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A SMOOTH LYAPUNOV FUNCTION FROM A CLASS-$\mathcal{KL}$ ESTIMATE INVOLVING TWO POSITIVE SEMIDEFINITE FUNCTIONS*

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Abstract. We consider differential inclusions where a positive semidefinite function of the solutions satisfies a class-$\mathcal{KL}$ estimate in terms of time and a second positive semidefinite function of the initial condition. We show that a smooth converse Lyapunov function, *i.e.*, one whose derivative along solutions can be used to establish the class-$\mathcal{KL}$ estimate, exists if and only if the class-$\mathcal{KL}$ estimate is robust, *i.e.*, it holds for a larger, perturbed differential inclusion. It remains an open question whether all class-$\mathcal{KL}$ estimates are robust. One sufficient condition for robustness is that the original differential inclusion is locally Lipschitz. Another sufficient condition is that the two positive semidefinite functions agree and a backward completability condition holds. These special cases unify and generalize many results on converse Lyapunov theorems for differential equations and differential inclusions that have appeared in the literature.

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Basic definitions

- Given a set $\mathcal{A}$, $\overline{\mathcal{A}}$ stands for the closure of $\mathcal{A}$, $\mathring{\mathcal{A}}$ stands for the interior set of $\mathcal{A}$, $\overline{\text{co}}\mathcal{A}$ stands for the closed convex hull of $\mathcal{A}$ and $\partial\mathcal{A}$ stands for the boundary of $\mathcal{A}$.
- The notation $x \to \partial\mathcal{A}^\infty$ indicates a sequence of points x belonging to $\mathcal{A}$ converging to a point on the boundary of $\mathcal{A}$ or, if $\mathcal{A}$ is unbounded, having the property $|x| \to \infty$.
- Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_\mathcal{A}$ denotes the distance from x to $\mathcal{A}$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to class-$\mathcal{K}_\infty$ if, in addition, it is unbounded.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$, and, for each $s \geq 0$, $\beta(s,\cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

The requirements imposed for a function to be of class-$\mathcal{KL}$ are slightly weaker than usual. In particular, $\beta(\cdot,t)$ is not required to be continuous or strictly increasing. See, also, Remark 3.

Keywords and phrases: Differential inclusions, Lyapunov functions, uniform asymptotic stability.

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1. Introduction

1.1. Background

In the same text [19] where Lyapunov introduced his famous sufficient conditions for asymptotic stability of the origin of a differential equation

$$\dot{\xi} = f(\xi, t), \tag{1}$$

we can find the first contribution ([19], Sect. 20, Th. II) to the converse question: what aspects of asymptotic stability and the function f guarantee the existence of a (smooth) function satisfying Lyapunov's sufficient conditions for asymptotic stability? The answers have proved instrumental, over the years, in establishing robustness of various stability notions and have served as the starting point for many nonlinear control systems design concepts.

One of the early important milestones in the pursuit of smooth converse Lyapunov functions was Massera's 1949 paper [21] which provided a semi-infinite integral construction for time-invariant, continuously differentiable systems with an asymptotically stable equilibrium. Later, in 1954, Malkin observed that Massera's construction worked even for time-varying systems as long as the asymptotic stability and the differentiability of f with respect to the state were uniform in time [20]. Regarding stability Malkin assumed, in effect, the existence of a class-$\mathcal{KL}$ function β such that the solutions $\xi(t, t_\circ, \xi_\circ)$ of the system (1), issued from $\xi_\circ$ at time $t_\circ$, satisfy

$$|\xi(t, t_\circ, \xi_\circ)| \le \beta(|\xi_\circ|, t - t_\circ) \qquad \forall t \ge t_\circ \ge 0 \tag{2}$$

at least for initial conditions $\xi_\circ$ sufficiently small. In [5], Barbashin and Krasovskii generalized Malkin's result to the case where (2) holds for all initial conditions. Both Massera [22] and Kurzweil [15], independently in the mid-1950's, weakened the assumptions made by Malkin, and Barbashin and Krasovskii about the function f. Kurzweil's contribution especially stands out because he was able to establish a converse theorem even when f is only continuous so that uniqueness of solutions is not guaranteed. In his work he made precise a notion of strong stability of the origin on an open neighborhood $\mathcal{G}$ of the origin which amounted to the existence of a function $\beta \in \mathcal{KL}$ and a locally Lipschitz, positive definite function $\omega : \mathcal{G} \to \mathbb{R}_{\ge 0}$, proper on $\mathcal{G}$, such that all solutions of the system (1) with $\xi_\circ \in \mathcal{G}$ satisfy

$$\omega(\xi(t, t_\circ, \xi_\circ)) \le \beta(\omega(\xi_\circ), t - t_\circ) \qquad t \ge t_\circ \ge 0. \tag{3}$$

Kurzweil showed that this strong stability and continuity of f imply the existence of a smooth converse Lyapunov function, *i.e.*, a function whose derivative along solutions can be used to deduce (3).

Much of the research in the 1960's focused on developing converse Lyapunov theorems for systems possessing asymptotically stable closed, not necessarily compact, sets. Taking this approach, the time-varying case can be subsumed into the time-invariant case by augmenting the state-space of (1) as:

$$\dot{x} = \frac{d}{dt}\begin{pmatrix} \xi \\ p \end{pmatrix} = \begin{pmatrix} f(\xi, p) \\ 1 \end{pmatrix} =: F(x), \qquad x_\circ = \begin{bmatrix} \xi_\circ \\ t_\circ \end{bmatrix}. \tag{4}$$

(One disadvantage in treating time-varying systems as time-invariant ones is that it usually leads to imposing stronger than necessary conditions on the time-dependence of the right-hand side, *e.g.*, continuity where only measurability is needed. An example where a converse theorem is developed for systems with right-hand sides measurable in time, and for Lyapunov and Lagrange stability, is [4].) A closed set $\mathcal{A}$ for (4) is said to be uniformly asymptotically stable if there exists of a function $\beta \in \mathcal{KL}$ such that all solutions of (4) with $|x_\circ|_{\mathcal{A}}$ sufficiently small exist for all forward time and satisfy

$$|\phi(t, x_\circ)|_{\mathcal{A}} \le \beta(|x_\circ|_{\mathcal{A}}, t) \qquad \forall t \ge 0. \tag{5}$$

Some of the early results on converse Lyapunov functions for set stability are summarized in [41]. Particularly noteworthy are the result of Hoppensteadt [14] who generated a C^1 converse Lyapunov function for parameterized differential equations and the result of Wilson [40] who provided a smooth converse Lyapunov function for uniform asymptotic stability of a set. Converse theorems for (compact) set stability for nonlinear difference equations are also now standard (see, *e.g.* [34], Th. 1.7.6).

Noting that every solution $\phi(\cdot, x_\circ)$ of (4) can be written as

$$\phi(t, x_\circ) = \begin{bmatrix} \xi(t + t_\circ, t_\circ, \xi_\circ) \\ t + t_\circ \end{bmatrix} \tag{6}$$

where $\xi(\cdot, t_\circ, \xi_\circ)$ is a solution of (1), it follows that (2, 3) and (5) are all particular cases of the estimate

$$\omega(\phi(t, x_\circ)) \le \beta(\omega(x_\circ), t) \qquad \forall t \ge 0 \tag{7}$$

where ω is a continuous, positive semidefinite function.

In the 1970's, Lakshmikantham and coauthors [17] (see also [16], Sect. 3.4) provided a Lipschitz converse Lyapunov function for Lipschitz ordinary differential equations of the form (4) under an assumption essentially the same as: given two continuous, positive semidefinite functions ω_1 and ω_2, there exists a function $\beta \in \mathcal{KL}$ such that, for all initial conditions $x_\circ$ with $\omega_2(x_\circ)$ sufficiently small, all solutions exist for all forward time and satisfy

$$\omega_1(\phi(t, x_\circ)) \le \beta(\omega_2(x_\circ), t) \qquad \forall t \ge 0. \tag{8}$$

The stability concept described by (8), apparently first introduced in [27] and often called stability with respect to two measures, generalizes (7) and thus includes the notions of local uniform asymptotic stability of a point, of a prescribed motion and of a closed set. It also covers the notion of local uniform partial asymptotic stability such as when, for $x_\circ$ sufficiently small,

$$|h(\phi(t, x_\circ))| \le \beta(|x_\circ|, t) \qquad \forall t \ge 0 \tag{9}$$

where h is a continuous (output) function of the state. A smooth converse Lyapunov theorem for the global version of (9) was recently derived in ([33], Th. 2) (see also [31,32]). (See [39] for a survey on the partial stability problem.)

Extensions of the above results to differential inclusions started to appear in the late 1970's with some of the most general results appearing only recently. Some motivations for the study of differential inclusions are: 1) they describe the solution set for ordinary differential equations with arbitrary, measurable bounded disturbances (see Cor. 1 and Cor. 2) they describe important notions of solutions for control systems that use discontinuous feedbacks (see [12], Sect. 8.3).

The results in [23] pertain to differential inclusions of the form

$$\dot{x} \in F(x) := \overline{\text{co}}\,\{v \in \mathbb{R}^n : v = f(x, d),\ d \in \mathcal{D}\} \tag{10}$$

where $\mathcal{D}$ is compact, f is continuous and continuously differentiable with respect to x, and asymptotic stability in the first approximation is assumed, *i.e.*, for the inclusion

$$\dot{x} \in \left\{ v \in \mathbb{R}^n : v = \frac{\partial f}{\partial x}(0, d)x,\ d \in \mathcal{D} \right\}, \tag{11}$$

an estimate of the form

$$|x(t)| \le k|x(0)| \exp(-\lambda t) \qquad k > 0,\ \lambda > 0 \tag{12}$$

is assumed for all solutions starting from sufficiently small initial conditions. The result ([23], Th. 2) states that this assumption implies the existence of a smooth converse Lyapunov function for local exponential stability and asymptotic stability on the basin of attraction of the origin for the inclusion (10). Related results for inclusions of the type (11) can also be found in [24–26].

In [18], Lin *et al.* considered the differential inclusion (10)[1], relaxing the continuous differentiability assumption of f with respect to x to a local Lipschitz assumption, and assumed the estimate (5), *i.e.*, the estimate (8) with $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$, for all initial conditions. They showed, under the additional assumption that either $\mathcal{A}$ is compact or all solutions exist for all backward time, that the estimate (5) for the differential inclusion (10) implies the existence of a smooth converse Lyapunov function. In [1], the authors combined the ideas of [18] with the idea of Kurzweil [15] establishing the existence of a smooth converse Lyapunov function for the differential inclusion (10) in the case of the existence of a compact set $\mathcal{A}$, a neighborhood $\mathcal{G}$ of $\mathcal{A}$ and function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz, positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{G}$ and a function $\beta \in \mathcal{KL}$ such that, for all $x_\circ \in \mathcal{G}$, the solutions of (10) satisfy

$$\omega(\phi(t, x_\circ)) \leq \beta(\omega(x_\circ), t) \qquad \forall t \geq 0. \tag{13}$$

The first results on smooth converse theorems for differential inclusions that are only upper semicontinuous (see Def. 1 below) appeared in [6]. In that work, Clarke *et al.* studied

$$\dot{x} \in F(x) \tag{14}$$

under the assumption that $F(x)$ is nonempty, compact and convex for each $x \in \mathbb{R}^n$ and F is upper semicontinuous. They assumed the estimate (8) with $\omega_1(x) = \omega_2(x) = |x|$, and showed that this implies the existence of a smooth converse Lyapunov function. A related result, for the case of uniform exponential stability for switching systems, can be found in [9].

Other interesting results on the existence of nondifferentiable converse Lyapunov functions can be found in [2], Chapter 6 and [36–38].

1.2. **Contributions**

In this paper, we consider differential inclusions

$$\dot{x} \in F(x) \tag{15}$$

that satisfy the conditions assumed in [6]. In particular, F is a set-valued map from an open set $\mathcal{G}$ to subsets of $\mathbb{R}^n$ that is upper semicontinuous on $\mathcal{G}$ (see Def. 1) and is such that $F(x)$ is nonempty, compact and convex for each $x \in \mathcal{G}$. The stability property we will assume for (15) we will refer to as "$\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$". Namely, given two continuous functions $\omega_1 : \mathcal{G} \to \mathbb{R}_{\geq 0}$ and $\omega_2 : \mathcal{G} \to \mathbb{R}_{\geq 0}$, we assume the existence of a class-$\mathcal{KL}$ function β such that all solutions of the differential inclusion (15) starting in $\mathcal{G}$ remain in $\mathcal{G}$ for all forward time and satisfy

$$\omega_1(\phi(t, x_\circ)) \leq \beta(\omega_2(x_\circ), t) \qquad \forall t \geq 0. \tag{16}$$

(See also Def. 6.) This is like the stability property considered in [17].

Our main result is (see Th. 1):

> *A smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) (see Def. 7) exists if and only if the $\mathcal{KL}$-stability is robust, i.e., it holds for a larger, perturbed differential inclusion (see Def. 8).*

[1] For a clarification, see the proof of Corollary 1.

This type of equivalence between robust stability and the existence of a Lyapunov function, reminiscent of the classical "total stability" results for ordinary differential equations ([13], Th. 56.4), is already present in the proofs of Kurzweil [15] and Clarke *et al.* [6].

It remains an open question whether $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is robust, in general. However, we will show (see Th. 2):

> *If the set-valued map defining the differential inclusion (15) is locally Lipschitz on $\mathcal{G}$ (see Def. 3) then $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$ is robust.*

This is the case for the problems considered by Lakshmikantham *et al.* [16], Lin *et al.* [18], and Sontag and Wang ([33], Th. 2).

We will also show (see Th. 3):

> *If the differential inclusion (15) is backward completable by ω-normalization (see Def. 9) then $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ is robust.*

This condition holds for the problems considered by Kurzweil [15] (see Cor. 2) and Clarke *et al.* [6] (see Cor. 3). It is also useful for generating smooth converse Lyapunov functions for compact, stable attractors. As an illustration, we provide a smooth converse Lyapunov function for finite time convergence to a compact set from a larger compact set (see Cor. 4). This result is useful for the problem of semiglobal practical asymptotic stabilization of nonlinear control systems as studied in [35], for example.

Our converse Lyapunov function is constructed in the following steps:

1. We imbed the original differential equation or differential inclusion into a larger, locally Lipschitz differential inclusion that still exhibits $\mathcal{KL}$-stability with respect to (ω_1, ω_2). This idea is due to Kurzweil [15] for the case of ordinary differential equations with continuous right-hand side under strong stability of the origin. It is due to Clarke *et al.* [6] for the case of nonempty, compact, convex, upper semicontinuous differential inclusions and global asymptotic stability of the origin. In general, it is possible if and only if the $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is robust.
2. We find class-$\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ such that $\widetilde{\alpha}_1(\beta(s,t)) \leq \widetilde{\alpha}_2(s)e^{-2t}$, where β quantifies $\mathcal{KL}$-stability with respect to (ω_1, ω_2) for the locally Lipschitz differential inclusion constructed in Step 1. A recent result by Sontag ([30], Prop. 7) shows that this is always possible.
3. We define a trial Lyapunov function $V_1(x)$ as the supremum, over time and solutions $\phi(\cdot, x)$ of the locally Lipschitz differential inclusion constructed in Step 1, of the quantity $\widetilde{\alpha}_1(\omega_1(\phi(t,x)))e^t$ where $\widetilde{\alpha}_1$ was constructed in Step 2. This is a classical construction once the estimate in Step 2 is available, at least for locally Lipschitz ordinary differential equations where the supremum over solutions is not needed. (See, *e.g.* [41], Sect. 19.) We show, using many of the tools used in [6] and [18], that this trial Lyapunov function has all of the desired properties except smoothness. However, it is locally Lipschitz. It would only be upper semicontinuous, in general, if the supremum were taken over solutions of the original differential inclusion.
4. We smooth the trial Lyapunov function using ideas that go back to Kurzweil [15] and that have been clarified, generalized and used over the years by, for example, Wilson [40], Lin *et al.* [18] and Clarke *et al.* [6].

The rest of the paper is organized as follows:

- In Section 2 we present some definitions related to set-valued maps and some properties of solutions to differential inclusions. These definitions are needed for understanding the statement of our main results.
- In Section 3 we give precise definitions of $\mathcal{KL}$-stability with respect to (ω_1, ω_2) (Def. 6) as well as the robust version of this property (Def. 8), and of a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) (Def. 7). Then we present our main results on the existence of a smooth converse Lyapunov function and relate these results to others that have appeared in the literature.
- Section 4 contains some technical prerequisites that are necessary for the proofs of our main results.
- We prove our main results in Section 5. Particularly noteworthy are Section 5.1.2, which contains the construction of our smooth converse Lyapunov function under the assumption of robust $\mathcal{KL}$-stability, and

Section 5.3 where we establish robust $\mathcal{KL}$-stability under the assumption of nominal $\mathcal{KL}$-stability with respect to (ω, ω) plus a backward completability assumption.
- Section 6 contains the proofs of some propositions that are used to make connections to other results that have appeared in the literature.
- To help the reader with many technicalities arising in the paper, we include in an addendum (elements of) the proofs of Lemmas 1, 8, 9, 16 and 17 and of Proposition 1. They can be found elsewhere in the literature, maybe with some minor modifications.

In this paper, we have borrowed many ideas and technicalities from our predecessors. We try to make this point clear in bibliographical notes.

2. Preliminaries

Throughout this paper F will be a set-valued map from $\mathcal{G}$ to subsets of $\mathbb{R}^n$ where $\mathcal{G}$ is an open subset of $\mathbb{R}^n$. Also $\mathcal{B}$ denotes the open unit ball in $\mathbb{R}^n$ and

$$F(x) + \varepsilon\mathcal{B} := \left\{ z \in \mathbb{R}^n : |z|_{F(x)} < \varepsilon \right\} \cdot \tag{17}$$

We review some definitions concerning set-valued maps (see also [12], Sects. 5.3, 7.2):

Definition 1. The set-valued map F is said to be *upper semicontinuous on* $\mathcal{G}$ if, given $x \in \mathcal{G}$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathcal{G}$ satisfying $|x - \xi| < \delta$ we have $F(\xi) \subseteq F(x) + \varepsilon\mathcal{B}$.

Definition 2. The set-valued map F is said to *satisfy the basic conditions on* $\mathcal{G}$ if it is upper semicontinuous on $\mathcal{G}$ and, for each $x \in \mathcal{G}$, $F(x)$ is nonempty, compact and convex.

We will need the following fact:

Lemma 1. *If the set-valued map F satisfies the basic conditions on $\mathcal{G}$ and $\rho : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a continuous function such that for all $x \in \mathcal{G}$, we have*

$$\{x\} + \rho(x)\overline{\mathcal{B}} \ \subset \ \mathcal{G}, \tag{18}$$

then the set-valued map

$$x \ \mapsto \ \overline{\mathrm{co}} \left(\bigcup_{\xi \in \{x\} + \rho(x)\overline{\mathcal{B}}} F(\xi) \right) \ + \ \rho(x)\overline{\mathcal{B}},$$

which we denote by $\overline{\mathrm{co}}F(\cdot + \rho(\cdot)\overline{\mathcal{B}}) + \rho(\cdot)\overline{\mathcal{B}}$, satisfies the basic conditions on $\mathcal{G}$.

Definition 3. Let $\mathcal{O}$ be an open subset of $\mathcal{G}$. The set-valued map F is said to be *locally Lipschitz on* $\mathcal{O}$ if, for each $x \in \mathcal{O}$, there exists a neighborhood $\mathcal{U} \subset \mathcal{O}$ of x and a positive real number L such that

$$x_1, x_2 \in \mathcal{U} \qquad \Longrightarrow \qquad F(x_1) \subseteq F(x_2) + L|x_1 - x_2|\mathcal{B}. \tag{19}$$

Given a set-valued map F, we can define a solution of the differential inclusion

$$\dot{x} \in F(x). \tag{20}$$

Definition 4. A function $x : [0, T] \to \mathcal{G}$ $(T > 0)$ is said to be a solution of the differential inclusion (20) if it is absolutely continuous and satisfies, for almost all $t \in [0, T]$,

$$\dot{\overbrace{x(t)}} \in F(x(t)). \tag{21}$$

A function $x : [0, T) \to \mathcal{G}$ $(0 < T \leq \infty)$ is said to be a maximal solution of the differential inclusion (20) if it does not have an extension which is a solution belonging to $\mathcal{G}$, *i.e.*, either $T = \infty$ or there does not exist a solution $y : [0, T_+] \to \mathcal{G}$ with $T_+ > T$ such that $y(t) = x(t)$ for all $t \in [0, T)$.

The following basic fact about the existence of maximal solutions is a combination of ([12], Sect. 7, Th. 1 and [29], Props. 1 and 2).

Lemma 2. *If F satisfies the basic conditions on $\mathcal{G}$ then for each $x_\circ \in \mathcal{G}$ there exist solutions of (20) for sufficiently small $T > 0$ satisfying $x(0) = x_\circ$. In addition, every solution can be extended into a maximal solution. Moreover, if a maximal solution $x(\cdot)$ is defined on a bounded interval $[0, T)$ then $x(t) \to \partial\mathcal{G}^\infty$ as $t \to T$.*

Henceforth, we will use $\phi(\cdot, x)$ to denote a solution of (20) starting at x and we will denote by $\mathcal{S}(x)$ or $\mathcal{S}(\mathcal{C})$ (respectively, $\mathcal{S}[0, T](x)$ or $\mathcal{S}[0, T](\mathcal{C})$) the set of maximal solutions (respectively, solutions defined on $[0, T]$) of the differential inclusion (20) starting at x or in the compact set $\mathcal{C}$. Note that with $\phi_1 \in \mathcal{S}[0, T_1](x)$ and $\phi_2 \in \mathcal{S}[0, T_2](\phi_1(T_1, x))$ and defining

$$\begin{aligned} \phi_3(t, x) &= \phi_1(t, x) && \text{if} \quad 0 \leq t \leq T_1, \\ &= \phi_2(t - T_1, \phi_1(T_1, x)) && \text{if} \quad T_1 \leq t \leq T_1 + T_2 \end{aligned} \tag{22}$$

we have $\phi_3 \in \mathcal{S}[0, T_1 + T_2](x)$.

Definition 5. The differential inclusion (20) is said to be *forward complete on* $\mathcal{G}$ if, for all $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ are defined (and remain in $\mathcal{G}$) for all $t \geq 0$. The differential inclusion (20) is said to be *backward complete on* $\mathcal{G}$ if the differential inclusion $\dot{x} \in -F(x)$ is forward complete on $\mathcal{G}$.

3. Main results

3.1. General statements

The stability concept we work with in this paper is called $\mathcal{KL}$-stability with respect to (ω_1, ω_2) where ω_1 and ω_2 are continuous, positive semidefinite functions. This concept is defined as follows:

Definition 6. Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The differential inclusion $\dot{x} \in F(x)$ is said to be *$\mathcal{KL}$-stable with respect to (ω_1, ω_2) on* $\mathcal{G}$ if it is forward complete on $\mathcal{G}$ and there exists $\beta \in \mathcal{KL}$ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$\omega_1(\phi(t, x)) \leq \beta(\omega_2(x), t) \qquad \forall t \geq 0. \tag{23}$$

As mentioned in the introduction, this stability concept was introduced in [27] and considered in [17] and [16]. It is often referred to as stability with respect to two measures. It covers standard stability notions like uniform global asymptotic stability of a closed set and partial asymptotic stability.

In the case where $\mathcal{A}$ is a closed set, $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$, and $\mathcal{G} = \mathbb{R}^n$, it has been shown in [18], Proposition 2.5 that $\mathcal{KL}$-stability is equivalent to the set $\mathcal{A}$ being uniformly globally stable and uniformly globally attractive. The technique used to prove ([18], Prop. 2.5) is used to prove the following generalization for $\mathcal{KL}$-stability with respect to (ω_1, ω_2):

Proposition 1. *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. The following are equivalent:*

1. *The differential inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*
2. *All of the following hold:*
    (a) *The differential inclusion $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$.*

(b) (Uniform stability and global boundedness)*: There exists a class-*$\mathcal{K}_\infty$ *function* γ *such that, for each* $x \in \mathcal{G}$*, all solutions* $\phi \in \mathcal{S}(x)$ *satisfy*

$$\omega_1(\phi(t,x)) \le \gamma(\omega_2(x)) \qquad \forall t \ge 0. \tag{24}$$

(c) (Uniform global attractivity)*: For each* $r > 0$ *and* $\varepsilon > 0$*, there exists* $T(r,\varepsilon) > 0$ *such that, for each* $x \in \mathcal{G}$*, all solutions* $\phi \in \mathcal{S}(x)$ *satisfy*

$$\omega_2(x) \le r, \quad t \ge T \qquad \Longrightarrow \qquad \omega_1(\phi(t,x)) \le \varepsilon. \tag{25}$$

$\mathcal{KL}$-stability with respect to (ω_1,ω_2) can be characterized in infinitesimal (with respect to time) terms *via* the existence of a smooth Lyapunov function:

Definition 7. Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\ge 0}$, $i = 1,2$, be continuous. A function $V : \mathcal{G} \to \mathbb{R}_{\ge 0}$ is said to be a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1,ω_2) on $\mathcal{G}$ for F if V is smooth on $\mathcal{G}$ and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that, for all $x \in \mathcal{G}$,

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{26}$$

and

$$\max_{w \in F(x)} \langle \nabla V(x), w\rangle \le -V(x). \tag{27}$$

The motivation for this definition is that (27) guarantees the derivative of V along solutions, denoted $\dot V(\phi(\cdot,x))$, satisfies

$$\dot V(\phi(t,x)) \le -V(\phi(t,x)) \tag{28}$$

for almost all t in the interval where $\phi(t,x)$ exists and belongs to $\mathcal{G}$. It follows that

$$V(\phi(t,x)) \le V(x)e^{-t} \tag{29}$$

on this interval and then, using (26) and assuming forward completeness on $\mathcal{G}$, we can deduce $\mathcal{KL}$-stability with respect to (ω_1,ω_2) on $\mathcal{G}$. (By relying on a result like ([18], Lem. 4.4), it is possible to deduce $\mathcal{KL}$-stability with respect to (ω_1,ω_2) on $\mathcal{G}$ when $V(x)$ on the right-hand side of (27) is replaced by any class-$\mathcal{K}_\infty$ function of $V(x)$.)

We are interested in whether $\mathcal{KL}$-stability with respect to (ω_1,ω_2) implies the existence of a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1,ω_2). This is still an open question, in general. What we will show here is that a smooth converse Lyapunov function exists if and only if the $\mathcal{KL}$-stability with respect to (ω_1,ω_2) is robust; that is $\mathcal{KL}$-stability with respect to (ω_1,ω_2) still holds for a set of differential inclusions given by supersets of F. This concept, which is present in the work of Kurzweil [15] and Clarke *et al.* [6], is defined more precisely as follows:

Definition 8. Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\ge 0}$, $i = 1,2$, be continuous. The differential inclusion $\dot x \in F(x)$ is said to be *robustly $\mathcal{KL}$-stable with respect to* (ω_1,ω_2) *on* $\mathcal{G}$ if there exists a continuous function $\delta : \mathcal{G} \to \mathbb{R}_{\ge 0}$ such that:

1. $\{x\} + \delta(x)\overline{\mathcal{B}} \subset \mathcal{G}$;
2. the differential inclusion

$$\dot x \in F_{\delta(x)}(x) := \overline{\mathrm{co}}F(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \tag{30}$$

is $\mathcal{KL}$-stable with respect to (ω_1,ω_2) on $\mathcal{G}$;

3. $\delta(x) > 0$ for all $x \in \mathcal{G}\backslash\mathcal{A}_\delta$ where

$$\mathcal{A}_\delta := \left\{ \xi \in \mathcal{G} : \sup_{t\geq 0, \phi\in\mathcal{S}_\delta(\xi)} \omega_1(\phi(t,\xi)) = 0 \right\} \tag{31}$$

and where $\mathcal{S}_\delta(\cdot)$ represents the set of maximal solutions to (30).

The main feature of the differential inclusion (30) is that its solution set includes the solution set of the differential inclusion $\dot{x} \in F(x)$ since $F(x) \subseteq F_{\delta(x)}(x)$. Note that even for ordinary differential equations, robust stability will be expressed in terms of stability for a corresponding differential inclusion.

The following theorem emphasizes that robust $\mathcal{KL}$-stability with respect to (ω_1, ω_2) is the key property for getting a smooth converse Lyapunov function.

Theorem 1. *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous and let F satisfy the basic conditions on $\mathcal{G}$. The following statements are equivalent:*

1. *The differential inclusion $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$ and there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$ for F.*
2. *The differential inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*

Proof. See Section 5.1. □

We now specify cases where robust $\mathcal{KL}$-stability is guaranteed. The first case is when the right-hand side of the differential inclusion is locally Lipschitz, at least on a specific subset of $\mathcal{G}$:

Theorem 2. *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous and let F satisfy the basic conditions on $\mathcal{G}$. If the differential inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ and F is locally Lipschitz on an open set containing $\mathcal{G}\backslash\mathcal{A}$ where*

$$\mathcal{A} := \left\{ \xi \in \mathcal{G} : \sup_{t\geq 0, \phi\in\mathcal{S}(\xi)} \omega_1(\phi(t,\xi)) = 0 \right\} \tag{32}$$

then the differential inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.

Proof. See Section 5.2. □

With the combination of Theorems 1 and 2 we obtain a smooth, global version of ([16], Th. 3.4.1) and we recover the converse Lyapunov function results of [18] and ([33], Th. 2). For instance, we have the following statement which includes ([18], Ths. 2.8, 2.9):

Corollary 1. *Consider the system*

$$\dot{x} = f(x, d(t)) \tag{33}$$

where $d(\cdot)$ belongs to the set $\mathcal{M}_\mathcal{D}$ of measurable functions taking values in a compact set $\mathcal{D}$ and f is continuous and locally Lipschitz in x uniformly in $d \in \mathcal{D}$. Let $\mathcal{A}$ be a closed, bounded set and define $\omega(x) := |x|_\mathcal{A}$. If there exists $\beta \in \mathcal{KL}$ such that, for each $d(\cdot) \in \mathcal{M}_\mathcal{D}$ and each $x \in \mathbb{R}^n$, the (unique) solution $\psi(\cdot, x, d)$ of (33) is defined on $[0, \infty)$ and satisfies

$$\omega(\psi(t, x, d)) \leq \beta(\omega(x), t) \qquad \forall t \geq 0. \tag{34}$$

Then there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathbb{R}^n$ for

$$x \mapsto F(x) := \overline{\mathrm{co}} \left\{ v \in \mathbb{R}^n : v = f(x, d),\ d \in \mathcal{D} \right\} \cdot \tag{35}$$

Remark 1. It follows that the backward completeness assumption in ([18], Th. 2.8) is not necessary.

Proof. Due to the assumptions on f and the compactness of $\mathcal{D}$, the set

$$F^\circ(x) := \{v \in \mathbb{R}^n : v = f(x,d),\ d \in \mathcal{D}\} \tag{36}$$

is nonempty and compact for each x and F° is locally Lipschitz. It follows from these properties for F° and Filippov's Lemma (see [11] or [7], Prob. 3.7.20) that to any solution $\phi(\cdot,x)$ of the differential inclusion

$$\dot{x} \in F^\circ(x) \tag{37}$$

we can associate a function $d(\cdot) \in \mathcal{M}_\mathcal{D}$ such that, for almost all t,

$$\dot{\overbrace{\phi(t,x)}} = f(\phi(t,x), d(t)). \tag{38}$$

This says that $\phi(\cdot,x)$ is a solution of (33) and so, from the assumption of the corollary, the differential inclusion (37) is $\mathcal{KL}$-stable with respect to (ω,ω) on $\mathbb{R}^n$.

Also from the properties of F° and [3] (Cor. 10.4.5), the closure of the solution set of $\dot{x} \in F^\circ(x)$ is exactly the solution set of $\dot{x} \in \overline{\text{co}}F^\circ(x) = F(x)$. With [18] (Prop. 5.1), the inclusion $\dot{x} \in F(x)$ is forward complete on $\mathbb{R}^n$ and thus, with the previous point, is $\mathcal{KL}$-stable with respect to (ω,ω) on $\mathbb{R}^n$.

By construction, $F(x)$ is nonempty, compact and convex for each x. Since F is the closed convex hull of the locally Lipschitz set-valued map F°, it follows from [2] (Sect. 1.1, Prop. 6). that F is also locally Lipschitz.

Theorem 2 now gives that the differential inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω,ω) on $\mathbb{R}^n$. Then the corollary follows from Theorem 1. □

Several important converse Lyapunov function results, like those due to Kurzweil [15] and Clarke *et al.* [6] are not covered by Theorem 2. However, they will be covered (see, respectively, Cor. 2 of Sect. 3.2 and Cor. 3 of Sect. 3.3) by our next set of sufficient conditions for robust $\mathcal{KL}$-stability. We will show that $\mathcal{KL}$-stability implies robust $\mathcal{KL}$-stability in the case where $\omega_1(x) = \omega_2(x) =: \omega(x)$ and the differential inclusion is backward completable by ω-normalization. The latter is defined as follows:

Definition 9. Let $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be continuous. The differential inclusion $\dot{x} \in F(x)$ is said to be *backward completable by ω-normalization* if there exists a continuous function $\kappa : \mathcal{G} \to [1,\infty)$, a class-$\mathcal{K}$ function γ and a positive real number c such that

$$\kappa(x) \leq \gamma(\omega(x)) + c \tag{39}$$

and the differential inclusion

$$\dot{x} \in \frac{1}{\kappa(x)} F(x) =: F_N(x) \tag{40}$$

is backward complete on $\mathcal{G}$.

In this definition, the existence of $\kappa(\cdot)$ making (40) backward complete on $\mathcal{G}$ is always guaranteed. Indeed, from [12] (Sect. 5, Lem. 15) or [3] (Th. 1.4.16), $\sup_{v\in F(x)} |v|$ can be upper bounded by a function $\kappa_\circ : \mathcal{G} \to [1,\infty)$ that is continuous on $\mathcal{G}$. So, for instance, in the case where $\mathcal{G} = \mathbb{R}^n$, by picking $\kappa(x) = \kappa_\circ(x)$ we get $\sup_{v\in F_N(x)} |v| \leq 1$ which implies that the differential inclusion (40) is backward complete on $\mathbb{R}^n$. The difficulty comes from requiring that κ simultaneously satisfies (39). However this difficulty disappears when either ω is proper on $\mathcal{G}$ or $\dot{x} \in F(x)$ is already backward complete, perhaps because the set-valued map F satisfies a linear growth condition: there exist positive real numbers ℓ and b such that

$$v \in F(x) \qquad \Longrightarrow \qquad |v| \leq \ell|x| + b. \tag{41}$$

More generally, when $\mathcal{G}$ is the Cartesian product $\mathcal{G} := \mathcal{G}_1 \times \mathbb{R}^{n_2}$, with $\mathcal{G}_1$ an open subset of $\mathbb{R}^{n_1}$ and by writing

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \in F(x), \tag{42}$$

we have:

Proposition 2. *If*

1. *F satisfies the basic conditions on $\mathcal{G}$;*
2. *we have*

$$\lim_{x_1 \to \partial \mathcal{G}_1^\infty} \inf_{x_2 \in \mathbb{R}^{n_2}} \omega(x_1, x_2) = \infty; \tag{43}$$

3. *there exist positive real numbers ℓ and b such that*

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in F(x) \qquad \Longrightarrow \qquad |v_2| \le \ell |x_2| + b \tag{44}$$

then the differential inclusion (42) is backward completable by ω-normalization.

Proof. See Section 6.1. □

With backward completability, $\mathcal{KL}$-stability with respect to (ω, ω) implies robust $\mathcal{KL}$-stability with respect to (ω, ω):

Theorem 3. *Let $\omega : \mathcal{G} \to \mathbb{R}_{\ge 0}$ be continuous and let F satisfy the basic conditions on $\mathcal{G}$. If the differential inclusion $\dot{x} \in F(x)$ is backward completable by ω-normalization and $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$ then it is robustly $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$.*

Proof. See Section 5.3. □

3.2. Results specialized to ordinary differential equations

To illustrate our results, we specialize them to ordinary differential equations

$$\dot{x} = F(x) \tag{45}$$

where $F : \mathcal{G} \to \mathbb{R}^n$ is continuous. As noted in the introduction, this covers the case of time-varying systems

$$\dot{\xi} = f(\xi, t) \tag{46}$$

where f is continuous in both variables (compare with [4]) by taking

$$x = \begin{pmatrix} \xi \\ p \end{pmatrix}, \quad F(x) = \begin{pmatrix} f(\xi, p) \\ 1 \end{pmatrix}. \tag{47}$$

A direct consequence of Theorem 2 or 3 and Theorem 1 is the following:

Corollary 2. *If either*

I. *the function F is locally Lipschitz on $\mathcal{G}$,*

or

II. i.) *F is continuous on $\mathcal{G}$,*
  ii.) *$\omega_1 = \omega_2 = \omega$, and*
  iii.) *the differential equation $\dot{x} = F(x)$ is backward completable by ω-normalization*

then the following two statements are equivalent:

1. $\dot{x} = F(x)$ *is* $\mathcal{KL}$*-stable with respect to* (ω_1, ω_2) *on* $\mathcal{G}$.
2. $\dot{x} = F(x)$ *is forward complete on* $\mathcal{G}$ *and there exists a smooth converse Lyapunov function for* $\mathcal{KL}$*-stability with respect to* (ω_1, ω_2) *on* $\mathcal{G}$ *for* F.

With condition (II) of this corollary, we recover Kurzweil's main result ([15], Th. 7) which applies to the case described by (47) and Proposition 2, with

$$\omega_1(x) = \omega_2(x) = |\xi| \tag{48}$$

when $\mathcal{G}_1 = \mathbb{R}^{n-1}$ and, otherwise,

$$\omega_1(x) = \omega_2(x) = \max\left\{|\xi|, \frac{1}{|\xi|_{\mathbb{R}^{n-1}\setminus\mathcal{G}_1}} - \frac{2}{|0|_{\mathbb{R}^{n-1}\setminus\mathcal{G}_1}}\right\}. \tag{49}$$

3.3. Results for compact attractors

The main result of this section is that $\mathcal{KL}$-stability with respect to (ω, ω), where ω is a type of indicator for a compact set $\mathcal{A}$, is equivalent to (local) stability of $\mathcal{A}$ plus attractivity. Uniform boundedness and uniform attractivity are guaranteed by the fact that the attractor is compact. Various applications of this observation are made including a corollary that recovers [6] (Th. 1.2).

Definition 10. Given a compact subset $\mathcal{A}$ of an open set $\mathcal{G}$, a function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is said to be a *proper indicator for* $\mathcal{A}$ *on* $\mathcal{G}$ if ω is continuous, $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and $\lim_{x\to\partial\mathcal{G}^\infty} \omega(x) = \infty$.

Remark 2. For each open set $\mathcal{G}$ and each compact set $\mathcal{A} \subset \mathcal{G}$, there exists a proper indicator function. When $\mathcal{G} = \mathbb{R}^n$ we can take $\omega(x) = |x|_{\mathcal{A}}$. Otherwise, we can take, for example,

$$\omega(x) = \max\left\{|x|_{\mathcal{A}},\ \frac{1}{|x|_{\mathbb{R}^n\setminus\mathcal{G}}} - \frac{2}{\operatorname{dist}(\mathcal{A}, \mathbb{R}^n \setminus \mathcal{G})}\right\}. \tag{50}$$

Note that the right-hand side of (49) is a function that is a proper indicator for the origin (in $\mathbb{R}^{n-1}$) on $\mathcal{G}_1$.

The properties of a proper indicator ω for $\mathcal{A}$ on $\mathcal{G}$ enforce that $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ implies that the set $\mathcal{A}$ is stable and all trajectories starting in $\mathcal{G}$ converge to $\mathcal{A}$. The first result of this subsection, which is similar to [15] (Th. 12), shows that the opposite is also true. Namely, for differential inclusions with right-hand side satisfying the basic conditions, the basin of attraction $\mathcal{G}$ for a stable, compact attractor $\mathcal{A}$ is open and, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$, the differential inclusion is $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$.

Proposition 3. *Let* F *satisfy the basic conditions on an open set* $\mathcal{O}$ *and let* $\mathcal{A} \subset \mathcal{O}$ *be compact. If the set* $\mathcal{A}$ *is stable and the set of points* $\mathcal{G}$ *from which* $\mathcal{A}$ *is strongly attractive contains a neighborhood of* $\mathcal{A}$*, i.e.,*

1. Stability: *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for each* $x \in \mathcal{O} \bigcap \left(\mathcal{A} + \delta\overline{\mathcal{B}}\right)$*, each solution* $\phi \in \mathcal{S}(x)$ *is defined and belongs to* $\mathcal{O}$ *for all* $t \geq 0$ *and satisfies*
$$|\phi(t, x)|_{\mathcal{A}} \leq \varepsilon \qquad \forall t \geq 0. \tag{51}$$
2. Attractivity: *the set* $\mathcal{G}$ *of points* $x \in \mathcal{O}$ *such that each solution* $\phi \in \mathcal{S}(x)$ *is defined and belongs to* $\mathcal{O}$ *for all* $t \geq 0$ *and satisfies* $\lim_{t\to\infty} |\phi(t, \xi)|_{\mathcal{A}} = 0$ *contains a neighborhood of* $\mathcal{A}$,

then the set $\mathcal{G}$ *is open and, for each function* ω *that is a proper indicator for* $\mathcal{A}$ *on* $\mathcal{G}$*, the differential inclusion* $\dot{x} \in F(x)$ *is* $\mathcal{KL}$*-stable with respect to* (ω, ω) *on* $\mathcal{G}$.

Proof. See Section 6.2. □

By combining Theorems 1, 3 and Proposition 2 with Proposition 3, we recover [6] (Th. 1.2):

Corollary 3. *Suppose F satisfies the basic conditions on $\mathbb{R}^n$ and the origin of the differential inclusion $\dot{x} \in F(x)$ is globally asymptotically stable, i.e.,*

- *for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|x| \leq \delta, \quad \phi \in \mathcal{S}(x) \qquad \Longrightarrow \qquad |\phi(t,x)| \leq \varepsilon \qquad \forall t \geq 0; \tag{52}$$

- *for each $x \in \mathbb{R}^n$, all solutions $\phi \in \mathcal{S}(x)$ are defined for all $t \geq 0$ and satisfy*

$$\lim_{t \to \infty} |\phi(t,x)| = 0.$$

Then, taking $\omega(x) = |x|$, there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathbb{R}^n$ for F.

Nontrivial compact attractors arise in various ways. One situation, which is commonly encountered in the semiglobal practical asymptotic stabilization of nonlinear control systems (see, for example [35]), is when:

Assumption 1. *There exist two compact sets $\mathcal{C}_1$, $\mathcal{C}_2$, two strictly positive real numbers ρ, T and an open set $\mathcal{O}$ such that*

- *$\mathcal{C}_1 + \rho\overline{\mathcal{B}} \subset \mathcal{C}_2 \subset \mathcal{O}$,*
- *F satisfies the basic conditions on $\mathcal{O}$ and is Lipschitz on $\mathcal{C}_1 + \rho\overline{\mathcal{B}}$,*
- *for all $x \in \mathcal{C}_2$, all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{O}$ for all $t \geq 0$ and belong to $\mathcal{C}_1$ for $t \geq T$.*

It can be shown that:

Proposition 4. *Under Assumption 1 the set*

$$\mathcal{A} := \{\xi \in \mathcal{C}_1 : \phi(t,\xi) \in \mathcal{C}_1, \ \forall \phi \in \mathcal{S}(\xi), \ \forall t \geq 0\} \tag{53}$$

is a nonempty, compact stable attractor with basin of attraction containing $\mathcal{C}_2$.

Proof. See Section 6.3. □

As a consequence, Proposition 3 applies for this set $\mathcal{A}$. Also, for each function ω that is a proper indicator for $\mathcal{A}$ on its strong domain of attraction, Proposition 2 allows us to apply Theorem 3 and then Theorem 1. So we can state the following converse Lyapunov function theorem, for finite-time convergence to a compact set from a larger compact set:

Corollary 4. *Under Assumption 1, there exist a compact set $\mathcal{A} \subseteq \mathcal{C}_1$ and an open set $\mathcal{G} \supset \mathcal{C}_2$ such that, for each function $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ that is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$, there exists a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for F.*

3.4. **Bibliographical Notes**

- Our main result, Theorem 1, is inspired by the observations of Kurzweil [15] and Clarke *et al.* [6] who recognized that robust stability, in the context of their specific problems, allowed them to construct a smooth converse Lyapunov function.
- The results of Section 3.3 for compact attractors are based on similar results in the special cases considered by Kurzweil [15] and Clarke *et al.* [6].

4. Technical prerequisites

The proofs of our main results will be based on several technical lemmas. They concern:

1. Sontag's lemma on $\mathcal{KL}$-estimates.
2. Solutions to differential inclusions satisfying the basic conditions.
3. Solutions to locally Lipschitz differential inclusions.
4. Derivatives of locally Lipschitz functions.
5. Smoothing continuous and locally Lipschitz functions.

4.1. Sontag's lemma on $\mathcal{KL}$-estimates

We recall a recent result of Sontag ([30], Prop. 7) that is one of the keys in our converse Lyapunov function construction. The lemma is a global version of a particular aspect of the well-known Massera lemma ([21], Sect. 12) (*cf.* [16], Lem. 3.4.1). We provide an alternative proof.

Lemma 3. *For each class-$\mathcal{KL}$ function β and each number $\lambda > 0$, there exist functions $\widetilde{\alpha}_1 \in \mathcal{K}_\infty$ and $\widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that $\widetilde{\alpha}_1$ is locally Lipschitz and*

$$\widetilde{\alpha}_1(\beta(s,t)) \leq \widetilde{\alpha}_2(s)e^{-\lambda t} \qquad \forall (s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \tag{54}$$

Proof. First we pick $\rho \in \mathcal{K}_\infty$ and a function $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ continuous and strictly decreasing with $\lim_{t\to\infty} \theta(t) = 0$ such that, for all $t \geq 0$, we have

$$\beta(\rho(t),t) \leq \theta(t). \tag{55}$$

To see that such functions exist, let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of strictly positive real numbers decreasing to zero. Since $\beta \in \mathcal{KL}$, there exists a sequence $\{t_k\}_{k=1}^\infty$ of strictly positive real numbers strictly increasing to infinity such that $\beta(k+1,t_k) \leq \varepsilon_k$. Define $t_0 = 0$ and $\varepsilon_0 = \max\{\beta(1,0), 2\varepsilon_1\}$. Then, choosing ρ to be any $\mathcal{K}_\infty$ function upper bounded by the piecewise constant curve $p_1(t) = j+1$ for $t \in [t_j, t_{j+1})$ and choosing θ to be any continuous, strictly decreasing to zero function that is lower bounded by the piecewise constant curve $p_2(t) = \varepsilon_j$ for $j \in [t_j, t_{j+1})$ and using that $\beta \in \mathcal{KL}$, we have, for each integer $j \geq 0$ and each $t \in [t_j, t_{j+1})$, $\beta(\rho(t),t) \leq \beta(j+1,t_j) \leq \varepsilon_j = p_2(t) \leq \theta(t)$.

Next, let θ^{-1} be the inverse of θ, which is defined and continuous on $(0,\theta(0)]$. It is also strictly decreasing with $\lim_{s\to 0}\theta^{-1}(s) = +\infty$. It follows that the function $e^{-2\lambda\theta^{-1}(\cdot)}$ is well-defined, continuous, positive, and strictly increasing on $(0,\theta(0)]$. Then we can find $\widetilde{\alpha}_1 \in \mathcal{K}_\infty$, locally Lipschitz and such that, for all $s \in (0,\theta(0)]$,

$$\widetilde{\alpha}_1(s) \leq e^{-2\lambda\theta^{-1}(s)}. \tag{56}$$

With (55), it follows, for all $t \geq 0$,

$$\widetilde{\alpha}_1(\beta(\rho(t),t))e^{2\lambda t} \leq \widetilde{\alpha}_1(\theta(t))e^{2\lambda t} \leq 1. \tag{57}$$

Now, using (57) and the fact that $\beta \in \mathcal{KL}$,

$$\begin{aligned}
0 < s \leq \rho(t) \quad \Rightarrow \quad \widetilde{\alpha}_1(\beta(s,t))e^{\lambda t} \quad &= \quad \sqrt{\widetilde{\alpha}_1(\beta(s,0))}\sqrt{\frac{\widetilde{\alpha}_1(\beta(s,t))}{\widetilde{\alpha}_1(\beta(s,0))}}\sqrt{\widetilde{\alpha}_1(\beta(s,t))e^{2\lambda t}}, && (58)\\
&\leq \quad \sqrt{\widetilde{\alpha}_1(\beta(s,0))}\sqrt{\widetilde{\alpha}_1(\beta(\rho(t),t))e^{2\lambda t}}, && (59)\\
&\leq \quad \sqrt{\widetilde{\alpha}_1(\beta(s,0))}. && (60)\\
\rho(t) \leq s \quad \Rightarrow \quad \widetilde{\alpha}_1(\beta(s,t))e^{\lambda t} \quad &\leq \quad \widetilde{\alpha}_1(\beta(s,0))e^{\lambda\rho^{-1}(s)}. && (61)
\end{aligned}$$

So (54) holds by taking $\widetilde{\alpha}_2 \in \mathcal{K}_\infty$ such that

$$\widetilde{\alpha}_2(s) \geq \max\left\{\sqrt{\widetilde{\alpha}_1(\beta(s,0))},\ \widetilde{\alpha}_1(\beta(s,0))e^{\lambda\rho^{-1}(s)}\right\}. \tag{62}$$

□

Remark 3. Even though we are assuming minimal continuity properties for class-$\mathcal{KL}$ functions, the preceding result shows that any class-$\mathcal{KL}$ function can always be upper bounded by a continuous class-$\mathcal{KL}$ function. In particular (54) can be rewritten as

$$\beta(s,t) \leq \alpha_1^{-1}\left(\alpha_2(s)e^{-\lambda t}\right). \tag{63}$$

Note that the right-hand side of (63) is of class-$\mathcal{KL}$ and is also continuous in (s,t).

4.2. **Solutions to inclusions satisfying the basic conditions**

For the differential inclusion $\dot{x} \in F(x)$ we denote the set of points reachable from a compact set $\mathcal{C} \subset \mathcal{G}$ in time $T > 0$ as

$$\mathcal{R}_{\leq T}(\mathcal{C}) := \{\xi \in \mathbb{R}^n : \xi = \phi(t,x),\ t \in [0,T],\ x \in \mathcal{C},\ \phi \in \mathcal{S}(x)\}\cdot \tag{64}$$

The following comes from [12] (Sect. 7, Th. 3) or [10] (Th. 7.1):

Lemma 4. *Let F satisfy the basic conditions on $\mathcal{G}$ and suppose the compact set $\mathcal{C} \subset \mathcal{G}$ and the strictly positive real number $T > 0$ are such that all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{G}$ for all $t \in [0,T]$. Then the set $\mathcal{R}_{\leq T}(\mathcal{C})$ is a compact subset of $\mathcal{G}$ and the set $\mathcal{S}[0,T](\mathcal{C})$ is a compact set in the metric of uniform convergence.*

A consequence of Lemma 4 is the following:

Lemma 5. *Let F satisfy the basic conditions on $\mathcal{G}$ and suppose $x \in \mathcal{G}$ is such that all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{G}$ for all $t \geq 0$. Then each sequence $\{\phi_n\}_{n=1}^\infty$ of solutions in $\mathcal{S}(x)$ has a subsequence converging to a function $\phi \in \mathcal{S}(x)$ and the convergence is uniform on each compact time interval.*

Proof. From Lemma 4, we know that for each integer k, the set $\mathcal{S}[0,k](x)$ is a compact set in the metric of uniform convergence. Since for all n and k, ϕ_n is in $\mathcal{S}[0,k](x)$, it follows that $\{\phi_n\}_{n=1}^\infty$ has a subsequence $\{\phi_{1m}\}_{m=1}^\infty$ converging uniformly on $[0,1]$ to a function $\phi_1 \in \mathcal{S}(x)$. Similarly $\{\phi_{1m}\}_{m=1}^\infty$ has a subsequence $\{\phi_{2m}\}_{m=1}^\infty$ converging uniformly on $[0,2]$ to a function $\phi_2 \in \mathcal{S}(x)$. And so on. The result follows by taking the subsequence given by the diagonal elements ϕ_{mm}. □

The next result is on "continuity" of solutions with respect to initial conditions and perturbations of the right-hand side. See [12] (Sect. 8, Cor. 2).

Lemma 6. *Suppose $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$, F satisfies the basic conditions on $\mathcal{G}$, and $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is continuous on $\mathcal{G}$. For each triple $(T,\varepsilon,\mathcal{C})$ where $T > 0$, $\varepsilon > 0$ and $\mathcal{C} \subset \mathcal{G}$ compact there exists $\delta > 0$ such that every maximal solution $\phi_\delta(\cdot,x_\delta)$ of*

$$\dot{x} \in F_\delta(x) := \overline{\mathrm{co}}F(x + \delta\overline{\mathcal{B}}) + \delta\overline{\mathcal{B}}, \tag{65}$$

with $x_\delta \in \mathcal{C} + \delta\overline{\mathcal{B}}$, remains in $\mathcal{G}$ for all $t \in [0,T]$ and there exists a solution $\phi(\cdot,x)$ of $\dot{x} \in F(x)$ with $x \in \mathcal{C}$ and $|x - x_\delta| \leq \varepsilon$ such that, for all $t \in [0,T]$,

$$|\omega(\phi_\delta(t,x_\delta)) - \omega(\phi(t,x))| \leq \varepsilon. \tag{66}$$

Remark 4. A useful remark is that if Lemma 6 holds for $\delta > 0$ then it holds for all $\bar{\delta} \in (0, \delta]$ since, in this case, $\mathcal{C} + \bar{\delta}\overline{\mathcal{B}} \subseteq \mathcal{C} + \delta\overline{\mathcal{B}}$ and $F_{\bar{\delta}}(x) \subseteq F_\delta(x)$.

Lemmas 4 and 6 can be used to show:

Lemma 7. *Suppose F satisfies the basic conditions on $\mathcal{G}$ and the differential inclusion*

$$\dot{x} \in F(x) \tag{67}$$

is forward complete on $\mathcal{G}$. Then there exists a continuous function $\delta : \mathcal{G} \to (0, \infty)$ such that

$$\{x\} + \delta(x)\overline{\mathcal{B}} \ \subset \ \mathcal{G}, \tag{68}$$

the differential inclusion

$$\dot{x} \in F_{\delta(x)}(x) := \overline{\text{co}} F(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \tag{69}$$

is forward complete on $\mathcal{G}$, and $F_{\delta(\cdot)}(\cdot)$ satisfies the basic conditions on $\mathcal{G}$.

Proof. In the following, let $\mathcal{R}_{\leq 1}(\mathcal{C})$, respectively $\mathcal{R}^{\delta(\cdot)}_{\leq 1}(\mathcal{C})$, denote the reachable set in time $t = 1$ from the compact set $\mathcal{C} \subset \mathcal{G}$ for the differential inclusion (67), respectively the differential inclusion (69).

According to Lemma 4, the forward completeness assumption on (67) implies that the reachable set in each finite time from each compact subset of $\mathcal{G}$ for the system (67) is a compact subset of $\mathcal{G}$. So, we can find a compact, countable covering $\mathcal{C}_i$, $i = 1, 2, \dots$, of $\mathcal{G}$ such that, for each i, there exists $\varepsilon_i > 0$ satisfying

$$\mathcal{C}_i \subset \mathcal{R}_{\leq 1}(\mathcal{C}_i) + \varepsilon_i\overline{\mathcal{B}} \subset \mathcal{C}_{i+1} \subset \mathcal{G}. \tag{70}$$

By applying Lemma 6 with the triple $(1, \varepsilon_i, \mathcal{C}_i)$ and $\omega(x) = x$ we get, for each i, the existence of

$$\delta_i \in (0, \varepsilon_i] \tag{71}$$

such that

$$\mathcal{R}^{\delta_i}_{\leq 1}(\mathcal{C}_i) \subset \mathcal{C}_{i+1}. \tag{72}$$

Without loss of generality, from Remark 4, we can assume that the sequence $\{\delta_i\}_{i=1}^{\infty}$ is nonincreasing. Define

$$\begin{aligned} i(x) \quad &:= \quad \inf_i \{x \in \mathcal{C}_i\} \\ \Delta(x) \quad &:= \quad \delta_{i(x)} \\ \delta(x) \quad &:= \quad \inf_{\xi \in \mathcal{G}} [\Delta(\xi) + |x - \xi|] \,. \end{aligned} \tag{73}$$

As in the proof of [7] (Th. 1.5.1), one can check that the function δ is well-defined and is Lipschitz on $\mathcal{G}$ with Lipschitz constant 1. We have also, for all $x \in \mathcal{G}$,

$$\delta(x) \leq \delta_{i(x)}, \qquad \delta(x) > 0 \tag{74}$$

and, from (70) and (71),

$$\{x\} + \delta(x)\overline{\mathcal{B}} \ \subset \ \mathcal{G}. \tag{75}$$

We will show that

$$\mathcal{R}^{\delta(\cdot)}_{\leq 1}(\mathcal{C}_i) \subset \mathcal{C}_{i+1} \quad \forall i. \tag{76}$$

From the definition of $i(x)$ and the fact that the sequence $\{\delta_i\}_{i=1}^{\infty}$ is nonincreasing, it follows that

$$x \in \partial\mathcal{C}_i \bigcup (\mathcal{G}\backslash\mathcal{C}_i) \qquad \Longrightarrow \qquad \delta(x) \leq \delta_i. \tag{77}$$

To establish a contradiction, suppose the existence of an integer j and $\xi \in \mathcal{R}^{\delta(\cdot)}_{\leq 1}(\mathcal{C}_j)$ such that $\xi \notin \mathcal{C}_{j+1}$, *i.e.*, there exists a solution ϕ of the differential inclusion (69) starting from a point $x \in \mathcal{C}_j$ that reaches, at time $\bar{t} \leq 1$, the point $\xi \notin \mathcal{C}_{j+1}$. Then, exploiting time-invariance, continuity of this solution with respect to time and compactness of $\mathcal{C}_j$, there exists $t_\circ \in [0, \bar{t})$ such that $\phi(t_\circ, x) \in \partial\mathcal{C}_j$, and $\phi(t, x) \in \partial\mathcal{C}_j \bigcup (\mathcal{G}\backslash\mathcal{C}_j)$ for all $t \in [t_\circ, \bar{t}]$. It follows from (77) that $\delta(\phi(t, x)) \leq \delta_j$ for all $t \in [t_\circ, \bar{t}]$. Therefore, on the interval $[t_\circ, \bar{t}]$, $\phi(\cdot, x)$ is a solution of the differential inclusion (69) with $\delta(x) = \delta_j$. But then, using (72) and the fact that $\bar{t} - t_\circ \leq 1$ and $\phi(t_\circ, x) \in \mathcal{C}_j$, it is impossible to have $\phi(\bar{t}, x) \notin \mathcal{C}_{j+1}$. This contradiction establishes that (76) holds.

Now, suppose that (69) is not forward complete. Then there exist an integer j, a point $x \in \mathcal{C}_j$, a solution ϕ of (69) starting at x and $t^* < \infty$ such that for each integer $m > j$ there exists $t_m < t^*$ such that $\phi(t_m, x) \notin \mathcal{C}_m$. On the other hand, (76) implies

$$\mathcal{R}^{\delta(\cdot)}_{\leq m-j}(\mathcal{C}_j) \subset \mathcal{C}_m \tag{78}$$

which implies that $m - j < t_m < t^*$ for all $m > j$. This is impossible. □

The final result of this subsection provides a link to the next subsection where solutions of locally Lipschitz differential inclusions are considered. The result is slight modification of [6] (Prop. 3.5).

Lemma 8. *Let $\mathcal{A}$ be such that $\mathcal{G}\backslash\mathcal{A}$ is open and $\Delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$ and such that $\{x\} + \Delta(x)\overline{\mathcal{B}} \subset \mathcal{G}$.*

If F satisfies the basic conditions on $\mathcal{G}$ then there exists a set-valued map F_L satisfying the basic conditions on $\mathcal{G}$, locally Lipschitz on $\mathcal{G}\backslash\mathcal{A}$, and such that

$$F(x) \subseteq F_L(x) \subseteq F_{\Delta(x)}(x) := \overline{\text{co}}F(x + \Delta(x)\overline{\mathcal{B}}) + \Delta(x)\overline{\mathcal{B}}. \tag{79}$$

4.3. Solutions to locally Lipschitz differential inclusions

We start with the following fact (see [7], Ex. 4.3.3.a):

Lemma 9. *Let $\mathcal{O}$ be an open subset of $\mathcal{G}$. If the set-valued map F is locally Lipschitz on $\mathcal{O}$, then for any compact set $K \subset \mathcal{O}$ there exists a positive real number L_K such that for any x_1 and x_2 in K, we have*

$$F(x_1) \subseteq F(x_2) + L_K|x_1 - x_2|\mathcal{B}. \tag{80}$$

The next result, on solutions to locally Lipschitz differential inclusions, is similar to [10] (Lem. 8.3 [7], Lem. 4.3.11) and [3] (Th. 10.4.1). (See also [6], proof of Lem. 4.9.) We provide a proof for completeness.

Lemma 10. *Let F satisfy the basic conditions on $\mathcal{G}$ and be locally Lipschitz on the open set $\mathcal{O} \subseteq \mathcal{G}$. For each $T > 0$ and each compact set $\mathcal{C} \subset \mathcal{O}$, there exist L and $\delta > 0$ such that, for each $x \in \mathcal{C}$, each $\phi \in \mathcal{S}(x)$ and each ξ satisfying $|x - \xi| \leq \delta$, there exists $\psi \in \mathcal{S}(\xi)$ with the property*

$$|\phi(t, x) - \psi(t, \xi)| \leq L|x - \xi| \qquad \forall t \in [0, T_x], \tag{81}$$

where $T_x \in [0, T]$ is such that $\phi(t, x) \in \mathcal{C}$ for all $t \in [0, T_x]$.

Proof. Preparation step: For each pair $(z, v) \in \mathcal{G} \times \mathbb{R}^n$, define $g(z, v)$ to be the unique (since $F(z)$ is closed and convex; see [12], Sect. 5, Lem. 2) closest point in $F(z)$ to v. Since F is locally Lipschitz, closed and convex it follows from [12] (Sect. 6, Lem. 8) that $g(\cdot, v)$ is continuous for each fixed v. Since $F(z)$ is closed and convex, $g(z, \cdot)$ is continuous for each fixed z. From [12] (Sect. 5, Lem. 15), for each compact subset $\mathcal{X}$ of $\mathcal{G}$ there exists a constant m such that $|g(z, v)| \leq m$ for all $(z, v) \in \mathcal{X} \times \mathbb{R}^n$.

Below, we will pick $x \in \mathcal{G}$ and $\phi \in \mathcal{S}(x)$ defined on $[0, T_{x\infty})$ and will define $w(t) := \dot{\overbrace{\phi(t, x)}} \in F(\phi(t, x))$ for almost all $t \in [0, T_{x\infty})$. (The function $w(\cdot)$ can be defined arbitrarily for those $t \in [0, T_{x\infty})$ where $\dot{\overbrace{\phi(t, x)}}$ is not defined.) Since $\phi(\cdot, x)$ is absolutely continuous, $w(\cdot)$ is measurable. Then we will define $g_x(z, t) := g(z, w(t))$. Since $w(\cdot)$ is measurable and g has the properties given above, $g_x : \mathcal{G} \times [0, T_{x\infty}) \to \mathbb{R}^n$ satisfies the Carathéodory conditions for existence of solutions to the ordinary differential equation $\dot{z} = g_x(z, t)$.

Core of the proof: Let $\mathcal{C} \subset \mathcal{O}$ and $T > 0$ be given. Since $\mathcal{C}$ is a compact subset of $\mathcal{O}$ which is open, there exists $\varepsilon > 0$ so that $\mathcal{C} + \varepsilon\overline{\mathcal{B}}$ is a compact subset of $\mathcal{O}$. Using Lemma 9, let K be a Lipschitz constant for F on $\mathcal{C} + \varepsilon\overline{\mathcal{B}}$. We choose

$$L = \exp(KT), \qquad \delta = \frac{\varepsilon}{2L} \cdot \tag{82}$$

Let x, $\phi \in \mathcal{S}(x)$, T_x and $T_{x\infty}$ be such that $T_x \in (0, T]$, $\phi(t, x) \in \mathcal{C}$ for all $t \in [0, T_x]$ and $\phi(\cdot, x)$ is right maximally defined on $[0, T_{x\infty})$. Necessarily $T_{x\infty} > T_x$.

(If $T_x = 0$ then, since $L \geq 1$, there is nothing to prove.) Let g_x be as above, defined on $\mathcal{G} \times [0, T_{x\infty})$. Pick ξ satisfying $|x - \xi| \leq \delta$. Then ξ is an interior point of $\mathcal{C} + \varepsilon\overline{\mathcal{B}}$. Let $\psi(\cdot, \xi)$ be a solution with values in $\mathcal{G}$ of

$$\dot{\overbrace{\psi(t, \xi)}} = g_x(\psi(t, \xi), t), \quad \psi(0, \xi) = \xi \tag{83}$$

right maximally defined on $[0, T_\xi)$. We have $T_\xi \leq T_{x\infty}$ and from the definition of g_x, $\psi(\cdot, \xi)$ is a solution on $[0, T_\xi)$ of the differential inclusion $\dot{x} \in F(x)$. Now, either $T_\xi = T_{x\infty}$ or there exists $\bar{t}_\circ \in [0, T_\xi)$ such that $\psi(\bar{t}_\circ, \xi) \notin \mathcal{C} + \varepsilon\overline{\mathcal{B}}$. We define

$$\bar{t} := \sup\left\{t \in [0, T_x] : \psi(s, \xi) \in \mathcal{C} + \varepsilon\overline{\mathcal{B}},\ \forall s \in [0, t]\right\} \cdot \tag{84}$$

We must have that $\bar{t} < T_\xi$ since in the case where $T_\xi = T_{x\infty}$ we have $\bar{t} \leq T_x < T_{x\infty} = T_\xi$ and in the case where $T_\xi < T_{x\infty}$ we have $\bar{t} \leq \bar{t}_\circ < T_\xi$. Thus $\psi(t, \xi)$ is well-defined for all $t \in [0, \bar{t}]$ and, by the continuity of $\psi(\cdot, \xi)$ and since ξ is an interior point of $\mathcal{C} + \varepsilon\overline{\mathcal{B}}$ and $T_x > 0$, we have $\bar{t} > 0$ and

$$\bar{t} < T_x \quad \Longrightarrow \quad \psi(\bar{t}, \xi) \in \partial(\mathcal{C} + \varepsilon\overline{\mathcal{B}}). \tag{85}$$

Then, from the definition of g_x we have, for almost all $t \in [0, \bar{t}]$,

$$\frac{d}{dt}|\phi(t, x) - \psi(t, \xi)| \leq |w(t) - g_x(\psi(t, \xi), t)| \tag{86}$$

$$= |w(t) - g(\psi(t, \xi), w(t))| \tag{87}$$

$$= \inf_{v \in F(\psi(t, \xi))} |w(t) - v|. \tag{88}$$

Then, since we have, from the Lipschitz property:

$$w(t) \in F(\phi(t, x)) \subset F(\psi(t, \xi)) + K\,|\phi(t, x) - \psi(t, \xi)|\,\mathcal{B}, \tag{89}$$

we conclude that, for almost all $t \in [0, \bar{t}]$,

$$\frac{d}{dt}|\phi(t,x) - \psi(t,\xi)| \ \leq \ K|\phi(t,x) - \psi(t,\xi)|. \tag{90}$$

Invoking a comparison theorem, we get, for all $t \in [0, \bar{t}]$,

$$|\phi(t,x) - \psi(t,\xi)| \leq \exp(KT)|x - \xi| = L|x - \xi| \leq L\delta = \frac{\varepsilon}{2} \cdot \tag{91}$$

Now if $\bar{t} = T_x$ we are done. Suppose $\bar{t} < T_x$. Since we have that $\phi(t,x) \in \mathcal{C}$ for all $t \in [0, T_x]$, (91) implies that $\psi(t,\xi)$ is in the interior of $\mathcal{C} + \varepsilon\overline{\mathcal{B}}$, for all $t \in [0, \bar{t}]$. But the latter contradicts (85). So we must have $\bar{t} = T_x$. □

Lemma 11. *Let F satisfy the basic conditions on $\mathcal{G}$ and let F be locally Lipschitz on a neighborhood of $x \in \mathcal{G}$. Then for each $v \in F(x)$ there exists a solution $\phi \in \mathcal{S}(x)$ satisfying*

$$\phi(t,x) = x + t(v + r(t)) \qquad \forall t \in [0, T) \tag{92}$$

for some $T > 0$ and for some function $r(\cdot)$ that is continuous on $[0, T)$ and satisfies $\lim_{t \to 0^+} r(t) = 0$.

Proof. (See also [6], proof of Lem. 4.8.) As in the previous proof, for each ξ in a neighborhood of x, let $g(\xi) \in F(\xi)$ be the unique closest point in the compact convex set $F(\xi)$ to the vector v. Again, the function g is well-defined and continuous on a neighborhood of x since F is locally Lipschitz on a neighborhood of x. Let $\phi(\cdot, x)$ be a solution to the differential equation

$$\dot{\xi} = g(\xi) \tag{93}$$

starting at x defined on $[0, T)$. Since $g(\xi) \in F(\xi)$, ϕ is also a solution of $\dot{x} \in F(x)$. Since $g(x) = v$, the result follows. □

4.4. Derivatives of locally Lipschitz functions

First we recall the definition of the Dini subderivate of a function $V : \mathcal{O} \to \mathbb{R}$ ($\mathcal{O}$ open), at a point $x \in \mathcal{O}$ in the direction $v \in \mathbb{R}^n$:

$$DV(x; v) := \liminf_{w \to v, \varepsilon \to 0^+} \frac{V(x + \varepsilon w) - V(x)}{\varepsilon} \cdot \tag{94}$$

From [7] (Ex. 3.4.1), we have:

Lemma 12. *If V is a locally Lipschitz function on an open set $\mathcal{O}$ of $\mathbb{R}^n$ then, for all $x \in \mathcal{O}$ such that the gradient of V (denoted ∇V) exists we have*

$$DV(x; v) = \langle \nabla V(x), v \rangle \cdot \tag{95}$$

The set of points where the gradient exists is characterized by Rademacher's Theorem (see [28], Def. VIII.3.2, Cor. VIII.3.1 and [7], p. 147):

Lemma 13. *If V is a locally Lipschitz function on an open set $\mathcal{O}$ of $\mathbb{R}^n$, it has a gradient ∇V at almost all points $x \in \mathcal{O}$.*

Finally, like in [6], to establish the Lipschitz property we use the following result from [8] (Cor. 3.7):

Lemma 14. *Let the function $V : \mathcal{O} \to (-\infty, \infty]$ be lower semicontinuous. Let $\mathcal{U} \subset \mathcal{O}$ be open and convex. The function V is Lipschitz with Lipschitz constant M on $\mathcal{U}$ if and only if*

$$DV(x; v) \leq M|v| \qquad \forall x \in \mathcal{U}, \ \forall v \in \mathbb{R}^n. \tag{96}$$

4.5. Smoothing continuous and locally Lipschitz functions

A standard approximation result is the following:

Lemma 15. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and let $\mu : \mathcal{O} \to (0, \infty)$ be continuous. Suppose $V : \mathcal{O} \to \mathbb{R}$ is continuous. Then there exists a smooth function $V_s : \mathcal{O} \to \mathbb{R}$ such that, for all $x \in \mathcal{O}$,*

$$|V(x) - V_s(x)| \leq \mu(x). \tag{97}$$

The next result is similar to [6] (Lem. 5.1) which is based on similar results in [15, 40] and [18] (Th. B.1).

Lemma 16. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and let the three functions $\alpha : \mathcal{O} \to \mathbb{R}$ and $\mu, \nu : \mathcal{O} \to (0, \infty)$ be continuous. Suppose $V : \mathcal{O} \to \mathbb{R}$ is locally Lipschitz on $\mathcal{O}$, and the set-valued map F satisfies the basic conditions on $\mathcal{O}$ and is locally Lipschitz on $\mathcal{O}$, and for almost all $x \in \mathcal{O}$,*

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \leq \alpha(x). \tag{98}$$

Then there exists a smooth function $V_s : \mathcal{O} \to \mathbb{R}$ such that, for all $x \in \mathcal{O}$,

$$|V(x) - V_s(x)| \leq \mu(x) \tag{99}$$

and

$$\max_{w \in F(x)} \langle \nabla V_s(x), w \rangle \leq \alpha(x) + \nu(x). \tag{100}$$

The last result is similar to [18] (Lem. 4.3).

Lemma 17. *Let $\mathcal{G}$ and $\mathcal{G}\backslash\mathcal{A}$ be open sets. Assume that $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is continuous, the restriction of V to the set $\mathcal{G}\backslash\mathcal{A}$ is smooth, $V(x) = 0$ for all $x \in \mathcal{A}$, and $V(x) > 0$ for all $x \in \mathcal{G}\backslash\mathcal{A}$. Then there exists a function $\rho \in \mathcal{K}_\infty$, smooth on $(0, \infty)$ and having derivative that is a class-$\mathcal{K}_\infty$ function satisfying $\rho(s) \leq s\rho'(s)$ for all $s \geq 0$, such that $V_s := \rho \circ V$ is smooth on $\mathcal{G}$.*

5. Proofs of main results

5.1. Proof of Theorem 1

5.1.1. *Forward completeness, smooth converse function $\Longrightarrow$ robust $\mathcal{KL}$-stability*

We start with the following observation:

Lemma 18. *Let F satisfy the basic conditions on $\mathcal{G}$ and suppose V is a smooth converse Lyapunov function for $\mathcal{KL}$-stability for (ω_1, ω_2) on $\mathcal{G}$ for F. There exists a continuous function $\delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$, positive on the set where $V(x) > 0$, such that:*

1. *$x \mapsto F_{\delta(x)}(x) := \overline{\mathrm{co}} F(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}}$ satisfies the basic conditions on $\mathcal{G}$;*
2. *$V(\cdot)^4$ is a smooth converse Lyapunov function for $\mathcal{KL}$-stability for (ω_1, ω_2) on $\mathcal{G}$ for $F_{\delta(\cdot)}(\cdot)$.*

Proof. We introduce two functions δ_1 and δ_2 as follows:

- Define

$$\delta_1(x) := \frac{V(x)}{4 \max\{1, |\nabla V(x)|\}}\,. \tag{101}$$

  This function is well-defined and continuous on $\mathcal{G}$ since V is smooth. Moreover, it is positive when $V(x)$ is positive.

- We define $\delta_2(x)$ as follows:
  - For $x \in \mathcal{G}$ such that $V(x) = 0$, define $\delta_2(x) = 0$.
  - For $x \in \mathcal{G}$ such that $V(x) > 0$, define $\delta_2(x)$ to be the supremum over all $\delta \leq 1$ such that $\{x\} + 2\delta\overline{\mathcal{B}} \subset \mathcal{G}$ and

$$\max_{w \in \overline{\text{co}} F(x+2\delta\overline{\mathcal{B}})} \langle \nabla V(x), w \rangle \leq -\frac{1}{2} V(x). \tag{102}$$

To see that this function is well-defined, first note that, from the upper semicontinuity of F, for each $x \in \mathcal{G}$ such that $V(x) > 0$ there exists $\bar{\delta}(x) > 0$ such that

$$F(x + 2\bar{\delta}(x)\overline{\mathcal{B}}) \subseteq F(x) + 2\delta_1(x)\overline{\mathcal{B}}. \tag{103}$$

Then, using the convexity of $F(x)$ and [12] (Sect. 5, Lem. 9),

$$\overline{\text{co}} F(x + 2\bar{\delta}(x)\overline{\mathcal{B}}) \subseteq \overline{\text{co}} \left[F(x) + 2\delta_1(x)\overline{\mathcal{B}} \right] = F(x) + 2\delta_1(x)\overline{\mathcal{B}}. \tag{104}$$

It follows from the definition of $\delta_1(x)$ that (102) is satisfied with $\delta = \min\{1, \bar{\delta}(x)\}$.

We claim that for each compact subset $\mathcal{C}$ of $\mathcal{G}$ such that $V(x) > 0$ for all $x \in \mathcal{C}$, we have $\inf_{x \in \mathcal{C}} \delta_2(x) > 0$. Suppose not. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$, with $x_n \in \mathcal{C}$ for each n, converging to $x^* \in \mathcal{C}$ such that

$$\max_{w \in \overline{\text{co}} F(x_n + \frac{1}{n}\overline{\mathcal{B}})} \langle \nabla V(x_n), w \rangle > -\frac{1}{2} V(x_n). \tag{105}$$

Using the upper semicontinuity of F and the convexity of $F(x^*)$, as in (104), we have, for n sufficiently large,

$$\overline{\text{co}} F(x_n + \frac{1}{n}\overline{\mathcal{B}}) \subseteq F(x^*) + \delta_1(x^*)\overline{\mathcal{B}} \tag{106}$$

and thus

$$-\frac{1}{2} V(x_n) < \max_{w \in \overline{\text{co}} F(x_n + \frac{1}{n}\overline{\mathcal{B}})} \langle \nabla V(x_n), w \rangle \leq \max_{w \in F(x^*) + \delta_1(x^*)\overline{\mathcal{B}}} \langle \nabla V(x_n), w \rangle \cdot \tag{107}$$

Since (101) gives

$$\max_{w \in F(x^*) + \delta_1(x^*)\overline{\mathcal{B}}} \langle \nabla V(x^*), w \rangle \leq -\frac{3}{4} V(x^*), \tag{108}$$

the continuity of V (107) and (108) provide a contradiction to (105).

We define

$$\Delta(x) := \min\{\delta_1(x), \delta_2(x)\} \tag{109}$$

which, from (101) and (102), gives

$$\max_{w \in \overline{\text{co}} F(x+\Delta(x)\overline{\mathcal{B}}) + \Delta(x)\overline{\mathcal{B}}} \langle \nabla V(x), w \rangle \leq -\frac{1}{4} V(x), \tag{110}$$

i.e.,

$$\max_{w\in\overline{\mathrm{co}}F(x+\Delta(x)\overline{\mathcal{B}})+\Delta(x)\overline{\mathcal{B}}}\langle\nabla(V(x)^4),w\rangle\leq -V(x)^4. \tag{111}$$

Now, we are ready to define the function δ of Lemma 18 as

$$\delta(x)=\inf_{\xi\in\mathcal{G}}\left[\Delta(\xi)+|x-\xi|\right]. \tag{112}$$

It is Lipschitz continuous and satisfies

$$\delta(x)\ \leq\ \Delta(x), \tag{113}$$

$$\Delta(x)>0\quad\Rightarrow\quad\delta(x)>0. \tag{114}$$

With such a function, Lemma 1 states that $F_{\delta(\cdot)}(\cdot)$ satisfies the basic conditions on $\mathcal{G}$. So, $\delta(\cdot)$ satisfies the conditions of Lemma 18. □

We assume that $\dot{x}\in F(x)$ is forward complete on $\mathcal{G}$ and we let V be a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1,ω_2) on $\mathcal{G}$ for F. Let δ_1 be given by Lemma 18 and let δ_2 be given by Lemma 7. Define

$$\delta(x)=\min\{\delta_1(x),\delta_2(x)\}\cdot \tag{115}$$

The function $\delta:\mathcal{G}\to\mathbb{R}_{\geq 0}$ is continuous, positive on the set where V is positive and $\{x\}+\delta(x)\overline{\mathcal{B}}\subset\mathcal{G}$. Using Lemma 7, the differential inclusion

$$\dot{x}\in F_{\delta(x)}(x):=\overline{\mathrm{co}}F(x+\delta(x)\overline{\mathcal{B}})+\delta(x)\overline{\mathcal{B}} \tag{116}$$

is forward complete on $\mathcal{G}$ and $F_{\delta(\cdot)}(\cdot)$ satisfies the basic conditions on $\mathcal{G}$. Using Lemma 18, there exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that for all maximal solutions $\phi(\cdot,x)$ of (116) and all $t\geq 0$, we have

$$\alpha_1(\omega_1(\phi(t,x)))\leq V(\phi(t,x))^4\leq\alpha_2(\omega_2(\phi(t,x))) \tag{117}$$

and

$$\overbrace{V(\phi(t,x))^4}^{\cdot}\leq -V(\phi(t,x))^4. \tag{118}$$

From a comparison theorem, (118) implies that

$$V(\phi(t,x))^4\leq V(x)^4e^{-t}\qquad\forall t\geq 0. \tag{119}$$

Then (117) gives

$$\omega_1(\phi(t,x))\leq\alpha_1^{-1}(\alpha_2(\omega_2(x))e^{-t})\qquad\forall t\geq 0, \tag{120}$$

i.e., the differential inclusion (116) is $\mathcal{KL}$-stable with respect to (ω_1,ω_2) on $\mathcal{G}$. Finally (119) and (117) give the following implications:

$$V(x)=0\quad\Longrightarrow\quad V(\phi(t,x))\equiv 0\quad\forall t\geq 0, \tag{121}$$

$$\Longrightarrow\quad\omega_1(\phi(t,x))\equiv 0\quad\forall t\geq 0. \tag{122}$$

So the set $\mathcal{V}^\circ := \{\xi : V(\xi) = 0\}$ is a subset of the set $\mathcal{A}_\delta$ defined by (31). Then since $\delta(x) > 0$ whenever $V(x) > 0$, we have

$$\delta(x) > 0 \quad \forall x \in \mathcal{G} \backslash \mathcal{V}^\circ \supseteq \mathcal{G} \backslash \mathcal{A}_\delta. \tag{123}$$

We conclude that the differential inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$. □

5.1.2. *Robust $\mathcal{KL}$-stability $\Longrightarrow$ forward completeness, smooth converse function*

Step 1: Preliminaries. If $\omega_1(x) = 0$ for all $x \in \mathcal{G}$ then the result is established by taking $V(x) = 0$ for all $x \in \mathcal{G}$. So, without loss of generality, we can assume that the set $\mathcal{G} \backslash \{\omega_1(x) = 0\}$ is a nonempty, open set.

Also without loss of generality, we can assume that ω_1 is locally Lipschitz on $\mathcal{G}$. Indeed, if ω_1 is only continuous on $\mathcal{G}$ then ω_1 can be smoothed on $\mathcal{G}$ using Lemma 15 and Lemma 17. In particular, we first get a function $\bar{\omega}_1$, continuous on $\mathcal{G}$ and smooth on $\mathcal{G} \setminus \{x : \omega_1(x) = 0\}$ and satisfying $|\bar{\omega}_1(x) - \omega_1(x)| \le \frac{1}{2}\omega_1(x)$ for all $x \in \mathcal{G}$. Then we get a function $\rho \in \mathcal{K}_\infty$ so that the function $\widetilde{\omega}_1 := \rho \circ \bar{\omega}_1$ is smooth on $\mathcal{G}$. In this way, the function $\widetilde{\omega}_1$ satisfies

$$\rho\left(\tfrac{3}{2}\omega_1(x)\right) \ge \widetilde{\omega}_1(x) \ \ge \ \rho\left(\tfrac{1}{2}\omega_1(x)\right). \tag{124}$$

These inequalities guarantee that the differential inclusion $\dot{x} \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to $(\widetilde{\omega}_1, \omega_2)$ on $\mathcal{G}$. The second inequality guarantees that if V is a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to $(\widetilde{\omega}_1, \omega_2)$ then it is also a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2).

Let $\delta : \mathcal{G} \to \mathbb{R}_{\ge 0}$ be the continuous function given by the robust $\mathcal{KL}$-stability assumption. From this δ let F_L, coming from Lemma 8, be a set-valued map satisfying the basic conditions on $\mathcal{G}$, locally Lipschitz on the open set

$$\mathcal{O} := \mathcal{G} \backslash \{\xi \in \mathcal{G} : \delta(\xi) = 0\} \tag{125}$$

and such that

$$F(x) \subseteq F_L(x) \subseteq F_{\delta(x)}(x). \tag{126}$$

From the property of $\dot{x} \in F_{\delta(x)}(x)$, we deduce that the differential inclusion $\dot{x} \in F_L(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$. We also have $\mathcal{G} \backslash \mathcal{A}_L \subseteq \mathcal{O}$ where

$$\mathcal{A}_L := \left\{ \xi \in \mathcal{G} : \sup_{t \ge 0, \phi \in \mathcal{S}_L(\xi)} \omega_1(\phi(t, \xi)) = 0 \right\} \tag{127}$$

where $\mathcal{S}_L(\cdot)$ represents the set of maximal solutions to $\dot{x} \in F_L(x)$. To see that this is the case, first note that, with $\mathcal{A}_\delta$ defined in (31), the set $\mathcal{G} \backslash \mathcal{A}_\delta$ is a subset of $\mathcal{O}$. Then, since $F_L(x) \subseteq F_{\delta(x)}(x)$, we have $\mathcal{A}_\delta \subseteq \mathcal{A}_L$. Combining, we have

$$\mathcal{G} \backslash \mathcal{A}_L \subseteq \mathcal{G} \backslash \mathcal{A}_\delta \subseteq \mathcal{O}. \tag{128}$$

Step 2: Construction of V_1 and its basic properties. Let $\beta \in \mathcal{KL}$ be such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}_L(x)$ satisfy

$$\omega_1(\phi(t, x)) \le \beta(\omega_2(x), t) \qquad \forall t \ge 0. \tag{129}$$

Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be given by Lemma 3 for this β and $\lambda = 2$, *i.e.*, for all $x \in \mathcal{G}$, $\phi \in \mathcal{S}_L(x)$ and $t \geq 0$,

$$\widetilde{\alpha}_1 \left(\omega_1(\phi(t,x)) \right) \leq \widetilde{\alpha}_2(\omega_2(x)) e^{-2t}. \tag{130}$$

Define, for each $x \in \mathcal{G}$,

$$V_1(x) := \sup_{t \geq 0, \phi \in \mathcal{S}_L(x)} \widetilde{\alpha}_1 \left(\omega_1(\phi(t,x)) \right) e^t. \tag{131}$$

The function V_1 has the following properties:

Proposition 5. *The function $V_1 : \mathcal{G} \to \mathbb{R}^n$ defined in (131) is continuous on $\mathcal{G}$, locally Lipschitz on $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$ and satisfies*

$$V_1(x) \; = \; 0 \quad \iff \quad x \in \mathcal{A}_L \tag{132}$$

$$\widetilde{\alpha}_1(\omega_1(x)) \leq V_1(x) \leq \widetilde{\alpha}_2(\omega_2(x)) \tag{133}$$

and, for almost all $x \in \{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$,

$$\max_{w \in F_L(x)} \langle \nabla V_1(x), w \rangle \leq -V_1(x). \tag{134}$$

Proof. We prove this proposition in three steps.

1. Property (132) and inequalities (133): (132) is a consequence of the definition (131) of V_1 and (127) of $\mathcal{A}_L$. The lower bound in (133) comes from

$$V_1(x) \geq \sup_{\phi \in \mathcal{S}_L(x)} \widetilde{\alpha}_1 \left(\omega_1(\phi(t,x)) \right) e^t \Big|_{t=0} = \widetilde{\alpha}_1(\omega_1(x)) \tag{135}$$

while the upper bound in (133) comes from using (130) to get

$$V_1(x) \leq \sup_{t \geq 0} \widetilde{\alpha}_2(\omega_2(x)) e^{-t} = \widetilde{\alpha}_2(\omega_2(x)). \tag{136}$$

2. Inequality (134): To establish (134) we will use the following claim:

Claim 1. *For all $x \in \mathcal{G}$, $\phi \in \mathcal{S}_L(x)$ and $t \geq 0$, we have*

$$V_1(\phi(t,x)) \leq V_1(x) e^{-t}. \tag{137}$$

Proof. The claim follows from

$$\begin{aligned} V_1(\phi(t,x)) \quad &= \quad \sup_{\tau \geq 0, \psi \in \mathcal{S}_L(\phi(t,x))} \widetilde{\alpha}_1 \left(\omega_1(\psi(\tau, \phi(t,x))) \right) e^\tau \\ &\leq \quad \sup_{\tau \geq t, \psi \in \mathcal{S}_L(x)} \widetilde{\alpha}_1 \left(\omega_1(\psi(\tau,x)) \right) e^{\tau - t} \\ &\leq \quad \sup_{\tau \geq 0, \psi \in \mathcal{S}_L(x)} \widetilde{\alpha}_1 \left(\omega_1(\psi(\tau,x)) \right) e^\tau e^{-t} \\ &= \quad V_1(x) e^{-t}. \end{aligned} \tag{138}$$

□

Since F_L is locally Lipschitz on $\mathcal{O}$, from Lemma 11, for each $x \in \mathcal{O}$ and each $w \in F_L(x)$ there exists $\phi \in \mathcal{S}_L(x)$ such that

$$\phi(t,x) = x + t(w + r(t)) \quad \forall t \in [0,T), \tag{139}$$

for some $T > 0$ and some continuous function r satisfying $\lim_{t\to 0^+} r(t) = 0$. Then from (137) we have, for t sufficiently small,

$$\frac{V_1(x + t(w + r(t))) - V_1(x)}{t} \le V_1(x)\frac{e^{-t} - 1}{t} \, . \tag{140}$$

With the definition (94), we have when this makes sense

$$DV_1(x;w) \le \liminf_{t\to 0^+} \frac{V_1(x + t(w + r(t))) - V_1(x)}{t} \le -V_1(x). \tag{141}$$

Since $w \in F_L(x)$ was arbitrary, for each $x \in \mathcal{O}$, we have when this makes sense

$$\sup_{w\in F_L(x)} DV_1(x;w) \le -V_1(x). \tag{142}$$

So with (132), if we can establish that V_1 is continuous on $\mathcal{G}$ and locally Lipschitz on $\mathcal{G}\backslash\mathcal{A}_L \subseteq \mathcal{O}$ then (134) will follow from Lemmas 12 and 13.

3. Local Lipschitz continuity: To establish the local Lipschitz continuity we will use the following result which states that the sup in the solutions used in the definition of V_1 is reached at a particular solution and that the sup in time is reached in a compact set:

Claim 2. *Let $x \in \mathcal{G}$ be such that $V_1(x) > 0$. Define*

$$T(x) := -\ln\left(\frac{V_1(x)}{\widetilde{\alpha}_2(\omega_2(x))}\right) + 1. \tag{143}$$

Then there exists $\hat{\phi}_x \in \mathcal{S}_L(x)$ such that

$$V_1(x) = \max_{t\in[0,T(x)]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right) e^t. \tag{144}$$

Proof. We have

$$\sup_{t\in[0,T(x)],\,\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right) e^t \;\le\; V_1(x) \tag{145}$$

and, with (130),

$$\begin{aligned} V_1(x) &= \max\left\{\sup_{t\in[0,T(x)],\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right) e^t,\; \sup_{t\ge T(x),\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right) e^t\right\} \\ &\le \max\left\{\sup_{t\in[0,T(x)],\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right) e^t,\; \widetilde{\alpha}_2(\omega_2(x))e^{-T(x)}\right\} \\ &\le \max\left\{\sup_{t\in[0,T(x)],\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right) e^t,\; V_1(x)\frac{1}{e}\right\} \end{aligned} \tag{146}$$

from which it follows that

$$V_1(x) = \sup_{t\in[0,T(x)],\phi\in\mathcal{S}_L(x)} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right)e^t = \sup_{\phi\in\mathcal{S}_L(x)}\max_{t\in[0,T(x)]} \widetilde{\alpha}_1\left(\omega_1(\phi(t,x))\right)e^t, \tag{147}$$

where we have used the continuity of $\omega_1(\phi(\cdot,x))$ to pass to the 'max'. Now let $\{\phi_k\}_{k=1}^{\infty}$ be a maximizing sequence of solutions in $\mathcal{S}_L(x)$ in (147), *i.e.*,

$$V_1(x) = \lim_{k\to\infty}\max_{t\in[0,T(x)]} \widetilde{\alpha}_1\left(\omega_1(\phi_k(t,x))\right)e^t. \tag{148}$$

From Lemma 4, a subsequence of $\{\phi_k(\cdot,x)\}_{k=1}^{\infty}$ converges uniformly on $[0,T(x)]$ to some solution $\hat{\phi}_x \in \mathcal{S}_L(x)$. From the continuity of ω_1 and $\widetilde{\alpha}_1$, we have

$$V_1(x) = \max_{t\in[0,T(x)]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right)e^t. \tag{149}$$

$\square$

3.1 Upper semicontinuity of V_1*:*
Now we show that V_1 is upper semicontinuous on $\mathcal{G}$. For the sake of getting a contradiction, suppose the existence of $x\in\mathcal{G}$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of points in $\mathcal{G}$ converging to $x\in\mathcal{G}$ such that

$$\limsup_{k\to\infty} V_1(x_k) > V_1(x) \geq 0. \tag{150}$$

Without loss of generality, we can assume that, for all k and some $\eta > 0$,

$$V_1(x_k) \geq \eta. \tag{151}$$

We define $\tau := \sup_k T(x_k)$. The condition (151), the continuity of $\widetilde{\alpha}_2\circ\omega_2$, and the definition of $T(\cdot)$ in (143) imply $\tau < \infty$. From Claim 2, let $\hat{\phi}_{x_k}\in\mathcal{S}_L(x_k)$ be such that

$$V_1(x_k) = \max_{t\in[0,T(x_k)]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_{x_k}(t,x_k))\right)e^t = \max_{t\in[0,\tau]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_{x_k}(t,x_k))\right)e^t. \tag{152}$$

For each $\varepsilon > 0$, Lemma 6 with the triple $(\tau,\varepsilon,\{x\})$ and the continuity of ω_1 and $\widetilde{\alpha}_1$ give the existence of k_ε so that for all $k\geq k_\varepsilon$, we can find $\psi_k\in\mathcal{S}_L(x)$ so that

$$V_1(x_k) = \max_{t\in[0,\tau]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_{x_k}(t,x_k))\right)e^t \quad\leq\quad \varepsilon + \max_{t\in[0,\tau]}\widetilde{\alpha}_1\left(\omega_1(\psi_k(t,x))\right)e^t \tag{153}$$

$$\leq\quad \varepsilon + V_1(x). \tag{154}$$

This implies

$$\limsup_{k\to\infty} V_1(x_k) \;\leq\; V_1(x). \tag{155}$$

This contradiction establishes the upper semicontinuity of V_1. Moreover, it also establishes continuity of V_1 at each point $x\in\{\xi\in\mathcal{G} : V_1(\xi)=0\}$ since, for each such x, we have

$$0 \leq \limsup_{z\to x} V_1(z) \leq V_1(x) = 0. \tag{156}$$

3.2 Local Lipschitz continuity of V_1:
To conclude the proof of Proposition 5, it is now sufficient to establish that V_1 is locally Lipschitz on

$$\mathcal{G}\backslash\mathcal{A}_L = \{\xi \in \mathcal{G} : V_1(\xi) \neq 0\} \cdot \tag{157}$$

We will do this by applying Lemma 14 to the lower semicontinuous function $-V_1$. Namely, we will show that for each $x \in \{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$ there is a neighborhood $\mathcal{U}$ of x and a constant $M > 0$ such that

$$D(-V_1)(\xi; v) \leq M|v| \qquad \forall \xi \in \mathcal{U},\ v \in \mathbb{R}^n. \tag{158}$$

To do this we will use the following claim:

Claim 3. *Let $V_1(x) > 0$ and let $T(x)$ and $\hat{\phi}_x(t,x)$ come from Claim 2. There exists $\widehat{T}(x) \in [0, T(x)]$ such that*

$$V_1(\hat{\phi}_x(t,x)) \geq V_1(x)e^{-T(x)} \qquad \forall t \in [0, \widehat{T}(x)], \tag{159}$$

$$V_1(x) = \max_{t\in[0,\widehat{T}(x)]} \widetilde{\alpha}_1(\omega_1(\hat{\phi}_x(t,x)))e^t. \tag{160}$$

Proof. First recall from (137) that

$$V_1(\hat{\phi}_x(T(x),x)) \ \leq \ V_1(x)\, e^{-T(x)}. \tag{161}$$

So the set

$$\mathcal{T} \ = \ \left\{ t \in [0, T(x)] \ : \ V_1(\hat{\phi}_x(t,x)) \leq V_1(x)e^{-T(x)} \right\} \tag{162}$$

is nonempty and we can define

$$\widehat{T}(x) \ := \ \inf\{t \in \mathcal{T}\} \cdot \tag{163}$$

Either $\widehat{T}(x) = 0$ and (159) holds from

$$V_1(\hat{\phi}_x(0,x)) \ = \ V_1(x) \ > \ V_1(x)\, e^{-T(x)}, \tag{164}$$

or $\widehat{T}(x) > 0$ and (159) holds for all $t \in [0, \widehat{T}(x))$. Also, with the upper semi-continuity of V_1 at $\hat{\phi}_x(\widehat{T}(x), x)$, we have

$$\begin{aligned} V_1(\hat{\phi}_x(\widehat{T}(x),x)) \ &\geq \limsup_{z\to\hat{\phi}_x(\widehat{T}(x),x)} V_1(z), && (165)\\ &\geq \limsup_{t\to\widehat{T}(x),\, t<\widehat{T}(x)} V_1(\hat{\phi}_x(t,x)) \geq V_1(x)e^{-T(x)}. && (166)\end{aligned}$$

So (159) holds also for $t = \widehat{T}(x)$.

Now we prove (160). If $\widehat{T}(x) = T(x)$ then there is nothing new to prove. If not, let $\{t_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of times in $[0, T(x)) \cap \mathcal{T}$ converging to $\widehat{T}(x)$. We have

$$\begin{aligned} V_1(x) \ &= \max\left\{ \max_{t\in[0,t_n]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right) e^t,\ \max_{t\in[t_n,T(x)]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right) e^t \right\}, \\ &\leq \max\left\{ \max_{t\in[0,t_n]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right) e^t,\ \sup_{t\geq 0,\, \psi\in\mathcal{S}_L(\hat{\phi}_x(t_n,x))} \widetilde{\alpha}_1\left(\omega_1(\psi(t, \hat{\phi}_x(t_n,x)))\right) e^t e^{t_n} \right\}, \qquad (167)\\ &\leq \max\left\{ \max_{t\in[0,t_n]} \widetilde{\alpha}_1\left(\omega_1(\hat{\phi}_x(t,x))\right) e^t,\ V_1(\hat{\phi}_x(t_n,x))e^{t_n} \right\} \end{aligned}$$

and also, since $t_n \in \mathcal{T}$ and $t_n < T(x)$,

$$V_1(\hat{\phi}_x(t_n, x))e^{t_n} \leq V_1(x)e^{-T(x)+t_n} < V_1(x). \tag{168}$$

Hence (167) and the continuity of $\widetilde{\alpha}_1$, ω_1 and $\hat{\phi}_x(\cdot, x)$ and the fact that t_n converges to $\widehat{T}(x)$ establishes (160). $\square$

To end our proof, we pick x arbitrarily in $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$. Let $\widehat{T}(x)$ and $\hat{\phi}_x(t, x)$ come from the previous claims. Remember that F_L is locally Lipschitz on $\mathcal{O}$ and that we have

$$\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\} = \mathcal{G} \backslash \mathcal{A}_L \subseteq \mathcal{O}. \tag{169}$$

From (159) we have that $V_1(\hat{\phi}_x(t, x)) \neq 0$ for all $t \in [0, \widehat{T}(x)]$ and so $\hat{\phi}_x(t, x) \in \mathcal{O}$ for all $t \in [0, \widehat{T}(x)]$. Then, by letting

$$T = \widehat{T}(x), \quad \mathcal{C} = \left\{ z \,:\, \exists t \in [0, \widehat{T}(x)] \,:\, z = \hat{\phi}_x(t, x) \right\} \tag{170}$$

and by invoking the local Lipschitz property of $\widetilde{\alpha}_1 \circ \omega_1(\cdot)$, we can apply Lemma 10 to get the existence of $\delta_x > 0$ and L_x such that for any v with $|v| \leq \delta_x$, there exists a solution $\psi \in \mathcal{S}_L(x + v)$ such that

$$\max_{t \in [0, \widehat{T}(x)]} \left| \widetilde{\alpha}_1(\omega_1(\hat{\phi}_x(t, x))) - \widetilde{\alpha}_1(\omega_1(\psi(t, x + v))) \right| \leq L_x |v|. \tag{171}$$

This yields

$$\begin{aligned} V_1(x) &= \max_{t \in [0, \widehat{T}(x)]} \widetilde{\alpha}_1(\omega_1(\hat{\phi}_x(t, x)))e^t \\ &\leq \sup_{t \geq 0} \widetilde{\alpha}_1(\omega_1(\psi(t, x + v)))e^t + \max_{t \in [0, \widehat{T}(x)]} |\widetilde{\alpha}_1(\omega_1(\hat{\phi}_x(t, x))) - \widetilde{\alpha}_1(\omega_1(\psi(t, x + v)))|e^t \\ &\leq V_1(x + v) + e^{T(x)} L_x |v|. \end{aligned} \tag{172}$$

Since v is arbitrary (but small enough in norm), this establishes that V_1 is lower semicontinuous on $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$. This, combined with the upper semicontinuity of V_1 on $\mathcal{G}$ and the continuity of V_1 on $\{\xi \in \mathcal{G} : V_1(\xi) = 0\}$ that we have already established, shows that V_1 is continuous on $\mathcal{G}$.

We are left with proving (158). Since we now know that $V_1(\cdot)$ and $T(\cdot)$ are continuous functions on $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$, there exists a compact subset $\mathcal{C}_0$ of $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$ which contains a neighborhood of x and is such that for all $z \in \mathcal{C}_0$, we have

$$T(z) \leq 2\,T(x), \qquad V_1(z)e^{-T(z)} \geq \frac{1}{2} V_1(x)e^{-T(x)}. \tag{173}$$

This and (159) imply that for all $z \in \mathcal{C}_0$, we have for all $t \in [0, \widehat{T}(z)]$,

$$V_1(\hat{\phi}_z(t, z)) \geq V_1(z)e^{-T(z)} \geq \frac{1}{2} V_1(x)e^{-T(x)}. \tag{174}$$

On the other hand, we know, from Lemma 4, that $\mathcal{R}_{\leq 2T(x)}(\mathcal{C}_0)$ is a compact subset of $\mathcal{G}$. Also, since V_1 is continuous on $\mathcal{G}$, the following is a compact subset of $\mathcal{G}$:

$$\mathcal{C} \;:=\; \mathcal{R}_{\leq 2T(x)}(\mathcal{C}_0) \bigcap \left\{ z \in \mathcal{G} \,:\, V_1(z) \geq \frac{1}{2} V_1(x)e^{-T(x)} \right\}. \tag{175}$$

From (173) and (174), we have that for all $z \in \mathcal{C}_0$, and all $t \in [0, \widehat{T}(z)]$,

$$\hat{\phi}_z(t, z) \in \mathcal{C}. \tag{176}$$

Since $\widetilde{\alpha}_1 \circ \omega_1(\cdot)$ is locally Lipschitz, we invoke Lemma 10 with $T = 2T(x)$ and $\mathcal{C}$ defined in (175), to get the existence of $\delta > 0$ and L such that for all $\bar{v} \in \mathbb{R}^n$ with $|\bar{v}| \leq \delta$, and all $z \in \mathcal{C}_0$, we have (following the same lines as for establishing (172)),

$$V_1(z) \ \leq \ V_1(z + \bar{v}) + e^{2T(x)} L |\bar{v}|. \tag{177}$$

It follows that for all $v \in \mathbb{R}^n$ and all $z \in \mathcal{C}_0$, we have

$$D(-V_1)(z; v) = \liminf_{w \to v, \varepsilon \to 0^+} \frac{V_1(z) - V_1(z + \varepsilon w)}{\varepsilon} \ \leq \ \liminf_{w \to v} e^{2T(x)} L |w| = e^{2T(x)} L |v|. \tag{178}$$

This establishes (158) and concludes the proof of Proposition 5. □

Step 3: Smoothing $V_1(x)$. The final step in the proof of Theorem 1 is to turn V_1, which is continuous on $\mathcal{G}$ and locally Lipschitz on the open set $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$, into a function that is smooth on $\mathcal{G}$, without losing the desirable properties of V_1. The first step in this task is to apply Lemma 16 with the open set $\{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$ and with $\alpha(x) = -V_1(x)$, $\mu(x) = \frac{1}{2} V_1(x)$ and $\nu(x) = \frac{1}{4} V_1(x)$. Let $V_2 : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be given by Lemma 16 for $x \in \{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$ and let $V_2(x) = 0$ for $x \in \{\xi \in \mathcal{G} : V_1(\xi) = 0\}$. With the help of (99) and (100), we can see that this function V_2 is continuous on $\mathcal{G}$, smooth on $\{\xi \in \mathcal{G} : V_2(\xi) \neq 0\}$, and satisfies, for all $x \in \mathcal{G}$,

$$\frac{1}{2} \widetilde{\alpha}_1(\omega_1(x)) \leq \frac{1}{2} V_1(x) \leq V_2(x) \leq \frac{3}{2} V_1(x) \leq \frac{3}{2} \widetilde{\alpha}_2(\omega_2(x)) \tag{179}$$

and, for all $x \in \{\xi \in \mathcal{G} : V_1(\xi) \neq 0\}$,

$$\max_{w \in F_L(x)} \langle \nabla V_2(x), w \rangle \leq -\frac{3}{4} V_1(x) \leq -\frac{1}{2} V_2(x). \tag{180}$$

Finally, to get a function that is smooth on $\mathcal{G}$ we apply Lemma 17 to the function V_2 which satisfies all of the assumptions of Lemma 17. Using the resulting function ρ, we take $V := (\rho \circ V_2)^2$ which is smooth by construction. From (179) it follows that (26) holds with

$$\alpha_1(s) := \rho\left(\frac{1}{2} \widetilde{\alpha}_1(s)\right)^2, \qquad \alpha_2(s) := \rho\left(\frac{3}{2} \widetilde{\alpha}_2(s)\right)^2. \tag{181}$$

Also, from (180) and the relation $\rho(s) \leq s\rho^{'}(s)$, it follows that

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \leq -2\rho(V_2(x))\rho^{'}(V_2(x)) \frac{1}{2} V_2(x) \leq -\rho(V_2(x))^2 = -V(x), \tag{182}$$

i.e. (27) holds. Thus V is a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$.

5.1.3. *Bibliographical Notes*

- The idea of imbedding the original differential equation or differential inclusion into a larger, locally Lipschitz differential inclusion that still exhibits $\mathcal{KL}$-stability with respect to (ω_1, ω_2) comes from Kurzweil [15], who did this for the case of ordinary differential equations with continuous right-hand side and the notion of strong stability of the origin, and from Clarke *et al.* [6], who did this for the case of nonempty, compact, convex, upper semicontinuous differential inclusions and global asymptotic stability of the origin.

- The particular construction of the trial Lyapunov function V_1 as the supremum, over time and solutions $\phi(\cdot, x)$ of the perturbed (locally Lipschitz) differential inclusion, of the quantity $\widetilde{\alpha}_1(\omega_1(\phi(t,x)))e^t$ where $\widetilde{\alpha}_1$ has a particular form is a combination of a classical construction, for locally Lipschitz ordinary differential equations with an exponentially stable equilibrium (see, *e.g.* [41], Sect. 19), and the construction in [18]. A suitable choice for $\widetilde{\alpha}_1$ is made possible by the recent result of Sontag ([30], Prop. 7).
- Claim 2 is a combination of [18] (Fact 2) and [6] (Lem. 4.5).
- The proof scheme for establishing that the trial Lyapunov function V_1 is locally Lipschitz with, in particular, the help of Lemma 14 is due to Clarke *et al.* [6].
- The smoothing technique used in the final step of the proof is borrowed from Kurzweil [15], Wilson [40], Lin *et al.* [18] and Clarke *et al.* [6].

5.2. **Proof of Theorem 2**

The assumption of Theorem 2 says that the differential inclusion $\dot{x} \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ and that F is locally Lipschitz on $\mathcal{G} \setminus \mathcal{A}$. So, from Sections 5.1.2 and 5.1.2 in the proof of the implication 2 $\Rightarrow$ 1 in the proof of Theorem 1, we know the existence of a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω_1, ω_2) on $\mathcal{G}$ for F. Then, since the differential inclusion $\dot{x} \in F(x)$ is assumed to be forward complete on $\mathcal{G}$, the implication 1 $\Rightarrow$ 2 of Theorem 1 allows us to conclude. $\square$

5.3. **Proof of Theorem 3**

The basic idea of this proof is the same as the one in the proof of Theorem 2. Namely, we will establish robust stability by first establishing the existence of a smooth converse Lyapunov function and then, since we are assuming forward completeness on $\mathcal{G}$, appealing to the implication 1 $\Rightarrow$ 2 of Theorem 1, *i.e.* "forward completeness plus smooth converse function implies robust stability".

Assume, temporarily, that we have established robust $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for

$$\dot{x} \in \frac{1}{\kappa(x)} F(x) =: F_N(x) \tag{183}$$

where κ is given by the backward completability by ω-normalization assumption and, thus, satisfies $\kappa(x) \geq 1$. Then, from the implication 2 $\Rightarrow$ 1 of Theorem 1, *i.e.* "robust stability implies smooth converse Lyapunov function", there exists a smooth converse Lyapunov function V for $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for F_N. In particular we have

$$\max_{w \in F_N(x)} \langle \nabla V(x), w \rangle \leq -V(x). \tag{184}$$

It follows directly that

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle = \kappa(x) \cdot \max_{w \in F_N(x)} \langle \nabla V(x), w \rangle \leq -\kappa(x) \cdot V(x) \leq -V(x) \tag{185}$$

which establishes that V is also a smooth converse Lyapunov function for $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for F. From the implication 1 $\Rightarrow$ 2 of Theorem 1, we get the robust $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for $\dot{x} \in F(x)$. So, to establish Theorem 3 it is sufficient to prove robust $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for $\dot{x} \in F_N(x)$. Hence, our task in the following is to exhibit a continuous function $\delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that

1. $\{x\} + \delta(x)\overline{\mathcal{B}} \subset \mathcal{G}$;
2. the differential inclusion

$$\dot{x} \in F_{N\delta(x)}(x) := \overline{\text{co}} F_N(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \tag{186}$$

   is $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$;

3. $\delta(x) > 0$ for all x in an open set containing $\mathcal{G}\backslash\mathcal{A}_\delta$ where

$$\mathcal{A}_\delta := \left\{\xi \in \mathcal{G} : \sup_{t\geq 0, \phi\in\mathcal{S}_{N\delta}(\xi)} \omega(\phi(t,\xi)) = 0\right\} \tag{187}$$

where $\mathcal{S}_{N\delta}(\cdot)$ represents the set of maximal solutions to (186).

With regard to the third point, we observe that if the second point holds then

$$\mathcal{A}_\delta = \Omega \ := \ \{\xi \in \mathcal{G} : \ \omega(\xi) = 0\} \cdot \tag{188}$$

Indeed, clearly $\mathcal{A}_\delta \subset \Omega$. On the other hand, $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for the differential inclusion (186) implies that, for all $x \in \mathcal{G}$, all $\phi_{N\delta} \in \mathcal{S}_{N\delta}(x)$, and all $t \geq 0$, we have

$$\omega(\phi_{N\delta}(t,x)) \ \leq \ \beta(\omega(x), t). \tag{189}$$

So, for all $x \in \Omega$, we obtain

$$\omega(\phi_{N\delta}(t,x)) \ = \ 0 \qquad \forall t \geq 0, \tag{190}$$

i.e., $x \in \mathcal{A}_\delta$.

To accomplish our task, we will first establish $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for $\dot{x} \in F_N(x)$. Then we will construct a function δ and finally establish that it has the needed properties.

5.3.1. *$\mathcal{KL}$-stability for F_N*

Lemma 19. *Let $\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ be continuous and let F satisfy the basic conditions on $\mathcal{G}$. If the differential inclusion $\dot{x} \in F(x)$ is backward completable by ω-normalization and $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$ then, with κ the function given by the backward completability assumption, the differential inclusion*

$$\dot{x} \in \frac{1}{\kappa(x)} F(x) =: F_N(x) \tag{191}$$

is backward complete on $\mathcal{G}$ and $\mathcal{KL}$-stable with respect to (ω, ω) on $\mathcal{G}$ and F_N satisfies the basic conditions on $\mathcal{G}$.

Proof. Since $1/\kappa(\cdot)$ is continuous, F_N satisfies the basic conditions on $\mathcal{G}$. Backward completeness follows by assumption. To establish forward completeness, we pick x arbitrarily in $\mathcal{G}$. Let $\phi_N(\cdot, x)$ be an arbitrary maximal solution of $\dot{x} \in F_N(x)$, right maximally defined on $[0, t_N)$. In this way, we have defined a continuous function $\kappa(\phi_N(\cdot, x))$ on $[0, t_N)$. Let $\tau(\cdot)$ be a solution belonging to $[0, t_N)$ of

$$\dot{\tau} = \kappa(\phi_N(\tau, x)) \qquad \tau(0) = 0, \tag{192}$$

right maximally defined on $[0, t_\tau)$. Since $\kappa(x) \geq 1$ for all $x \in \mathcal{G}$, $\tau(\cdot)$ is strictly increasing and we have, with the right maximality of $[0, t_\tau)$,

$$t \ \leq \ \tau(t), \tag{193}$$

$$t_\tau \ \leq \ \lim_{t\to t_\tau} \tau(t) \ = \ t_N. \tag{194}$$

Since $\phi_N(\cdot, x)$ is absolutely continuous and $\tau(\cdot)$ is C^1 and strictly increasing, the function $\phi_N(\tau(\cdot), x)$ is absolutely continuous (see [28], Th. 3, p. 245) and we have, for almost all $t \in [0, t_\tau)$,

$$\begin{aligned} \dot{\overbrace{\phi_N(\tau(t), x)}} &\in \frac{1}{\kappa(\phi_N(\tau(t), x))} F(\phi_N(\tau(t), x))\, \dot{\tau}(t) \\ &\in F(\phi_N(\tau(t), x)). \end{aligned} \tag{195}$$

Therefore, with the forward completeness of $\dot{x} \in F(x)$, $\phi_N(\tau(\cdot), x)$ is a solution of $\dot{x} \in F(x)$ on $[0, t_\tau)$.

Then, assume for the time being that $t_N < +\infty$. From (194) this implies $t_\tau < +\infty$. From the forward completeness of $\dot{x} \in F(x)$, there exists $\phi(\cdot, x)$, a solution of $\dot{x} \in F(x)$, defined and continuous on $[0, +\infty)$, such that

$$\phi(t, x) \;=\; \phi_N(\tau(t), x) \qquad \forall t \in [0, t_\tau). \tag{196}$$

Also the reachable set $\mathcal{R}_{\leq t_\tau}(x)$ for $\dot{x} \in F(x)$ is a compact subset of $\mathcal{G}$. With (196) and continuity of ϕ, this implies that, by letting $\phi_N(t_N, x) := \phi(t_\tau, x)$, with (194), ϕ_N, as a solution of $\dot{x} \in F_N(x)$, can be extended to $[0, t_N]$ and therefore, with Lemma 2, beyond t_N. This contradicts the definition of t_N and implies

$$t_N \;=\; +\infty. \tag{197}$$

So the differential inclusion $\dot{x} \in F_N(x)$ is forward complete on $\mathcal{G}$.

To complete our proof of $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for $\dot{x} \in F_N(x)$, we observe that, since this property holds for $\dot{x} \in F(x)$, we have

$$\omega(\phi_N(\tau(t), x)) \;\leq\; \beta(\omega(x), t) \qquad \forall t \in [0, t_\tau). \tag{198}$$

On the other hand, from the backward completability assumption, we know the existence of a class-$\mathcal{K}$ function γ and a constant c such that

$$\kappa(\phi_N(\tau(t), x)) \leq \gamma(\omega(\phi_N(\tau(t), x))) + c \qquad \forall t \in [0, t_\tau). \tag{199}$$

Together with (198), this yields

$$\kappa(\phi_N(\tau(t), x)) \;\leq\; \gamma(\beta(\omega(x), t)) \;+\; c \;\leq\; \gamma(\beta(\omega(x), 0)) \;+\; c \qquad \forall t \in [0, t_\tau). \tag{200}$$

It follows by integration that

$$\tau(t) \leq [\gamma(\beta(\omega(x), 0)) + c]\, t \tag{201}$$

for all $t \in [0, t_\tau)$. With (194) and (197), this implies

$$t_\tau \;=\; +\infty. \tag{202}$$

It follows that τ is a class-$\mathcal{K}_\infty$ function and its inverse τ^{-1} satisfies

$$\tau^{-1}(s) \geq \frac{s}{\gamma(\beta(\omega(x), 0)) + c}\,. \tag{203}$$

We then have, from (198), for all $s \in [0, +\infty)$,

$$\omega(\phi_N(s, x)) \;\leq\; \beta(\omega(x), \tau^{-1}(s)) \tag{204}$$

$$\leq\; \beta_N(\omega(x), s), \tag{205}$$

where the function

$$\beta_N(r,s) \;:=\; \beta\left(r, \frac{s}{\gamma(\beta(r,0))+c}\right) \tag{206}$$

is of class-$\mathcal{KL}$. This establishes $\mathcal{KL}$-stability with respect to (ω,ω) on $\mathcal{G}$ for $\dot{x} \in F_N(x)$. □

5.3.2. *Construction of* $\delta(\cdot)$

Applying Lemma 7 successively to the differential inclusions $\dot{x} \in F_N(x)$ and $\dot{x} \in -F_N(x)$, we get a strictly positive continuous function Δ_0 such that the differential inclusion

$$\dot{x} \in F_0(x) := \overline{\mathrm{co}} F_N(x+\Delta_0(x)\overline{\mathcal{B}}) + \Delta_0(x)\overline{\mathcal{B}} \tag{207}$$

is backward and forward complete on $\mathcal{G}$ and F_0 satisfies the basic conditions on $\mathcal{G}$.

Let β_N be the class-$\mathcal{KL}$ given by the $\mathcal{KL}$-stability of $\dot{x} \in F_N(x)$. According to Remark 3 we can, without loss of generality, assume that β_N is continuous. We define, recursively, a set of strictly positive real numbers ω_i, ε_i and T_i, for $i = 0, \pm 1, \pm 2, \ldots$ as follows:

- First we define ω_i to satisfy

$$\omega_0 \;=\; 1, \qquad \omega_{i+1} = 2\,\beta_N(\omega_i, 0). \tag{208}$$

  Since the definition of $\mathcal{KL}$-stability with respect to (ω,ω) implies

$$\beta_N(s,0) \geq s, \tag{209}$$

  the ω_i's are well-defined, strictly increasing and satisfy

$$\begin{aligned} \omega_i \quad &\geq \quad 2^i \qquad \forall i \geq 0, &(210)\\ &\leq \quad 2^i \qquad \forall i \leq 0. &(211) \end{aligned}$$

  So the intervals $[\omega_i, \omega_{i+1})$ cover $\mathbb{R}_{>0}$.
- Next we define ε_i to satisfy

$$\beta_N(\omega_{i-1}+\varepsilon_i, 0) + \varepsilon_i < \omega_i \tag{212}$$

  and

$$\varepsilon_i < \frac{\omega_{i-2}}{2}\,. \tag{213}$$

- Finally, we choose T_i to satisfy

$$\beta_N(\omega_{i+1}+\varepsilon_{i+2}, T_i) + \varepsilon_{i+2} < \omega_i. \tag{214}$$

  The fact that β_N is a class-$\mathcal{KL}$ function implies that T_i is well-defined.

With these numbers, we define now the sets

$$\Omega_i := \{x \in \mathcal{G} : \omega_i \leq \omega(x) < \omega_{i+1}\}\,. \tag{215}$$

With the set Ω, defined in (188), for each $\xi \in \mathcal{G}\backslash\Omega$, there is a unique i, denoted $i(\xi)$ such that $\xi \in \Omega_{i(\xi)}$. Now we define, for each $\xi \in \mathcal{G}\backslash\Omega$,

$$\varepsilon(\xi) \;:=\; \min_{j\in\{-2,-1,\ldots,2\}} \varepsilon_{i(\xi)+j+2}, \qquad T(\xi) \;:=\; \max_{j\in\{-2,-1,\ldots,2\}} T_{i(\xi)+j}. \tag{216}$$

Since for any point ξ and integer k related by

$$\xi \in \bigcup_{j=k-2}^{k+2} \Omega_j \tag{217}$$

there exists $j \in \{-2, -1, \dots, 2\}$ such that

$$k = i(\xi) + j, \tag{218}$$

it follows from (216) that

$$\mathcal{X} \subset \bigcup_{j=k-2}^{k+2} \Omega_j \qquad \Longrightarrow \qquad \left\{ \begin{array}{lcl} \inf\limits_{\xi\in\mathcal{X}} T(\xi) & \geq & T_k \\ \sup\limits_{\xi\in\mathcal{X}} \varepsilon(\xi) & \leq & \varepsilon_{k+2}. \end{array} \right. \tag{219}$$

This fact will be used later.

We denote by $\mathcal{C}(\xi)$ the reachable set in time $T(\xi)$ from $\xi \in \mathcal{G} \setminus \Omega$ for the differential inclusion $\dot{x} \in -F_0(x)$. From Lemma 4 and the forward completeness of $\dot{x} \in -F_0(x)$, $\mathcal{C}(\xi)$ is a compact[2] subset of $\mathcal{G}$. Since a solution of $\dot{x} \in F_\circ(x)$ in backward time is also a solution of $\dot{x} \in -F_\circ(x)$ in forward time, $\mathcal{C}(\xi)$ is also the set of points from which ξ can be reached in time $T(\xi)$ for the differential inclusion (207).

With the above, to each $\xi \in \mathcal{G}\backslash\Omega$, we have associated a triple $(T(\xi), \varepsilon(\xi), \mathcal{C}(\xi))$. Then, from Lemma 6, there exists a strictly positive real number $\overline{\Delta}_1(\xi)$ satisfying $\mathrm{P}(\overline{\Delta}_1(\xi), \xi)$ where $P(\cdot,\cdot)$ is the property:

$\mathrm{P}(\Delta, \xi)$: every maximal solution ϕ_1 of

$$\dot{x} \in F_1(x) =: \overline{\mathrm{co}} F_N(x + \Delta\overline{\mathcal{B}}) + \Delta\overline{\mathcal{B}}, \tag{220}$$

with initial condition $x_1 \in \mathcal{C}(\xi)$, remains in $\mathcal{G}$ for all $t \in [0, T(\xi)]$ and there exists a solution ϕ_N of $\dot{x} \in F_N(x)$, with initial condition $x_N \in \mathcal{G}$ such that, for all $t \in [0, T(\xi)]$, we have

$$|\omega(\phi_1(t, x_1)) - \omega(\phi_N(t, x_N))| \leq \varepsilon(\xi). \tag{221}$$

Then we choose

$$\Delta_1(\xi) = \min\left\{1, \tfrac{1}{2}\sup\{\Delta \ : \ \mathrm{P}(\Delta, \xi) \text{ holds}\}\right\} \cdot \tag{222}$$

To complete the definition of the function Δ_1 on $\mathcal{G}$, we let $\Delta_1(\xi) = 0$ when $\xi \in \Omega$. We claim that the function Δ_1 defined this way is bounded away from 0 on compact subsets of $\mathcal{G}\backslash\Omega$. Indeed, consider a compact set $K \subset \mathcal{G}\backslash\Omega$. Since ω is continuous, the definitions of Ω, $\varepsilon(\xi)$ and $T(\xi)$ imply the existence of $\eta > 0$ and $\tau > 0$ such that $\varepsilon(\xi) \geq \eta$ and $T(\xi) \leq \tau$ for all $\xi \in K$. Also, from backward completeness, the set of points from which the set K can be reached in time τ is a compact set $\mathcal{C}_K$ and we have $\mathcal{C}(\xi) \subset \mathcal{C}_K$ for all $\xi \in K$. Applying Lemma 6 with the triple $(\tau, \eta, \mathcal{C}_K)$ gives the existence of $\Delta_K > 0$ such that $\mathrm{P}(\Delta_K, \xi)$ holds for all $\xi \in K$. From the definition of $\Delta_1(\xi)$, we have $\Delta_1(\xi) \geq \min\left\{1, \frac{\Delta_K}{2}\right\} > 0$ for all $\xi \in K$.

With all these preliminaries, we can now give our definition of δ. We let

$$\delta(x) \ = \ \inf_{\xi\in\mathcal{G}} \Delta(\xi) + |\xi - x| \tag{223}$$

[2]This is the only point where backward completeness is used and, hence, normalization is needed. Actually, in the following, we do not need the set $\mathcal{C}(\xi)$ to be compact but only a subset of $\mathcal{C}(\xi)$ given by the closure of

$$\mathcal{C}(\xi) \bigcap \left\{x \in \mathcal{G} \,:\, \omega_{i(\xi)-2} \leq \omega(x) \leq \omega_{i(\xi)+2}\right\} \cdot$$

So, for example, if ω is proper in $\mathcal{G}$, the proof of Theorem 3 can be carried out without first normalizing F. This is the approach taken in [6].

where

$$\Delta(x) = \min\{\Delta_0(x), \Delta_1(x)\} \cdot \tag{224}$$

This function δ is continuous on $\mathcal{G}$, such that

$$\delta(x) \ \leq \ \Delta(x), \tag{225}$$

and bounded away from 0 on compact subsets of $\mathcal{G}\backslash\Omega$. The latter follows from the fact that $\delta(x) = 0$ implies $\Delta(x) = 0$. Let us collect the main properties obtained with this function δ.

Proposition 6. *We have the following properties:*

1. *The differential inclusion*

$$\dot{x} \in F_{N\delta}(x) := \overline{\mathrm{co}}F_N(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \tag{226}$$

is forward and backward complete on $\mathcal{G}$ and $F_{N\delta}$ satisfies the basic conditions on $\mathcal{G}$.

2. *Any solution of (226) is a solution of (207).*
3. *If $\mathcal{X}$ is a compact subset of $\mathcal{G}\backslash\Omega$ such that $\bigcap_{\xi\in\mathcal{X}}\mathcal{C}(\xi)$ is nonempty and we let*

$$\begin{aligned} T_s &:= \inf_{\xi\in\mathcal{X}} T(\xi) \ > \ 0 && (227)\\ \varepsilon_s &:= \sup_{\xi\in\mathcal{X}} \varepsilon(\xi) \ < \ \infty && (228)\\ \delta_s &:= \sup_{\xi\in\mathcal{X}} \delta(\xi) \ < \ \infty && (229)\end{aligned}$$

then if $x_s \in \bigcap_{\xi\in\mathcal{X}}\mathcal{C}(\xi)$ and ϕ_s is any maximal solution of

$$\dot{x} \in F_s(x) =: \overline{\mathrm{co}}F_N(x + \delta_s\overline{\mathcal{B}}) + \delta_s\overline{\mathcal{B}}, \tag{230}$$

starting from x_s, then ϕ_s is defined on $[0, T_s]$ and there exists a solution ϕ_N of $\dot{x} \in F_N(x)$, with initial condition $x_N \in \mathcal{G}$ such that, for all $t \in [0, T_s]$,

$$|\omega(\phi_s(t, x_s)) - \omega(\phi_N(t, x_N))| \leq \varepsilon_s. \tag{231}$$

Proof.

- Point 1 is a consequence of $\delta \leq \Delta_0$ and Lemma 1.
- Point 2 is a consequence of $\delta \leq \Delta_0$.
- Point 3 is established as follows. First we observe from the definitions of $T(\xi)$ and $\varepsilon(\xi)$ that, since $\mathcal{X}$ is a compact subset of $\mathcal{G}\backslash\Omega$, we do have $T_s > 0$ and $\varepsilon_s < \infty$. Also since δ is continuous, there exists $\xi^* \in \mathcal{X}$ such that:

$$\delta_s \ = \ \delta(\xi^*). \tag{232}$$

Note that we have trivially

$$\bigcap_{\xi\in\mathcal{X}} \mathcal{C}(\xi) \subset \mathcal{C}(\xi^*). \tag{233}$$

Also, since we have

$$\delta(\xi^*) \ \leq \ \Delta_1(\xi^*), \tag{234}$$

a solution ϕ_s of (230) is a solution of (220) with $\Delta = \Delta_1(\xi^*)$. These facts together with the property $\mathrm{P}(\Delta_1(\xi^*), \xi^*)$ imply that if $x_s \in \bigcap_{\xi\in\mathcal{X}} \mathcal{C}(\xi)$ then every maximal solution $\phi_s(\cdot, x_s)$ of (230) is defined on $[0, T(\xi^*)]$ and there exists a solution ϕ_N of $\dot{x} \in F_N(x)$, with initial condition $x_N \in \mathcal{G}$, such that, for all $t \in [0, T(\xi^*)]$, we have

$$|\omega(\phi_s(t, x_s)) - \omega(\phi_N(t, x_N))| \le \varepsilon(\xi^*). \tag{235}$$

Since, from the definitions of ε_s and T_s, we have

$$\varepsilon(\xi^*) \ \le \ \varepsilon_s, \qquad T(\xi^*) \ \ge \ T_s, \tag{236}$$

the result follows.

□

5.3.3. *Robust $\mathcal{KL}$-stability for F_N*

Lemma 20. *Given an integer k, for each $x_{N\delta} \in \Omega_k$ and each maximal solution $\phi_{N\delta}$ of (226) starting from $x_{N\delta}$, there exists $t_{N\delta} \in [0, T_k]$ such that:*

$$\omega(\phi_{N\delta}(t, x_{N\delta})) \le \omega_{k+2} \qquad \forall t \in [0, t_{N\delta}] \tag{237}$$

and

$$\omega(\phi_{N\delta}(t_{N\delta}, x_{N\delta})) < \omega_k. \tag{238}$$

Proof. Define

$$\tau := \sup\left\{t \in [0, T_k] \ : \ \omega(\phi_{N\delta}(s, x_{N\delta})) \in [\omega_{k-2}, \omega_{k+2}]\,,\ \forall s \in [0, t]\right\}\cdot \tag{239}$$

Since the right-hand side of (226) satisfies the basic conditions on $\mathcal{G}$ and is forward complete on $\mathcal{G}$, τ is well-defined. From the continuity of $\phi_{N\delta}$, if $\tau < T_k$, then either $\omega(\phi_{N\delta}(\tau, x_{N\delta})) = \omega_{k-2}$ or $\omega(\phi_{N\delta}(\tau, x_{N\delta})) = \omega_{k+2}$. Also the set

$$\mathcal{X} \ = \ \{\xi \ : \ \exists t \in [0, \tau] : \ \xi = \phi_{N\delta}(t, x_{N\delta})\} \tag{240}$$

is a compact subset of $\displaystyle\bigcup_{j=k-2}^{k+2} \Omega_i$ and we have, using (219),

$$T_s \ := \ \inf_{\xi\in\mathcal{X}} T(\xi) \ \ge \ T_k \tag{241}$$

$$\varepsilon_s \ := \ \sup_{\xi\in\mathcal{X}} \varepsilon(\xi) \ \le \ \varepsilon_{k+2} \tag{242}$$

$$\delta_s \ := \ \sup_{\xi\in\mathcal{X}} \delta(\xi) \ \ge \ \delta(\phi_{N\delta}(t, x_{N\delta})) \quad \forall\, t \in [0, \tau]. \tag{243}$$

Now, from point 2 in Proposition 6, $\phi_{N\delta}$ is a solution of (207). Since, for all $t \in [0, \tau]$, we have

$$T(\phi_{N\delta}(t, x_{N\delta})) \ge T_k \ge \tau, \tag{244}$$

it follows that, for all $t \in [0, \tau]$, $x_{N\delta}$ belongs to the set of points from which $\phi_{N\delta}(t, x_{N\delta})$ can be reached in time $T(\phi_{N\delta}(t, x_{N\delta}))$ for the differential inclusion (207), *i.e.*, for each $t \in [0, \tau]$, we have $x_{N\delta} \in \mathcal{C}(\phi_{N\delta}(t, x_{N\delta}))$. Consequently, with (243), $\phi_{N\delta}$, restricted to the interval $[0, \tau]$, is a solution of (230). It follows from point 3 in

Proposition 6 together with (241) and (244) that there exists a solution ϕ_N of $\dot{x} \in F_N(x)$, with initial condition $x_N \in \mathcal{G}$ satisfying

$$\omega(x_N) \le \omega(x_{N\delta}) + \varepsilon_s \le \omega_{k+1} + \varepsilon_s, \tag{245}$$

such that, with the $\mathcal{KL}$-stability of $\dot{x} \in F_N(x)$, for all $t \in [0, \tau]$,

$$\begin{aligned} \omega(\phi_{N\delta}(t, x_{N\delta})) &\le \omega(\phi_N(t, x_N)) + \varepsilon_s && (246)\\ &\le \beta_N(\omega(x_N), t) \; + \; \varepsilon_s. && (247) \end{aligned}$$

With (247, 245, 242), and (212), it follows that, for all $t \in [0, \tau]$,

$$\begin{aligned} \omega(\phi_{N\delta}(t, x_{N\delta})) &\le \beta_N(\omega(x_N), t) \; + \; \varepsilon_{k+2}, && (248)\\ &\le \beta_N(\omega_{k+1} + \varepsilon_{k+2}, t) \; + \; \varepsilon_{k+2}, && (249)\\ &\le \beta_N(\omega_{k+1} + \varepsilon_{k+2}, 0) \; + \; \varepsilon_{k+2}, && (250)\\ &< \omega_{k+2}. && (251) \end{aligned}$$

This implies that:

– either $\tau < T_k$ and then

$$\omega(\phi_{N\delta}(\tau, x_{N\delta})) \; = \; \omega_{k-2} \; < \; \omega_k. \tag{252}$$

So the lemma holds with $t_{N\delta} = \tau$;
– or $\tau = T_k$ and (with (214) and (249))

$$\begin{aligned} \omega(\phi_{N\delta}(T_k, x_{N\delta})) &\le \beta_N(\omega_{k+1} + \varepsilon_{k+2}, T_k) \; + \; \varepsilon_{k+2}, && (253)\\ &< \omega_k. && (254) \end{aligned}$$

So the lemma holds with $t_{N\delta} = T_k$.

□

With this lemma available, we can show that the conditions given in Proposition 1 that imply $\mathcal{KL}$-stability with respect to (ω, ω) on $\mathcal{G}$ for $\dot{x} \in F_{N\delta}(x)$, *i.e.*, the inclusion (226), are satisfied. We let $\mathcal{S}_{N\delta}(\cdot)$ denote the set of maximal solutions of the inclusion (226).

Condition 2a of the proposition, *i.e.*, forward completeness on $\mathcal{G}$, has already been established. To establish conditions 2b and 2c, we observe the following:

- For any $x \in \Omega = \{\xi \in \mathcal{G} : \omega(\xi) = 0\}$ and any solution $\phi_{N\delta}$ in $\mathcal{S}_{N\delta}(x)$, we have

$$\omega(\phi_{N\delta}(t, x)) \; = \; 0 \qquad \forall t \ge 0. \tag{255}$$

Indeed, assume this is not the case, *i.e.*, there exist $x \in \Omega$, a solution $\phi_{N\delta}$ and a time $\tau_0 > 0$ such that

$$\omega(\phi_{N\delta}(\tau_0, x)) \; > \; 0. \tag{256}$$

Then there exists i such that $\phi_{N\delta}(\tau_0, x) \in \Omega_i$. Let

$$\tau_1 \; = \; \sup\left\{t \le \tau_0 \; : \; \omega(\phi_{N\delta}(t, x)) \le \omega_{i-3}\right\}. \tag{257}$$

Since $\phi_{N\delta}(\cdot, x)$ is continuous, τ_1 is well-defined and we have $\phi_{N\delta}(\tau_1, x) \in \Omega_{i-3}$. So from Lemma 20, there exists $t_{N\delta}$ such that

$$\omega(\phi_{N\delta}(t, x)) \le \omega_{i-1} \qquad \forall t \in [\tau_1, \tau_1 + t_{N\delta}] \tag{258}$$

and

$$\omega(\phi_{N\delta}(\tau_1 + t_{N\delta}, x)) < \omega_{i-3}. \tag{259}$$

Hence either $\tau_1 + t_{N\delta} \ge \tau_0$ and (258) contradicts the fact that $\phi_{N\delta}(\tau_0, x) \in \Omega_i$, or $\tau_1 + t_{N\delta} < \tau_0$ and, with the continuity of $\phi_{N\delta}(\cdot, x)$ (259) contradicts the definition of τ_1. So (255) is established.

- For any $x \in \mathcal{G}\backslash\Omega$, x is in $\Omega_{i(x)}$ and, from Lemma 20, for any solution $\phi_{N\delta} \in \mathcal{S}_{N\delta}(x)$, there exists $t_\circ \le T_{i(x)}$ such that

$$\omega(\phi_{N\delta}(t, x)) \ \le \ \omega_{i(x)+2} \qquad \forall t \in [0, t_\circ] \tag{260}$$

and

$$\omega(\phi_{N\delta}(t_\circ, x)) \ < \ \omega_{i(x)}. \tag{261}$$

So, again from Lemma 20, there exists $t_1 \le T_{i(x)-1}$ such that

$$\omega(\phi_{N\delta}(t_\circ + t, x)) \ \le \ \omega_{i(x)+1} \qquad \forall t \in [0, t_1] \tag{262}$$

and

$$\omega(\phi_{N\delta}(t_\circ + t_1, x)) \ < \ \omega_{i(x)-1}. \tag{263}$$

And so on ...

So, regarding condition 2b of Proposition 1, let $\gamma \in \mathcal{K}_\infty$ upper bound the piecewise constant function $p(s)$ defined by $p(s) = \omega_{k+2}$ for all $s \in [\omega_k, \omega_{k+1})$. Since $\omega_k \to 0$ as $k \to -\infty$, such a function γ can be found. Then for any $x \in \mathcal{G}$ and any solution $\phi_{N\delta} \in \mathcal{S}_{N\delta}(x)$, we have, if x is in Ω

$$\omega(\phi_{N\delta}(t, x)) = 0 = \gamma(\omega(x)) \quad \forall t \ge 0 \tag{264}$$

and, if not, we have $\omega(x) \in [\omega_{i(x)}, \omega_{i(x)+1})$ and

$$\omega(\phi(t, x)) \le \omega_{i(x)+2} = p(\omega(x)) \le \gamma(\omega(x)) \quad \forall t \ge 0. \tag{265}$$

This establishes condition 2b.

Finally, regarding condition 2c, for any $r > 0$ and $\varepsilon > 0$, we define $T(r, \varepsilon)$ as follows: let the integer i be such that $\max\{r, \varepsilon\} \in [\omega_i, \omega_{i+1})$, let the integer j be such that $\min\{r, \varepsilon\} \in [\omega_j, \omega_{j+1})$ (note that $i \ge j$), and define

$$T(r, \varepsilon) := \sum_{m=j-3}^{i} T_m. \tag{266}$$

From the previous observation, we see that, for any $x \in \mathcal{G}$ such that $\omega(x) \le r$ and any solution $\phi_{N\delta} \in \mathcal{S}_{N\delta}(x)$, we have

$$\omega(\phi(t, x)) \le \varepsilon \qquad \forall t \ge T(r, \varepsilon). \tag{267}$$

This establishes condition 2c. □

5.3.4. *Bibliographical Notes*

- The proof of Lemma 20 incorporates calculations modeled after those in the proof of [6] (Lem. 7.2).

6. Proofs of Propositions 2–4

6.1. **Proof of Proposition 2**

The proof has the following steps:

1. Using the second assumption of the proposition, we will find a smooth function ρ that is positive and proper on $\mathcal{G}_1$, a class-$\mathcal{K}_\infty$ function $\gamma_\circ$ and a constant $c_\circ \geq 0$ such that

$$\rho(x_1) \leq \gamma_\circ(\omega(x)) + c_\circ. \tag{268}$$

2. We will find class-$\mathcal{K}_\infty$ functions γ_1 and γ_2 and a constant $c_1 \geq 0$ such that

$$\max_{w \in F(x)} \langle \nabla\rho(x_1), w_1 \rangle \leq \max\left\{\gamma_1(\rho(x_1)), \gamma_2(|x_2|), c_1\right\} \cdot \tag{269}$$

3. Using the first and third assumption of the proposition, we will show that normalizing with

$$\kappa(x) := \max\left\{1, \gamma_1(\rho(x_1))\right\} \leq 1 + \gamma_1(2c_\circ) + \gamma_1(2\gamma_\circ(\omega(x))), \tag{270}$$

which gives

$$\max_{w \in F_N(x)} \langle \nabla\rho(x_1), w_1 \rangle \leq \max\left\{1, c_1, \gamma_2(|x_2|)\right\}, \tag{271}$$

guarantees backward completeness.

6.1.1. *Construction of ρ and its upper bound*

Let x_1^* be a point in $\mathcal{G}_1$. If $\mathcal{G}_1 = \mathbb{R}^{n_1}$ then, for all $x_1 \in \mathcal{G}_1$, we define

$$\rho_0(x_1) := 1 + |x_1 - x_1^*|. \tag{272}$$

Otherwise, for all $x_1 \in \mathcal{G}_1$, we define:

$$\rho_0(x_1) := 1 + \max\left\{|x_1 - x_1^*|\,,\ \frac{1}{|x_1|_{\mathbb{R}^{n_1}\setminus\mathcal{G}_1}} - \frac{2}{|x_1^*|_{\mathbb{R}^{n_1}\setminus\mathcal{G}_1}}\right\}. \tag{273}$$

In each case the function ρ_0 is continuous, positive and proper on $\mathcal{G}_1$. Let ρ be given by Lemma 15 with $\mathcal{O} = \mathcal{G}_1$, $V(x_1) = \rho_\circ(x_1)$ and $\mu(x_1) = \frac{1}{2}\rho_0(x_1)$. This function is smooth, positive and proper on $\mathcal{G}_1$ since we have

$$\rho(x_1) \geq \frac{1}{2}\rho_0(x_1) \qquad \forall x_1 \in \mathcal{G}_1. \tag{274}$$

Moreover, note that $\rho(x_1^*) \leq \dfrac{3}{2}\rho_0(x_1^*) = \dfrac{3}{2}$.

Let us show that, for each $r \geq 0$, there exists $R(r) \geq 0$ such that, $\forall (x_1, x_2) \in \mathcal{G}_1 \times \mathbb{R}^{n_2}$:

$$\omega(x_1, x_2) \leq r \quad \Longrightarrow \quad \rho(x_1) \leq R. \tag{275}$$

If this were not the case, there would be some $r \geq 0$ and a sequence $\{(x_{1n}, x_{2n})\}_{n=1}^{\infty}$ of points in $\mathcal{G}_1 \times \mathbb{R}^{n_2}$ such that

$$\inf_{x_2 \in \mathbb{R}^{n_2}} \omega(x_{1n}, x_2) \leq \omega(x_{1n}, x_{2n}) \leq r \tag{276}$$

and

$$\rho(x_{1n}) > n. \tag{277}$$

The latter implies that x_{1n} goes either to the boundary of $\mathcal{G}_1$ or to infinity. With (43), this contradicts (276). So (275) is established.

Without loss of generality, we can assume that the function $R(\cdot)$ is nondecreasing. Then we pick $\gamma_\circ \in \mathcal{K}_\infty$ and $c_\circ \geq 0$ such that, for all $s \geq 0$,

$$R(s) \leq \gamma_\circ(s) + c_\circ. \tag{278}$$

Then, from (275) and (278), it follows that

$$\rho(x_1) \leq R(\omega(x_1, x_2)) \leq \gamma_\circ(\omega(x_1, x_2)) + c_\circ. \tag{279}$$

6.1.2. *Bounding the derivative of ρ along solutions*

Let

$$\bar{\gamma}_1(s) := \sup_{\rho(x_1) \leq s+2} |\nabla \rho(x_1)|\,. \tag{280}$$

Since ρ is smooth and proper, and $\rho(x_1^*) \leq \frac{3}{2}$, $\bar{\gamma}_1$ is well-defined on $\mathbb{R}_{\geq 0}$. It is nondecreasing and satisfies

$$|\nabla \rho(x_1)| \leq \bar{\gamma}_1(\rho(x_1)). \tag{281}$$

Next, from [12] (Sect. 5, Lem. 15), the functions

$$\bar{\gamma}_2(s) := \sup_{\{(x_1,x_2):\, |x_2| \leq \rho(x_1) \leq s+2\}} \ \sup_{w \in F(x_1,x_2)} |w_1| \tag{282}$$

$$\bar{\gamma}_3(s) := \sup_{\{(x_1,x_2):\, \rho(x_1) \leq |x_2| \leq s+2\}} \ \sup_{w \in F(x_1,x_2)} |w_1| \tag{283}$$

are well-defined, nondecreasing functions on $\mathbb{R}_{\geq 0}$ and we have, for all $(x_1, x_2) \in \mathcal{G}_1 \times \mathbb{R}^{n_2}$,

$$w \in F(x) \qquad \Longrightarrow \qquad |w_1| \leq \max\{\bar{\gamma}_2(\rho(x_1)), \bar{\gamma}_3(|x_2|)\}\,\cdot \tag{284}$$

With (281) and (284), we have (using $ab \leq \max\{|a|^2, |b|^2\}$)

$$\max_{w \in F(x)} \langle \nabla \rho(x_1), w_1 \rangle \leq \bar{\gamma}_1(\rho(x_1)) \max\{\bar{\gamma}_2(\rho(x_1)), \bar{\gamma}_3(|x_2|)\} \tag{285}$$

$$\leq \max\left\{\bar{\gamma}_1(\rho(x_1))\bar{\gamma}_2(\rho(x_1)), \bar{\gamma}_1(\rho(x_1))^2, \bar{\gamma}_3(|x_2|)^2\right\}\cdot \tag{286}$$

Then, picking a constant $c_1 \geq 0$ and class-$\mathcal{K}_\infty$ functions γ_1 and γ_2 such that, for all $s \geq 0$,

$$\max\left\{\bar{\gamma}_1(s)\bar{\gamma}_2(s), \bar{\gamma}_1^2(s)\right\} \leq \max\{\gamma_1(s), c_1\} \tag{287}$$

and

$$\bar{\gamma}_3^2(s) \le \max\{\gamma_2(s), c_1\}, \tag{288}$$

we have

$$\max_{w\in F(x)} \langle \nabla\rho(x_1), w_1\rangle \le \max\{\gamma_1(\rho(x_1)), \gamma_2(|x_2|), c_1\}\cdot \tag{289}$$

6.1.3. *Construction of κ and backward completeness*

We define

$$\kappa(x) := \max\{1, \gamma_1(\rho(x_1))\} \tag{290}$$

which, from (279), satisfies (using $\gamma_1(a+b) \le \gamma_1(2a) + \gamma_1(2b)$)

$$\kappa(x) \le 1 + \gamma_1(2c_\circ) + \gamma_1(2\gamma_\circ(\omega(x))). \tag{291}$$

We need to show that the differential inclusion $\dot{x} \in -F_N(x)$ is forward complete on $\mathcal{G}$ where

$$F_N(x) := \frac{1}{\kappa(x)} F(x). \tag{292}$$

Suppose this is not the case. Since ρ is proper on $\mathcal{G}_1$ and F_N satisfies the basic conditions on $\mathcal{G}_1 \times \mathbb{R}^{n_2}$, it follows from Lemma 2 that there exists a maximal solution $\phi(\cdot, x)$ of $\dot{x} \in -F_N(x)$ and a time $\bar{t} < \infty$ such that

$$\lim_{t\to\bar{t}} \max\{\rho(\phi_1(t,x)), |\phi_2(t,x)|\} \;=\; \infty. \tag{293}$$

Since $\kappa(x) \ge 1$ and (44) holds, it follows from the Gronwall Lemma that $\phi_2(\cdot, x)$ is bounded on $[0, \bar{t})$. Thus we must have that $\rho(\phi_1(\cdot, x))$ is unbounded on $[0, \bar{t})$. But we have that, for almost all $t \in [0, \bar{t})$

$$\overset{\cdot}{\overbrace{\rho(\phi_1(t,x))}} \;\le \max_{w\in -F_N(\phi(t,x))} \langle \nabla\rho(\phi_1(t,x)), w_1\rangle \tag{294}$$

and, from (289, 290) and (292) we obtain, for all $x \in \mathcal{G}$,

$$\max_{w\in -F_N(x)} \langle \nabla\rho(x_1), w_1\rangle \le \max\{1, c_1, \gamma_2(|x_2|)\}\cdot \tag{295}$$

This implies that $\overset{\cdot}{\overbrace{\rho(\phi_1(\cdot,x))}}$ is bounded on $[0, \bar{t})$. Thus, so is $\rho(\phi_1(\cdot, x))$. This contradiction establishes that the differential inclusion $\dot{x} \in -F_N(x)$ is forward complete on $\mathcal{G}$. □

6.2. Proof of Proposition 3

To prove Proposition 3, we will use the following claim:

Claim 4. *Under the assumptions of Proposition 3, for each $x \in \mathcal{G}$ and each $\varepsilon > 0$ there exists $T_x(\varepsilon) > 0$ such that*

$$\phi \in \mathcal{S}(x),\ t \ge T_x(\varepsilon) \qquad \Longrightarrow \qquad |\phi(t,x)|_{\mathcal{A}} \le \varepsilon. \tag{296}$$

Proof. Since $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{O}$ for all $t \geq 0$. Hence Lemma 5 applies. To establish the claim by contradiction, suppose the existence of ε and $x \in \mathcal{G}$ such that, for each integer $n \geq 0$ there exists $t_n \geq n$ and $\phi_n \in \mathcal{S}(x)$ satisfying $|\phi_n(t_n, x)|_{\mathcal{A}} > \varepsilon$, *i.e.* (51) does not hold. So, from the stability assumption, there exists $\delta > 0$ such that we have, for each n,

$$|\phi_n(t, x)|_{\mathcal{A}} > \delta \qquad \forall t \in [0, n]. \tag{297}$$

Now, using Lemma 5 $\{\phi_n\}_{n=1}^{\infty}$ has a subsequence, still denoted $\{\phi_n\}_{n=1}^{\infty}$, that converges uniformly on compact time intervals to some solution $\phi \in \mathcal{S}(x)$. Then, from the definition of $\mathcal{G}$, since $x \in \mathcal{G}$ there exists $\tau > 0$ such that

$$|\phi(t, x)|_{\mathcal{A}} \leq \delta/2 \qquad \forall t \geq \tau. \tag{298}$$

But there exists also an integer $n \geq \tau$ such that

$$|\phi_n(\tau, x) - \phi(\tau, x)| \leq \delta/2. \tag{299}$$

This contradicts (297, 298). □

We are now ready to prove Proposition 3.

6.2.1. *The set $\mathcal{G}$ is open*

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n \backslash \mathcal{G}$ converging to a point $x^* \in \mathbb{R}^n$. We will show that $x^* \notin \mathcal{G}$ so that $\mathbb{R}^n \backslash \mathcal{G}$ is closed, *i.e.*, $\mathcal{G}$ is open. To prove this claim by contradiction, suppose $x^* \in \mathcal{G}$. Since $\mathcal{G}$ contains a neighborhood of $\mathcal{A}$, there exists a strictly positive real number ρ_1 such that $\mathcal{A} + \rho_1 \overline{\mathcal{B}}$ is a compact subset of $\mathcal{G}$. Then, from Claim 4, we get $T := T_{x^*}(\rho_1/2)$ such that

$$\phi \in \mathcal{S}(x^*),\ t \geq T \qquad \Longrightarrow \qquad |\phi(t, x^*)|_{\mathcal{A}} \leq \frac{\rho_1}{2} \cdot \tag{300}$$

According to Lemma 4, the set $\mathcal{R}_{\leq T}(x^*)$ is a compact subset of $\mathcal{O}$. So the set $\mathcal{C}_{x^*}$ defined as

$$\mathcal{C}_{x^*} := \mathcal{R}_{\leq T}(x^*) \bigcup \left(\mathcal{A} + \rho_1 \overline{\mathcal{B}}\right) \tag{301}$$

is also a compact subset of $\mathcal{O}$. Also, we have established that, for any solution $\phi \in \mathcal{S}(x^*)$, $\phi(t, x^*)$ is in $\mathcal{R}_{\leq T}(x^*)$, for $t \leq T$ and in $\left(\mathcal{A} + \rho_1 \overline{\mathcal{B}}\right)$ for $t \geq T$, *i.e.*,

$$\phi \in \mathcal{S}(x^*),\ t \geq 0 \qquad \Longrightarrow \qquad \phi(t, x^*) \in \mathcal{C}_{x^*}. \tag{302}$$

Now, since $\mathcal{O}$ is open, there exist a strictly positive real number ν and a continuous function $\ell : \mathcal{O} \to [0, 1]$ such that $\mathcal{C}_{x^*} + 2\nu\overline{\mathcal{B}}$ is a compact subset of $\mathcal{O}$ and $\ell \equiv 1$ on $\mathcal{C}_{x^*} + \nu\overline{\mathcal{B}}$ and $\ell \equiv 0$ on $\mathcal{O} \setminus \left(\mathcal{C}_{x^*} + 2\nu\overline{\mathcal{B}}\right)$. It follows that the set-valued map $x \mapsto F_\ell(x) := \ell(x)F(x)$ satisfies the basic conditions on $\mathcal{O}$, the differential inclusion $\dot{x} \in F_\ell(x)$ is forward complete on $\mathcal{O}$ and $F_\ell(x) = F(x)$ for all $x \in \mathcal{C}_{x^*} + \nu\overline{\mathcal{B}}$. Moreover, we claim that (300) and (302) hold for the inclusion $\dot{x} \in F_\ell(x)$, *i.e.*,

$$\phi_\ell \in \mathcal{S}_\ell(x^*),\ t \geq T \qquad \Longrightarrow \qquad |\phi_\ell(t, x^*)|_{\mathcal{A}} \leq \frac{\rho_1}{2} \tag{303}$$

and

$$\phi_\ell \in \mathcal{S}_\ell(x^*),\ t \geq 0 \qquad \Longrightarrow \qquad \phi_\ell(t, x^*) \in \mathcal{C}_{x^*}. \tag{304}$$

The relations (303) and (304) follow from the fact that

$$\phi_\ell \in \mathcal{S}_\ell(x^*) \qquad \Longrightarrow \qquad \phi_\ell \in \mathcal{S}(x^*) \tag{305}$$

together with (300) and (302). The relation (305) holds because if not, from the definition of ℓ, there would exists a time $\bar{t}$ such that

$$\phi_\ell(\bar{t}, x^*) \notin \mathcal{C}_{x^*} \tag{306}$$

and, from the continuity of $\phi_\ell(\cdot, x^*)$,

$$\phi_\ell(t, x^*) \in \mathcal{C}_{x^*} + \nu\overline{\mathcal{B}} \qquad \forall t \in [0, \bar{t}]. \tag{307}$$

It follows from (307) and the definition of ℓ that the restriction of $\phi_\ell(\cdot, x^*)$ to the interval $[0, \bar{t}]$ is a solution of $\dot{x} \in F(x)$. Thus (306) contradicts (302).

For each i sufficiently large x_i is in $\mathcal{C}_{x^*} + \frac{\nu}{2}\overline{\mathcal{B}}$, but also $x_i \notin \mathcal{G}$. So there exist a time t_i and a solution $\phi_i \in \mathcal{S}(x_i)$ such that

1. $\phi_i(t, x_i) \in C_{x^*} + \nu\overline{\mathcal{B}}$ for all $t \in [0, t_i]$ and
2. either
   (a) $t_i \le T$ and $\phi_i(t_i, x_i) \in \partial\left(\mathcal{C}_{x^*} + \nu\overline{\mathcal{B}}\right)$ or
   (b) $t_i = T$ and $|\phi_i(t, x_i)|_{\mathcal{A}} > \rho_1$ for all $t \in [0, t_i]$.

It follows from point 1 and the definition of ℓ that the restriction of $\phi_i(\cdot, x_i)$ to the interval $[0, t_i] \subset [0, T]$ is a solution of $\dot{x} \in F_\ell(x)$, *e.g.*, point 2 holds for a solution $\phi_{i,\ell} \in \mathcal{S}_\ell(x_i)$. On the other hand, from (303, 304) and Lemma 4 applied to $\dot{x} \in F_\ell(x)$ with the triple $(T, \frac{\min\{\rho_1, \nu\}}{2}, \{x^*\})$, we have also, for each i sufficiently large, $\phi_{i,\ell}(t, x_i) \in \mathcal{C}_{x^*} + \frac{\nu}{2}\overline{\mathcal{B}}$ for all $t \in [0, T]$ and $|\phi_{i,\ell}(T, x_i)|_{\mathcal{A}} \le \rho_1$. This contradiction of point 2 above (with $\phi_{i,\ell}$ in place of ϕ_i) establishes that $x^* \notin \mathcal{G}$, *i.e.* $\mathcal{G}$ is open.

6.2.2. *$\mathcal{KL}$-stability with respect to (ω, ω)*

We will show that the conditions of Proposition 1 hold. For this, we observe that, since ω is a proper indicator for $\mathcal{A}$ on $\mathcal{G}$ we have that:

- the set

$$\mathcal{C}(r) := \{x \in \mathcal{G} : \omega(x) \le r\} \tag{308}$$

  is a compact subset of $\mathcal{G}$ for each $r \ge 0$;
- Let $\rho > 0$ be such that $\mathcal{A} + \rho\overline{\mathcal{B}} \subset \mathcal{G}$. Define the functions

$$\begin{aligned} \bar{\alpha}_1(s) &:= \inf_{x \in \mathcal{G}:\, s \le |x|_{\mathcal{A}}} \omega(x) \\ \bar{\alpha}_2(s) &:= \sup_{x \in \mathcal{G}:\, |x|_{\mathcal{A}} \le \min\{\rho, s\}} \omega(x). \end{aligned} \tag{309}$$

  There exists class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1(s) \le \bar{\alpha}_1(s), \qquad \bar{\alpha}_2(s) \le \alpha_2(s). \tag{310}$$

  So, for all $x \in \mathcal{G}$, we have

$$\alpha_1(|x|_{\mathcal{A}}) \le \omega(x) \tag{311}$$

$$|x|_{\mathcal{A}} \le \rho \qquad \Longrightarrow \qquad \omega(x) \le \alpha_2(|x|_{\mathcal{A}}). \tag{312}$$

Forward completeness:
That the differential inclusion $\dot{x} \in F(x)$ is forward complete on $\mathcal{G}$ follows from the definition of $\mathcal{G}$ and the fact that this set is forward invariant. Indeed, for all $x \in \mathcal{G}$, all $\phi \in \mathcal{S}(x)$ and all $s \geq 0$, we have by concatenation of solutions that all $\psi \in \mathcal{S}(\phi(s,x))$ are defined and belong to $\mathcal{O}$ for all $t \geq 0$ and satisfy $\lim_{t\to\infty} |\psi(t,\phi(s,x))|_{\mathcal{A}} = 0$, *i.e.*, $\phi(s,x) \in \mathcal{G}$.

Uniform global attractivity:
To establish this property by contradiction, suppose the existence of $r > 0$ and $\varepsilon > 0$ such that, for each integer $i \geq 0$, we can find a point $x_i \in \mathcal{C}(r)$, a solution $\phi_i \in \mathcal{S}(x_i)$ and a time $t_i \geq i$ such that $\omega(\phi_i(t_i,x_i)) > \varepsilon$. So, from (312), we have either $|\phi_i(t_i,x_i)|_{\mathcal{A}} > \rho$ or $|\phi_i(t_i,x_i)|_{\mathcal{A}} > \alpha_2^{-1}(\varepsilon)$. From the stability assumption and (311), we get $\delta > 0$ such that, for each i,

$$\omega(\phi_i(t,x_i)) > \alpha_1(\delta) \qquad \forall t \in [0,i]. \tag{313}$$

Without loss of generality we can assume that δ is such that $\alpha_2^{-1}\left(\frac{\alpha_1(\delta)}{2}\right) \leq \rho$. Since the set $\mathcal{C}(r)$ is compact, there exists a subsequence of $\{x_i\}_{i=1}^{\infty}$ converging to a point $x^* \in \mathcal{G}$. Let $T := T_{x^*}\left(\alpha_2^{-1}(\frac{\alpha_1(\delta)}{2})\right)$ come from Claim 4. With (312), we have

$$\phi \in \mathcal{S}(x^*),\ t \geq T \qquad \Longrightarrow \qquad \omega(\phi(t,x^*)) \leq \frac{\alpha_1(\delta)}{2}\,. \tag{314}$$

With Lemma 4 with the triple $(T, \frac{\alpha_1(\delta)}{2}, \{x^*\})$, this implies that, for i sufficiently large, $\omega(\phi_i(T,x_i))) \leq \alpha_1(\delta)$. This contradicts (313), and thus establishes that for each $r > 0$ and $\varepsilon > 0$ there exists $T(r,\varepsilon) > 0$ such that

$$x \in \mathcal{C}(r),\ \phi \in \mathcal{S}(x),\ t \geq T \qquad \Longrightarrow \qquad \omega(\phi(t,x)) \leq \varepsilon. \tag{315}$$

Uniform stability and global boundedness:
Let $\rho_1 > 0$ be as above such that $\mathcal{A} + \rho_1\overline{\mathcal{B}} \subset \mathcal{G}$. We have just seen in (315) that, for each $r > 0$, there exists T such that

$$x \in \mathcal{C}(r),\ \phi \in \mathcal{S}(x),\ t \geq T \qquad \Longrightarrow \qquad |\phi(t,x)|_{\mathcal{A}} \leq \rho_1. \tag{316}$$

So, as in the proof of the fact that $\mathcal{G}$ is open, we have

$$x \in \mathcal{C}(r),\ \phi \in \mathcal{S}(x),\ t \geq 0 \qquad \Longrightarrow \qquad \phi(t,x) \in \left(\mathcal{R}_{\leq T}(C(r)) \bigcup \left(\mathcal{A} + \rho_1\overline{\mathcal{B}}\right)\right) \subset \mathcal{G}. \tag{317}$$

This can be rewritten

$$\mathcal{R}_{<\infty}(\mathcal{C}(r)) \subset \mathcal{R}_{\leq T}(C(r)) \bigcup \left(\mathcal{A} + \rho_1\overline{\mathcal{B}}\right) \subset \mathcal{G}. \tag{318}$$

But, since $\mathcal{R}_{\leq T}(C(r))$ is compact, this establishes that the set $\overline{\mathcal{R}_{<\infty}(\mathcal{C}(r))}$ is a compact subset of $\mathcal{G}$.

Define the function

$$R(r) := \max_{\xi \in \overline{\mathcal{R}_{<\infty}(\mathcal{C}(r))}} \omega(\xi). \tag{319}$$

We have

$$\omega(\phi(t,x)) \leq R(\omega(x)) \tag{320}$$

for all $x \in \mathcal{G}$ and all $\phi \in \mathcal{S}(x)$. Also this function is nondecreasing and $\lim_{r\to 0} R(r) = 0$. The latter follows from (312) and the stability assumption. Indeed for each $\varepsilon > 0$, there exists $r > 0$ such that

$$x \in \mathcal{C}(r),\ t \geq 0 \qquad \Longrightarrow \qquad \omega(\phi(t,x)) \leq \varepsilon \tag{321}$$

and therefore $R(r) \leq \varepsilon$. The function γ of Point 2b of Proposition 1 is obtained by choosing $\gamma \in \mathcal{K}_\infty$ to satisfy $R(s) \leq \gamma(s)$ for all $s \geq 0$. □

6.2.3. *Bibliographical Notes*

Our proof of Proposition 3 combines the ideas of [15] (pp. 71-72) and the proof of [6] (Prop. 2.2).

6.3. Proof of Proposition 4

6.3.1. $\mathcal{A}$ *is nonempty*

To establish by contradiction that the set $\mathcal{A}$ is nonempty, assume the contrary, *i.e.*, for each $x \in \mathcal{C}_1$, there exists $\phi \in \mathcal{S}(x)$ and $t > 0$ such that

$$\phi(t,x) \notin \mathcal{C}_1. \tag{322}$$

Note that, from Assumption 1, we must have $t < T$. Pick x_0 arbitrarily in $\mathcal{C}_1$ and ϕ_0 arbitrarily in $\mathcal{S}(x_0)$. From Assumption 1, we have

$$x_1 := \phi_0(T, x_0) \in \mathcal{C}_1 \tag{323}$$

and from (322), there exists $\phi_1 \in \mathcal{S}(x_1)$ and $0 < t_1 < T$ such that

$$\phi_1(t_1, x_1) \notin \mathcal{C}_1. \tag{324}$$

But the function $\phi(\cdot, x_0)$ defined as

$$\begin{aligned} \phi(t,x_0) &:= \phi_0(t,x_0) && \forall t \in [0,T), \\ &:= \phi_1(t-T,x_1) && \forall t \in [T,+\infty) \end{aligned} \tag{325}$$

is a solution which does not satisfy Assumption 1. So $\mathcal{A}$ must be nonempty.

Note also that, from its definition, $\mathcal{A}$ has also the following two properties:

1. $x \in \mathcal{A}$, $\phi \in \mathcal{S}(x)$ imply $\phi(t,x) \in \mathcal{A}$ for all $t \geq 0$, *i.e.*, $\mathcal{A}$ is forward invariant;
2. for all $x \in \mathcal{C}_2$ and all $\phi \in \mathcal{S}(x)$, we have

$$|\phi(t,x)|_{\mathcal{A}} = 0 \quad \forall t \geq T. \tag{326}$$

6.3.2. $\mathcal{A}$ *is compact*

$\mathcal{A}$ is bounded since it is a subset of $\mathcal{C}_1$ which is compact. Suppose it is not closed. Then there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of points in $\mathcal{A}$ converging to a point $x \notin \mathcal{A}$. Since $\mathcal{C}_1$ is compact, we must have $x \in \mathcal{C}_1$. Since $x \notin \mathcal{A}$ there exists $\phi \in \mathcal{S}(x)$ and $T_x \in (0,T)$ such that

$$\phi(T_x, x) \notin \mathcal{C}_1 \tag{327}$$

and, from the continuity of $\phi(\cdot, x)$, we can impose

$$\phi(t,x) \in \mathcal{C}_1 + \rho\overline{\mathcal{B}} \qquad \forall t \in [0, T_x], \tag{328}$$

where ρ is given by Assumption 1. Since F is assumed to be Lipschitz on $\mathcal{C}_1 + \rho\overline{\mathcal{B}}$, with (328) and Lemma 10, we know the existence of a sequence of solutions $\{\phi_i(t, x_i)\}_{i=1}^{\infty}$ converging to $\phi(t, x)$ on $[0, T_x]$. But this, together with (327), contradicts $x_i \in \mathcal{A}$ for large i. This implies that x must be in $\mathcal{A}$.

6.3.3. *$\mathcal{A}$ is stable*

We want to show that point 1 in Proposition 3 holds. Let $\rho > 0$ be given by Assumption 1 and let $\ell : \mathcal{O} \to [0, 1]$ be a continuous function such that $\ell(x) = 1$ on $\mathcal{C}_1 + \frac{\rho}{2}\overline{\mathcal{B}}$ and $\ell(x) = 0$ on $\mathcal{O}\backslash\left(\mathcal{C}_1 + \rho\overline{\mathcal{B}}\right)$. The set-valued map $x \mapsto F_\ell(x) := \ell(x)F(x)$ satisfies the basic conditions on $\mathcal{O}$ and the differential inclusion $\dot{x} \in F_\ell(x)$ is forward complete on $\mathcal{O}$. Moreover, for the inclusion $\dot{x} \in F_\ell(x)$, the set $\mathcal{A}$ is forward invariant. If not there would exist $x \in \mathcal{A}$, a solution $\phi_\ell \in \mathcal{S}_\ell(x)$ and a time $\bar{t} > 0$ such that

$$\phi_\ell(\bar{t}, x) \notin \mathcal{A} \tag{329}$$

and

$$\phi_\ell(t, x) \in \mathcal{C}_1 + \frac{\rho}{2}\overline{\mathcal{B}} \qquad \forall t \in [0, \bar{t}]. \tag{330}$$

It follows from (330) and the definition of $\ell(x)$ that the restriction of $\phi_\ell(\cdot, x)$ to the interval $[0, \bar{t}]$ is a solution of $\dot{x} \in F(x)$. But then (329) contradicts the fact that the set $\mathcal{A}$ is forward invariant for the inclusion $\dot{x} \in F(x)$.

Let $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < \frac{\rho}{2}$. With Lemma 6 applied to the differential inclusion $\dot{x} \in F_\ell(x)$ with the triple $(T, \varepsilon, \mathcal{A})$ we get the existence of $\delta > 0$ such that, with the forward invariance of $\mathcal{A}$,

$$|x|_\mathcal{A} \le \delta,\ \phi_\ell \in \mathcal{S}_\ell(x) \qquad \Longrightarrow \qquad |\phi_\ell(t, x)|_\mathcal{A} \le \varepsilon \qquad \forall t \in [0, T]. \tag{331}$$

We claim, for the differential inclusion $\dot{x} \in F(x)$, that

$$|x|_\mathcal{A} \le \delta,\ \phi \in \mathcal{S}(x) \qquad \Longrightarrow \qquad |\phi(t, x)|_\mathcal{A} \le \varepsilon \qquad \forall t \ge 0. \tag{332}$$

If this is not the case then, with (326) and using that $\varepsilon < \frac{\rho}{2}$, there exist $x \in \mathcal{G}$ with $|x|_\mathcal{A} \le \delta$, $\phi \in \mathcal{S}(x)$ and $\tau \in [0, T]$ such that

$$|\phi(\tau, x)|_\mathcal{A} > \epsilon \tag{333}$$

and

$$|\phi(t, x)|_\mathcal{A} \le \frac{\rho}{2} \qquad \forall t \in [0, \tau]. \tag{334}$$

It follows from (334), $\mathcal{A} \subset \mathcal{C}_1$ and the definition of $\ell(x)$ that the restriction of $\phi(\cdot, x)$ to the interval $[0, \tau] \subset [0, T]$ is a solution of the inclusion $\dot{x} \in F_\ell(x)$. Thus (333) contradicts (331). □

A. Addendum

A.1. **Proof of Lemma 1**

Let

$$\mathcal{F}(x) \ = \bigcup_{\xi \in \{x\} + \rho(x)\overline{\mathcal{B}}} F(\xi). \tag{335}$$

- Let us show by contradiction that $\mathcal{F}$ is upper semicontinuous on $\mathcal{G}$, *i.e.*, for all $x \in \mathcal{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for all x' satisfying $|x-x'| \leq \delta$ and all y' satisfying $|y'| \leq \rho(x')$ we have $F(x'+y') \subset \mathcal{F}(x)+\varepsilon\mathcal{B}$. So assume the existence of $x \in \mathcal{G}$, $\varepsilon > 0$, a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x, a sequence $\{y_n\}_{n=1}^{\infty}$ satisfying $|y_n| \leq \rho(x_n)$, a sequence $\{v_n\}_{n=1}^{\infty}$ of vectors in $F(x_n + y_n)$ such that $v_n \notin \mathcal{F}(x) + \varepsilon\mathcal{B}$. Since ρ is continuous and x_n converges to x, y_n has a cluster point y^* satisfying $|y^*| \leq \rho(x)$. From (18), $x + y^*$ is in $\mathcal{G}$ and, from the upper semicontinuity of F, there exists $\delta > 0$ such that, for all $\xi \in \mathcal{G}$ satisfying $|(x + y^*) - \xi| \leq \delta$, we have $F(\xi) \subset F(x + y^*) + \varepsilon\mathcal{B} \subset \mathcal{F}(x) + \varepsilon\mathcal{B}$. But from the convergence properties, we can find n such that $|(x + y^*) - (x_n + y_n)| \leq \delta$. This implies $F(x_n + y_n) \subset F(x + y^*) + \varepsilon\mathcal{B} \subset \mathcal{F}(x) + \varepsilon\mathcal{B}$ and contradicts the property of v_n. So the upper semicontinuity property of $\mathcal{F}$ is established and actually the same holds for the closure $\overline{\mathcal{F}}$.
- Secondly, we observe that [12] (Sect. 5, Lem. 15) implies that the set $\mathcal{F}(x)$ is bounded. So $\overline{\mathcal{F}}(x)$ is compact and nonempty.
- The proof is concluded invoking [12] (Sect. 5, Lem. 16).

□

A.2. **Proof of Proposition 1**

The implication 1⇒ 2 follows from the properties of $\mathcal{KL}$ functions.

To establish the implication 2 ⇒ 1, we remark from (25) that, without loss of generality, we can assume that:

- for each $r > 0$, the function $T(r, \cdot)$ is nonincreasing on $\mathbb{R}_{>0}$ and, with (24), we have $\lim_{\epsilon \to +\infty} T(r, \epsilon) = 0$;
- for each $\epsilon > 0$, the function $T(\cdot, \epsilon)$ is nondecreasing on $\mathbb{R}_{>0}$.

Let $\psi_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function which is strictly decreasing and onto $\mathbb{R}_{>0}$ (hence invertible on $\mathbb{R}_{>0}$) and satisfies

$$\psi_r(s) \geq T(r, s) \qquad \forall s > 0. \tag{336}$$

Claim 5. *For each $r > 0$, each $x \in \mathcal{G}$ and each $\phi \in \mathcal{S}(x)$, we have*

$$\omega_2(x) \leq r \qquad \Longrightarrow \qquad \omega_1(\phi(t, x)) \leq \psi_r^{-1}(t) \quad \forall t > 0. \tag{337}$$

Proof. From condition 2a, the differential inclusion is forward complete so that $\phi(\cdot, x)$ is defined for all $t \geq 0$. Then, for each $t > 0$, we let $\varepsilon := \psi_r^{-1}(t)$. From (336) we have $\psi_r(\varepsilon) = t \geq T(r, \varepsilon)$. Therefore, from condition 2c, we have $\omega_1(\phi(t, x)) \leq \epsilon = \psi_r^{-1}(t)$. □

For each $r > 0$, we let $\psi_r^{-1}(0) := +\infty$. Then we define

$$\beta(s, t) := \min\left\{\gamma(s), \inf_{r \in (s,\infty)} \psi_r^{-1}(t)\right\} \tag{338}$$

where $\gamma \in \mathcal{K}_\infty$ comes from condition 2b. We have $\beta \in \mathcal{KL}$ since $\gamma \in \mathcal{K}_\infty$ and for each r the function $\psi_r^{-1}(\cdot)$ is strictly decreasing to zero. Also, from the claim and condition 2b it follows that, for all $x \in \mathcal{G}$ and all $\phi \in \mathcal{S}(x)$,

$$\omega_1(\phi(t, x)) \leq \beta(\omega_2(x), t). \tag{339}$$

□

A.3. **Proof of Lemma 8**

Define the function $\delta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ as

$$\delta(x) := \inf_{\xi \in \mathcal{G}} \left[\Delta(\xi) + |\xi - x|\right]. \tag{340}$$

As in the proof of [7] (Th. 1.5.1), one can check that this function is Lipschitz on $\mathcal{G}$ with Lipschitz constant 1 and is positive on $\mathcal{G}\backslash\mathcal{A}$. It also satisfies

$$\delta(x) \leq \Delta(x). \tag{341}$$

For every $x \in \mathcal{G}\backslash\mathcal{A}$ define

$$\mathcal{W}_x = \left\{\xi \in \mathcal{G} : |\xi - x| < \frac{1}{3}\delta(x)\right\}. \tag{342}$$

These sets form an open covering of $\mathcal{G}\backslash\mathcal{A}$. Let $\{\mathcal{U}_i\}$ be a locally finite refinement of this cover and let $\{\psi_i\}$ be a smooth partition of unity on $\mathcal{G}\backslash\mathcal{A}$ subordinate[3] to $\{\mathcal{U}_i\}$. For each i, choose x_i such that $\mathcal{U}_i \subseteq \mathcal{W}_{x_i}$. Then define

$$\begin{array}{rcll} F_L(x) & := & \sum_i \psi_i(x)\overline{\mathrm{co}}F(x_i + \frac{1}{3}\delta(x_i)\overline{\mathcal{B}}) & \forall x \in \mathcal{G}\backslash\mathcal{A}, \\ & := & F(x) & \forall x \in \mathcal{A}. \end{array} \tag{343}$$

It can be verified that F_L satisfies the basic conditions on $\mathcal{G}$ and F_L is locally Lipschitz on $\mathcal{G}\backslash\mathcal{A}$.

For $x \in \mathcal{G}\backslash\mathcal{A}$ consider i such that

$$\psi_i(x) > 0 \tag{344}$$

which means that $x \in \mathcal{U}_i \subseteq \mathcal{W}_{x_i}$ so that $|x - x_i| < \frac{1}{3}\delta(x_i)$. This implies $F(x) \subseteq F(x_i + \frac{1}{3}\delta(x_i)\mathcal{B})$ for every i such that (344) holds. This implies $F(x) \subseteq F_L(x)$. Since δ is Lipschitz with Lipschitz constant 1 we have

$$\delta(x_i) - \delta(x) \leq |x_i - x| < \frac{1}{3}\delta(x_i) \tag{345}$$

or

$$\frac{2}{3}\delta(x_i) < \delta(x) \tag{346}$$

which implies

$$\{x_i\} + \frac{1}{3}\delta(x_i)\overline{\mathcal{B}} \subseteq \{x\} + \frac{2}{3}\delta(x_i)\overline{\mathcal{B}} \subseteq \{x\} + \delta(x)\overline{\mathcal{B}}. \tag{347}$$

We conclude that

$$\overline{\mathrm{co}}F(x_i + \frac{1}{3}\delta(x_i)\overline{\mathcal{B}}) \subseteq \overline{\mathrm{co}}F(x + \delta(x)\overline{\mathcal{B}}) \tag{348}$$

for every i satisfying (344). Thus, with (341), we get finally

$$F_L(x) \subseteq \overline{\mathrm{co}}F(x + \delta(x)\overline{\mathcal{B}}) \subset F_{\Delta(x)}(x). \tag{349}$$

[3] We remind the reader that this means:
- for each $x \in \mathcal{G}\backslash\mathcal{A}$, there is a neighborhood of x that intersects only a finite number of the sets $\mathcal{U}_i$,
- each ψ_i is smooth on $\mathcal{G}\backslash\mathcal{A}$, takes values in $[0,1]$, $\psi_i(x) > 0$ implies $x \in \mathcal{U}_i$, and $\sum_i \psi_i(x) = 1$.

A.4. **Proof of Lemma 9**

For each $x \in \mathcal{O}$ let $r(x) > 0$ be such that $\{x\} + r(x)\mathcal{B}$ is contained in the neighborhood $\mathcal{U}$ of x given by the assumption that F is locally Lipschitz on $\mathcal{O}$. Let $L(x)$ be the corresponding Lipschitz constant. Then $\left\{\{x\} + \frac{1}{2}r(x)\mathcal{B}\right\}_{x\in K}$ is an open covering of K which is compact. So we can extract a finite covering $\left\{\{x_n\} + \frac{1}{2}r(x_n)\mathcal{B}\right\}_{n\in\{1,\dots,N\}}$. Let (see [12], Sect. 5, Lem. 15)

$$L = \max_{n\in\{1,\dots,N\}} L(x_n), \quad r = \min_{n\in\{1,\dots,N\}} \frac{r(x_n)}{2}, \quad M = \max_{n\in\{1,\dots,N\}} r(x_n), \quad R = 3 \sup_{x\in K, v\in F(x)} |v|. \tag{350}$$

Note that we have, for all i and j

$$F(x_i) \subset F(x_j) + R\mathcal{B}. \tag{351}$$

Then pick two points x and x' in K.

If $|x - x'| < r$ $(\le \frac{r_i}{2})$ and $|x - x_i| < r_i/2$, then we have $|x' - x_i| < (r + r_i/2) \le r_i$. So x and x' are in $\{x_i\} + r(x_i)\mathcal{B}$ and we have

$$F(x) \ \subset\ F(x') + L_i|x - x'|\mathcal{B} \ \subset\ F(x') + L|x - x'|\mathcal{B}.$$

If $|x - x'| \ge r$ and $|x - x_{i_1}| < r_{i_1}/2$, $|x' - x_{i_2}| < r_{i_2}/2$. Then we can find a path joining x to x' and visiting (only once!) the points x_{i_j}. To facilitate the notations, say that we have $i_1 < \ldots \le i_j \le \ldots < i_2$. This gives

$$\begin{aligned}
F(x) \ &\subset\ F(x_{i_1}) + L|x - x_{i_1}|\mathcal{B} && (352)\\
&\subset\ F(x_{i_1+1}) + [R + L|x - x_{i_1}|]\,\mathcal{B} && (353)\\
&\ \vdots && (354)\\
&\subset\ F(x_{i_2}) + [(i_2 - i_1)R + L|x - x_{i_1}|]\,\mathcal{B} && (355)\\
&\subset\ F(x') + [L|x' - x_{i_2}| + (i_2 - i_1)R + L|x - x_{i_1}|]\,\mathcal{B} && (356)\\
&\subset\ F(x') + \left[L\frac{r_{i_2} + r_{i_1}}{2} + (N-1)R\right]\mathcal{B} && (357)\\
&\subset\ F(x') + \frac{LM + (N-1)R}{r}|x - x'|\mathcal{B}. && (358)
\end{aligned}$$

So (80) holds with

$$L_K \ =\ \frac{LM + (N-1)R}{r}\,. \tag{359}$$

A.5. **Proof of Lemma 16**

Let $\psi : \mathbb{R}^n \to [0, \infty)$ be smooth, vanish outside of the unit disk and satisfy $\int \psi(\xi)d\xi = 1$. Let $\sigma \in (0, 1]$ and, for those x such that $\{x\} + \sigma\overline{\mathcal{B}} \subset \mathcal{O}$, define

$$V_\sigma(x) := \int V(x + \sigma\xi)\psi(\xi)d\xi \tag{360}$$

and

$$\alpha_\sigma(x) := \int \alpha(x + \sigma\xi)\psi(\xi)d\xi. \tag{361}$$

Claim 6. *For each compact set $\mathcal{C} \subset \mathcal{O}$ and $\varepsilon > 0$, there exists $\sigma_\circ > 0$ such that, for all $\sigma \in (0, \sigma_\circ)$, the functions V_σ and α_σ are smooth on $\mathcal{C}$ and for all $x \in \mathcal{C}$, we have*

$$\max_{w \in F(x)} \langle \nabla V_\sigma(x), w \rangle \leq \alpha(x) + \varepsilon \tag{362}$$

and

$$|V(x) - V_\sigma(x)| \leq \varepsilon, \qquad |\alpha(x) - \alpha_\sigma(x)| \leq \frac{\varepsilon}{2} \cdot \tag{363}$$

Proof. The smoothness of V_σ and α_σ as well as (363) are standard results of analysis (see, for example [42], Prop. I.8). To obtain (362), we follow the proof of [6] (Lem. 5.1) which is modeled after the proof of [18] (Lem. B.5). Let ρ be such that $\mathcal{C} + \rho\overline{\mathcal{B}} \subset \mathcal{O}$ and let L be a Lipschitz constant for F on $\mathcal{C} + \rho\overline{\mathcal{B}}$. Let $M_1 > 0$ satisfy $|\xi| \leq M_1$ for all $\xi \in \mathcal{C}$ and let $M_2 > 0$ satisfy $|\nabla V(\xi)| \leq M_2$ for almost all $\xi \in \mathcal{C} + \rho\overline{\mathcal{B}}$. Define

$$\sigma_\circ := \min\left\{\rho, \frac{\varepsilon}{2LM_1M_2}\right\} \tag{364}$$

and consider $\sigma \in (0, \sigma_\circ)$. Let $x \in \mathcal{C}$ and $v \in F(x)$. Given $\xi \in \overline{\mathcal{B}}$, let $g_\sigma(\xi)$ be the closest point in $F(x + \sigma\xi)$ to v. Then $g_\sigma : \overline{\mathcal{B}} \to \mathbb{R}^n$ is continuous and

$$g_\sigma(\xi) \in F(x + \sigma\xi), \qquad |g_\sigma(\xi) - v| \leq L\sigma|\xi| \quad \forall \xi \in \overline{\mathcal{B}}. \tag{365}$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\langle \nabla V_\sigma(x), v \rangle = \int \langle \nabla V(x + \sigma\xi), v \rangle \psi(\xi) d\xi. \tag{366}$$

Then we have

$$\begin{aligned}
\langle \nabla V_\sigma(x), v \rangle &= \int \langle \nabla V(x + \sigma\xi), v \rangle \psi(\xi) d\xi \\
&= \int \langle \nabla V(x + \sigma\xi), g_\sigma(\xi) \rangle \psi(\xi) d\xi + \int \langle \nabla V(x + \sigma\xi), v - g_\sigma(\xi) \rangle \psi(\xi) d\xi \\
&\leq \int \alpha(x + \sigma\xi) \psi(\xi) d\xi + L\sigma \int |\nabla V(x + \sigma\xi)|\, |\xi| \psi(\xi) d\xi \\
&\leq \alpha(x) + \tfrac{\varepsilon}{2} + LM_1M_2\sigma \\
&\leq \alpha(x) + \varepsilon.
\end{aligned} \tag{367}$$

□

From here we follow the proof of [18] (Th. B.1).

Let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a locally finite open cover of $\mathcal{O}$ with $\overline{\mathcal{U}}_i$ a compact subset of $\mathcal{O}$ and let $\{\kappa_i\}_{i=1}^\infty$ be a smooth partition of unity on $\mathcal{O}$ subordinate[4] to $\{\mathcal{U}_i\}$. Note that, since $\sum_{i=1}^\infty \kappa_i(x) = 1$ for all $x \in \mathcal{O}$, we have

$$\sum_{i=1}^\infty \langle \nabla \kappa_i, v \rangle = 0 \qquad \forall (x, v) \in \mathcal{O} \times \mathbb{R}^n. \tag{368}$$

[4]See Footnote 3.

Define

$$\varepsilon_i := \inf_{\xi \in \mathcal{U}_i} \min\{\mu(\xi), \nu(\xi)\} \tag{369}$$

and

$$q_i := \max_{\xi \in \overline{\mathcal{U}}_i, v \in F(\xi)} |\nabla \kappa_i(\xi)||v|. \tag{370}$$

From Claim 6, for each i, there exists σ_i such that V_{σ_i} is smooth on $\overline{\mathcal{U}}_i$ and such that, for each $x \in \overline{\mathcal{U}}_i$,

$$|V(x) - V_{\sigma_i}(x)| \leq \frac{\varepsilon_i}{2^{(i+1)}(1+q_i)} \tag{371}$$

and, for each $v \in F(x)$,

$$\langle \nabla V_{\sigma_i}(x), v\rangle \leq \alpha(x) + \frac{\varepsilon_i}{2}\,\cdot \tag{372}$$

Define $V_s(x) := \sum_{i=1}^{\infty} \kappa_i(x) V_{\sigma_i}(x)$. V_s is smooth on $\mathcal{O}$. Also, defining, for each $x \in \mathcal{O}$, $\mathcal{I}_x = \{j : x \in \mathcal{U}_j\}$, we have

$$|V(x) - V_s(x)| \leq \sum_{i=1}^{\infty} \kappa_i(x)|V(x) - V_{\sigma_i}(x)| \leq \max_{j \in \mathcal{I}_x} \varepsilon_j \leq \mu(x). \tag{373}$$

Finally, with (368), we get, for all $x \in \mathcal{O}$,

$$\begin{aligned}
\langle \nabla V_s(x), v\rangle &= \sum_{i=1}^{\infty} \langle \nabla \kappa_i(x), v\rangle V_{\sigma_i}(x) + \sum_{i=1}^{\infty} \kappa_i(x) \langle \nabla V_{\sigma_i}(x), v\rangle \\
&= \sum_{i=1}^{\infty} \langle \nabla \kappa_i(x), v\rangle \left(V_{\sigma_i}(x) - V(x)\right) + \sum_{i=1}^{\infty} \kappa_i(x) \langle \nabla V_{\sigma_i}(x), v\rangle \\
&\leq \sum_{i \in \mathcal{I}_x} \left[\frac{q_i \varepsilon_i}{2^{(i+1)}(1+q_i)} + \kappa_i(x)\left(\alpha(x) + \frac{\varepsilon_i}{2}\right)\right] \\
&\leq \alpha(x) + \max_{j \in \mathcal{I}_x} \varepsilon_j \leq \alpha(x) + \nu(x).
\end{aligned} \tag{374}$$

Since $v \in F(x)$ was arbitrary, the result follows. □

A.6. **Proof of Lemma 17**

In this paragraph we prove Lemma 17. This Lemma is a straightforward extension of [18] (Lem. 4.3). The only differences are that we want a result on $\mathcal{G}$ which is only an open subset of $\mathbb{R}^n$ and we want ρ' to be a class-$\mathcal{K}_\infty$ function. So here we reproduce the proof of [18] (Lem. 4.3) with the modifications needed to handle the above differences. Nevertheless to simplify the presentation we establish only that $\rho \circ V$ is C^1. The smooth (C^∞) result follows exactly as in [18] by modifying the definitions of c_i and d_i below and estimating the higher derivatives in terms of these numbers.

A.6.1. *Construction of* ρ

For our expression of ρ, we need several ingredients:

- Let $\{\mathcal{C}_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of $\mathcal{G}$ such that each $x \in \mathcal{G}$ is in some $\mathcal{C}_i$ and $\mathcal{C}_i$ is contained in the interior of $\mathcal{C}_{i+1}$, *i.e.*,

$$\mathcal{C}_i \subset \overset{\circ}{\overbrace{\mathcal{C}_{i+1}}}. \tag{375}$$

  Note that, since V is continuous on $\mathcal{G}$, the set $\mathcal{C}_i \bigcap \left\{x \in \mathcal{G} : \frac{1}{i+2} \leq V(x) \leq \frac{1}{i}\right\}$ is a compact subset of $\mathcal{G} \backslash \mathcal{A}$.
- Let $\varphi : \mathbb{R} \to [0,1]$ be a C^∞ function satisfying
  - $\varphi(t) = 1$ if $t \leq \frac{1}{2}$,
  - $\varphi(t) = 0$ if $1 \leq t$,
  - $\varphi'(t) < 0$ if $\frac{1}{2} < t < 1$.

  We denote $\bar{\varphi} = \sup_{t\in\mathbb{R}} |\varphi'(t)|$. Note that all the derivatives of φ must be 0 at $t = \frac{1}{2}$ and $t = 1$.
- For each integer i, let I_i be the following open interval

$$I_i = \left(\frac{1}{i+2}, \frac{1}{i}\right). \tag{376}$$

- For $i \geq 1$, let π_i be the C^∞ function defined as follows:
  - $\pi_i(t) = 0$ if $t \notin I_i$,
  - $\pi_i(t) = 1 - \varphi\left(\frac{(i+1)(i+2)}{2}t - \frac{i}{2}\right)$ if $\frac{1}{i+2} \leq t \leq \frac{1}{i+1}$,
  - $\pi_i(t) = \varphi\left(\frac{i(i+1)}{2}t - \frac{i-1}{2}\right)$ if $\frac{1}{i+1} \leq t \leq \frac{1}{i}$.

  Note that

$$\pi_{i-1}(t) + \pi_i(t) = 1 \qquad \forall t \in \left[\frac{1}{i+1}, \frac{1}{i}\right]. \tag{377}$$

  For $i = 0$, let $\pi_0 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a C^∞ function which is class-$\mathcal{K}_\infty$ on $\mathbb{R}_{\geq 0}$.
- Let c_i satisfy
  - $c_i \geq 1$,
  - $c_i \geq |\nabla V(x)|$ for all x in $\mathcal{C}_i \bigcup \left\{x \in \mathcal{G} : \frac{1}{i+2} \leq V(x) \leq \frac{1}{i}\right\}$,
  - $c_i \geq \sup_{t\geq 0} |\pi_i^{(k)}(t)|$, for all $k \leq i$.

  We choose d_i satisfying

$$0 < d_i < \min\left\{\frac{d_{i-1}}{2}, \frac{2}{(i+2)c_{i+1}(2c_i + c_{i+1})}\right\}. \tag{378}$$

- Let π be the function

$$\begin{aligned} \pi(t) &= \sum_{i=1}^{\infty} d_i \pi_i(t) + \pi_0(t - (1/3)) \qquad \forall t > 0, \\ \pi(0) &= 0. \end{aligned} \tag{379}$$

Since each $t > 0$ is in at most two sets I_i, $\pi(t)$ is the sum of at most three terms. So π is well-defined and continuous on $[0, +\infty)$ and C^∞ on $(0, +\infty)$. For $i \geq 3$, $\frac{1}{i+1} \leq t \leq \frac{1}{i}$ and $k \leq i-1$, we have, using (378),

$$\begin{aligned} |\pi^{(k)}(t)| &= \left| d_{i-1}\pi_{i-1}^{(k)}(t) \;+\; d_i \pi_i^{(k)}(t) \right| && (380)\\ &\leq d_{i-1}c_{i-1} \;+\; d_i c_i, && (381)\\ &\leq d_{i-1}\left(c_{i-1} + \frac{c_i}{2}\right), && (382)\\ &\leq \frac{1}{i+1}. && (383) \end{aligned}$$

This implies

$$\lim_{t \to 0_+} \pi^{(k)}(t) \;=\; 0 \qquad \forall k. \tag{384}$$

Also, for $i \geq 3$, $\frac{1}{i+1} < t < \frac{1}{i}$, we get from (377),

$$\pi'(t) \;=\; [d_{i-1} - d_i]\pi'_{i-1}(t) \tag{385}$$

and so, since $d_i \leq \frac{d_{i-1}}{2}$ and $\varphi'(s) < 0$ for $s \in \left(\frac{1}{2}, 1\right)$,

$$\pi'(t) \;>\; 0. \tag{386}$$

With the properties of π_0, this implies that π is a class-$\mathcal{K}_\infty$ function.

With these data, we define on $[0, +\infty)$

$$\rho(t) \;=\; \int_0^t \pi(s)ds. \tag{387}$$

This function and its derivatives belong to class-$\mathcal{K}_\infty$. The function is C^∞ on $(0, +\infty)$ and all its derivatives tend to 0 as t tends to 0.

A.6.2. *Verification of the required properties*

To conclude our proof we show that $\rho \circ V$ is C^1 on $\mathcal{G}$.

<u>On $\mathcal{G} \backslash \{x : V(x) \neq 0\}$</u>, $\rho \circ V$ is C^1 since V and ρ are C^∞ on $\mathcal{G} \backslash \{x : V(x) \neq 0\}$ and $(0, +\infty)$ respectively.

<u>On $\overbrace{\{x : V(x) = 0\}}^{\circ} \cap \mathcal{G}$</u>, $\rho \circ V$ is C^1 since it is identically zero.

<u>Finally, let $x \in \mathcal{G} \cap \partial\{\zeta : V(\zeta) = 0\}$</u>. There exists ℓ so that $x \in \mathcal{C}_\ell$. And, for all $k \geq \ell$, there exists $\delta > 0$ such that, for each ξ satisfying $|\xi - x| \leq \delta$, we have $\xi \in \mathcal{G}$ and $V(\xi) \leq \frac{1}{k}$. Also, for each such ξ satisfying also $V(\xi) \neq 0$, there exists $i \geq k$ such that

$$\xi \;\in\; \mathcal{C}_i \cap \left\{ x \in \mathcal{G} : \frac{1}{i+1} \leq V(x) \leq \frac{1}{i} \right\}. \tag{388}$$

With (378), this yields:

$$
\begin{aligned}
|\nabla\rho\circ V(\xi)| &= |d_{i-1}\pi_{i-1}(V(\xi)) + d_i\pi_i(V(\xi))|\ |\nabla V(\xi)|\,, && (389)\\
&\le d_{i-1}\left(c_{i-1}+\frac{c_i}{2}\right)|\nabla V(\xi)|\,, && (390)\\
&\le d_{i-1}\left(c_{i-1}+\frac{c_i}{2}\right)c_i, && (391)\\
&\le \frac{1}{i+1}\ \le\ V(\xi)\ \le\ \frac{1}{k}\cdot && (392)
\end{aligned}
$$

On the other hand, the continuity of $\rho\circ V$ implies the existence of $s^*\in(0,1)$ such that

$$
\begin{aligned}
\rho(V(x+s(\xi-x))) &\ge \frac{\rho(V(\xi))}{2}\quad \forall s\in[s^*,1], && (393)\\
\rho(V(x+s^*(\xi-x))) &= \frac{\rho(V(\xi))}{2}\cdot && (394)
\end{aligned}
$$

Since (392) holds for all points $x+s(\xi-x)$ with $s\in[s^*,1]$, the Mean Value Theorem gives

$$
0\le\rho(V(\xi))\ =\ 2\Big(\rho(V(\xi))-\rho(V(x+s^*(\xi-x)))\Big)\ \le\ 2\frac{1}{k}\,(1-s^*)\,|\xi-x|. \tag{395}
$$

Collecting all of the above, we have established:

For each $\varepsilon\in(0,\frac{1}{\ell})$, there exists $\delta>0$ such that, for each $h\in(0,\delta]$ and each unit vector u, we have

$$
\left|\frac{\rho(V(x+hu))-\rho(V(x))}{h}\right|\ \le\ 2\,\varepsilon. \tag{396}
$$

This establishes that $\rho\circ V$ is differentiable at x and that its derivative is continuous at this point. □

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