X_k - X) = 0.$$

Then, $\lim_{k \rightarrow \infty} X_k = X$.

3.3. Localisation method

We have proved in Section 3.2 that if (u_ϵ) converges weakly in $W_0^{1,p}(\Omega)$ to a function u which is non identically zero, then this limit is a positive solution of (4). We denote by (H1) this assumption.

Let (H2) be the following condition:

$$K(N, p)^p \lambda |f(x)|^{1-p/N} < 1$$

at every point x in Ω , where $f(x) > 0$.

We prove in this section that (H1) follows from (H2). For that aim, we adapt the isometry-concentration method used by Hebey [10] and Demengel-Hebey [4] (see also [3] and [12]).

In what follows, we assume that (u_ϵ) converges weakly to 0 in $W_0^{1,p}(\Omega)$.

Let $P \in \bar{\Omega}$, $\delta > 0$, and $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, such that:

$$\eta(x) = \begin{cases} 1 & \text{if } x \in B_P(\delta/2) \\ 0 & \text{if } x \notin B_P(\delta). \end{cases}$$

Let us multiply equation (12) by $\eta^p u_\epsilon$ and integrate over Ω . One obtains

$$\frac{\epsilon}{p} \int_\Omega \nabla u_\epsilon \cdot \nabla (\eta^p u_\epsilon) + \int_\Omega \left(|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla (\eta^p u_\epsilon) + a(x) |u_\epsilon|^p \eta^p \right) = \mu_\epsilon \int_\Omega f(x) |u_\epsilon|^{p^*} \eta^p. \tag{14}$$

On the one hand,

$$\frac{\epsilon}{p} \int_\Omega \nabla u_\epsilon \cdot \nabla (\eta^p u_\epsilon) = \frac{\epsilon}{p} \int_\Omega \eta^p |\nabla u_\epsilon|^2 + \epsilon \int_\Omega \eta^{p-1} u_\epsilon \nabla u_\epsilon \cdot \nabla \eta = o(1) \quad (\epsilon \rightarrow 0).$$

On the other hand,

$$\int_\Omega a(x) |u_\epsilon|^p \eta^p = o(1) \quad (\epsilon \rightarrow 0)$$

since $u_\epsilon \rightarrow 0$ in $L^p(\Omega)$. Let us now treat the second term in the left-hand side of (14):

$$\int_{\Omega} |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla (\eta^p u_\epsilon) = \int_{\Omega} \eta^p |\nabla u_\epsilon|^p + p \int_{\Omega} \eta^{p-1} |\nabla u_\epsilon|^{p-2} u_\epsilon \nabla u_\epsilon \cdot \nabla \eta. \tag{15}$$

First, one has

$$\begin{aligned} \left| p \int_{\Omega} \eta^{p-1} u_\epsilon |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \cdot \nabla \eta \right| &\leq \|\eta\|_{\infty}^{p-1} \|\nabla \eta\|_{\infty} \int_{\Omega} u_\epsilon |\nabla u_\epsilon|^{p-1} \\ &\leq \|\eta\|_{\infty}^{p-1} \|\nabla \eta\|_{\infty} \left(\int_{\Omega} |u_\epsilon|^p \right)^{1/p} \left(\int_{\Omega} |\nabla u_\epsilon|^p \right)^{1-1/p} = o(1) \quad (\epsilon \rightarrow 0). \end{aligned} \tag{16}$$

(Here, K denotes a generic positive constant.)

In addition,

$$\int_{\Omega} |\nabla (\eta u_\epsilon)|^p = \int_{\Omega} |\eta \nabla u_\epsilon + u_\epsilon \nabla \eta|^p. \tag{17}$$

Taking $X = u_\epsilon \nabla \eta$ and $Y = \eta \nabla u_\epsilon$ in the following inequality, valid for vectors in \mathbb{R}^N :

$$||X + Y|^p - |Y|^p| \leq p (|X|^{p-1} + |Y|^{p-1}) |X|$$

one gets

$$p \int_{\Omega} |X|^p = p \int_{\Omega} |u_\epsilon|^p |\nabla \eta|^p = o(1) \quad (\epsilon \rightarrow 0) \tag{18}$$

$$p \int_{\Omega} |Y|^{p-1} |X| = p \int_{\Omega} \eta^{p-1} u_\epsilon |\nabla \eta| |\nabla u_\epsilon|^{p-1} = o(1) \quad (\epsilon \rightarrow 0) \tag{19}$$

by (16). Using (15–18), and (19), one finally gets

$$\int_{\Omega} |\nabla (\eta u_\epsilon)|^p + o(1) = \int_{\Omega} \mu_\epsilon f(x) |u_\epsilon|^{p^*} \eta^p.$$

Assume first that $f(P) < 0$. Then, by choosing δ small enough, one has that f is negative on $B_P(\delta)$. Hence

$$\overline{\lim}_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla (\eta u_\epsilon)|^p = 0.$$

Assume now that $f(P) \geq 0$. Writing that

$$\begin{aligned} \mu_\epsilon \int_{\Omega} f(x) u_\epsilon^{p^*} \eta^p &\leq \mu_\epsilon \sup_{B_P(\delta)} |f|^{p/p^*} \int_{\Omega} |\eta u_\epsilon|^p |f(x)|^{1-p/p^*} |u_\epsilon|^{p^*-p} \\ &\leq \mu_\epsilon \sup_{B_P(\delta)} |f|^{p/p^*} \left(\int_{\Omega} |\eta u_\epsilon|^{p^*} \right)^{p/p^*} \left(\int_{B_P(\delta)} |f(x)| |u_\epsilon|^{p^*} \right)^{1-p/p^*} \end{aligned}$$

and using the definition of $K(N, p)$, one has that

$$\left(\int_{\Omega} |\eta u_\epsilon|^{p^*} \right) \leq K(N, p)^p \int_{\Omega} |\nabla (\eta u_\epsilon)|^p.$$

Thus,

$$\mu_\epsilon \int_\Omega f(x)|u_\epsilon|^{p^*} \eta^p \leq \mu_\epsilon \sup_{B_P(\delta)} |f|^{p/p^*} K(N,p)^p \left(\int_{B_P(\delta)} |f(x)||u_\epsilon|^{p^*} \right)^{1-p/p^*} \int_\Omega |\nabla(\eta u_\epsilon)|^p.$$

Now, if $f(P) = 0$, by choosing δ small enough we have

$$\mu_\epsilon \sup_{B_P(\delta)} |f|^{p/p^*} K(N,p)^p \left(\int_{B_P(\delta)} |f(x)||u_\epsilon|^{p^*} \right)^{1-p/p^*} < 1.$$

Hence,

$$\overline{\lim}_{\epsilon \rightarrow 0} \int_\Omega |\nabla(\eta u_\epsilon)|^p = 0.$$

In the same manner, if $f(P) > 0$, one can choose δ small enough so that f is positive in $B_P(\delta)$. Now, assuming that

$$\lambda K(N,p)^p f(P)^{1-N/p} \overline{\lim}_{\epsilon \rightarrow 0} \left(\int_{B_P(\delta)} |f(x)||u_\epsilon|^{p^*} \right)^{1-p/p^*} < 1$$

once more we get

$$\overline{\lim}_{\epsilon \rightarrow 0} \int_\Omega |\nabla(\eta u_\epsilon)|^p = 0.$$

We have obtained the following result

Lemma 3.4. *Assume that (H1) does not hold and that, for every point P in $\overline{\Omega}$ such that $f(P) > 0$, there exists some real $\delta_P > 0$ satisfying*

$$\lambda K(N,p)^p f(P)^{1-p/N} \overline{\lim}_{\epsilon \rightarrow 0} \left(\int_{B_P(\delta_P)} |f(x)||u_\epsilon|^{p^*} \right)^{1-p/p^*} < 1.$$

Then,

$$\lim_{\epsilon \rightarrow 0} \int_\Omega |\nabla(\eta_P u_\epsilon)|^p = 0$$

where η_P denotes the cut-off function defined at the beginning of the section.

We can now prove that (H1) follows from (H2). According to the assumptions of Lemma 3.4, for all $x \in \overline{\Omega}$, there exists $\delta_x > 0$, and a cut-off function η_x , verifying

$$\lim_{\epsilon \rightarrow 0} \int_\Omega |\nabla(\eta_x u_\epsilon)|^p = 0.$$

By the compactness of $\overline{\Omega}$, one can find a finite number of points $(x_i)_{1 \leq i \leq k}$ and reals $(\delta_{x_i})_{1 \leq i \leq k}$ such that

$$\overline{\Omega} = \bigcup_{i=1}^k B_{x_i}(\delta_{x_i}).$$

By convexity, one gets

$$\int_{\Omega} |\nabla u_{\epsilon}|^p \leq k^{p-1} \sum_{i=1}^k \int_{\Omega} |\nabla (\eta_{x_i} u_{\epsilon})|^p$$

thus,

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla u_{\epsilon}|^p = 0$$

which contradicts $u_{\epsilon} \in \mathcal{H}_{p^*}$, since $\int_{\Omega} f|u_{\epsilon}|^{p^*} = 1, \forall \epsilon > 0$.

Consequently, there exists $P \in \overline{\Omega}$, such that $f(P) > 0$, which verifies

$$\forall \delta > 0, \quad K(N, p)^p \lambda f(P)^{p/p^*} \overline{\lim}_{\epsilon \rightarrow 0} \left(\int_{B_P(\delta)} |f(x)| |u_{\epsilon}|^{p^*} \right)^{1-p/p^*} \geq 1.$$

Furthermore, one can choose δ small enough in order to have

$$1 \geq \int_{B_P(\delta)} |f(x)| |u_{\epsilon}|^{p^*} + \int_{f \leq 0} f(x) |u_{\epsilon}|^{p^*}.$$

The previous computations proves that (u_{ϵ}) converges strongly in L^{p^*} in the neighbourhood of every point where f is nonnegative, and then, one obtains that

$$\overline{\lim}_{\epsilon \rightarrow 0} \int_{B_P(\delta)} |f(x)| |u_{\epsilon}|^{p^*} \leq 1.$$

This contradicts (H2), and then Lemma 3.4 implies that the limit u cannot be identically zero. To complete the proof of Theorem 2.2, it remains to show that (a subsequence of) (u_{ϵ}) converges strongly to a solution u .

Proposition 3.6. *Up to a subsequence, (u_{ϵ}) converges strongly to u in $W_0^{1,p}(\Omega)$ as ϵ tends to 0.*

Proof. We have proved before that every subsequence of (u_{ϵ}) which converges weakly in $W_0^{1,p}(\Omega)$ has a limit u which is positive in Ω . Let us show that, in fact, the convergence is strong. For that aim, it is sufficient to show that the L^p -norm of the gradient (∇u_{ϵ}) converge to $\|\nabla u\|_{L^p}$. According to Proposition 3.5, u is a solution of

$$-\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) + a(x) |u|^{p-2} u = \lambda f(x) |u|^{p^*-2} u$$

then, multiplying by u and using Green's formula, one gets

$$\int_{\Omega} (|\nabla u|^p + a(x) |u|^p) = \lambda \int_{\Omega} f(x) |u|^{p^*}.$$

First, let us prove that $\lambda = J_0(u)$.

Set $v = \left(\int f(x) |u|^{p^*} \right)^{-1/p^*} u$. The equation above writes

$$\left(\int_{\Omega} f(x) |u|^{p^*} \right)^{p/p^*} \int_{\Omega} (|\nabla v|^p + a(x) |v|^p) = \lambda \int_{\Omega} f(x) |u|^{p^*}.$$

Since $v \in \mathcal{W}_{p^*}$, $J_0(v) \geq \lambda$, and then

$$\int_{\Omega} f(x)|u|^{p^*} \geq 1.$$

Furthermore, since $u_{\epsilon} \rightarrow u$ in $W_0^{1,p}(\Omega)$, one can write

$$\lambda \int_{\Omega} f(x)|u|^{p^*} = \int_{\Omega} (|\nabla u|^p + a(x)|u|^p) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \int_{\Omega} |\nabla u_{\epsilon}|^2 + \int_{\Omega} (|\nabla u_{\epsilon}|^p + a(x)|u_{\epsilon}|^p) \leq \liminf_{\epsilon \rightarrow 0} \lambda_{\epsilon} \leq \lambda$$

which gives the result. Actually, $J_{\epsilon}(u)$ tends to $J_0(u)$, then

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla u_{\epsilon}|^p = \int_{\Omega} |\nabla u|^p$$

and the proof of Theorem 2.2 is completed. □

4. A CONTINUITY RESULT

In this section, we are interested in the continuity of the set of the solutions for equation (4). Here, we prove that $u \equiv 0$ is isolated among the solutions of (4), in some sense which will be precised later. First, we give a result concerning the isolation of 0 in a strong sense.

Lemma 4.1. *Let $u \in W_0^{1,p}(\Omega)$ be a solution of (4).*

- *If $\int_{\Omega} f(x)|u|^{p^*} = 0$, then $u \equiv 0$.*
- *If this integral is not zero, then we have $\int_{\Omega} f(x)|u|^{p^*} \geq \lambda^{N/p}$.*

Proof. Suppose that $\int_{\Omega} f(x)|u|^{p^*} = 0$, multiply by u and integrate over Ω , to obtain

$$J(u) = \int_{\Omega} (|\nabla u|^p + a(x)|u|^p) = 0$$

thus $u \equiv 0$ by coercivity of L .

Suppose now that $u \not\equiv 0$, so that $\int_{\Omega} f(x)|u|^{p^*} > 0$. Defining

$$w = \left(\int_{\Omega} f(x)|u|^{p^*} \right)^{-1/p^*} u$$

one has, by construction, that $w \in \mathcal{W}_{p^*}(G)$, hence

$$\lambda \leq \left(\int_{\Omega} f(x)|u|^{p^*} \right)^{-p/p^*} J(u) \leq \left(\int_{\Omega} f(x)|u|^{p^*} \right)^{1-p/p^*}$$

and actually

$$\int_{\Omega} f(x)|u|^{p^*} \geq \lambda^{N/p}.$$

□

Remark. Here, one can see that λ represents a minimal energy level for non zero solutions of (4).

Now, we prove a stronger result which prevents, under some assumption on the energy level of considered solutions, from concentration phenomena that could happen if the sequence converges weakly but not strongly in $W_0^{1,p}(\Omega)$.

In what follows, we assume that the existence's condition of Theorem 2.2 holds, that is on every point $x \in \overline{\Omega}$ where $f(x) > 0$,

$$\lambda f(x)^{1-p/N} K(N,p)^p < 1.$$

Theorem 4.1. *Let (u_n) be a bounded sequence (in $W_0^{1,p}(\Omega)$) of solutions of (4) which are non identically zero. We assume that*

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x) |u_n|^{p^*} = \lambda^{N/p}.$$

Then, if (u_n) converges weakly to u , u is a non identically zero solution of (4) (In fact, (u_n) converges strongly to u in $W^{1,p}(\Omega)$.)

Remark. This result can be extended to positive and nodal solutions, when the data Ω , a and f present some symmetries.

Proof. The fact that u is a solution of (4) immediately follows from Section 3.2. It is sufficient to show that u is not the trivial function.

For that aim, we use the localisation method in Section 3.3. Let $P \in \overline{\Omega}$, δ be a positive real, and η be the cut-off function defined in Section 3.3. Following the proof of Lemma 3.4, and assuming that $(u_n) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, one obtains that

$$\int_{\Omega} |\nabla(\eta u_n)|^p + o(1) = \int_{\Omega} f(x) |u_n|^{p^*} \eta^p \quad (n \rightarrow \infty).$$

- If $f(P) < 0$, one has, for δ small enough, that $\int_{\Omega} |\nabla(\eta u_n)|^p$ tends to 0 as n tends to ∞ .
- If $f(P) \geq 0$, one has the following estimate:

$$\int_{\Omega} f(x) |u_n|^{p^*} \eta^p \leq \sup_{B_P(\delta)} |f|^{p/p^*} K(N,p)^p \left(\int_{B_P(\delta)} |f(x) |u_n|^{p^*} \right)^{1-p/p^*} \int_{\Omega} |\nabla(\eta u_n)|^p. \quad (20)$$

Hence,

- * for $f(P) = 0$, the conclusion is the same;
- * for $f(P) > 0$, one assumes that

$$K(N,p)^p f(P)^{1-p/N} \overline{\lim}_{n \rightarrow \infty} \left(\int_{B_P(\delta)} |f(x) |u_n|^{p^*} \right)^{1-p/p^*} < 1$$

which yields that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla(\eta u_n)|^p = 0.$$

Then, if (20) holds at every point P where $f(P) > 0$, one has by compactness that $u_n \rightarrow 0$ in $W_0^{1,p}(\Omega)$, which is a contradiction with the estimate

$$\int_{\Omega} f(x)|u_n|^{p^*} \geq \lambda^{N/p} > 0.$$

It follows that there exists some $P \in \overline{\Omega}$ such that $F(P) > 0$ and

$$K(N, p)^p f(P)^{1-p/N} \overline{\lim}_{n \rightarrow \infty} \left(\int_{B_P(\delta)} |f(x)||u_n|^{p^*} \right)^{p/N} \geq 1$$

and then, by the condition in Theorem 2.2,

$$\overline{\lim}_{n \rightarrow \infty} \left(\int_{B_P(\delta)} |f(x)||u_n|^{p^*} \right)^{p/N} > \lambda. \tag{21}$$

Noting that for δ small enough,

$$\int_{\Omega} f(x)|u_n|^{p^*} \geq \int_{B_P(\delta)} |f(x)||u_n|^{p^*} + \int_{f \leq 0} f(x)|u_n|^{p^*}$$

and using the fact that the second integral of the right-hand side tends to 0, and the left-hand side to $\lambda^{N/p}$, one obtains

$$\lambda \geq \overline{\lim}_{n \rightarrow \infty} \left(\int_{B_P(\delta)} |f(x)||u_n|^{p^*} \right)^{p/N}$$

which is a contradiction with (21).

Finally, with the same arguments as in Proposition 3.6, one obtains the strong convergence of the sequence (u_n) to u , which completes the proof. \square

5. THE CASE WHERE Ω IS INVARIANT UNDER SYMMETRIES

This section is devoted to the case where the domain Ω is invariant under the action of some subgroup G of the orthogonal group $O_N(\mathbb{R})$. We give the analogous stability results for positive and nodal extremal functions (a function is said to be nodal if it changes sign).

Let G be a subgroup of $O_N(\mathbb{R})$ and σ be an involution of $O_N(\mathbb{R})$, $\sigma \neq Id$. We assume that G is compact (if it is not the case, one may take the closure \overline{G} of G , since every G -invariant function is \overline{G} -invariant) Let us also define the sets

$$\mathcal{W}_{p^*}(G) = \left\{ v \in W_0^{1,p}(\Omega); \forall \tau \in G, v \circ \tau = v, \int_{\Omega} f(x)|v|^{p^*} = 1 \right\},$$

$$\mathcal{W}_{p^*}^{\sigma}(G) = \left\{ v \in W_0^{1,p}(\Omega); \forall \tau \in G, v \circ \tau = v, v \circ \sigma = -v, \int_{\Omega} f(x)|v|^{p^*} = 1 \right\},$$

$$\mathcal{W}_{p^*}(G) = \mathcal{H}_{p^*}(G) \cap W_0^{1,p}(\Omega), \quad \mathcal{W}_{p^*}^{\sigma}(G) = \mathcal{H}_{p^*}^{\sigma}(G) \cap W_0^{1,p}(\Omega),$$

and, for $\epsilon \geq 0$, the quantities

$$\lambda_\epsilon(G) = \inf_{v \in \mathcal{H}_{p^*}^\sigma(G)} J_\epsilon(v), \quad (22)$$

$$\lambda_\epsilon^\sigma(G) = \inf_{v \in \mathcal{H}_{p^*}^\sigma(G)} J_\epsilon(v). \quad (23)$$

Finally, one defines

$$\lambda(G) = \inf_{v \in \mathcal{W}_{p^*}(G)} J_0(v), \quad (24)$$

$$\lambda^\sigma(G) = \inf_{v \in \mathcal{W}_{p^*}^\sigma(G)} J_0(v). \quad (25)$$

Arguing as in the proof of Lemma 3.1, one can show that $\lambda_0(G) = \lambda(G)$ and $\lambda_0^\sigma(G) = \lambda^\sigma(G)$.

Then, the results are the following:

Theorem 5.1. *We assume that Ω , a , and f are G -invariant and that, for every point $x \in \overline{\Omega}$ such that $f(x) > 0$,*

$$\lambda(G)f(x)^{1-\frac{p}{N}}K(N,p)^p < (\text{Card}O_G(x))^{\frac{p}{N}}. \quad (26)$$

Then,

1) *for every $\epsilon > 0$, there exists a nonnegative function $u_\epsilon \in \mathcal{H}_{p^*}(G)$ which realizes the infimum in (22). In addition, u_ϵ is a weak solution of the following equation:*

$$-\frac{\epsilon}{p}\Delta u_\epsilon - \text{div}\left(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\right) + a(x)|u_\epsilon|^{p-2}u_\epsilon = \mu_\epsilon(G)f(x)|u_\epsilon|^{p^*-2}u_\epsilon \quad (27)$$

where

$$\mu_\epsilon(G) = \lambda_\epsilon(G) + \epsilon\left(\frac{1}{p} - \frac{1}{2}\right)\int_\Omega |\nabla u_\epsilon|^2.$$

2) *Up to a subsequence, (u_ϵ) converges strongly in $W_0^{1,p}(\Omega)$ to a positive and G -invariant solution u of equation (4). Furthermore, u is an extremal function for (24).*

Theorem 5.2. *We assume that Ω , a , and f are H -invariant. Suppose that*

$$\exists x \in \overline{\Omega}; f(x) > 0 \text{ and } \sigma(O_G(x)) \cap O_G(x) = \emptyset \quad (28)$$

and that, for every $x \in \overline{\Omega}$, verifying $f(x) > 0$,

$$\lambda^\sigma(G)f(x)^{1-\frac{p}{N}}K(N,p)^p < (\text{Card}O_H(x))^{\frac{p}{N}}. \quad (29)$$

Then, we have

1) *for every $\epsilon > 0$, there exists a function $u_\epsilon \in \mathcal{H}_{p^*}^\sigma(G)$ which realizes the infimum in (23). In addition, u_ϵ is a weak solution of the equation:*

$$-\epsilon\Delta u_\epsilon - \text{div}\left(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\right) + a(x)|u_\epsilon|^{p-2}u_\epsilon = \mu_\epsilon^\sigma(G)f(x)|u_\epsilon|^{p^*-2}u_\epsilon \quad (30)$$

where

$$\mu_\epsilon^\sigma(G) = \lambda_\epsilon^\sigma(G) + \epsilon \left(\frac{1}{p} - \frac{1}{2} \right) \int_\Omega |\nabla u_\epsilon|^2.$$

2) Up to a subsequence, (u_ϵ) converges strongly in $W_0^{1,p}(\Omega)$ to a G -invariant and σ -antisymmetrical solution u of equation (4). Furthermore, u is an extremal function for (25).

The assumption (28) provides that the set $\mathcal{W}_{p^*}^\sigma(G)$ is not empty. For Theorems 5.1 and 5.2, all the arguments developed in Section 3 remain valid, under the condition that there exists a symmetrization of functions for each problem, which leaves the functionals invariant. The symmetrizations are the following: for a function v , one takes

$$v_G(x) = \int_G v \circ \tau d\mu(\tau)$$

where μ is the Haar measure of G . (It exists since G is locally compact.) For the problem of nodal solutions, one takes

$$v_G^\sigma(x) = v_G \circ \sigma(x) - v_G(x)$$

with v_G defined as above.

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