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RANK–2 DISTRIBUTIONS SATISFYING THE GOURSAT CONDITION: ALL THEIR LOCAL MODELS IN DIMENSION 7 AND 8*

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Abstract. We study the rank–2 distributions satisfying so-called Goursat condition (GC); that is to say, codimension–2 differential systems forming with their derived systems a flag. Firstly, we restate in a clear way the main result of [7] giving preliminary local forms of such systems. Secondly – and this is the main part of the paper – in dimension 7 and 8 we explain which constants in those local forms can be made 0, normalizing the remaining ones to 1. All constructed equivalences are explicit. The complete list of local models in dimension 7 contains 13 items, and not 14, as written in [7], while the list in dimension 8 consists of 34 models (and not 41, as could be concluded from some statements in [7]). In these dimensions (and in lower dimensions, too) the models are eventually discerned just by their small growth vector at the origin.

Résumé. Nous étudions les distributions de rang 2 vérifiant la condition de Goursat ; c'est-à-dire, les systèmes différentiels de co-rang 2 formant, avec leurs systèmes dérivés, un drapeau. Nous donnons d'abord un énoncé clair du résultat principal de Kumpera et Ruiz sur des formes locales préliminaires de ces systèmes. Puis, dans la partie principale de l'article, en dimension 7 et 8, nous expliquons quelles constantes dans les formes préliminaires de Kumpera et Ruiz peuvent passer à 0, en normalisant simultanément les constantes restantes à 1. Toutes les équivalences proposées sont explicites. La liste complète des modèles locaux en dimension 7 contient 13 objets (et non 14 énoncés dans [7]), tandis que celle en dimension 8 comporte 34 modèles (et non 41 comme on pouvait le déduire de [7]). En ces dimensions (et en dimensions inférieures aussi) on n'utilise pour distinguer les modèles que leur petit vecteur de croissance à l'origine.

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1. Introduction

Let D be a distribution of an arbitrary dimension k on a given manifold M. We set $D_1 = D,\ D_2 = D + [D, D], \dots, D_{l+1} = D_l + [D, D_l]$; $\quad D^{(0)} = D,\ D^{(1)} = D + [D, D], \dots, D^{(l+1)} = D^{(l)} + [D^{(l)}, D^{(l)}]$.
By the *small growth vector* of D at p we understand the sequence $[n_1, n_2, n_3, \dots]$ of dimensions at $p \in M$ of an *ascending* flag of modules of vector fields $D_1 \subset D_2 \subset D_3 \subset \dots$ $(n_1 = k)$, and by the *big growth vector* of

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D at p — the sequence $[m_0, m_1, m_2, \dots]_\mathrm{B}$ of dimensions at p of the ascending flag $D^{(0)} \subset D^{(1)} \subset D^{(2)} \subset \dots$ ($m_0 = k$). In what follows we shall most frequently just write "small/big gr. v.". (With these related is the notion of the *first derived system* $S^{(1)}$ of a given constant rank differential system S: $(D^\perp)^{(1)}$ is the annihilator of $D^{(1)}$ whenever $m_1(\cdot)$ is locally constant.)

It is important that the small and big growth vectors of D at any point p are invariants of the local equivalence class (under smooth local diffeomorphisms) of D around p.

Definition 1. ([1]). Let D be a 2–distribution on an $(n+2)$–dimensional manifold. We say that D satisfies the *Goursat Condition* (GC for short **in the sequel**) if D has, at every point of the manifold, the big growth vector $[2, 3, \dots, n+1, n+2]_\mathrm{B}$ (see also [14]).

This condition is also sometimes called the Cartan–Goursat condition. Two basic works [7] and [4] treat GC. They deal with Pfaffian differential systems having the annihilator satisfying GC. Such systems are called there to form (or, using the French terminology, to be in –) a flag. The growth vectors do not show up explicitly there, and this is not surprising, as GC has been – historically – introduced for differential systems.

In dimension 3 ($n = 1$) and 4 ($n = 2$), GC is an open condition (if non-generic) among all the distributions, fulfilled – for every distribution typical in the sense of Thom – at typical points.

In these low dimensions GC entirely characterizes a distribution locally (up to a diffeomorphism of the underlying space). The local models are well-known and named after Darboux and Engel, respectively.

In dimensions bigger than 4, GC is no longer an open condition among all smooth distributions; it imposes, and moreover at every point, a severely deccelerated growth of the big vector, and hence is of codimension infinity. Nevertheless, GC is important in applications, in particular in the control theory; see, in this respect [6, 8, 9], and also Chapter 6 of the present paper.

Independently of that, possible local classification of GC would have applications in the problem of feedback classification of affine control systems. Also in sub-Riemannian geometry, in different situations, local normal forms are proving to be useful.

The authors of the note [5] were the first to have observed that, in dimension 5, GC admits two non-equivalent local models — correcting thus one old statement of E. Cartan[1] ([2], p. 119, the case b) IV corresponding to $[2, 3, 4, 5]_\mathrm{B}$). Let us note that [10] gave, independently, another simple derivation of the same normal forms. Here is this result.

Theorem 2. [5, 10]. *Let D be a 2–distribution on $\mathbb{R}^5$ satisfying GC. Then D is locally equivalent, around any given point p, either to the Goursat Normal Form, i.e., the germ at $0 \in \mathbb{R}^5(x^1, \dots, x^5)$ of the distribution $\left(\frac{\partial}{\partial x^1},\ \frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^4} + x^4\frac{\partial}{\partial x^5}\right)$, or else to the exceptional model, i.e., the germ at 0 of the distribution spanned by $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial x^2} + x^1x^3\frac{\partial}{\partial x^4} + x^1x^4\frac{\partial}{\partial x^5}$.*
The small gr. v. at p is: $[2, 3, 4, 5]$ *in the first case, and* $[2, 3, 4, 4, 5]$ *in the second.*

In dimension 6 there are already 5 non-equivalent local models of GC ([7], p. 227). In dimension 7 the authors of [7] claimed to have found 14 non-equivalent local models. Actually, two items on their list appear to be the same (equivalent), and the total number of models is 13 (see our Ths. 16 and 17)[2].

Let us mention at last that:

- the paper [12] also treats one topic pertinent to [7] (and settled completely there in Th. 9.2, except for the fact – mentioned already above – that the authors did not use the notion of the small gr. v.). It concerns the local description, the *Goursat Normal Form* in an arbitrary dimension, of an additional condition that the small gr. v. coincide everywhere with the big one of GC;
- the important work [13] analyzes in depth a condition dual to GC that the small gr. v. be the same at all points and grow always by 1 (*i.e.*, that the small vector is as the big one in GC).

[1]Repeated after Cartan by E. Goursat as well.
[2]Added in revision: after submitting the present work, the authors learned about the existence of a note by Gaspar ([3], now included into references) in which she had much earlier corrected that error.

In this paper, at first, we state again and explicitly the local (preliminary) forms of a 2–distribution in arbitrary dimension satisfying GC. This is a reformulation of the main result of [7], given, possibly, not transparently enough in general dimension $n+2$ in that work of reference. Henceforth we call those forms *KR pseudo–normal forms*.

Secondly, we treat the dimension 7, filling the gap of [7] mentioned above.

Thirdly, we treat the subsequent dimension 8 and establish the full list of 34 pairwise non-equivalent local models of GC in that dimension.

We conclude by putting forward an open question as to the local classification of GC in higher dimensions in terms of a kinematic model satisfying GC, and alluded to earlier in this Introduction.

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2. Local forms – found by Kumpera and Ruiz – revisited, and their coding

Let us consider a 2−distribution on $\mathbb{R}^{n+2}$ satisfying GC. Kumpera and Ruiz have already sketched the local forms of such distributions. We put their classification as a separate theorem (Th. 3), with our proof using different arguments, possibly less algebraic and more explicit. Those KR pseudo–normal forms have a serious disadvantage: starting from the dimension 6 constants come up which à priori are not known to be reducible to 0 or 1. To be precise, and this is noted in [7], in dimension 6 and 7 a rescaling of the variables suffices to obtain such a reduction. But already in dimension 8 there are local forms with constants resisting direct rescalings.

Also, once a rescaling is achieved, it does not necessarily mean that one deals with a separate local model. The first instance of such phenomenon shows up in dimension 7 (see Th. 17), and in dimension 8 it starts to be a commonplace (see the theorems of Chap. 5).

The problem of finding the exact local models in dimensions exceeding 8 has its own complexity and is actually worked upon (*cf.* [11], and also Open question of Chap. 6).

Theorem 3. [7]. *Let S be a codimension 2 Pfaffian system on an $(n+2)$−dimensional manifold, $n \geq 1$, such that its annihilator $S^{\perp}$ satisfies* GC. *Then S can be written locally around any fixed point p of the manifold as the germ at $0 \in \mathbb{R}^{n+2}(x^1, x^2, \dots, x^{n+2})$ of a system of the type*

$$\begin{cases} \omega^1 = dx^{i_1} + x^3 dx^{j_1}, & (i_1, j_1) = (2, 1) \\ \omega^2 = dx^{i_2} + x^4 dx^{j_2}, & (i_2, j_2) = (3, j_1) \\ \omega^3 = dx^{i_3} + x^5 dx^{j_3}, & (i_3, j_3) \in \{(4, j_2), (j_2, 4)\} \\ \omega^4 = dx^{i_4} + X^6 dx^{j_4}, & (i_4, j_4) \in \{(5, j_3), (j_3, 5)\} \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ \omega^n = dx^{i_n} + X^{n+2} dx^{j_n}, & (i_n, j_n) \in \{(n+1, j_{n-1}), (j_{n-1}, n+1)\}, \end{cases}$$

where, for $6 \leq l \leq n+2$, $X^l = x^l$ if $(i_{l-2}, j_{l-2}) = (j_{l-3}, l-1)$ and $X^l = x^l + c^l$ in the opposite case of $(i_{l-2}, j_{l-2}) = (l-1, j_{l-3})$. The $c^6, c^7, \dots, c^{n+2}$ are real constants.

On top of that, one gets, around that point p, the successive derived systems of S by always removing the one last (bottommost) Pfaffian equation.

At that, if $j_3 = j_4 = \dots = j_l = 1$ (i.e., $i_3 = 4$, $i_4 = 5, \dots, i_l = l+1$) for certain $4 \leq l \leq n$, then the constants $c^6, c^7, \dots, c^{l+2}$ can be taken equal to 0 keeping the unchanged writing of $\omega^1, \omega^2, \omega^3, \omega^{l+1}, \dots, \omega^n$ (for $l = n$ one thus gets the Goursat Normal Form – GNF*).*

Remark 4. All pairs of sequences $\{i_l\}$ and $\{j_l\}$ $(l = 1, \dots, n)$ fulfilling the conditions worded in Theorem 3, and arbitrary real constants $c^6, c^7, \dots, c^n$ (when applicable) are permitted and always give 2–distributions satisfying GC.

This remark may serve as a supplement to Theorem 3, and its justification will be straightforward once proven this theorem.

Before giving a proof, we want to organize somehow the family of local forms given by Theorem 3.

Definition 5. Our indexing of the local forms starts in the first interesting dimension 5 (to recall the originating note [5]): the system realizing the first alternative $(i_3, j_3) = (4, j_2)(= (4, 1))$ we name *1*, and we name *3* that of the second alternative $(i_3, j_3) = (j_2, 4)$ $(= (1, 4))$.

In turn, inductively, we index the system obtained by adding the last equation $\omega^n = 0$ by prolonging to the right the code of the previous (derived) system by a dot followed by:

1 in the case of the first alternative for ω^n, and when $c^{n+2} = 0$,
2 in the case of the first alternative, when $c^{n+2} \neq 0$,
3 in the case of the second alternative.

In this way, a given symbol "*2*" in a code sequence serves all non-zero values of the respective constant. That is, it does not entirely specify the Pfaffian equation related to its place in the code (and, consequently, the code of a system forgets about the specific non-zero values of system's constants).

Definition 6. If we deal with a system being, in the vicinity of $0 \in \mathbb{R}^{n+2}$, under the form of Theorem 3, with a given non-zero constant normalized to 1, we shall write **2** instead of *2* in the appropriate place of its code.

Example 7. The five pairwise non-equivalent local systems existing in dimension 6, listed in ([7], p. 227), are getting indexed, from the left to right: *1.1*, *1.3*, *3.3*, *3.1*, *3.*__2__. (They are written again in Th. 14 of the present paper.)

Remark 8. The codes, just defined, of the KR pseudo–normal forms given by Theorem 3, always start with a string of *1*'s (that can be void) followed by a *3*, unless one has coded by *1.1 . . . 1* the GNF in the respective dimension – *cf.* the last statement of Theorem 3.

Proof of Theorem 3. We start with an elementary

Observation 9. For any pair of sequences $\{i_l\}$ et $\{j_l\}$ fulfilling the conditions formulated in Theorem 3, for $1 \le l \le n$,

a) $i_l \le l + 1,\ j_l \le l + 1,\ i_l \neq j_l$,
b) $i_l \notin \{i_1, \dots, i_{l-1}\}$,
c) $j_l \notin \{i_1, \dots, i_l\}$.

Proof. One shows simultaneously a), b) and c) by induction on l, with an evident beginning of the induction.

Suppose that a), b) and c) hold for $l-1$ and recall that $(i_l, j_l) \in \{(l+1, j_{l-1}),\ (j_{l-1}, l+1)\}$. It follows that a) holds for l.

If $i_l = l+1$, since $\{i_1, .., i_{l-1}\} \subset \{1, 2, .., l\}$ by a), then $i_l \notin \{i_1, .., i_{l-1}\}$. In turn, $j_l = j_{l-1} \notin \{i_1, .., i_{l-1}\}$ (by the hypothesis c)) and, by a), $j_l \le l < i_l$. Thus b) – c) still hold.

If $i_l = j_{l-1}$, then $i_l \notin \{i_1, .., i_{l-1}\}$ by the hypothesis c). In this case $j_l = l+1$ exceeds – by a) – all the $i_1, .., i_l$, and c) holds again. □

Corollary 10. *The indices $i_1, \dots, i_n$ are all distinct and do not exceed $n+1$.*

Corollary 11. *The member of the set $\{1, 2, \dots, n\}$ that is missing in $\{i_1, \dots, i_{n-1}\}$, is j_{n-1}.*

Proof. By Observation 9 a) there is $j_{n-1} \le n$, and one applies Corollary 10 and Observation 9 c). □

In order to prove Theorem 3, we need

Lemma 12. *Let V be a dimension 3 distribution on $\mathbb{R}^{m+1}$ having at all points the big gr. v. $[3,\,4,\ldots,\,m+1]_{\mathrm{B}}$. Then V is locally equivalent, in a neighbourhood of any given point p, to the germ at $0\in\mathbb{R}^{m+1}$ of the distribution $(\widetilde{V},\ \frac{\partial}{\partial x^{m+1}})$, where $\widetilde{V}$ is a distribution defined on $\mathbb{R}^m(x^1,\,x^2,\ldots,\,x^m)$, having at every point the big gr. v. $[2,\,3,\ldots,\,m-1,\,m]_{\mathrm{B}}$, and $\widetilde{V}$ is understood on $\mathbb{R}^{m+1}(x^1,\ldots,\,x^m,\,x^{m+1})$.*

Proof. Let us consider a local basis $\{X,\,Y,\,Z\}$ of V around p such that $X,\,Y,\,Z$ and $[X,\,Y]$ are linearly independent. We search for a characteristic vector field of V in the form $\widetilde{Z}=Z+fX+gY$: $[X,\,\widetilde{Z}]=[X,\,Z]+X(f)X+X(g)Y+g[X,\,Y]$ and $[Y,\,\widetilde{Z}]=[Y,\,Z]+Y(f)X+Y(g)Y-f[X,\,Y]$. Write $[X,\,Z]=\alpha X+\beta Y+\gamma Z+\delta[X,\,Y]$ and $[Y,\,Z]=\lambda X+\mu Y+\nu Z+\rho[X,\,Y]$. On taking $f=\rho$ and $g=-\delta$, $[X,\,\widetilde{Z}]\in V$ and $[Y,\,\widetilde{Z}]\in V$. Therefore, $\widetilde{Z}$ is a characteristic vector field of V: $[\widetilde{Z},\,V]\subset V$.

Take now local coordinates $x^1,\ldots,\,x^{m+1}$ around p such that $\widetilde{Z}=\frac{\partial}{\partial x^{m+1}}$. Denoting by ϕ^t the flow of this characteristic field $\frac{\partial}{\partial x^{m+1}}$, it is well-known that

$$\phi^t_*V=V\quad\forall t\in\mathbb{R}\tag{1}$$

(V does not depend on x^{m+1}). Let us write now $V=(X,\,Y,\,\frac{\partial}{\partial x^{m+1}})$ in such a way that the v. f. X and Y have no components in the direction $\frac{\partial}{\partial x^{m+1}}$. One can suppose further that these two generators do not depend on x^{m+1}:
Putting $\widetilde{X}(x^1,\,x^2,\ldots,\,x^{m+1})=X(x^1,\,x^2,\ldots,\,x^m,\,0)$ and $\widetilde{Y}(x^1,\,x^2,\ldots,\,x^{m+1})=Y(x^1,\,x^2,\ldots,\,x^m,\,0)$, $\widetilde{X}$ and $\widetilde{Y}$ are ϕ^t–invariant, and – with $\frac{\partial}{\partial x^{m+1}}$ – generate V on the level $x^{m+1}=0$. By (1), they generate V everywhere.

It follows directly from this expression for V that $\widetilde{V}=(\widetilde{X},\,\widetilde{Y})$ is defined on $\mathbb{R}^m(x^1,..,x^m)$ and has everywere the big gr. v. $[3-1,\,4-1,\ldots,\,m+1-1]_{\mathrm{B}}$. □

We need also an important

Observation 13. In the wording of Theorem 3, the minor of the coefficients of $dx^{i_1},\,dx^{i_2},\ldots,\,dx^{i_{n-1}}$ entering $\omega^1(0),\,\omega^2(0),\ldots,\,\omega^{n-1}(0)$, is always non-zero.

Proof. One shows by induction on l that the minor of the coefficients of $dx^{i_1},\ldots,\,dx^{i_l}$ entering $\omega^1(0),\ldots,\,\omega^l(0)$ is non-zero: $l=1$ — obvious.

Suppose this for $l-1\le n-1$. If $i_l<j_l$, then $\omega^l(0)=dx^{i_l}$ and one applies Obs. 9. b). In the opposite case $\omega^l(0)=dx^{i_l}+c^{l+2}dx^{j_l}$, and Obs. 9. b)−c) suffices to conclude. □

Now we return to the proof of Theorem 3 which goes by induction on $n\ge 1$.
For $n=1$, this is the classical Darboux−type local description of the contact structures on $\mathbb{R}^3$.

Suppose now that the theorem holds in dimension $n+1\ge 3$ (*i.e.*, for the length of the flag $n-1\ge 1$), and take a Pfaffian system S on $\mathbb{R}^{n+2}$ such that $D=S^{\perp}$ possesses at every point the big gr. v. $[2,\,3,\ldots,\,n+1,\,n+2]_{\mathrm{B}}$. As a consequence, $D^{(1)}$ has $[3,\,4,\ldots,n+1,\,n+2]_{\mathrm{B}}$ at every point, and one may apply Lemma 12. In this way, in the vicinity of p, $D^{(1)}$ turns out to be equivalent to the germ at $0\in\mathbb{R}^{n+1}$ of $(\Delta,\,\frac{\partial}{\partial x^{n+2}})$, where Δ is defined already on $\mathbb{R}^{n+1}(x^1,\ldots,\,x^{n+1})$ and has $[2,\,3,\ldots,n,\,n+1]_{\mathrm{B}}$ at every point.

It is to Δ in the vicinity of $0\in\mathbb{R}^{n+1}$ that we apply the induction hypothesis – the germ at $0\in\mathbb{R}^{n+1}$ of Δ starts to be described by $n-1$ Pfaffian equations using only the coordinates $x^1,\,x^2,\ldots,\,x^{n+1}$. Simultaneously, the 3–distribution $(\Delta,\,\frac{\partial}{\partial x^{n+2}})$ on $\mathbb{R}^{n+2}(x^1,\ldots,\,x^{n+1},\,x^{n+2})$ obtains, in a neighbourhood of $0\in\mathbb{R}^{n+2}$, the same description. The Pfaffian equations read

$$\begin{cases} \omega^1 = dx^{i_1} + x^3 dx^{j_1}, & (i_1,\, j_1) = (2,1) \\ \omega^2 = dx^{i_2} + x^4 dx^{j_2}, & (i_2,\, j_2) = (3,\, j_1) \\ \omega^3 = dx^{i_3} + x^5 dx^{j_3}, & (i_3,\, j_3) \in \{(4,\, j_2),\ (j_2,\, 4)\} \\ \omega^4 = dx^{i_4} + X^6 dx^{j_4}, & (i_4,\, j_4) \in \{(5,\, j_3),\ (j_3,\, 5)\} \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ \omega^{n-1} = dx^{i_{n-1}} + X^{n+1} dx^{j_{n-1}}, & (i_{n-1},\, j_{n-1}) \in \{(n,\, j_{n-2}),\ (j_{n-2},\, n)\}, \end{cases}$$

with the meaning of X^l given in the wording of Theorem 3, and with the additional reduction of constants when $j_3 = j_4 = \cdots = j_l = 1$ for certain $4 \le l \le n-1$. Consequently, the germ of S at p is equivalent to the germ at $0 \in \mathbb{R}^{n+2}$ of a system $(\omega^1, \omega^2, \dots, \omega^{n-1}, \omega^n)$, with a 1–form ω^n not yet precised. In view of Observation 13 and Corollary 11, one can take without loss of generality

$$\omega^n \;=\; a^{j_{n-1}} dx^{j_{n-1}} + a^{n+1} dx^{n+1} + a^{n+2} dx^{n+2}. \tag{2}$$

Convention. Let us return to the distribution Δ introduced above. One knows, by the induction hypothesis on the level of $\mathbb{R}^{n+1}(x^1, \dots, x^{n+1})$, that, removing consecutively (always one at a time) the equations $\omega^{n-1} = 0$, $\omega^{n-2} = 0, \dots$, one obtains the families of generators of derived systems $(\Delta^{(1)})^\perp$, $(\Delta^{(2)})^\perp$, etc.
The same remains true on the level of $\mathbb{R}^{n+2}(x^1, \dots, x^{n+1}, x^{n+2})$ for $(\Delta, \frac{\partial}{\partial x^{n+2}})$, because $(\Delta, \frac{\partial}{\partial x^{n+2}})^{(h)} = (\Delta^{(h)}, \frac{\partial}{\partial x^{n+2}})$ for $h = 1, 2, \dots, n-1$. Throughout the rest of the proof we identify the two locally equivalent 3–distributions $D^{(1)}$ and $(\Delta, \frac{\partial}{\partial x^{n+2}})$.

We are going to say and use then, that the derived systems of $(D^{(1)})^\perp$ are locally generated by the respective sub-families of $\{\omega^1, \omega^2, \dots, \omega^{n-1}\}$.

The remaining of the proof of Theorem 3 consists of four steps.
First step: $d\omega^l \wedge \omega^1 \wedge \cdots \wedge \omega^{l+1} = 0$ for every $l = 1, 2, \dots, n-1$.
Indeed: by the convention, $(D^{(1)})^\perp = (\omega^1, \omega^2, \dots, \omega^{n-1})$. On top of that, as $D^{(n-l)} = (D^{(1)})^{(n-l-1)}$, and, for $l \le n-2$, $D^{(n-l-1)} = (D^{(1)})^{(n-l-2)}$, by the induction hypothesis $(D^{(n-l-1)})^\perp = (\omega^1, \dots, \omega^{l+1})$ and $(D^{(n-l)})^\perp = (\omega^1, \dots, \omega^l)$.

Since $D^{(n-l-1)} + [D^{(n-l-1)}, D^{(n-l-1)}] = D^{(n-l)}$, then by the classical Cartan identity $d\omega^l|_{D^{(n-l-1)}} = 0$.
Second step: a^{n+2} in the equation (2) vanishes identically.
To show this, one uses the first step for $l = n-1$:
$0 = dx^{n+1} \wedge dx^{j_{n-1}} \wedge (dx^{i_1} + x^3 dx^{j_1}) \wedge (dx^{i_2} + x^4 dx^{j_2}) \wedge (dx^{i_3} + x^5 dx^{j_3}) \wedge (dx^{i_4} + X^6 dx^{j_4}) \wedge \cdots \wedge (dx^{i_{n-3}} + X^{n-1} dx^{j_{n-3}} \wedge (dx^{i_{n-2}} + X^n dx^{j_{n-2}}) \wedge (dx^{i_{n-1}} + X^{n+1} dx^{j_{n-1}}) \wedge (a^{j_{n-1}} dx^{j_{n-1}} + a^{n+1} dx^{n+1} + a^{n+2} dx^{n+2})$.

The first factor on the RHS of this equation allows to skip the summand $a^{n+1} dx^{n+1}$ in the last factor. Similarily, the factor $dx^{j_{n-1}}$ allows to skip the summands having $dx^{j_{n-1}}$ in the one before last and last factors, and so to have $a^{n+2} dx^{n+2}$, and also $dx^{i_{n-1}}$, as FACTORS.

Since j_{n-2} equals either i_{n-1} or j_{n-1}, one can skip further the summands having $dx^{j_{n-2}}$ in the ALL respective factors, obtaining thus: 1) $dx^{i_{n-2}}$ as a factor, and 2) the possibility to skip the summands having $dx^{j_{n-3}}$ in ALL the – remaining – respective factors (j_{n-3} being either i_{n-2} or j_{n-2}).

Continuing this process to its end, the equation in question eventually assumes the form

$$dx^{n+1} \wedge dx^{j_{n-1}} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-2}} \wedge dx^{i_{n-1}} \wedge a^{n+2} dx^{n+2} \;=\; 0,$$

and we are done by Corollary 11.

After this step one thus has

$$\omega^n \;=\; a^{j_{n-1}}dx^{j_{n-1}} + a^{n+1}dx^{n+1}. \tag{3}$$

Third step:

$$\left(a^{n+1}\frac{\partial a^{j_{n-1}}}{\partial x^{n+2}} - a^{j_{n-1}}\frac{\partial a^{n+1}}{\partial x^{n+2}}\right)(0) \neq 0. \tag{4}$$

Indeed: it follows directly from the first step that $d\omega^l \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n = 0 \quad$ for $l = 1, 2, \ldots, n-1$. But D is not involutive, so that necessarily

$$d\omega^n \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n|_0 \neq 0. \tag{5}$$

Let us calculate explicitly:
$d\omega^n \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n = (da^{j_{n-1}} \wedge dx^{j_{n-1}} + da^{n+1} \wedge dx^{n+1}) \wedge (dx^{i_1} + x^3 dx^{j_1}) \wedge (dx^{i_2} + x^4 dx^{j_2}) \wedge (dx^{i_3} + x^5 dx^{j_3}) \wedge \cdots \wedge (dx^{i_{n-2}} + X^n dx^{j_{n-2}}) \wedge (dx^{i_{n-1}} + X^{n+1} dx^{j_{n-1}}) \wedge (a^{j_{n-1}}dx^{j_{n-1}} + a^{n+1}dx^{n+1})$
$= (-1)^{n-1}(da^{j_{n-1}} \wedge dx^{j_{n-1}} \wedge a^{n+1}dx^{n+1} + da^{n+1} \wedge dx^{n+1} \wedge a^{j_{n-1}}dx^{j_{n-1}}) \wedge (dx^{i_1} + x^3 dx^{j_1}) \wedge (dx^{i_2} + x^4 dx^{j_2}) \wedge \cdots \wedge (dx^{i_{n-2}} + X^n dx^{j_{n-2}}) \wedge (dx^{i_{n-1}} + X^{n+1} dx^{j_{n-1}})$
$= (-1)^{n-1}(a^{n+1}da^{j_{n-1}} - a^{j_{n-1}}da^{n+1}) \wedge dx^{j_{n-1}} \wedge dx^{n+1} \wedge (dx^{i_1} + x^3 dx^{j_1}) \wedge (dx^{i_2} + x^4 dx^{j_2}) \wedge \cdots \wedge (dx^{i_{n-2}} + X^n dx^{j_{n-2}}) \wedge (dx^{i_{n-1}} + X^{n+1} dx^{j_{n-1}})$.

Now it is visible that one can consecutively skip in the $n-1$ last factors of the last RHS (analogously to the justification of the second step) the respective summands $X^{l+2}dx^{j_l}$, $l = 1, 2, \ldots, n-1$, obtaining eventually $(-1)^{n-1}(a^{n+1}da^{j_{n-1}} - a^{j_{n-1}}da^{n+1}) \wedge dx^{j_{n-1}} \wedge dx^{n+1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{n-2}} \wedge dx^{i_{n-1}}$.
The inequality (4) follows by the way of (5), using again Corollary 11.

Fourth step depends on whether $n = 2$ or $n \geq 3$:
When $n = 2$, one can assume without loss of generality that $a^3(0) \neq 0$. Indeed, if $a^3(0) = 0$ then $a^1(0) \neq 0$, and in the new coordinates $(\bar{x}^1, \bar{x}^2, x^3, x^4)$, $x^1 = \bar{x}^1 + x^3$, $x^2 = \bar{x}^2 - \frac{1}{2}(x^3)^2$, the coefficient at dx^3 is already invertible as the germ at 0, while the Darboux writing of $\omega^1 = 0$ keeps hold. That is, one can assume to be in the situation •• specified below, and so proceed along the lines therein.

For $n \geq 3$ a similar trick can no longer be performed with the function a^{n+1} in (3). Consequently, two situations may happen in (4):
• $a^{n+1}(0) = 0$.
In this case, obviously, $a^{j_{n-1}}(0) \neq 0$, and one could have simplified, before the third step, the writing of the last 1–form (in the vicinity of the origin): $\omega^n = dx^{j_{n-1}} + a^{n+1}dx^{n+1}$. Let us suppose to have done this, and remained, surely enough, within •. But then still, by (4), $\frac{\partial a^{n+1}}{\partial x^{n+2}}(0) \neq 0$. Consequently, taking $(x^1, x^2, \ldots, x^{n+1}, a^{n+1})$ as a new system of coordinates around $0 \in \mathbb{R}^{n+2}$, $\quad \omega^n = dx^{j_{n-1}} + x^{n+2}dx^{n+1} \quad$ (*i.e.*, $(i_n, j_n) = (j_{n-1}, n+1)$).
•• $a^{n+1}(0) \neq 0$.
One could then have $\omega^n = dx^{n+1} + a^{j_{n-1}}dx^{j_{n-1}}$ before the third step (remaining now, obviously, within ••). This time (4) yields $\frac{\partial a^{j_{n-1}}}{\partial x^{n+2}}(0) \neq 0$. On writing $a^{j_{n-1}}(0) = c^{n+2}$ and taking new coordinates around 0 $(x^1, x^2, \ldots, x^{n+1}, a^{j_{n-1}} - c^{n+2})$, $\quad \omega^n = dx^{n+1} + (c^{n+2} + x^{n+2})dx^{j_{n-1}} \quad$ (*i.e.*, $(i_n, j_n) = (n+1, j_{n-1})$).
Let us observe also that our very inductive procedure has been based on the fact that the skipping of the last equation $\omega^n = 0$ signified the passing to the first derived system $(D^{(1)})^\perp$ of $D^\perp$. Therefore – and that has already been derived from the induction hypothesis in the course of the first step – the skipping of any given number h of the last (bottommost) equations yields the h-th derived system $(D^{(h)})^\perp$.

Observe also that after the fourth step the constant c^4 (when $n = 2$) and c^5 (when $n = 3$ and $j_3 = 1$) may be present and of arbitrary value.

What remains now to be proved in Theorem 3 are the following two things:
– show that eventually the constants c^4, c^5 can be got rid of (*i.e.*, can, always when applicable, be taken 0);
– show the additional (last in Th. 3) statement concerning the situation $j_3 = j_4 = \cdots = j_l = 1$ for a certain $4 \le l \le n$.
As the whole proof, including this statement, goes by induction on n, we shall minimize our twofold task by the following trick: let us think that we are proving the same Theorem 3 – with x^3, x^4, x^5 replaced by $x^3 + c^3$, $x^4 + c^4$, $x^5 + c^5$ (respectively) and with $1 \le l \le n$ instead of $4 \le l \le n$ in the last statement of the theorem: if $j_1 = j_2 = \cdots = j_l = 1$ for certain $1 \le l \le n$, then $c^3, c^4, \dots, c^{l+2}$ can be taken 0 keeping $\omega^{l+1}, \omega^{l+2}, \dots, \omega^n$ untouched. This, of course, will not change the theorem.

The additional statement thus extended gives easily in by the same induction:
$n = 1$: the equation $\omega^1 = 0$ is, by Darboux theorem, with no constant c^3.
$n-1 \Rightarrow n$: if $l \le n-1$ then all has already been done, as the local form, in the vicinity of that fixed point p, of the first derived system of S, obtained from the induction premise is not changed at steps 1 through 4. When $l = n$ (the situation going to be called GNF), the equations $\omega^j = 0$, $j = 1, 2, \dots, n-1$ are written without constants by the induction hypothesis, and after the fourth step of the induction step $\omega^n = dx^{n+1} + (c^{n+2} + x^{n+2})dx^1$. Then one changes the coordinates around $0 \in \mathbb{R}^{n+2}$ as follows:

$$x^1 = \bar{x}^1,\ x^j = \bar{x}^j + (-1)^{n-j}\tfrac{c^{n+2}}{(n+2-j)!}(\bar{x}^1)^{n+2-j} \text{ for } j = 2, 3, \dots, n+1, \quad x^{n+2} = \bar{x}^{n+2}.$$

This keeps the equations $j = 1, 2, \dots, n-1$ untouched:

$$\omega^j = dx^{j+1} + x^{j+2}dx^1 = d\bar{x}^{j+1} + (-1)^{n-1-j}\tfrac{c^{n+2}}{(n-j)!}(\bar{x}^1)^{n-j}d\bar{x}^1 + \left(\bar{x}^{j+2} + (-1)^{n-j}\tfrac{c^{n+2}}{(n-j)!}(\bar{x}^1)^{n-j}\right)d\bar{x}^1 = d\bar{x}^{j+1} + \bar{x}^{j+2}d\bar{x}^1,$$

while the last equation assumes the desired form:

$$\omega^n = dx^{n+1} + (x^{n+2} + c^{n+2})dx^1 = d\bar{x}^{n+1} - c^{n+2}d\bar{x}^1 + (\bar{x}^{n+2} + c^{n+2})d\bar{x}^1 = d\bar{x}^{n+1} + \bar{x}^{n+2}d\bar{x}^1.$$

The last statement of the theorem – artifically extended in order to establish the eventual writing of ω^2 and ω^3 as well – is now proved by the induction that governs the whole proof of the theorem.

This concludes the proof of Theorem 3. □

Justification of Remark 4:
Any system of equations taken from the wording of Theorem 3 defines – even globally – a 2–distribution D on $\mathbb{R}^{n+2}(x^1, \dots, x^{n+1}, x^{n+2})$, because Observation 13 (taken for $n+1$ instead of n) guarantees the independence of equations. Let us fix an arbitrary integer l between 0 and $n-1$, and make the following two remarks:

1) For every $1 \le j \le l$ the $(l+3)$–form $d\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^{l+1}$ is written uniquely in the forms $dx^1, dx^2, \dots, dx^{l+2}$, and as such vanishes identically.

2) The $(l+3)$–form $d\omega^{l+1} \wedge \omega^1 \wedge \cdots \wedge \omega^{l+1}$ can be calculated as in the proof of (4), with the only exception that this time the last 1–form ω^{l+1} is known explicitly. Therefore, this $(l+3)$–form equals $dx^{l+3} \wedge dx^{j_{l+1}} \wedge dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_{l+1}} = \pm\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{l+2} \wedge dx^{l+3} \neq 0$ (by Cor. 11), and this at every point.

1) and 2) taken for $l = n-1$ say that $\dim(D^{(1)}/D^{(0)}) = 1$ at every point AND that $(D^{(1)})^\perp = (\omega^1, \dots, \omega^{n-1})$. In turn, $l = n-2$ yields $\dim(D^{(2)}/D^{(1)}) = 1$ at every point AND $(D^{(2)})^\perp = (\omega^1, \dots, \omega^{n-2})$. By the decreasing induction one thus obtains that D has at every point the big gr. v. $[2, 3, \dots, n+1, n+2]_{\mathrm{B}}$.

3. Local classification of GC in dimension 5 and 6

We group in this chapter, for completeness, the pertinent results of [4, 5, 7], using our compact coding of Definitions 5, 6.

Theorem 14. *In dimension* 5 *there exist precisely two non-equivalent local models of* GC*:*
1, having (at $0 \in \mathbb{R}^5$ *and everywhere else) the small gr. v.* $[2, 3, 4, 5]$*,*
and

3, having (at $0 \in \mathbb{R}^5$ *and at all points of* $\{x^5 = 0\}$*) the small gr. v.* [2, 3, 4, 4, 5]*, and having* [2, 3, 4, 5] *elsewhere.*

1 is just GNF*, while 3 is called the exceptional model.*

In dimension 6 *there exist precisely five non-equivalent local models of* GC*. They are as follows, written along with their small gr. v. at* $0 \in \mathbb{R}^6$ *(being all different):*

1.1	$[2, 3, 4, 5, 6]$,
1.3	$[2, 3, 4_2, 5_2, 6]$,
3.1	$[2, 3, 4, 5_3, 6]$,
*3.***2**	$[2, 3, 4, 5_2, 6]$,
3.3	$[2, 3, 4_2, 5_3, 6]$.

Attention: the subscripts in the growth vectors show how many times the respective integers occur in the actual vectors. In the sequel, we shall ONLY use this type of notation.

4. Local classification of GC in dimension 7

In the motivation of all constructions going to be made explicit in this and the following chapters, we are going to use many times one handy piece of information encompassing all 2–distributions satisfying GC, and having its roots in the paper [15].

Observation 15. For $n \geq 2$, any local diffeomorphism of $(\mathbb{R}^{n+2}, 0)$ into itself, conjugating near $0 \in \mathbb{R}^{n+2}$ two 2–distributions being both under the form given by Theorem 3, preserves their common linear subdistribution $(\frac{\partial}{\partial x^{n+2}})$.

Proof. If E is a 2–distribution under the form of Theorem 3 then, for $n \geq 2$, in view of Remark 4, E fulfils the condition (9.1) of [15] which gives rise to a linear subdistribution $L_E \subset E$ ([15], Prop. 9.1). On top of that (and easily enough), $L_E = (\frac{\partial}{\partial x^{n+2}})$ then. The note concluding the paragraph 9.4 of [15] gives an invariant characterization of L_E for any bracket generating 2–distribution E fulfilling (9.1) of that paper, and all the GC is bracket generating. □

The classification of GC in dimension 7 had, up to one important exception, been already given in ([7], p. 233), including the normalization (possible in this dimension) of the constants c^6 and c^7. After a careful study – as there was no explicit proof in the respective chapter 8 of [7] – the authors of the present paper have arrived at the conclusion that the only thing missing there had been the interrelation between the items No 9 and 10 (claimed non-equivalent) in the Kumpera–Ruiz list ([7], p. 233). They have turned out to be the same thing – see Theorem 17 below (and also [3], as explained in footnote 2 on p. 138).

Taking this into account, here is the eventual local classification of GC in dimension 7 (we are using the codes put forward in Defs. 5, 6):

Theorem 16. *In dimension* 7 *there exist the following* 13 *non-equivalent local models of* GC*, listed below alongside with their small gr. v. at* $0 \in \mathbb{R}^7$ *which are all different:*

1.1.1	$[2, 3, 4, 5, 6, 7]$
1.1.3	$[2, 3, 4_2, 5_2, 6_2, 7]$
1.3.1	$[2, 3, 4, 5_3, 6_3, 7]$
*1.3.***2**	$[2, 3, 4, 5_2, 6_2, 7]$
1.3.3	$[2, 3, 4_2, 5_3, 6_3, 7]$
3.1.1	$[2, 3, 4, 5, 6_4, 7]$
*3.1.***2**	$[2, 3, 4, 5, 6_3, 7]$
3.1.3	$[2, 3, 4_2, 5_2, 6_5, 7]$
*3.***2***.1*	$[2, 3, 4, 5, 6_2, 7]$
*3.***2***.3*	$[2, 3, 4_2, 5_2, 6_4, 7]$
3.3.1	$[2, 3, 4, 5_3, 6_4, 7]$
*3.3.***2**	$[2, 3, 4, 5_2, 6_3, 7]$
3.3.3	$[2, 3, 4_2, 5_3, 6_5, 7]$

Theorem 17. *3.***2***.1* $\equiv$ *3.***2.2**.

Proof. We search for a local diffeomorphism $g = (g^1, g^2, \ldots, g^7)$ conjugating in the vicinity of $0 \in \mathbb{R}^7$ the given 2–distributions. On writing down the explicit formulas for the coordinate functions of g, the proof would have been extremely short. Alternatively, we do want to supply some motivations – some general rules g should comply with. Those rules will turn out to be precise enough as regards the quest of g.

Applying Observation 15 to this situation, one gets that $g^1, \ldots, g^6$ depend only on $x^1, x^2, \ldots, x^6$. At that, $\frac{\partial}{\partial x^7}$ is a characteristic vector field for the (common) Lie square of the distributions, so that this common first derived system is actually defined, as a 2–distribution, on $\mathbb{R}^6(x^1, \ldots, x^6)$, and suspended only in the direction of $\frac{\partial}{\partial x^7}$. This system reduced to $\mathbb{R}^6$ is nothing but *3.***2**, and $(g^1, \ldots, g^6)$ is one of its automorphisms. On applying Observation 15 again, $g^1, \ldots, g^5$ depend only on $x^1, \ldots, x^5$. One can repeat this two steps further (passing to the second derived systems, then to third), and obtain additionally that g^4 depends only on $x^1, \ldots, x^4$, and g^1, g^2, g^3 – only on x^1, x^2, x^3.

Now the nature of g^5 can be further precised. A direct calculation shows that $\{x^5 = 0\}$ is, identically for both distributions in question, the locus of points at which the small gr. v. of either of them is $[2, 3, 4, 5, 6_2, 7]$.

(Outside this hyperplane their small vectors coincide with the unique and constant big one *cf.* Def. 1.).

As this set should be preserved by g, we certainly have $g^5(x^1, .., x^5) = x^5 G(x^1, .., x^5)$. Before writing down the equations for the sollicited g, let us adopt, and that until the end of the present paper, a

Notation: We shall write simply g^4_3 for $\frac{\partial g^4}{\partial x^3}$, G_5 for $\frac{\partial G}{\partial x^5}$, etc. When we want to evaluate an expression φ at 0, we will write $\varphi\ |0$. For instance, the equation $\varphi(0) = c^7$ will be written down as $\varphi\ |0 = c^7$, **no matter how long an expression for** φ**.**

We apply g_* to a (written in a natural way from the Pfaffian equations) second generator of the distribution $($*3.***2***.1*$)^\perp$, getting a combination of generators of $($*3.***2.2**$)^\perp$; then take all that at point $g(x^1, \ldots, x^7)$. We only write down SIX equations obtained by equalling the coefficients at $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^6}$, remembering about limitations that $g^1, \dots, g^6$ are subject to:

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 & 0 & 0 & 0 \\ g^2_1 & g^2_2 & g^2_3 & 0 & 0 & 0 \\ g^3_1 & g^3_2 & g^3_3 & 0 & 0 & 0 \\ g^4_1 & g^4_2 & g^4_3 & g^4_4 & 0 & 0 \\ x^5G_1 & x^5G_2 & x^5G_3 & x^5G_4 & G + x^5G_5 & 0 \\ g^6_1 & g^6_2 & g^6_3 & g^6_4 & g^6_5 & g^6_6 \end{pmatrix} \begin{pmatrix} -x^5 \\ x^3\,x^5 \\ x^4\,x^5 \\ 1 \\ -1 - x^6 \\ -x^7 \end{pmatrix} = f \begin{pmatrix} -x^5\,G \\ g^3\,x^5\,G \\ g^4\,x^5\,G \\ 1 \\ -1 - g^6 \\ -1 - g^7 \end{pmatrix} \tag{6}$$

where f is an invertible (germ of a) function, *i.e.*, in our notation, $f\ |0 \neq 0$. This last inequality comes from the fact that, by Observation 15, the line subdistribution $(\frac{\partial}{\partial x^7})$ is preserved by g.

Here is a first instance of some power hidden in this system of equations: watching the fourth equation and having the already mentioned information,

f depends only on $x^1, \dots, x^5$.

The equations "1", "2" and "3" of (6), after dividing them by x^5 (which is not a zero divisor in the ring of germs at 0 of functions on $\mathbb{R}^5(x^1, \dots, x^5)$), assume the form

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 \\ g^2_1 & g^2_2 & g^2_3 \\ g^3_1 & g^3_2 & g^3_3 \end{pmatrix} \begin{pmatrix} -1 \\ x^3 \\ x^4 \end{pmatrix} = f \begin{pmatrix} -G \\ g^3G \\ g^4G \end{pmatrix} \tag{7}$$

We are aiming at having as neat a system as possible, so that we make now a radical assumption that $f\,G = 1$[3] Under this assumption, (7) becomes

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 \\ g^2_1 & g^2_2 & g^2_3 \\ g^3_1 & g^3_2 & g^3_3 \end{pmatrix} \begin{pmatrix} -1 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} -1 \\ g^3 \\ g^4 \end{pmatrix}. \tag{8}$$

Now the following two objectives have to be met:

a) an appropriate g^4 (and g^3, g^2, g^1 – *cf.* (8) – too) should stand behing such f, PRODUCING IT by the way of equation "4" of (6);

b) the corresponding $G = f^{-1}$ and $g^5 = x^5G$ ought to produce, by the way of "5" of (6), a function g^6 capable to stand the test offered by the 0–jets of "6" of (6)[4].

Ad b) Assume for simplicity that $f\ |0 = 1$ $(= G\ |0)$. The equation "5" of (6) will then be guaranteed on the level of 0–jets, and g^6 will be defined once g^5 is known. Under this assumption the equation "6" means on the 0–jet level $g^6_5 - g^6_4\ |0 = 1$. On the other hand, solving "5" of (6) for g^6, we compute directly that $g^6_4\ |0 = G_4 - f_4\ |0$ and $g^6_5\ |0 = -f_5 - G_4 + 2G_5\ |0$. Hence our requirement reads

$$f_4 - f_5 + 2G_5 - 2G_4\ |0 = 1. \tag{9}$$

Watching (9), after a couple of trials (meaning also the meeting of a)), we find it purposeful to seek f in the form $1 + a\,x^5$, with a being a real constant (a decisive moment for the proof). Then $G_5\ |0 = -a$, $f_5\ |0 = a$, and $a = -\frac{1}{3}$ makes (9) hold.

Ad a) Having $f = 1 - \frac{1}{3}x^5$ already, we guess consecutively (in a kind of the domino effect) that: $g^4_1 = \frac{1}{3}$, $g^4_4 = 1$, and so $g^4 = \frac{1}{3}x^1 + x^4$ does.

This suggests (see "3" of (8)) $g^3_1 = -\frac{1}{3}x^1$, $g^3_3 = 1$, and further $g^3 = -\frac{1}{6}(x^1)^2 + x^3$.

[3]This assumption does in dimension 7, as well as in all but one cases in dimension 8. It is not implied by (7), however, and the assumption will have to be abandoned in Chap. 5; see Remark 30.

[4]Here resides the core of the problem, as we try to conjugate zero and non-zero constants of *3*.**2**.*1* and *3*.**2**.**2**.

This in turn suggests ("2" of (8)) $g^2_1 = \frac{1}{6}(x^1)^2$, $g^2_2 = 1$, hence we guess $g^2 = \frac{1}{18}(x^1)^3 + x^2$.

At last, in "1" of (8), there suffices to have $g^1_1 = 1$, so that we put $g^1 = x^1$.

Having thus met a), we pass to the higher coordinates, noting in the meantime that $g^5 = x^5(1 - \frac{1}{3}x^5)^{-1}$.

Now the equation "5" of (6), secured until now on the 0–jet level, yields easily $g^6 = (1 + x^6)(1 - \frac{1}{3}x^5)^{-3} - 1$, making it possible to write "6" of (6) as

$$(1 + x^6)\, g^6_5 + x^7 g^6_6 \;=\; (1 - \frac{1}{3}x^5)(1 + g^7).$$

We are nearly done, for, computing directly,

$$g^7 \;=\; \frac{x^7}{(1 - \frac{1}{3}x^5)^4} + \frac{(1 + x^6)^2}{(1 - \frac{1}{3}x^5)^5} - 1.$$

All seven coordinate functions of g are now given explicitly. Looking at their expressions – no doubt that g is a local diffeomorphism of $(\mathbb{R}^7, 0)$ into itself, assuring, by the very construction, the equivalence in question. □

Corollary 18. *In dimension* 8, *3*.**2**.**2**.*2 is equivalent either to 3*.**2**.*1*.*1 or to 3*.**2**.*1*.*2.*

Proof. A system of the form *3*.**2**.*1*.*1* or *3*.**2**.*1*.*2* displays $X^8 = x^8 + \tilde{c}^8$, where $\tilde{c}^8 \in \mathbb{R}$ (*cf.* Defs. 5, 6). We search for a diffeo $g = (g^1, \ldots, g^7, g^8)$, with $g^1, \ldots, g^7$ constructed in the proof of Theorem 17, conjugating near $0 \in \mathbb{R}^8(x^1, .., x^8)$ certain such system to a GIVEN system *3*.**2**.**2**.*2* displaying a non-zero constant c^8. Stipulating conjugation now, we extend the system of equations (6) by one more equation "7":

$$-\, g^7_1 x^5 + g^7_2 x^3\, x^4 + g^7_3 x^4\, x^5 + g^7_4 - g^7_5(1 + x^6) - g^7_6 x^7 - g^7_7(\tilde{c}^8 + x^8) = -\, f(c^8 + g^8)$$

with unchanged $f = 1 - \frac{1}{3}x^5$. We secure "7" on the 0–jet level first (as we did with "6" in the previous proof), by writing $g^7_4 - g^7_5 - \tilde{c}^8 g^7_7 \,|0 = -\, c^8$, hence by picking up $\tilde{c}^8 = c^8 - \frac{5}{3}$.

Now that the constants match, we are able to solve "7" for a vanishing at $0 \in \mathbb{R}^8(x^1, .., x^8)$ function g^8,

$$g^8 \;=\; \frac{x^8}{(1 - \frac{1}{3}x^5)^5} + g^8(x^1, .., x^7, 0).$$

Surely enough, $(g^1, \ldots, g^7, g^8)$ is a local diffeomorphism near 0.

Observe, that for a specific value of c^8 $(= \frac{5}{3})$ we get $\tilde{c}^8 = 0$. This explains the formulation (a bit strange) of the corollary. □

Corollary 19. *In dimension* 8, *3*.**2**.**2**.*1* $\equiv$ *3*.**2**.*1*.*2*.

Proof. Put $c^8 = 0$ in the previous proof. The desired system is produced from the one with $\tilde{c}^8 = -\frac{5}{3}$. □

Remark 20. Here is an alternative proof of Corollary 18:

Starting from *3*.**2**.**2**.*2*, one knows (from Th. 3) that its first derived system is *3*.**2**.**2** suspended in the direction $\frac{\partial}{\partial x^8}$. Now apply to it the diffeomorphism g constructed in the proof of Theorem 17 extended to 8 dimensions by taking $g^8 = x^8$. Plug *3*.**2**.**2**.*2*, but changed by this diffeomorphism!, into the proof of Theorem 3 at the moment of writing (2)[5]. The system resulting NOW from that proof is, surely enough, *3*.**2**.*1*.*k*, with $k \in \{1, 2, 3\}$. The proof ends by noting that $k \neq 3$ in virtue of the old argument of [7], p. 234: $k = 3$ corresponds to the second

[5] $n = 6$, $j_5 = 4$.

case $dh + l\,dg$ on that page, implying $\mathrm{rg}\,\kappa_{(2)} = 0$ at the origin, $\kappa_{(2)}$ – so-called second reduced tensor of Kumpera and Ruiz. On the other hand, *3*.**2**.**2**.*2* falls within the first case ($dg + l\,dh$) which implies $\mathrm{rg}\,\kappa_{(2)} = 1$ at the origin[6].

(One could also use the small gr. v. at 0, instead of the tensor $\kappa_{(2)}$.)

5. Local classification of GC in dimension 8

We start the analysis of this dimension by a remark on constants.

Remark 21. Local forms in dimension 8 coming out of Theorem 3 often possess constants (c^6, c^7 or c^8). In several cases these constants can be normalized to 1 by just rescaling certain coordinates, as Kumpera and Ruiz did in dimension 6 and 7, with the only exceptions of *3.2.2.2* and *3.2.1.2*, when possible reductions include *3*.**2**.**2**.*2*, and *3*.**2**.*1*.*2*, respectively. This is left to the reader as an exercise, and will be used in proving the main Theorem 23 (INCLUDING these mentioned reductions in the two most stubborn cases in dimension 8).
We underline, however, that it is just a technical simplification intended to keep the proofs as transparent as possible. Besides, the normalization of a constant(s) need not mean arriving at a definitive local model[7].

The shortest proof has

Theorem 22. *3*.**2**.**2**.*3* $\equiv$ *3*.**2**.*1*.*3*.

Proof. The alternative proof of Corollary 18, given in Remark 20 applies also here, with *3*.**2**.**2**.*3* instead of *3*.**2**.**2**.*2*, with the only change: *3*.**2**.**2**.*3* has $\mathrm{rg}\,\kappa_{(2)} = 0$ at 0, so the output *3*.**2**.*1*.*k* has also $\mathrm{rg}\,\kappa_{(2)} = 0$ at 0, implying $k = 3$. □

We arrive now at our main theorem reporting – locally – on the status of GC in dimension 8. Each item on this final list is accompanied by its small gr. v. at 0, making thus clear that all the models are really distinct.

Theorem 23. GC *in dimension* 8 *locally materializes itself as the germ at* $0 \in \mathbb{R}^8(x^1, .., x^8)$ *of precisely one among the following* 34 *systems. Alongside with the local models written are their small growth vectors at* $0 \in \mathbb{R}^8$ *that discern the models univocally.*

[6]One should read that page carefully, the evoked argument being correct, and the sentence "Par contre, si S_1 n'est pas homogène, il donnera naissance à deux systèmes S non-équivalents ..." – being wrong in general (from the dimension 7 – the length 5 – onwards), as shows our Theorem 17.

[7]For example, *3*.*1*.**2**.**2** $\equiv$ *3*.*1*.**2**.*1* — Theorem 27.

1.1.1.1	[2, 3, 4, 5, 6, 7, 8]
1.1.1.3	[2, 3, 4_2, 5_2, 6_2, 7_2, 8]
1.1.3.1	[2, 3, 4, 5_3, 6_3, 7_3, 8]
*1.1.3.***2**	[2, 3, 4, 5_2, 6_2, 7_2, 8]
1.1.3.3	[2, 3, 4_2, 5_3, 6_3, 7_3, 8]
1.3.1.1	[2, 3, 4, 5, 6_4, 7_4, 8]
*1.3.1.***2**	[2, 3, 4, 5, 6_3, 7_3, 8]
1.3.1.3	[2, 3, 4_2, 5_2, 6_5, 7_5, 8]
*1.3.***2***.1*	[2, 3, 4, 5, 6_2, 7_2, 8]
*1.3.***2***.3*	[2, 3, 4_2, 5_2, 6_4, 7_4, 8]
1.3.3.1	[2, 3, 4, 5_3, 6_4, 7_4, 8]
*1.3.3.***2**	[2, 3, 4, 5_2, 6_3, 7_3, 8]
1.3.3.3	[2, 3, 4_2, 5_3, 6_5, 7_5, 8]
3.1.1.1	[2, 3, 4, 5, 6, 7_5, 8]
*3.1.1.***2**	[2, 3, 4, 5, 6, 7_4, 8]
3.1.1.3	[2, 3, 4_2, 5_2, 6_2, 7_7, 8]
*3.1.***2***.1*	[2, 3, 4, 5, 6, 7_3, 8]
*3.1.***2***.3*	[2, 3, 4_2, 5_2, 6_2, 7_6, 8]
3.1.3.1	[2, 3, 4, 5_3, 6_3, 7_7, 8]
*3.1.3.***2**	[2, 3, 4, 5_2, 6_2, 7_5, 8]
3.1.3.3	[2, 3, 4_2, 5_3, 6_3, 7_8, 8]
*3.***2***.1.1*	[2, 3, 4, 5, 6, 7_2, 8]
*3.***2***.1.3*	[2, 3, 4_2, 5_2, 6_2, 7_4, 8]
*3.***2***.3.1*	[2, 3, 4, 5_3, 6_3, 7_6, 8]
*3.***2***.3.***2**	[2, 3, 4, 5_2, 6_2, 7_4, 8]
*3.***2***.3.3*	[2, 3, 4_2, 5_3, 6_3, 7_6, 8]
3.3.1.1	[2, 3, 4, 5, 6_4, 7_5, 8]
*3.3.1.***2**	[2, 3, 4, 5, 6_3, 7_4, 8]
3.3.1.3	[2, 3, 4_2, 5_2, 6_5, 7_7, 8]
*3.3.***2***.1*	[2, 3, 4, 5, 6_2, 7_3, 8]
*3.3.***2***.3*	[2, 3, 4_2, 5_2, 6_4, 7_6, 8]
3.3.3.1	[2, 3, 4, 5_3, 6_4, 7_7, 8]
*3.3.3.***2**	[2, 3, 4, 5_2, 6_3, 7_5, 8]
3.3.3.3	[2, 3, 4_2, 5_3, 6_5, 7_8, 8]

Proof. It has already begun in Chap. 4 (Cor. 18, 19 and Th. 22), the departure point having been Theorem 3 enhanced in this dimension 8 by Remark 21. The rest of it is built out of four separated statements (being of some interest in their own right): Lemma 24, Lemma 25, Lemma 26 and Theorem 27.

Lemma 24. *1.3.***2.2** $\equiv$ *1.3.***2.***1*.

Lemma 25. *3.3.***2.2** $\equiv$ *3.3.***2.***1*.

Lemma 26. *3.***2.***1.2* $\equiv$ *3.***2.***1.1*.

Theorem 27. *3.1.***2.2** $\equiv$ *3.1.***2.***1*.

At first we sketch the proof of Lemma 24, its technique being but that of Theorem 17 (*cf.* also Rem. 28): We search now a diffeo $g = (g^1, \dots, g^5, x^6G, g^7, g^8)$ sending $(1.3.\mathbf{2}.1)^\perp$ to $(1.3.\mathbf{2.2})^\perp$. The piece of information $g^6 = x^6G$ comes from the fact that now $\{x^6 = 0\}$ is, the same for both systems, the locus of the small gr. v. differing from the standard one of GNF (in the occurrence, it is $[2, 3, 4, 5, 6_2, 7_2, 8]$), and g has to preserve this set.

Using several times Observation 15, one knows that g^4 depends on $x^1, .., x^4$, and g^1, g^2, g^3 – on x^1, x^2, x^3. A similar to (6) system of – seven now – equations can be written, subsuming the relations necessary to hold in the directions $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^7}$, with an invertible function factor f, as previously.

The first 4 of those equations are, consequently, divisible by x^6, so let us have them divided by x^6 already. Imitating further the proof of Theorem 17, we stipulate again $f\,G = 1$. Then, omitting this unit factor, the first four equations read

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 & 0 \\ g^2_1 & g^2_2 & g^2_3 & 0 \\ g^3_1 & g^3_2 & g^3_3 & 0 \\ g^4_1 & g^4_2 & g^4_3 & g^4_4 \end{pmatrix} \begin{pmatrix} -1 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = \begin{pmatrix} -1 \\ g^3 \\ g^4 \\ g^5 \end{pmatrix}. \tag{10}$$

The analogy with the situation in Theorem 17 is far reaching, with x^l and g^l, $l \in \{4, 5, 6, 7\}$ from its proof being now replaced by x^{l+1} and g^{l+1}, respectively. (Then "5" and "6" of (6) become the equations number "6" and "7" now.) We put $f = 1 - \frac{1}{3}x^6$, $g^6 = \frac{x^6}{1-\frac{1}{3}x^6}$. Then the analogy yields automatically $g^7 = \frac{1+x^7}{(1-\frac{1}{3}x^6)^3} - 1$, and

$$g^8 \;=\; \frac{x^8}{(1-\frac{1}{3}x^6)^4} + \frac{(1+x^7)^2}{(1-\frac{1}{3}x^6)^5} - 1.$$

Along with a guess solving (10) and modelled on the previous one for (8), we are done:
$g^5 = x^5 + \frac{1}{3}x^1$, $\quad g^4 = x^4 - \frac{1}{6}(x^1)^2$, $\quad g^3 = x^3 + \frac{1}{18}(x^1)^3$, $\quad g^2 = x^2 - \frac{1}{72}(x^1)^4$, $\quad g^1 = x^1$. □

Remark 28. Examining the proofs of Theorem 17 and Lemma 24, one easily sees how to construct an explicit equivalence *1 . . . 1.3.***2.***1* $\equiv$ *1 . . . 1.3.***2.2** in each dimension exceeding 8.

Proof of Lemma 25. About a diffeo g sending $(3.3.\mathbf{2}.1)^\perp$ to $(3.3.\mathbf{2.2})^\perp$, we know for sure that $g^5 = x^5G^5$ and $g^6 = x^6G^6$. This is so because in the present situation both $\{x^5 = 0\}$ and $\{x^6 = 0\}$ can be invariantly characterized, and that – for both distributions at a time, in terms of the small gr. v. We skip the particulars of that, mentioning only that the two variables x^5 and x^6 are distinguished in the both systems: their first appearances in these Pfaffian systems are coded with "*3*", thus making important their 0–level sets (*cf.* also [7], p. 234). Alternatively, the discussion of these variables boils down to the second derived systems, being both (the suspension of) *3.3*. In that low dimension 6 one can calculate by hand the small vectors at ALL points (see also Th. 14), thus finding the relevance of the two mentioned hyperplanes.

Let us write, for the reader's convenience, the starting system of seven equations, implied – as usual – by Observation 15.

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 & 0 & 0 & 0 & 0 \\ g^2_1 & g^2_2 & g^2_3 & 0 & 0 & 0 & 0 \\ g^3_1 & g^3_2 & g^3_3 & 0 & 0 & 0 & 0 \\ g^4_1 & g^4_2 & g^4_3 & g^4_4 & 0 & 0 & 0 \\ x^5G^5_1 & x^5G^5_2 & x^5G^5_3 & x^5G^5_4 & G^5+x^5G^5_5 & 0 & 0 \\ x^6G^6_1 & x^6G^6_2 & x^6G^6_3 & x^6G^6_4 & x^6G^6_5 & G^6+x^6G^6_6 & 0 \\ g^7_1 & g^7_2 & g^7_3 & g^7_4 & g^7_5 & g^7_6 & g^7_7 \end{pmatrix} \begin{pmatrix} x^5\,x^6 \\ -\,x^3\,x^5\,x^6 \\ -\,x^4\,x^5\,x^6 \\ -\,x^6 \\ 1 \\ -1-x^7 \\ -\,x^8 \end{pmatrix} = f \begin{pmatrix} x^5\,G^5\,x^6\,G^6 \\ -\,g^3\,x^5\,G^5\,x^6\,G^6 \\ -\,g^4\,x^5\,G^5\,x^6\,G^6 \\ -\,x^6\,G^6 \\ 1 \\ -1-g^7 \\ -1-g^8 \end{pmatrix}, \tag{11}$$

$f\;|0 \neq 0$. The equations "1" – "4" of (11), after dividing them by x^6, read

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 & 0 \\ g^2_1 & g^2_2 & g^2_3 & 0 \\ g^3_1 & g^3_2 & g^3_3 & 0 \\ g^4_1 & g^4_2 & g^4_3 & g^4_4 \end{pmatrix} \begin{pmatrix} x^5 \\ -\,x^3\,x^5 \\ -\,x^4\,x^5 \\ -\,1 \end{pmatrix} \;=\; f\,G^6 \begin{pmatrix} x^5\,G^5 \\ -\,g^3\,x^5\,G^5 \\ -\,g^4\,x^5\,G^5 \\ -\,1 \end{pmatrix}. \tag{12}$$

In turn, the equations "1" – "3" of (12) get divided by x^5. On stipulating $(f\,G^6)\,G^5 = 1$, they become, after multiplying both sides by -1, just (8). Therefore, we take $g^1, \dots, g^4$ as in the proof of Theorem 17, after which "4" of (12) gives us $f\,G^6 = 1 - \frac{1}{3}x^5$, and G^5 – the inverse of it. Thus g^5 is also being taken as in Theorem 17.

Now the equation "5" of (11) brings in $f = \frac{1}{(1-\frac{1}{3}x^5)^2}$. In consequence we have $G^6 = (1-\frac{1}{3}x^5)^3$, $g^6 = x^6(1-\frac{1}{3}x^5)^3$, and "6" of (11) is satisfied on the 0–jet level, $G^6\;|0 = f\;|0$, yielding then

$$g^7 \;=\; (1+x^7)(1-\frac{1}{3}x^5)^5 + x^6(1-\frac{1}{3}x^5)^4 - 1.$$

We arrive finally at the conjugation of constants, and write explicitly the equation "7" of (11):

$$g^7_5 - g^7_6(1+x^7) - g^7_7x^8 \;=\; -\,f\,(c^8+g^8). \tag{13}$$

The proof is not yet finished, because nothing guarantees that c^8 be 1. Nevertheless, in view of Remark 21, any non-zero value of c^8 would do, and this is our case: calculating g^7_5, g^7_6, g^7_7 we see that the value of c^8 securing (13) on the 0–jet level is $\frac{8}{3}$. With this constant, we solve (13) for g^8. For the curious,

$g^8 \;=\; x^8(1-\frac{1}{3}x^5)^7 + \frac{8}{3}(1+x^7)(1-\frac{1}{3}x^5)^6 + \frac{4}{3}x^6(1-\frac{1}{3}x^5)^5 - \frac{8}{3}$. □

Proof of Lemma 26. As in proving Theorem 17 (with which there will be many resemblances), we want in the first place to give motivations. Again, $\{x^5 = 0\}$ is invariant, as the locus of the small gr. v. [2, 3, 4, 5, 6, 7_2, 8] for both $(3.\mathbf{2}.1.1)^\perp$ and $(3.\mathbf{2}.1.2)^\perp$[8]. Therefore a diffeo g conjugating near $0 \in \mathbb{R}^8$ the former distribution to the latter should have its fifth coordinate function g^5 of the form x^5G, precisely as in Theorem 17. And more, given the well-known limitations (based on Obs. 15) concerning variables being only essential in other coordinates of

[8] The small vector coincides with the big one, for either distribution, off $\{x^5 = 0\}$.

g, the equations "1" – "5" of (6) hold true in the present situation. The equation "6" of (6) remains true after skipping the -1 on its RHS (should be just $-f\,g^7$, as $c^7 = 0$ now), and we get a new equation comparing the coefficients in the $\frac{\partial}{\partial x^7}$ −direction (after multiplying its both sides by -1):

$$x^5 g^7_1 - x^3 x^5 g^7_2 - x^4 x^5 g^7_3 - g^7_4 + (1 + x^6)\, g^7_5 + x^7 g^7_6 + x^8 g^7_7 = f\,(c^8 + g^8). \tag{6′}$$

On top of that, we still stipulate $f\,G = 1,\ f\ |0 = 1$. The complexity of the proof – in comparison to that of Theorem 17 – grows, as one should meet now:
a) from the proof of Theorem 17;
b') $G = f^{-1}$ produces a g^6 satisfying now $g^6_5 - g^6_4\ |0 = 0$ which means $2G_5 - 2G_4 + f_4 - f_5\ |0 = 0$ ("5" of (6) holding true, the LHS of (9) is still $g^6_5 - g^6_4\ |0$, and $c^7 = 0$ now);
c) g^6 produces a g^7 capable to firstly satisfy (6') on the 0–jet level: $g^7_5 - g^7_4\ |0 = c^8$.
Therefore, there are two musts dealing with constants now — b)' and c) — and c) descends deeper into the functions f and G, as one sees in what follows[9].

Beginning with c), the most demanding is just to express $g^7_4\ |0$ and $g^7_5\ |0$ in function of f and G. In order to calculate these derivatives, we express g^7 by g^6, then restrict to $\{x^6 = x^7 = 0\}$, obtaining $f^{-1}(x^5 g^6_1 - x^3 x^5 g^6_2 - x^4 x^5 g^6_3 - g^6_4 + g^6_5)$. Therefore, $g^7_4\ |0 = (f^{-1}(-\,g^6_4 + g^6_5))_4\ |0 = -\,g^6_{44} + g^6_{45}\ |0$ (we have used b') already), and $g^7_5\ |0 = g^6_1 - g^6_{45} + g^6_{55}\ |0$. The requirement c) assumes the form

$$g^6_1 + g^6_{44} - 2g^6_{45} + g^6_{55}\ |0 = c^8. \tag{14}$$

By "5" of (6), $g^6_1\ |0 = (f^{-1}G)_1\ |0 = G_1 - f_1\ |0$. Turning to g^6_4, we write this function putting $x^6 = 0$ and skipping its terms included in m^2, m – the maximal ideal of germs of functions at $0 \in \mathbb{R}^5(x^1, .., x^5)$, as we only want to compute $g^6_{44}\ |0$ and $g^6_{45}\ |0$ (we write $\equiv$ instead of $=$):
$g^6_4 \equiv f^{-1}(-\,x^5 G_{44} + G_4 + x^5 G_{45}) - f_4 f^{-2}(-\,x^5 G_4 + G + x^5 G_5)$.
Therefore,

$$\begin{aligned} g^6_{44}\ |0 &= -\,2f_4 G_4 + G_{44} - f_{44} + 2(f_4)^2\ |0,\\ g^6_{45}\ |0 &= -\,f_5 G_4 - G_{44} + 2G_{45} + f_4 G_4 - 2f_4 G_5 - f_{45} + 2f_4 f_5\ |0. \end{aligned}$$

Proceeding analogously with g^6_5,

$$\begin{aligned} g^6_5 &\equiv f^{-1}(2\,x^5 G_1 - G_4 - x^5 G_{45} + 2G_5 + x^5 G_{55}) - f_5 f^{-2}(-\,x^5 G_4 + G + x^5 G_5),\\ g^6_{55}\ |0 &= 2f_5\, G_4 - 4f_5 G_5 + 2G_1 - 2G_{45} + 3G_{55} - f_{55} + 2(f_5)^2\ |0. \end{aligned}$$

All in all, (14) can now be written as

$$\begin{aligned} &\underline{3G_1} - \underline{f_1} - 4f_4 G_4 + 3G_{44} - f_{44} + 2(f_4)^2 + 4f_5 G_4 - \underline{6G_{45}} + 4f_4 G_5 + \underline{2f_{45}} + \\ &\quad -\,4f_4 f_5 - 4f_5 G_5 + 3G_{55} - f_{55} + 2(f_5)^2\ |0 \ = \ c^8. \end{aligned} \tag{15}$$

Sticking to b') and having in mind to meet a), it is not quite easy to choose a promising form of f that together with $G = f^{-1}$ would guarantee (15). In fact, it was a simultaneous looking at a) and (15) that has led us to suppose that $f = 1 + a\,x^1 - 3a\,x^4 x^5$, with a being a real parameter (see later a bit more on that).

As regards (15), only the underlined summands are then non-zero, with $f_1\ |0 = a$, $G_1\ |0 = -\,a$, $G_{45}\ |0 = 3a$, $f_{45}\ |0 = -\,3a$. Consequently, (15) becomes $-\,3a - a - 18a - 6a = c^8$, or else $a = -\frac{c^8}{28}$.

[9]For a presumable *3*.**2**.*1*.*1*.*1* $\equiv$ *3*.**2**.*1*.*1*.*2* in dimension 9, the LHS's of (9) and (15) would equal 0, and $g^8_5 - g^8_4\ |0 = c^9$ would get expressed by the 3–jets of f and G, *etc.*

b') is easily met, too.

As for a) – or else the three equations (8) plus the equation "4" of (6) producing f – all of them are also met, although the domino effect is less transparent than in dimension 7. After a while we guess:

$$\begin{cases} g^1 = x^1, \\ g^2 = (1 + a\,x^1)\,x^2, \\ g^3 = -\,a\,x^2 + (1 + a\,x^1)\,x^3, \\ g^4 = -\,2a\,x^3 + (1 + a\,x^1)\,x^4. \end{cases}$$

We are done already, because the constants in the equations defining g^6, g^7, g^8 are adjusted (including "5" of (6), meaning on the 0–jet level $G\,|0 = f\,|0$, a very obvious thing in our approach), and the remaining is just the matter of consecutive solving of equations: "5" of (6) for g^6, "6" of (6) with the RHS without -1 – for g^7, $(6')$ – for g^8. Looking at the coefficients at the highest variables x^l in g^l, we get altogether a diffeo $(\mathbb{R}^8, 0) \hookleftarrow$. ☐

We skip writing down the explicit formulas for $g^5, \ldots, g^8$ corresponding to the found value of a $(= -\frac{c^8}{28})$, preferring to give again – in a compact form – our solution, pertinent for this chapter, of the decisive system of equations (8):

$$\begin{pmatrix} 1 & 0 & 0 \\ a\,x^2 & 1 + a\,x^1 & 0 \\ a\,x^3 & -\,a & 1 + a\,x^1 \end{pmatrix} \begin{pmatrix} -1 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} -1 \\ -\,a\,x^2 + (1 + a\,x^1)\,x^3 \\ -\,2a\,x^3 + (1 + a\,x^1)\,x^4 \end{pmatrix}.$$

The function g^4 written above produces, by the way of "4" of (6), the functional coefficient f put forward in the present proof.

Remark 29. One could wonder why not – instead of splitting into Corollary 18 and Lemma 26 – to prove directly that *3*.**2**.**2**.*2* ≡ *3*.**2**.*1*.*1*?

The answer is that the method of proving Theorem 17 ($f = 1 + a\,x^5$, $f\,G = 1$) applied directly to the equivalence in question would choose just one value of a $(= -\frac{1}{3})$ good for producing $c^7 = 1$, and with that a one could NOT produce an arbitrary c^8, but only, as actually has been computed in the proof of Corollary 18, $c^8 = \frac{5}{3}$ (for a general c^7 that would be $c^8 = \frac{5}{3}(c^7)^2$). *Cf.* Remark 21.

Proof of Theorem 27. We suppose that a local diffeo g: $(\mathbb{R}^8, 0) \hookleftarrow$ sends $(3.1.\mathbf{2}.1)^\perp$ to $(3.1.\mathbf{2}.2)^\perp$. By analyzing the limitations it has to be subject to, we will eventually construct such a g.

Attention: we have written *2* in the end of the latter system' code, instead of **2**, because the non-zero constant c^8 that would appear in the course of the proof, would NOT yet be normalized. The theorem will only follow by Remark 21.

The two distributions in question have the small gr. v. [2, 3, 4, 5, 6, 7_3, 8] at points of $\{x^5 = x^6 = 0\}$, and the small vector [2, 3, 4, 5, 6, 7_2, 8] at points of $\{x^5 = 0,\ x^6 \neq 0\}$. Needless to say, then, that $\{x^5 = 0\}$ should be preserved by g also in this situation, and that we could write safely $g^5 = x^5 G$ (G–certain function of $x^1, \ldots, x^5$). Notwithstanding certain analogies with the situations already discussed, let us write down the full respective system of seven equations (with the well-known simplifications already introduced), as minor changes in the formulas hide here some essential differences:

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 & 0 & 0 & 0 & 0 \\ g^2_1 & g^2_2 & g^2_3 & 0 & 0 & 0 & 0 \\ g^3_1 & g^3_2 & g^3_3 & 0 & 0 & 0 & 0 \\ g^4_1 & g^4_2 & g^4_3 & g^4_4 & 0 & 0 & 0 \\ x^5\,G_1 & x^5\,G_2 & x^5\,G_3 & x^5\,G_4 & G + x^5\,G_5 & 0 & 0 \\ g^6_1 & g^6_2 & g^6_3 & g^6_4 & g^6_5 & g^6_6 & 0 \\ g^7_1 & g^7_2 & g^7_3 & g^7_4 & g^7_5 & g^7_6 & g^7_7 \end{pmatrix} \begin{pmatrix} -\,x^5 \\ x^3x^5 \\ x^4x^5 \\ 1 \\ -\,x^6 \\ -1 - x^7 \\ -\,x^8 \end{pmatrix} = f \begin{pmatrix} -\,x^5\,G \\ g^3x^5G \\ g^4x^5G \\ 1 \\ -\,g^6 \\ -1 - g^7 \\ -\,c^8 - g^8 \end{pmatrix}, \tag{16}$$

$f\,|0 \neq 0$. Observe that "5" of (16) gives $g^6 = x^5\alpha(x^1,..,x^5) + x^6\beta(x^1,..,x^5)$[10], where $\alpha = f^{-1}(x^5G_1 - x^3x^5G_2 - x^4x^5G_3 - G_4)$, and $\beta = f^{-1}(G + x^5G_5)$.

The equation "6" of (16) would produce a precise g^7 provided that "6" were satisfied on the 0–jet level $g^6_6 - g^6_4\,|0 = f\,|0$. Here $g^6_4\,|0 = 0$ and $g^6_6\,|0 = \frac{G}{f}\,|0$, whence the requirement

$$G\,|0 \;=\; (f)^2\,|0. \tag{17}$$

Under (17), g^7 may now be supposed found (or else: expressed in terms of g^6), and we may pass to the 0–jet level of "7" of (16), where lies the core of the problem:

$$g^7_6 - g^7_4\,|0 \;=\; c^8\,f\,|0. \tag{18}$$

During the computing of $g^7_4\,|0$ one may substitute $x^5 = x^6 = x^7 = 0$ to g^7, while for $g^7_6\,|0$ it is allowed to have $x^5 = x^7 = 0$, remembering also that f does not depend on x^6. Making these calculations carefully,

$g^7_4\,|0 \;=\; -\,f_4f^{-2}\beta + f^{-1}\beta_4\,|0, \quad g^7_6\,|0 \;=\; -\,f^{-1}\beta_4 + f^{-1}\alpha\,|0,$

after which the LHS of (18) becomes

$$\begin{aligned} &-\frac{2}{f}\beta_4 + \frac{1}{f}\alpha + \frac{f_4}{(f)^2}\beta\,|0 \\ &= \quad -\frac{2}{f}\frac{G_4f - G\,f_4}{(f)^2} - \frac{G_4}{(f)^2} + \frac{f_4}{(f)^2}\frac{G}{f}\,|0 \\ &= \quad -\frac{2G_4}{(f)^2} + \frac{2f_4}{f} - \frac{G_4}{(f)^2} + \frac{f_4}{f}\,|0 \\ &= \quad \frac{3}{f}\left(f_4 - \frac{G_4}{f}\right)\,|0, \end{aligned}$$

giving (18) a new look

$$c^8 = 3\,f^{-2}(f_4 - f^{-1}G_4)\,|0. \tag{19}$$

Let us make a sidekick (that turned out important in the course of proving Th. 27):

Remark 30. The simplifying trick $f\,G = 1$, working in all previous situations, is impossible in the situation of Theorem 27.

Justification. Suppose to have $f\,G = 1$. (17) has to hold anyway, hence $f\,|0 = G\,|0 = 1$. But then also $G_4\,|0 = -\,f_4\,|0$, so that (19) assumes the form $6\,f_4\,|0 = c^8$. Yet, under the assumption made, the old reduced system (8) holds, yielding g^4 as an AFFINE function of x^4 (here and in the sequel we mean this in the STRONG sense that the coefficients are but functions of x^1, x^2, x^3). This gives quickly $f_4\,|0 = 0$, as f is, by "4" of (16), the sum of g^4_4 and a multiple of x^5. Contradiction. □

Taking Remark 30 into account, from now on we write $\varphi = f\,G$. The equations "1" – "3" of (16) (the analog of (8) for the system (6)), divided sidewise by x^5, are the following

$$\begin{pmatrix} g^1_1 & g^1_2 & g^1_3 \\ g^2_1 & g^2_2 & g^2_3 \\ g^3_1 & g^3_2 & g^3_3 \end{pmatrix} \begin{pmatrix} -\,1 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} -\,\varphi \\ \varphi\,g^3 \\ \varphi\,g^4 \end{pmatrix}. \tag{20}$$

Let us focus on two facts, yielded by "1" and "3" of (20):

[10] This form of g^6 is understandable, given the joint locus of the small gr. v. [2, 3, 4, 5, 6, 7_3, 8], *cf.* above.

⋆ φ is affine with respect to x^4;

⋆⋆ $\varphi\, g^4$ is affine with respect to x^4.

Now the expression (19) for c^8 can be further clarified, which will bring in a precise indication of the way to follow. As $G_4\;|0 = f^{-1}\varphi_4 - f^{-2}\,\varphi\, f_4\;|0$,
$f_4 - f^{-1}G_4\;|0 = -\,f^{-2}\varphi_4 + f_4(1 + f^{-3}\varphi)\;|0 = -\,f^{-2}\varphi_4 + 2\,g^4_{44}\;|0$ (by (17)).
Equation "1" of (20) yields $\varphi_4 = -\,g^1_3$, while ⋆⋆ helps to compute $g^4_{44}\;|0$: differentiating "3" of (20) two times wrt x^4 and then evaluating at 0, $0 = -\,2\,g^1_3\,g^4_4 + \varphi\,g^4_{44}\;|0$. Thus
$f_4 - f^{-1}G_4\;|0 = f^{-2}g^1_3 + 2\cdot 2\,\varphi^{-1}g^1_3\,g^4_4\;|0 = f^{-3}(f\,g^1_3 + 4\,g^1_3\,g^4_4)\;|0 = 5\,f^{-2}\,g^1_3\;|0$, and (19) assumes eventually the form

$$c^8 = 15\,f^{-4}\,g^1_3\;|0. \tag{21}$$

Therefore, we have to arrange for $g^1_3\;|0 \neq 0$. The joint interpretation of ⋆ and ⋆⋆ leads to a proper guess concerning g^4, then φ, then g^1. Namely, writing $g^4(0,0,0,x^4) = x^4\gamma(x^4)$ and substituting for a while $x^1 = x^2 = x^3 = 0$ to "3" of (20), that equation "3" becomes $(a + b\,x^4)\,x^4\gamma(x^4) = A + B\,x^4$, with a, b, A, B – certain constants. This implies $A = 0$. Thus

$$g^4(0,\,0,\,0,\,x^4) = \frac{B\,x^4}{a + b\,x^4}\cdot$$

A NON-restricted g^4 of this form, for instance $g^4 = \frac{x^4}{1+x^4}$ (picking $a = b = B = 1$) does the job: $\varphi = 1 + x^4$ comes from, for instance, $g^1_1 = 1$ and $g^1_3 = -\,1$ (*cf.* "1" of (20)). So it is purposeful to take $g^1 = x^1 - x^3$. In turn, as the RHS of "3" of (20) is just x^4, it is natural to have $g^3_1 = 0$, $g^3_2 = 0$, $g^3_3 = 1$, and take $g^3 = x^3$. At last, can we produce such g^3 by the way of "2" of (20), with the φ already proposed? In other words, we have to uncover $g^2_1(-1) + g^2_2x^3 + g^2_3x^4 = (1 + x^4)x^3 = x^3 + x^3x^4$. Here it imposes by itself to have $g^2_1 = 0$, $g^2_2 = 1$, $g^2_3 = x^3$, and take, finally, $g^2 = x^2 + \frac{1}{2}(x^3)^2$.

The rest of the proof is a matter of automatic verifications: "4" of (16) yields $f = (1 + x^4)^{-2}$, and this, in turn, $G = \varphi\, f^{-1} = (1 + x^4)^3$. The coordinate function g^5 becomes thus known, and "5" of (16) produces g^6. Later, with (17) being obviously satisfied, "6" of (16) brings forth g^7:

$$\begin{cases} g^5 = x^5(1 + x^4)^3, \\ g^6 = x^6(1 + x^4)^5 - 3\,x^5(1 + x^4)^4, \\ g^7 = (1 + x^7)(1 + x^4)^7 - 8\,x^6(1 + x^4)^6 + 12\,x^5(1 + x^4)^5 - 1. \end{cases}$$

Coming back to the constant c^8, (21) says that $c^8 = 15(-\,1) = -\,15$ (one could also compute c^8 directly from (18)). This is that non-zero value of c^8 we are able to produce – *cf.* the beginning of the proof. For the sake of completeness, we also write down g^8 issuing from "7" of (16) with $c^8 = -\,15$:

$$g^8 \;=\; x^8(1 + x^4)^9 - 15(1 + x^7)(1 + x^4)^8 + 60\,x^6(1 + x^4)^7 - 60\,x^5(1 + x^4)^6 + 15.$$

There is no doubt that $g = (g^1, \ldots, g^8)$ is a local diffeomorphism of $\mathbb{R}^8$ in the vicinity of 0.

We note that, with this Theorem 27 proved, the whole proof of Theorem 23 is already concluded.

6. GC in control systems of cars with N trailers

As mentioned in introduction, the 2–distributions describing in appropriate configuration spaces the motion of a car drawing a variable quantity, n, of passive trailers, always satisfy GC [8, 9]. Recently Jean gave in [6] precise recursive formulas for computing the small gr. v. of those distributions at every point – for different angles between consecutive trailers in the string[11].

[11] For the minimal numbers of Lie brackets necessary to span at least a given dimension, in fact, this obviously determining the small gr. v.

Making the statements of [6] explicit in the dimensions not exceeding 8 (*i.e.*, for n not greater than 5), it turns out that in these dimensions the number of non-equivalent local behaviours of the car system coincides with the total of existing local models of GC. That is to say: 1 in dimension 4, 2 – in 5, 5 – in 6, 13 – in 7, 34 – in dimension 8. In Observation 34 below we are putting this in a more organized way.

Saying differently, for the mentioned numbers of trailers the lists giving the local classification of GC – Theorems 14, 16, 23 – could be composed of local behaviours of the car systems, or else: car systems exhaust the whole GC.

We want to conclude the present work by posing a question whether it is likewise in the higher dimensions. Observe first that the numeration 1., 2., 3. of the situations listed in the main Theorem 3.1 in [6] allows one to code – by sequences with values in $\{1,\,2,\,3\}$, as well – the different regions in the configuration space, governed by different rules as regards the (recursive) computing of the small gr. v. In the proposed definition we are using the very notations of [6]:

Definition 31. A given point $(x,\,y,\,\theta^0,\,\theta^1,\ldots,\,\theta^n)$ belongs to the region coded by the following sequence: write utmost left the number of the situation in ([6], Th. 3.1, N° of trailers 2) subsuming the point $(x,\,y,\,\theta^0,\,\theta^1,\theta^2)$. Follow this to the right by the number of the situation in (Th. 3.1, N° of trailers 3) subsuming the point $(x,\,y,\,\theta^0,\,\theta^1,\theta^2,\,\theta^3)$. Continue this recursively until writing utmost right the number of the situation in (Th. 3.1, N° of trailers n) subsuming the departure point $(x,\,y,\,\theta^0,\,\theta^1,\ldots,\,\theta^n)$.

Example 32. (for $n=5$). Following [6] in writing $a^1=\frac{\pi}{2}$, $a^{l+1}=\arctan(\sin(a^l))$ for $l\geq 1$, the points satisfying:

a) $\theta^5-\theta^4=\pm\,a^1$, $\theta^4-\theta^3=\pm\,a^2$, $\theta^3-\theta^2=\pm\,a^1$, $\theta^2-\theta^1\neq\pm\,a^1$, form the region coded 3.1.2.1;
b) $\theta^5-\theta^4=\pm\,a^1$, $\theta^4-\theta^3\neq\pm\,a^1,\,\pm\,a^2$, $\theta^3-\theta^2=\pm\,a^1$, $\theta^2-\theta^1\neq\pm\,a^1$ — form the region 3.1.3.1.

In order to better compare our codes of the KR pseudo–normal forms given in Th. 3, and the codes of regions in the configuration space for the car system (Def. 31), we make two, rather cosmetic, changes.

Convention. 1) Agree that in our codes entering Theorems 14, 16, 23, all *1*'s forming a sequence placed in the beginning (utmost left), or directly following any *2*, are replaced univocally by 2's (for istance, *1.1.3.1* is being replaced by 2.2.*3.1*, *1.3.2.1* – by 2.*3.2*.2, *3.2.1.1* – by *3.2*.2.2, whereas *3.1.1.2* remains unchanged);
2) agree that the situations listed in ([6], Th. 3.1) are numbered 3., 1., 2. instead of 1., 2., 3., thus changing accordingly the codes introduced an instant ago in Definition 31 (for instance, the region a) in Example 32 obtains a new code 2.3.1.3, while the region b) – 2.3.2.3.).

Proposition 33. *Once the convention adopted, the regions of the configuration space for a car system with* $n\geq 2$ *trailers are coded by the sequences of length* $n-1$ *with values in* $\{1,\,2,\,3\}$*, beginning NOT with "1" and such that never a "2" is followed by a "1".*

Proof. The sequences do not begin with a "1" – corresponding to the Jean situation 2. – because that situation does not occur for any point of the form $(x,\,y,\,\theta^0,\,\theta^1,\theta^2)$ (*cf.* our Def. 31).

Suppose to the contrary that in the code of a certain region there is, after the convention, a "2" followed by a "1". This means that before the convention there existed a "3" followed by a "2". We analyze the meaning of "3"; say that this "3" is at the l-th place in that code sequence ($l\leq n-2$). This means that the angles of any point of the region in question do not satisfy neither 1. nor 2. of (Th. 3.1, N° of trailers $l+1$). We want to be more explicit at this point, and denote THAT condition 1. by $(1;\,l+1)$. In turn, THAT condition 2. is given in [6] as the alternative of certain $l-1$ conditions – denoted HERE by $(2;\,l+1;\,p)$ – corresponding to $p=1,\ldots,\,l-1$.

We repeat, then, that the angles do not meet neither $(1;\,l+1)$ nor any of $(2;\,l+1;\,p)$, $p=1,..,\,l-1$. Having a "2" at the following place $l+1$ (*i.e.*, for the number of trailers $l+2$), we see in (Th. 3.1) that, for any point $(x,\,y,\,\theta^0,\,\theta^1,\ldots,\,\theta^n)$ in the discussed region, there holds the alternative of the following conditions (for the numbers $a^1,\,a^2,\,\ldots$ – see Ex. 32):

$(\theta^{l+2} - \theta^{l+1} = \pm\, a^2$ and $(1;\, l+1))$,
$(\theta^{l+2} - \theta^{l+1} = \pm\, a^3$ and $(2;\, l+1;\, l-1))$,
• • •
$(\theta^{l+2} - \theta^{l+1} = \pm\, a^{l+1}$ and $(2;\, l+1;\, 1))$.

This contradicts the precedent statement on the violation of all the $(1;\, l+1)$, $(2;\, l+1;\, p)$, $p = 1, \ldots, l-1$.

That there is no other limitations on the sequences coding the regions — we leave as a (similar to the above) exercise in the form of Jean's conditions. □

Proposition 33 says that the set of Jean's codes – after Convention – coincides with the set of our codes of Definition 5, after Convention. The following observation subsumes the whole output of the present paper.

Observation 34. For $n \le 5$ the germ of the car system with n trailers at any point having a given Jean's code (after Convention) is equivalent to **any** KR pseudo–normal form in dimension $n+3$ having **after** Convention that same code.

Proof. One computes, applying recursively ([6], Th. 3.1), the small gr. v., one and the same at any point of the region given by the code. Then identifies the obtained vector on the respective list of Theorem 14, or 16, or 23, and subjects to Convention the code of the local model having that small vector at the origin, always arriving at the departure code. On the other hand, by the evoked theorems, in dimensions not exceeding 8 the small vector at a point characterizes the germ of a GC distribution at that point up to local equivalence. □

Open question. What remains true of Observation 34 in higher dimensions – from 9 onwards? (In particular, does the car system locally exhaust Goursat Condition in dimension $n+3 \ge 9$?)

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