

## DYNAMIC STABILIZATION OF SYSTEMS VIA DECOUPLING TECHNIQUES

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**Abstract.** We give sufficient conditions which allow the study of the exponential stability of systems closely related to the linear thermoelasticity systems by a decoupling technique. Our approach is based on the multipliers technique and our result generalizes (from the exponential stability point of view) the earlier one obtained by Henry *et al.*

**Résumé.** Nous donnons des conditions suffisantes qui permettent l'étude de la stabilité exponentielle de systèmes similaires à celui de la thermoélasticité en utilisant une technique de découplage. Notre approche est basée sur la technique des multiplicateurs et notre résultat généralise (du point de vue de la stabilité exponentielle) celui obtenu par Henry *et al.*

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## 1. INTRODUCTION

In stabilizability theory, a significant number of papers have appeared dealing with the stabilization by means of “static” stabilizers of the system

$$\begin{aligned} u' &= \mathcal{A}u + \mathcal{B}v \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

( $\mathcal{A}$  and  $\mathcal{B}$  being unbounded operators acting on Hilbert spaces). It consists in finding an operator  $K$  such that if the control function  $v$  is given in the feedback form

$$v = \mathcal{K}u$$

the energy

$$\mathcal{E}(t) = \frac{1}{2} \| u(t) \|^2$$

associated to the system (1.1) decays to zero for an arbitrary initial data  $u_0$  (this is the *asymptotic stability* of the system) or, in much of the cases, decays to zero uniformly in the initial data (this is the *exponential*

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*stability*). Motivated by works in automatic field, we are interested by a different kind of feedback law. It consists in coupling the system (1.1) with a second system

$$v' = -\mathcal{B}^*u + Cv \quad (1.2)$$

describing another dynamic and to find  $C$  such that the coupled system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (1.3)$$

is asymptotically or exponentially stable. This is what we will mean by *dynamic stabilization* and we call the pair  $(\mathcal{B}, C)$  a dynamic stabilizer. The concept of *dynamic stabilization* has been introduced by the automaticians for systems governed by ordinary differential equations (see for instance [11, 12]).

This approach has largely been discussed by Russell for elastic mechanisms ([25] and [26]). In this last paper, what is defined as *direct damping* by Russell corresponds to “static” stabilizers while *indirect damping* corresponds to “dynamic” stabilizers.

We are interested, in this paper, by the dynamic stabilization of elastic structures. So we consider systems of the form

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & B \\ 0 & -B^* & -C \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{in } D(A^{1/2}) \times H \times G \quad (1.4)$$

where  $A$  is a positive self-adjoint operator on the Hilbert space  $H$ ,  $C$  is a positive self-adjoint operator on the Hilbert space  $G$  and  $B$  is a linear operator acting from  $G$  to  $H$ . This kind of indirect damping is referred by Russell (see [26]) as the *velocity coupled dissipator*. Now, while Russell looks for conditions insuring the analyticity of the associated semigroup, we would like to describe classes of operators which lead to the exponential stability of this semigroup.

Some works in this direction may be found in [1–3]. In the first two papers, the coupling operator  $B$  was assumed to be boundedly invertible. This restriction does not allow the use, in case of systems described by partial differential equations, of a coupling on subdomains of the domain in consideration. So, one of the mathematical motivations of this work is to remove this restriction.

The second question of interest for us is to clarify the relation between static and dynamic stabilization (or between direct and indirect damping). Considering the system (1.4) in the finite dimensional case (*i.e.*:  $A, B$  and  $C$  are matrices), direct computations prove that its exponential stability amounts to that of the system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -BC^{-1}B^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } D(A^{1/2}) \times H. \quad (1.5)$$

In other words, the dynamic stabilizer  $C$  is in correspondance with the static stabilizer  $-BC^{-1}B^*$ . It is then natural to ask if such an equivalence remains true in infinite dimensional situations. The answer is no (and this is not surprising), as one can see in the following example. Let's assume that  $H = G$ , that  $A$  is a boundedly invertible positive self-adjoint operator with compact resolvent and that  $C = B = I$  (the identity operator in  $H$ ). In this case,  $BC^{-1}B^* = I$  and it is well-known that the energy of the system (1.5) decays exponentially while the one of the system (1.4) does not (see for instance [1] or Sect. 3 of this paper). However, Henry *et al.* [17] provided sufficient conditions on the unbounded operators  $A, B$  and  $C$  insuring this equivalence. This decoupling method is applied successfully in [27, 29] and the difficulties related to the noncommutativity of the operators due to the boundary conditions is overcome by means of hidden regularity results. It is our aim, in this work, to give, using a method based on the multiplier technique, a wider class of operators than the one in [17] for which this decoupling technique is valid. We will also show, with the help of an example, that the

conditions we propose are far to be optimal. Results of this kind, that do not cover ours, can also be found in Engel [13], Liu and Yong [20].

The plan of the paper is the following. In the second section, we study the well-posedness of (1.4). The second section is devoted to the statement of our main result and to the discussion of our assumptions. In the third section, we prove our main result. Applications to systems governed by partial differential equations are given in the fourth and last section.

## 2. PRELIMINARIES

In the sequel,  $H$  and  $G$  will be two separable Hilbert spaces. We consider the operator on  $X = H \times G$  defined in matrix form by

$$L = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & \mathcal{C} \end{pmatrix} \tag{2.1}$$

$$D(L) = (D(\mathcal{A}) \cap D(\mathcal{B}^*)) \times (D(\mathcal{B}) \cap D(\mathcal{C})). \tag{2.2}$$

We will assume that  $\mathcal{A}$  and  $\mathcal{C}$  are m-dissipative operators on  $H$  and  $G$  respectively and that  $\mathcal{B}$  is a densely defined closable operator from  $G$  to  $H$ .

**Proposition 2.1.** Assume that

$$\overline{D(L)} = X \tag{2.3}$$

$$\overline{(D(\mathcal{A}^*) \cap D(\mathcal{B}^*)) \times (D(\mathcal{B}) \cap D(\mathcal{C}^*))} = X. \tag{2.4}$$

Then  $L$  is closable and its closure is m-dissipative.

*Proof.* It is readily verified that  $L$  is dissipative and, since it is densely defined, it is closable. Now,  $L^*$  is an extension of the operator

$$O = \begin{pmatrix} \mathcal{A}^* & -\overline{\mathcal{B}} \\ \mathcal{B}^* & \mathcal{C}^* \end{pmatrix}$$

$$D(O) = (D(\mathcal{A}^*) \cap D(\mathcal{B}^*)) \times (D(\mathcal{B}) \cap D(\mathcal{C}^*)).$$

Since  $O$  is densely defined and dissipative, it follows that  $L^*$  is also densely defined and dissipative. So the conclusion follows from the Lumer-Phillip theorem. □

**Remark 1.** For the operator  $L$ , Engel [13] has another kind of assumptions for well-posedness, namely: if  $\mathcal{A}$  and  $\mathcal{C}$  are dissipative and boundedly invertible and

$$\operatorname{Re}(\mathcal{B}\mathcal{B}^*u, \mathcal{A}^{-1}u) \leq 0 \quad \text{for all } u \in D(\mathcal{B}\mathcal{B}^*)$$

$$\operatorname{Re}(\mathcal{B}^*\mathcal{B}v, \mathcal{C}^{-1}v) \leq 0 \quad \text{for all } v \in D(\mathcal{B}\mathcal{B}^*)$$

then  $L$  defined by (2.1) and (2.2) is densely defined, closable and its closure is m-dissipative. □

In some cases, the closure of  $L$  and its adjoint operator can be explicitly computed. Indeed, one has

**Proposition 2.2.** [4] In the situation of Proposition 2.1, assume moreover that  $\mathcal{C}$  is boundedly invertible, that  $\mathcal{B} \in \mathcal{L}(D(\mathcal{C}), H)$  and that  $\mathcal{C}^{-1}\mathcal{B}^*$  extends to a bounded linear operator (we denote its closure with the same symbol). Then  $L$  is closed if and only if  $\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*$  is closed and one has

$$\begin{aligned} \overline{L} &= \begin{pmatrix} I & \mathcal{B}\mathcal{C}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \overline{\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*} & 0 \\ 0 & \mathcal{C} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathcal{C}^{-1}\mathcal{B}^* & I \end{pmatrix} \\ D(\overline{L}) &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in D\left(\overline{\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*}\right) \times G, \mathcal{C}^{-1}\mathcal{B}^*u + v \in D(\mathcal{C}) \right\}. \end{aligned}$$

For the adjoint operator, one can state

**Proposition 2.3.** Under the assumptions of Proposition 2.2, the adjoint operator is given by the relation

$$L^* = \begin{pmatrix} I & -\overline{\mathcal{B}}(\mathcal{C}^*)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} (\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^* & 0 \\ 0 & \mathcal{C}^* \end{pmatrix} \begin{pmatrix} I & 0 \\ (\mathcal{B}\mathcal{C}^{-1})^* & I \end{pmatrix} \tag{2.5}$$

$$D(L^*) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in D((\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^*)^*) \times G, (\mathcal{B}\mathcal{C}^{-1})^* u + v \in D(\mathcal{C}) \right\}. \tag{2.6}$$

*Proof.* The operator  $L$  can be factored in the Frobenius-Schur sense:

$$L = \begin{pmatrix} I & \mathcal{B}\mathcal{C}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* & 0 \\ 0 & \mathcal{C} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\mathcal{C}^{-1}\mathcal{B}^* & I \end{pmatrix} = PSQ. \tag{2.7}$$

Now, since  $P, Q \in L(X)$  and are boundedly invertible, it amounts to show that

$$(PSQ)^* = Q^*S^*P^*.$$

It is sufficient to show that

$$(SQ)^* = Q^*S^* \tag{2.8}$$

since it is classical that for any bounded operator  $P$  the relation  $(PS)^* = S^*P^*$ . Note first that  $D(SQ) = D(S)$ . This implies that that

$$f \in D((SQ)^*) \iff f \in D(S^*)$$

and, then (2.8) is clearly true and (2.5) follows. □

Note that this last proposition provides a second proof for Proposition 2.2 since  $(L^*)^* = \overline{L}$ .

### 3. STATEMENT OF THE MAIN RESULT

In this section, we will assume that  $A : D(A) \subset H_1 \rightarrow H_1$  where  $H_1$  is a Hilbert space, and  $C : D(C) \subset G \rightarrow G$  are self adjoint strictly positive operators with compact inverses, while  $B : D(B) \subset G \rightarrow H_1$  is a closable operator and we consider system (1.4) that we recall here:

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & B \\ 0 & -B^* & -C \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ in } D(A^{1/2}) \times H_1 \times G$$

completed with initial conditions. With respect to the notations adopted in the previous section, we have

$$H = D(A^{1/2}) \times H_1, \\ \mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \text{ and } \mathcal{C} = -C.$$

We introduce the operator

$$M = \mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* = \begin{pmatrix} 0 & I \\ -A & -\mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* \end{pmatrix}. \tag{3.1}$$

Let's describe now briefly the result in [17] we have already mentioned. The assumptions were

$$D(C^{\frac{1}{2}}) \subset D(B), \quad D(A^{\frac{1}{2}}) \subset D(B^*) \tag{3.2}$$

$$A^{\frac{1}{2}}BC^{-1} \text{ is densely defined and bounded from } G \text{ to } H_1. \tag{3.3}$$

In this situation, it can be deduced from the results in [17] that the semigroup  $e^{Lt}$  is exponentially stable if and only if the semigroup  $e^{Mt}$  is since the equivalence from the asymptotic stability point of view is clearly true because for the two semigroups, it amounts to the condition

$$\begin{cases} w'' = -Aw & \text{in } D(A^{\frac{1}{2}}) \times H \\ B^*w' = 0 & \text{in } H \end{cases} \implies w \equiv 0 \tag{3.4}$$

(see [17]). This result was derived from the fact that  $e^{Lt} - e^{Mt}$  is compact as the cited authors proved.

To generalize this result, we use a different characterization of the exponential stability of a semigroup. Indeed, the semigroup  $e^{Lt}$  is exponentially stable if and only if there exists a nonnegative self-adjoint operator  $P \in L(X)$  such that

$$PL + L^*P = -I \quad \text{on } D(L) \tag{3.5}$$

or, equivalently

$$\operatorname{Re}(PLx, x) = -\|x\|^2 \quad \forall x \in D(L) \tag{3.6}$$

$(.,.)$  denoting the inner product of  $X$  and  $\|.\|$  the associated norm.  $P$  is the Liapunov operator and it is unique when it exists (see *e.g.* [28]). It is in fact given by

$$P = \int_0^\infty e^{L^*t} e^{Lt} dt$$

when  $e^{Lt}$  is exponentially stable.

Let's now state our main result. Let  $B : D(B) \subset Y \rightarrow X$  be a densely defined linear operator such that

$$D(B) \supset D(C) \tag{3.7}$$

$$M \text{ is densely defined and closed} \tag{3.8}$$

$$A^{\frac{1}{2}}BC^{-\frac{3}{2}} \in L(Y, X). \tag{3.9}$$

Then:

**Theorem 3.1.** *Under the assumptions (3.7, 3.8) and (3.9),  $(e^{Lt})$  is exponentially stable if and only if  $(e^{Mt})$  is.*

The example we treat now will show that Theorem 3.1 is far to be optimal. Let  $\alpha, \beta$  be in  $\mathbb{R}^+$ ,  $H_1 = G$  and consider the operator

$$L_{\alpha,\beta} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & A^\alpha \\ 0 & -A^\alpha & -A^\beta \end{pmatrix};$$

$$D(L_{\alpha,\beta}) = D(A) \times \left( D(A^\alpha) \cap D(A^{\frac{1}{2}}) \right) \times (D(A^\beta) \cap D(A^\alpha)). \tag{3.10}$$

Let's also define on  $H = D(A^{1/2}) \times H_1$  the operator

$$M_{\alpha,\beta} = \begin{pmatrix} 0 & I \\ -A & -A^{2\alpha-\beta} \end{pmatrix}$$

$$D(M_{\alpha,\beta}) = D(A) \times \left( D(A^{1/2}) \cap D(A^{2\alpha-\beta}) \right).$$

It is not difficult to see that  $M_{\alpha,\beta}$  is closed if and only if  $2\alpha - \beta \leq \frac{1}{2}$ . Then, if moreover  $\alpha \leq \beta$ , applying Proposition 2.2 with  $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ ,  $\mathcal{B} = \begin{pmatrix} 0 \\ A^\alpha \end{pmatrix}$  and  $\mathcal{C} = -A^\beta$ , it appears that  $L_{\alpha,\beta}$  is closed too. On the other hand, if  $\max\left(\frac{1}{2}, \beta\right) \leq \alpha$ , then applying Proposition 2.2 with  $\mathcal{A} = 0$ ,  $\mathcal{B} = \begin{pmatrix} I & 0 \end{pmatrix}$  and  $\mathcal{C} = \begin{pmatrix} 0 & A^\alpha \\ -A^\alpha & -A^\beta \end{pmatrix}$  with  $D(\mathcal{C}) = D(A^\alpha) \times D(A^\alpha)$  ( $\mathcal{C}$  is closed since it is boundedly invertible on  $H_1 \times H_1$ ), it appears that  $\mathcal{A} + \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^* = -A^{\beta-2\alpha+1}$  and that  $L_{\alpha,\beta}$  is closed. It holds also true that  $L_{\alpha,\beta}$  is closed if  $\beta \leq \alpha \leq \frac{1}{2}$  again by applying Proposition 2.2 with  $\mathcal{C} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ ,  $\mathcal{B}^* = \begin{pmatrix} 0 \\ A^\alpha \end{pmatrix}$  and  $\mathcal{A} = -A^\beta$ . In all other cases (*i.e.* if  $\alpha < \beta < 2\alpha - \frac{1}{2}$ ),  $L_{\alpha,\beta}$  is not closed but its closure and its adjoint may be explicitly computed by using the results of the previous section. We will prove in the next section the following result

**Theorem 3.2.** *Assume that  $\alpha, \beta \geq 0$ . Then*

1. *The semigroup of contractions  $e^{L_{\alpha,\beta}t}$  is exponentially stable if and only if*

$$\max(1 - 2\alpha, 2\alpha - 1) \leq \beta \leq 2\alpha. \tag{3.11}$$

2. *The equivalence*

$$e^{L_{\alpha,\beta}t} \text{ exponentially stable} \Leftrightarrow e^{M_{\alpha,\beta}t} \text{ exponentially stable} \tag{3.12}$$

*holds true in (and only in) the  $(\alpha, \beta)$ -region of  $\mathbb{R}^2$*

$$\{1 - 2\alpha \leq \beta\} \cup \{2\alpha < \beta\}. \tag{3.13}$$

**Remark 2.** Note that the second point of this theorem provides also an area of non equivalence from the exponential stability point of view, namely:

$$0 \leq \beta \leq \min(2\alpha, 1 - 2\alpha).$$

With this example in mind, one could easily visualize the areas covered by the different conditions imposed in [17,20] or by Theorem 3.1 to insure (3.12). However, it should be noted that the entries of  $L$  do not commute each other and  $B$  is not assumed to be boundedly invertible in all of these cited results.

Note that Munoz Rivera and Racke [24] considered the system associated to  $L_{\alpha,\beta}$  from the smoothing property point of view. They give the connexion with thermoelastic systems:  $\alpha = \beta = \frac{1}{2}$  lead to a *thermoelastic-plate* system while  $\beta = 1, \alpha = \frac{1}{2}$  lead to the one-dimensional *thermoelastic* system and  $\alpha = 0, \beta = \frac{1}{2}$  correspond to a *viscoelastic* system. They prove that strict inequalities in (3.11) induce smoothness.

#### 4. PROOFS

*Proofs of Theorem 3.2.* Let's set

$$T = \begin{pmatrix} A^{1/2} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

where  $T$  is a bounded invertible operator from  $D(A^{1/2}) \times H_1 \times H_1$  in  $(H_1)^3$ .

$$\tilde{L}_{\alpha,\beta} = TL_{\alpha,\beta}T^{-1} = \begin{pmatrix} 0 & A^{1/2} & 0 \\ -A^{1/2} & 0 & A^\alpha \\ 0 & -A^\alpha & -A^\beta \end{pmatrix} \tag{4.1}$$

$$D(\tilde{L}_{\alpha,\beta}) = D(A^{1/2}) \times (D(A^{1/2}) \cap D(A^\alpha)) \times (D(A^\beta) \cap D(A^\alpha)). \tag{4.2}$$

Clearly,  $e^{\tilde{L}_{\alpha,\beta}t}$  is exponentially stable on  $(H_1)^3$  if and only if  $e^{L_{\alpha,\beta}t}$  is exponentially stable on  $D(A^{1/2}) \times H_1 \times H_1$ . But the operator

$$P_{\alpha,\beta} = \begin{pmatrix} \frac{1}{2}A^{2\alpha-\beta-1} + A^{\beta-2\alpha} + A^{1-\beta-2\alpha} & \frac{1}{2}A^{-1/2} & \frac{1}{2}A^{\alpha-\beta-1/2} - A^{-\alpha-\beta+1/2} \\ \frac{1}{2}A^{-1/2} & A^{\beta-2\alpha} + A^{1-\beta-2\alpha} + A^{-\beta} & A^{-\alpha} \\ \frac{1}{2}A^{\alpha-\beta-1/2} - A^{-\alpha-\beta+1/2} & A^{-\alpha} & \frac{3}{2}A^{-\beta} \end{pmatrix}$$

is self-adjoint on  $(H_1)^3$  and it is bounded if and only if (3.11) is satisfied. Now, direct computations show that  $P_{\alpha,\beta}$  is nonnegative and satisfies the Liapunov equation (3.6). So, if (3.11) is satisfied,  $e^{\tilde{L}_{\alpha,\beta}t}$  (and then  $e^{L_{\alpha,\beta}t}$ ) is exponentially stable.

Conversly, since  $A$  has a compact inverse, it admits a sequence  $(\mu_n)_{n \geq 1}$  of positive eigenvalues such that  $\mu_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . The corresponding family of eigenvectors  $(\phi_n)_{n \geq 1}$  is an orthonormal basis of  $H_1$ . It can then be showed that  $\tilde{L}_{\alpha,\beta}$  admits a sequence of eigenvalues which satisfies the algebraic equations

$$\lambda^3 + \mu_n^\beta \lambda + (\mu_n^{2\alpha} + \mu_n) \lambda + \mu_n^{1+\beta} = 0, \quad n = 1, 2, \dots \tag{4.3}$$

It can be established just as in [1] that if (3.11) is not satisfied, then there exists a sequence  $(\lambda_n)$  of solutions of (4.3) such that  $\text{Re } \lambda_n \rightarrow 0$  when  $n \rightarrow +\infty$ . So, from the Hille-Yosida theorem,  $e^{\tilde{L}_{\alpha,\beta}t}$  is not exponentially stable and this proves the first point.

The second point is a simple consequence of the fact that, in this case,  $e^{M_{\alpha,\beta}t}$  is exponentially stable if and only if (see [9])

$$2\alpha - 1 \leq \beta \leq 2\alpha.$$

□

*Proofs of Theorem 3.1.* An easy computation shows that

$$\text{Re}(LY, Y) = - \| C^{\frac{1}{2}}w \|^2 \tag{4.4}$$

for all  $Y = (u, v, w) \in D(L)$ .

**1/- Sufficiency.** Let's suppose that  $(e^{Mt})$  is exponentially stable. In view of Theorem 3.5, there exists  $P = (P_{ij})_{1 \leq i, j \leq 2} \in L(D(A^{\frac{1}{2}}) \times X)$  such that  $P = P^* > 0$  and

$$PM + M^*P = -I \quad (4.5)$$

where  $I$  is the identity operator in  $D(A^{\frac{1}{2}}) \times X$ . From this last equality, we are going to built a suitable multiplier for system (4.2). Let's set

$$Q = \begin{pmatrix} P & PBC^{-1} \\ C^{-1}B^*P & \frac{1}{2}C^{-1} \end{pmatrix} \quad (4.6)$$

where  $B = \begin{pmatrix} 0 \\ B \end{pmatrix}$ . Clearly,  $Q$  is bounded, positive and selfadjoint on  $H$ . We introduce the following function

$$\rho_\varepsilon(Y) = \|Y\|^2 + \varepsilon(QY, Y) \quad Y \in H.$$

We have

**Lemma 4.1.** For a sufficiently small  $\varepsilon > 0$ , one has

$$\begin{aligned} (i) & a_0 \|Y\|^2 \leq \rho_\varepsilon(Y) \leq a_1 \|Y\|^2, \quad \forall Y \in H \\ (ii) & \end{aligned}$$

$$\rho_\varepsilon(Y(t)) \leq -\exp(-a_2 t) \rho_\varepsilon(Y_0) \quad \forall t \geq 0$$

where  $a_i, i = 0, 1, 2$  are various positive real constants and  $Y$  is the solution of system (4.2). □

*Proof.* (i) It follows immediately from

$$-\|Q\| \|Y\|^2 \leq (QY, Y) \leq \|Q\| \|Y\|^2 \quad \forall Y \in H$$

and by taking  $\varepsilon < 1/\|Q\|$ .

(ii) First, for  $Y(t) = (u(t), v(t), w(t))$  solution of system (1.4) (corresponding to an initial data  $Y_0 \in D(L)$ ), one has

$$\frac{d}{dt} \rho_\varepsilon(Y(t)) = 2 \operatorname{Re}(LY(t), Y(t)) + \varepsilon((QL + L^*Q)Y(t), Y(t)) \quad t > 0.$$

Now, from (4.4)

$$2 \operatorname{Re}(LY(t), Y(t)) = -2 \|C^{\frac{1}{2}}w\|^2.$$

Using the definition of  $M$  (see (3.1)), the definition of  $Q$  (see (4.6)) and identity (4.5), one gets after some computations

$$\begin{aligned} ((QL + L^*Q)Y(t), Y(t)) &= -\|Y\|^2 + 2 \operatorname{Re}(-(A^{\frac{1}{2}}P_{22}BC^{-1}w, A^{\frac{1}{2}}u) \\ &\quad + (AP_{12}BC^{-1}w, v) + \frac{1}{2}(BC^{-1}w, v) + (C^{-1}B^*P_{22}Bw, w)). \end{aligned} \quad (4.7)$$



But (4.5), when developed, amounts to the four relations

$$P_{12}A + P_{21} = I \tag{4.8}$$

$$P_{11} - P_{22} = P_{12}BC^{-1}B^* \tag{4.9}$$

$$AP_{11} - P_{22}A = BC^{-1}B^*P_{21} \tag{4.10}$$

$$P_{21} + AP_{12} = -I + BC^{-1}B^*P_{22} + P_{22}BC^{-1}B^*. \tag{4.11}$$

Using (4.9), one deduces

$$\begin{aligned} A^{\frac{1}{2}}P_{22}BC^{-\frac{3}{2}} &= A^{\frac{1}{2}}P_{11}BC^{-\frac{3}{2}} - A^{\frac{1}{2}}P_{12}BC^{-1}B^*BC^{-\frac{3}{2}} \\ &= (A^{\frac{1}{2}}P_{11}A^{-\frac{1}{2}})(A^{\frac{1}{2}}BC^{-\frac{3}{2}}) - (A^{\frac{1}{2}}P_{12})(BC^{-1})(B^*A^{-\frac{1}{2}})(A^{\frac{1}{2}}BC^{-\frac{3}{2}}). \end{aligned}$$

But the boundedness of  $P$  implies that  $P_{11} \in L(D(A^{\frac{1}{2}}))$  or equivalently  $A^{\frac{1}{2}}P_{11}A^{-\frac{1}{2}} \in L(X)$  and, on the other hand,  $P_{12} \in L(X, D(A^{\frac{1}{2}}))$  or  $AP_{12} \in L(X)$ . We conclude from assumptions (3.7, 3.8) and (3.9) that

$$A^{\frac{1}{2}}P_{22}BC^{-\frac{3}{2}} \in L(Y, X). \tag{4.12}$$

From (4.11), it follows that

$$\begin{aligned} AP_{12}BC^{-\frac{3}{2}} &= -P_{21}BC^{-\frac{3}{2}} - BC^{-\frac{3}{2}} + BC^{-1}B^*P_{22}BC^{-\frac{3}{2}} + P_{22}BC^{-1}B^*BC^{-\frac{3}{2}} \\ &= -(P_{21}A^{-\frac{1}{2}})(A^{\frac{1}{2}}BC^{-\frac{3}{2}}) + BC^{-1}(B^*A^{-\frac{1}{2}})(A^{\frac{1}{2}}P_{22}A^{-\frac{1}{2}})A^{\frac{1}{2}}BC^{-\frac{3}{2}} \\ &\quad + P_{22}BC^{-1}(B^*A^{-\frac{1}{2}})A^{\frac{1}{2}}BC^{-\frac{3}{2}} - BC^{-\frac{3}{2}}. \end{aligned}$$

From (4.9), the boundedness of  $P$  and (3.7, 3.8), one gets that

$$A^{\frac{1}{2}}P_{22}A^{-\frac{1}{2}} = A^{\frac{1}{2}}P_{11}A^{-\frac{1}{2}} - A^{\frac{1}{2}}P_{12}(BC^{-1})B^*A^{-\frac{1}{2}} \in L(X).$$

It follows again from (3.9) that

$$AP_{12}BC^{-\frac{3}{2}} \in L(Y, X). \tag{4.13}$$

And last, arguing as for the two preceding operators, we deduce that

$$C^{-\frac{3}{2}}B^*P_{22}BC^{-\frac{1}{2}} = C^{-\frac{3}{2}}B^*A^{\frac{1}{2}}(A^{-\frac{1}{2}}P_{22}A^{\frac{1}{2}})A^{-\frac{1}{2}}BC^{-\frac{1}{2}} \in L(Y). \tag{4.14}$$

Using (4.12, 4.13) and (4.14), there exist positive real constants  $\alpha < 1, \beta = \beta(\alpha)$  such that

$$((QL + L^*Q)Y(t), Y(t)) \leq -(1 - \alpha) \| Y \|^2 + \beta(\alpha) \| C^{\frac{1}{2}}w \|^2 .$$

Thus

$$\frac{d}{dt}\rho_\varepsilon(Y(t)) \leq -\varepsilon(2 - \alpha) \| Y(t) \|^2 - (2 - \varepsilon\beta(\alpha)) \| C^{\frac{1}{2}}w \|^2 \leq -\varepsilon(2 - \alpha) \| Y(t) \|^2 \leq -\frac{\varepsilon(2 - \alpha)}{a_1}\rho_\varepsilon(Y(t)) \tag{4.15}$$

by taking  $\varepsilon < \frac{2}{\beta(\alpha)}$  and using (i) of this lemma. Inequality (4.15) implies clearly (ii) with  $a_2 = \frac{\varepsilon(2-\alpha)}{a_1}$ .  $\square$

To conclude the proof of the sufficiency part of our theorem, we use (i) and (ii) of the lemma to obtain

$$\| Y(t) \|^2 \leq c \exp(-a_2 t) \| Y_0 \|^2 \quad \forall t \geq 0$$

with  $c = \frac{a_1}{a_0}$ .

**2/- Necessity.** If  $(e^{Lt})$  is uniformly stable, then again by Theorem 3.5, there exists a positive selfadjoint operator  $P \in L(H)$  such that

$$PL + L^*P = -I \quad \text{in } D(L). \tag{4.16}$$

We write  $P = (P_{ij})_{1 \leq i, j \leq 2}$  with

$$P_{11} = (R_{ij})_{1 \leq i, j \leq 2} \in L(D(A^{\frac{1}{2}}) \times X);$$

$$P_{12} = (P_{12}^1, P_{12}^2) \in L(Y, D(A^{\frac{1}{2}}) \times X);$$

$$P_{21} = (P_{21}^1, P_{21}^2) \in L(D(A^{\frac{1}{2}}) \times X, Y).$$

With these notations and if  $M_0(u, v) = (v, -Au)$  with  $D(M_0) = D(A) \times D(A^{\frac{1}{2}})$ ,  $\mathcal{B} = (0, B)$ , (4.16) amounts to the following identities

$$P_{11}M_0 - P_{12}\mathcal{B}^* - M_0P_{11} - \mathcal{B}P_{21} = -I \tag{4.17}$$

$$P_{11}\mathcal{B} - P_{12}C - M_0P_{12} - \mathcal{B}P_{22} = 0 \tag{4.18}$$

$$P_{21}M_0 - P_{22}\mathcal{B}^* + \mathcal{B}^*P_{11} - CP_{21} = 0 \tag{4.19}$$

$$P_{21}\mathcal{B} - P_{22}C + \mathcal{B}^*P_{12} - CP_{22} = -I. \tag{4.20}$$

We will prove, in a first step, that the decoupled system

$$\begin{cases} u_{tt} = -Au - BC^{-1}B^*u_t \\ \theta_t = -B^*u_t - C\theta \end{cases} \tag{4.21}$$

with initial conditions is uniformly stable in  $H$ . The associated linear operator is defined by

$$L_0(u, v, \theta) = (v, -Au - BC^{-1}B^*v, -B^*v - C\theta)$$

$$D(L_0) = D(L).$$

Let  $Y = (u, u_t, \theta)$  the solution of system (4.21). We introduce the following function

$$\psi_\varepsilon(Y(t)) = \| Y(t) \|^2 + \varepsilon(QY(t), Y(t))$$

where  $Q \in L(H)$  is defined by

$$Q = \begin{pmatrix} P_{11} & P_{12} - P_{11}BC^{-1} \\ P_{21} - C^{-1}B^*P_{11} & P_{22} - C^{-1}B^*P_{12} - P_{11}BC^{-1} \end{pmatrix}.$$

We then have

**Lemma 4.2.** *For a sufficiently small  $\varepsilon > 0$ , one has*

$$\begin{aligned} (i) & c_0 \|Y\|^2 \leq \psi_\varepsilon(Y) \leq c_1 \|Y\|^2, \quad \forall Y \in H \\ (ii) & \end{aligned}$$

$$\psi_\varepsilon(Y(t)) \leq -\exp(-c_2 t) \psi_\varepsilon(Y_0) \quad \forall t \geq 0$$

where  $c_i, i = 0, \dots, 2$  are various positive real constants and  $Y$  is the solution of system (4.21).

*Proof.* The point (i) follows from the boundedness of  $Q$  exactly as in the proof of Lemma 4.1. Let's prove (ii). One has, if  $Y(t) = (u(t), v(t), \theta(t))$  is a solution of (4.21)

$$\frac{d}{dt} \psi_\varepsilon(Y(t)) = 2 \operatorname{Re}(L_0 Y(t), Y(t)) + \varepsilon((QL_0 + L_0^*Q)Y(t), Y(t)).$$

But

$$\begin{aligned} \operatorname{Re}(L_0 Y(t), Y(t)) &= -\|C^{-\frac{1}{2}}B^*v\|^2 - \|C^{\frac{1}{2}}\theta\|^2 - \operatorname{Re}(B^*v, \theta) \\ &= -\|C^{-\frac{1}{2}}B^*v\|^2 - \|C^{\frac{1}{2}}\theta\|^2 - \operatorname{Re}(C^{-\frac{1}{2}}B^*v, C^{\frac{1}{2}}\theta) \\ &\leq -\frac{1}{2}\|C^{-\frac{1}{2}}B^*v\|^2 - \frac{1}{2}\|C^{\frac{1}{2}}\theta\|^2. \end{aligned}$$

Furthermore,  $Q$  was in fact chosen so that (direct computations show this)

$$QL_0 + L_0^*Q = \begin{pmatrix} -I & S \\ S^* & -I + T \end{pmatrix}$$

where

$$S = BP_{21}BC^{-1} + M_0P_{11}BC^{-1} + BC^{-1}B^*P_{11}BC^{-1}$$

$$T = C^{-1}B^*P_{12}C + CP_{21}BC^{-1}.$$

After computing all these quantities, we derive the following differential inequality

$$\begin{aligned} \frac{d}{dt} \psi_\varepsilon(Y(t)) &\leq -\|C^{-\frac{1}{2}}B^*v\|^2 - \|C^{\frac{1}{2}}\theta\|^2 - \varepsilon\|Y\|^2 \\ &\quad + 2\varepsilon \operatorname{Re} \left\{ (A^{\frac{1}{2}}R_{22}BC^{-1}\theta, A^{\frac{1}{2}}u) - (AR_{12}BC^{-1}\theta, v) \right\} \\ &\quad + \operatorname{Re} \left\{ (BC^{-1}B^*R_{22}BC^{-1}\theta, v) + (BP_{21}^2BC^{-1}\theta, v) \right\} \\ &\quad + \operatorname{Re}(CP_{21}^2BC^{-1}\theta, \theta). \end{aligned} \tag{4.22}$$

Developping equation (4.17), we see that we have in particular

$$R_{22} = R_{11} - P_{12}^1 B^*.$$

Since  $R_{11} \in L(D(A^{\frac{1}{2}}))$  and  $P_{12}^1 \in L(Y, D(A^{\frac{1}{2}}))$ , it follows that  $A^{\frac{1}{2}} R_{22} A^{-\frac{1}{2}} \in L(X)$ . Thus, using assumptions (3.7) and (3.9)

$$\operatorname{Re}(A^{\frac{1}{2}} R_{22} B C^{-1} \theta, A^{\frac{1}{2}} u) = \operatorname{Re}(A^{\frac{1}{2}} R_{22} A^{-\frac{1}{2}} (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) C^{\frac{1}{2}} \theta, A^{\frac{1}{2}} u) \leq \alpha \|A^{\frac{1}{2}} u\|^2 + c(\alpha) \|C^{\frac{1}{2}} \theta\|^2 \quad (4.23)$$

and

$$\begin{aligned} \operatorname{Re}(B C^{-1} B^* R_{22} B C^{-1} \theta, v) &= \operatorname{Re}(C^{-\frac{1}{2}} B^* A^{-\frac{1}{2}} (A^{\frac{1}{2}} R_{22} A^{-\frac{1}{2}}) (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) C^{\frac{1}{2}} \theta, C^{-\frac{1}{2}} B^* v) \\ &\leq \alpha \|C^{-\frac{1}{2}} B^* v\|^2 + c(\alpha) \|C^{\frac{1}{2}} \theta\|^2 \end{aligned} \quad (4.24)$$

where  $\alpha$  is any positive real number and  $c(\alpha) > 0$ .

From equation (4.16), it follows that

$$L^* P L^{-1} = -L^{-1} - P \in L(H).$$

This means that  $P \in L(D(L), D(L^*))$  and, in particular,  $P_{21}^2 \in L(D(A^{\frac{1}{2}}), D(C))$ , thus  $C^{\frac{1}{2}} P_{21}^2 A^{-\frac{1}{2}} \in L(Y, X)$ . Again by assumption (3.9), one gets

$$\operatorname{Re}(B P_{21}^2 B C^{-1} \theta, v) = \operatorname{Re}(C^{\frac{1}{2}} P_{21}^2 A^{-\frac{1}{2}} (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) C^{\frac{1}{2}} \theta, C^{-\frac{1}{2}} B^* v) \leq \alpha \|C^{\frac{1}{2}} \theta\|^2 + c(\alpha) \|C^{-\frac{1}{2}} B^* v\|^2 \quad (4.25)$$

and

$$\operatorname{Re}(C P_{21}^2 B C^{-1} \theta, \theta) = \operatorname{Re}(C^{\frac{1}{2}} P_{21}^2 A^{-\frac{1}{2}} (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) C^{\frac{1}{2}} \theta, C^{\frac{1}{2}} \theta) \leq \beta \|C^{\frac{1}{2}} \theta\|^2 \quad (4.26)$$

where  $\beta > 0$ . It remains to estimate one term in the right-hand member of inequality (4.17). From (4.19), after developping, it appears that

$$A R_{12} = R_{21} + B P_{21}^2 + P_{12}^2 B^* - I.$$

Thus

$$\begin{aligned} A R_{12} B C^{-\frac{3}{2}} &= R_{21} B C^{-\frac{3}{2}} + B P_{21}^2 B C^{-\frac{3}{2}} + P_{12}^2 B^* B C^{-\frac{3}{2}} - B C^{-\frac{3}{2}} \\ &= R_{21} B C^{-\frac{3}{2}} + B P_{21}^2 B C^{-\frac{3}{2}} + P_{12}^2 B^* A^{-\frac{1}{2}} (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) - B C^{-\frac{3}{2}}. \end{aligned} \quad (4.27)$$

Moreover

$$(B P_{21}^2 B C^{-\frac{3}{2}} (C^{\frac{1}{2}} \theta), v) = (C^{\frac{1}{2}} P_{21}^2 A^{-\frac{1}{2}} (A^{\frac{1}{2}} B C^{-\frac{3}{2}}) (C^{\frac{1}{2}} \theta), C^{-\frac{1}{2}} B^* v). \quad (4.28)$$

From (4.27) and (4.28), it is immediate that

$$-\operatorname{Re}(A R_{12} B C^{-1} \theta, v) \leq \alpha \|C^{\frac{1}{2}} \theta\|^2 + c(\alpha) \|C^{-\frac{1}{2}} B^* v\|^2. \quad (4.29)$$

The conclusion of this lemma will follow by using (4.23–4.26) and (4.29) in (4.22) by arguing exactly as in the proof of Lemma 4.1.  $\square$

So,  $(S_{L_0}(t))$  is exponentially stable. To conclude the proof of the necessity part, we need the following lemma

**Lemma 4.3.**  $(S_{L_0}(t))$  is exponentially stable if and only if  $(e^{Mt})$  is.

*Proof.* For  $\lambda \in \rho(L_0)$  the resolvent set of  $L_0$ , one has

$$(\lambda - L_0)^{-1} = \begin{pmatrix} (\lambda - M)^{-1} & 0 \\ -(\lambda + C)^{-1}B^*(\lambda - M)^{-1} & (\lambda + C)^{-1} \end{pmatrix}.$$

The assumptions on  $C$  imply that

$$\sup_{\operatorname{Re} \lambda \geq 0} \|(\lambda + C)^{-1}\| < \infty, \quad \sup_{\operatorname{Re} \lambda \geq 0} \|C(\lambda + C)^{-1}\| < \infty$$

$L_0$  and  $M$  being dissipative, the result follows from Huang’s result [16]

The conclusion of the theorem is then achieved. □

## 5. APPLICATIONS

### 5.1. A linear model of well-reservoir coupling

Let  $\Omega = (0, 1) \times (0, 1)$ ,  $\Gamma_0 = ]0, 1[ \times \{0\}$  and  $\Gamma_1 = \partial\Omega/\overline{\Gamma_0}$ . We consider the problem

$$\begin{cases} u_{tt}(t, x) = u_{xx}(t, x) + \frac{\partial w}{\partial y}(t, x, 0) & (t, x) \in \mathbf{R}^+ \times (0, 1) \\ w_t(t, x, y) = \Delta w(t, x, y) & (t, x, y) \in \mathbf{R}^+ \times \Omega \\ u(t, 0) = u(t, 1) = 0 & t \in \mathbf{R}^+ \\ w(t, x, y) = \begin{cases} u_t(t, x) & (t, x, y) \in \mathbf{R}^+ \times \Gamma_0 \\ 0 & (t, x, y) \in \mathbf{R}^+ \times \Gamma_1. \end{cases} \end{cases} \tag{5.1}$$

This system is equivalent to the one studied in [8] which is itself the linearization of a model of well-reservoir coupling considered by Bourgeat [7].  $\Omega$  is the reservoir region and  $\Gamma_0$  is the wellbore. System (4.15) may be interpreted as follows:  $u$  is the well pressure,  $u_x$  is the well acceleration (up to the sign) and  $w$  is the derivative (in time) of the reservoir pressure (see [8] for more details).

In order to set the abstract formulation of system (5.1), let’s recall some definitions, notations and classical results. We first define the strictly positive self-adjoint operator  $A : L^2(0, 1) \rightarrow L^2(0, 1)$

$$\begin{aligned} A &= -\frac{d^2}{dx^2}, & D(A) &= H^2(0, 1) \cap H_0^1(0, 1) \\ D(A^{1/2}) &= H_0^1(0, 1) \end{aligned} \tag{5.2}$$

and introduce the Dirichlet realization of the Laplace operator

$$C = \Delta, \quad D(C) = H^2(\Omega) \cap H_0^1(\Omega).$$

Recall that

$$D((-C)^\alpha) = \begin{cases} H^{2\alpha}(\Omega) & \text{if } \alpha \in ]0, 1/4[ \\ \{u \in H^{2\alpha}(\Omega) : u = 0 \text{ on } \partial\Omega\} & \text{if } \alpha \in ]1/4, 1[. \end{cases}$$

Let us introduce the Dirichlet mapping

$$v \mapsto Dv = \theta \Leftrightarrow \begin{cases} \Delta\theta = 0 & \text{in } \Omega \\ \theta = \begin{cases} v & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1. \end{cases} \end{cases} \quad (5.3)$$

Classical results yield that

$$D \in \mathcal{L}(H^s(0,1); H^{s+1/2}(\Omega))$$

and using a Green's formula (see for instance [19]), the operator  $D^*C \in \mathcal{L}(H_0^{3/2+2\varepsilon}(\Omega); L^2(0,1))$  ( $\varepsilon > 0$ ) and  $\forall w \in D((-C)^{3/4+\varepsilon})$

$$D^*Cw = \frac{\partial w}{\partial \nu} |_{\Gamma_0}.$$

Following Balakrishnan [6], the problem

$$\begin{cases} w_t(t, x, y) = \Delta w(t, x, y) & (t, x, y) \in \mathbf{R}^+ \times \Omega \\ w(t, x, y) = \begin{cases} u_t(t, x) & (t, x, y) \in \mathbf{R}^+ \times \Gamma_0 \\ 0 & (t, x, y) \in \mathbf{R}^+ \times \Gamma_1 \end{cases} \end{cases}$$

has, as an abstract formulation (set  $w(t) = w(t, \cdot, \cdot)$ )

$$w_t = Cw - CDu_t \quad \text{in } D(C)'$$

where  $D(C)'$  stands for the dual space of  $D(C)$  with respect to  $L^2(\Omega)$  inner product ( $D(C) \hookrightarrow L^2(\Omega) \hookrightarrow D(C)'$ ). Setting  $u(t) = u(t, \cdot)$ , and denoting by  $H$  the Hilbert space

$$H = H_0^1(0,1) \times L^2(0,1) \times L^2(\Omega),$$

the abstract form of System (5.1) is then

$$\frac{d}{dt} \begin{pmatrix} u \\ u_t \\ w \end{pmatrix} = L \begin{pmatrix} u \\ u_t \\ w \end{pmatrix}$$

with

$$L = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & D^*C \\ 0 & -CD & C \end{pmatrix}$$

which, in view of the notations of the previous sections, corresponds to

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ D^*C \end{pmatrix}$$

$$D(\mathcal{A}) = D(A) \times D(A^{1/2}), \quad D(\mathcal{B}) = D((-C)^{3/4+\varepsilon}) \quad (\varepsilon \in ]0, 1/4[).$$

Now, (3.7) is satisfied. The operator  $M = \mathcal{A} + \mathcal{B}C^{-1}\mathcal{B}^*$  is then given by

$$M = \begin{pmatrix} 0 & I \\ -A & K \end{pmatrix}$$

where  $K := D^*CD = (\frac{\partial}{\partial v} \circ D)|_{\Gamma_0} : D(K) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ .  $K$  is the *capacity operator* of  $\Gamma_0$  (see for instance [10], p. 477) and its explicit expression in this case is:

$$Kv = - \sum_{n \geq 1} n\pi \coth(n\pi)v_n\phi_n$$

where  $\phi_n = \sqrt{2}\sin(n\pi x)$  and  $v_n = \int_0^1 v\phi_n dx$ . Clearly  $D(K) = H_0^1(0, 1)$  and  $A^{1/2} \leq -K \leq \coth(\pi)A^{1/2}$  where  $A$  is defined in (4.18). Thus,  $M$  satisfies assumption (3.8) (it can be readily verified by direct computations). Moreover,  $M$  is boundedly invertible and is the generator of an analytic semigroup of contractions (see [9]) and then its semigroup is exponentially stable. In order to verify assumption (3.9), we have:

$$A^{1/2}B(-C)^{-3/2} = A^{1/2}(D^*C)(-C)^{-3/2}$$

and it appears that this operator is bounded from  $L^2(\Omega)$  in  $L^2(0, 1)$  since  $(-C)^{-3/2} \in \mathcal{L}(L^2(\Omega), H^3(\Omega) \cap H_0^1(\Omega))$  and  $D^*C \in \mathcal{L}(H^3(\Omega) \cap H_0^1(\Omega), H_0^1(0, 1))$ . Applying our main result, it follows that the semigroup associated to system (5.1) is exponentially stable.

### 5.2. Exact controllability and dynamic stabilization

Haraux [14] proved the following result:

**Theorem 5.1.** ([14], Props. 1 and 2). *Let  $H$  be a Hilbert space,  $A = A^* \geq 0$  an unbounded linear operator and  $B = B^* \geq 0$  a bounded linear operator on  $H$ . Then the semigroup associated to the system*

$$\begin{cases} y_{tt} + Ay + By_t = 0, & t \in R, \\ y(0) = y_0 \end{cases}$$

is exponentially stable if and only if there exist  $T > 0, a > 0$  such that

$$\| A^{\frac{1}{2}}\varphi_0 \|^2 + \|\varphi_1\|^2 \leq a \int_0^T (BC^{-1}B^*\varphi_t, \varphi_t)dt \quad \forall (\varphi_0, \varphi_1) \in D(A^{\frac{1}{2}}) \times X$$

where  $\varphi$  is the solution of

$$\begin{cases} \varphi_{tt} = -A\varphi \\ \varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1. \end{cases} \tag{5.4}$$

We get the immediate consequence:

**Corollary 5.2.** *Under the assumptions of Theorem 3.1, and if moreover  $BC^{-1}B^*$  extends to a bounded linear operator,  $(e^{Lt})$  is exponentially stable if and only if there exist  $T > 0, a > 0$  such that*

$$\| A^{\frac{1}{2}}\varphi_0 \|^2 + \|\varphi_1\|^2 \leq a \int_0^T (BC^{-1}B^*\varphi_t, \varphi_t)dt \quad \forall (\varphi_0, \varphi_1) \in D(A^{\frac{1}{2}}) \times X$$

where  $\varphi$  is the solution of (5.4).

This last proposition reduces the study of the uniform stability of system (1.4) to the study of the exact controllability of (5.4) by  $BC^{-1}B^*$ . The assumptions in Proposition 5.2 are in particular satisfied by the linear thermoelasticity system. As an illustration of this result, let's consider the following example.

**Example 5.3.** Let  $\Omega$  be a bounded open set of  $R^n$  with smooth boundary  $\partial\Omega$ . Let us consider the following system

$$\begin{cases} u_{tt} - \Delta u - a(x)(-\Delta)^{\frac{1}{2}}\theta = 0 & \text{in } \Omega \times (0, \infty) \\ \theta_t - \Delta\theta + (-\Delta)^{\frac{1}{2}}a(x)u_t = 0 & \text{in } \Omega \times (0, \infty) \\ u = \theta = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \quad (5.5)$$

with initial conditions

$$\begin{cases} u(x, 0) = u_0(x) & u_t(x, 0) = u_1(x) & \text{in } \Omega \\ \theta(x, 0) = \theta_0 & & \text{in } \Omega \end{cases} \quad (5.6)$$

where  $a \in C^\infty(\Omega)$  and  $A := -\Delta$  with  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ . System (5.5–5.6) is well-posed in the energy space  $H = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ . This system is closely related to the linear thermoelasticity system. In fact, if one sets  $\theta = (-\Delta)^{\frac{1}{2}}w$ ,  $w \in H_0^1(\Omega)$ , (5.5) transforms as

$$\begin{cases} u_{tt} - \Delta u + a(x)\Delta w = 0 & \text{in } \Omega \times (0, \infty) \\ w_t - \Delta w + a(x)u_t = 0 & \text{in } \Omega \times (0, \infty) \\ u = w = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \quad (5.7)$$

which is well-posed in the energy space  $E = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ . The tie with thermoelasticity is explained in Lebeau and Zuazua [21] where the null controllability is studied. Returning to system (5.5–5.6), one verifies easily the assumptions of Proposition 5.2 and thus

**Theorem 5.4.** *If  $\omega$ , the support of  $a$ , satisfies the geometric condition of Bardos-Lebeau-Rauch (see [5]), then system (5.5–5.6) is exponentially stable in  $H$ . The same conclusion is true for system (5.7) in  $E$ .*

Our last remark is that, when  $A$ ,  $B$  and  $C$  are bounded (in particular in the finite dimensional case), our main result leads to the following criterion:

**Proposition 5.5.** If  $A$ ,  $B$  and  $C$  are bounded, the following assertions are equivalent:

- (i) the pair  $(B, C)$  is a dynamic stabilizer for  $A$ .
- (ii) System (4.3) is stable.
- (iii) The pair  $(A, BC^{-1}B^*)$  is controllable.

(See [28] for the definition of the controllability in the present case.)

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