ESAIM: Control, Optimisation and Calculus of Variations August 2000, Vol. 5, 425–444
URL: http://www.emath.fr/cocv/

ON SOME OPTIMAL CONTROL PROBLEMS FOR THE HEAT RADIATIVE TRANSFER EQUATION *

Sandro Manservisi[1, 2] and Knut Heusermann[2]

Abstract. This paper is concerned with some optimal control problems for the Stefan-Boltzmann radiative transfer equation. The objective of the optimisation is to obtain a desired temperature profile on part of the domain by controlling the source or the shape of the domain. We present two problems with the same objective functional: an optimal control problem for the intensity and the position of the heat sources and an optimal shape design problem where the top surface is sought as control. The problems are analysed and first order necessity conditions in form of variation inequalities are obtained.

AMS Subject Classification. 49N50, 80A23.

Received August 3, 1999. Revised June 2, 2000.

1. Introduction

Radiative heat exchange can be found in many natural and engineering processes. At low temperature this exchange is small but at high temperature could be the leading form of heat transfer and in many situations, like in some forming processes, it is appropriate to model the heat exchange with the pure radiation equation.

Motivated by the desire to remove hot spots and control the temperature distribution we present a systematic mathematical theory for the optimisation of systems which are appropriately described by this model. We consider radiative heat exchange in a two-dimensional domain Ω with boundary Γ and radiative source f governed by the integral boundary equations [16]

$$\begin{array}{rcll} u(s) & = & \sigma T^4(s) & s \in \Gamma \\ u(s) - \int_\Gamma K(s,s')\, u(s')\, ds' & = & f(s) & s \in \Gamma, \end{array} \tag{1.1}$$

where T is the temperature, u is the radiosity, σ is the Stefan-Boltzmann constant and $K(s,s')$ is the kernel representing the radiative heat exchange between s and s' [16].

Keywords and phrases: Optimal control, heat radiative transfer, optimal shape design.

* *S.M. was supported by European community under grant XCT-97-0117.*

[1] DIENCA - Universitá degli studi di Bologna, Via dei Colli 16, 40136 Bologna, Italy;
e-mail: sandro.manservisi@mail.ing.unibo.it

[2] ITWM - Kaiserslautern University, Erwin-Schrödinger-Strasse, 67663 Kaiserslautern, Germany.

In particular we focus on the control of the temperature on some parts of the boundary $\Gamma_2 \subset \Gamma$. If T_0 is the desired distribution of temperature on Γ_2 we minimise the functional

$$\mathcal{J} = \int_{\Gamma_2} (T - T_0)^2 d\Gamma \tag{1.2}$$

under various constraints. This leads to a consistent mathematical optimal control problem and a corresponding first order necessity condition in form of variational inequality.

The plan of the rest of the paper is as follows. In Section 2 we formulate the heat radiation exchange model in a convenient and precise mathematical way. In Section 3, a simple model for radiating sources is discussed and the optimal source control problem is stated; then, we will prove the existence of optimal control solutions and characterize them by deriving the first order necessity conditions associated with the problem. Finally, in Section 4, an optimal shape problem is formulated where the control is assumed to be the top part of the domain; then the existence of controls are proved and the corresponding first order necessity conditions are derived.

2. Model and notations

2.1. **Model**

In this section we discuss briefly the exchange radiation model assumed in this paper. In particular we focus on the problem of pure radiation with diffuse grey surfaces which implies uniform emission and reflection in every direction and the same spectral emissivity ϵ and absorptivity a for all wave lengths. The reflectivity is denoted by ρ and under the above assumptions we have $\epsilon = a = 1 - \rho$.

We consider a bounded domain Ω with boundary $\partial\Omega$. Let $\mathcal{S}$ be the surrounding of Ω, with boundary $\partial\mathcal{S}$. The intersection between $\partial\mathcal{S}$ and $\partial\Omega$ defines the radiating boundary Γ.

On Γ the heat balance is stated as

$$\Phi - u + \mu = 0, \tag{2.1}$$

where Φ is the surface heat flux, u is the radiosity (radiative energy leaving the surface) and μ is the irradiation (incoming radiative energy).

For grey and diffuse surfaces the total radiosity is the sum of the black body emission (corresponding to the Stefan-Boltzmann radiation law) and the reflected part of the irradiation, *i.e.*,

$$u = \epsilon\, \sigma\, T^4 + (1 - \epsilon)\, \mu, \tag{2.2}$$

where σ denotes the Stefan-Boltzmann constant ($\sigma = 5.670 \times 10^{-8}$ W/m^{2}K^4).

The irradiation on Γ depends of the radiation emitted by all visible parts and the external radiation source u_∞. We have

$$\mu(s) = \int_\Gamma K(s, s')\, u(s')\, ds' + u_\infty(s) \quad \forall s \in \Gamma, \tag{2.3}$$

where $K(s, s')$ is the view factor between s and s' (see for example [16]). By combining (2.2) and (2.3) we obtain the equation for the radiosity u in the following form

$$u = \epsilon\, \sigma\, T^4 + (1 - \epsilon) \left(\int_\Gamma K(s, s')\, u(s')\, ds' + u_\infty \right) \tag{2.4}$$

and for the heat flux Φ as

$$\Phi = u - \int_\Gamma K(s,s')\, u(s')\, ds' - u_\infty. \tag{2.5}$$

The stationary heat equation for the conductive body S can be written as

$$-k\nabla^2 T = 0 \quad \text{ in } \mathcal{S},$$

where k denotes the heat conductivity. On some parts of the boundary of $\partial\mathcal{S}$ we assume Neumann boundary conditions, namely a perfect insulating boundary, and elsewhere radiating boundary conditions. For simplicity we assume $\mathcal{S}$ to be isolated on $\partial\mathcal{S}/\Gamma$. Then the complete combined conduction radiation boundary value problem is stated as

$$\begin{cases} -k\nabla^2 T = 0 & \text{in } \mathcal{S} \\ k\dfrac{\partial T}{\partial n} = 0 & \text{on } \partial\mathcal{S}/\Gamma, \\ k\dfrac{\partial T}{\partial n} = \Phi & \text{on } \Gamma \end{cases} \tag{2.6}$$

where Φ is given by (2.5). If we neglect the conduction in $\mathcal{S}$ and just consider the radiative heat transport on Γ ($\Phi = 0$) then we have that the radiosity u is equal to σT^4 and satisfies a Fredholm integral equation of second kind with weakly singular kernel K and known right hand side f, namely (2.5) becomes

$$u(s) - \int_\Gamma K(s,s')\, u(s')\, ds' = f(s) \qquad \forall s \in \Gamma. \tag{2.7}$$

In the rest of the paper we neglect the conduction on $\mathcal{S}$ and identify the domain surrounding Ω with the boundary Γ.

2.2. Notations

Let $\mathcal{O}$ be an open bounded set which is either the domain Ω, or its boundary Γ, or part of its boundary. We denote the space of measurable functions ϕ of order $p \geq 1$ by $L^p(\mathcal{O})$ and the standard Sobolev space of order $s \in \mathbb{R}$ by $H^s(\mathcal{O})$. Whenever m is a nonnegative integer, the inner product over $H^m(\mathcal{O})$ is denoted by $(f,g)_m$ and (f,g) indicates the inner product over $H^0(\mathcal{O}) = L^2(\mathcal{O})$. Hence we associates to $H^m(\mathcal{O})$ its natural norm $\|f\|_{m,\mathcal{O}} = \sqrt{(f,g)_m}$. Whenever it is possible, we will neglect the domain label.

For vector-valued functions and spaces we use boldface notation. For example, $\mathbf{H}^s(\mathcal{O}) = [H^s(\mathcal{O})]^n$ denotes the space of $\mathbb{R}^n$-valued functions such that each component belongs to $H^s(\mathcal{O})$. Of special interest is the space

$$H^1(\mathcal{O}) = \left\{ v \in L^2(\mathcal{O}) \,\Big|\, \frac{\partial v}{\partial x} \in L^2(\mathcal{O}) \right\}$$

equipped with the norm $\|v\|_1 = (\int_\mathcal{O} (|v|^2 + |v_x|^2) dx)^{1/2}$. For $\partial\mathcal{O}_s \subset \partial\mathcal{O}$ with nonzero measure, we also consider the subspace

$$H^1_{\partial\mathcal{O}_s}(\mathcal{O}) = \{ v \in H^1(\mathcal{O}) \,\big|\, v = 0 \text{ on } \partial\mathcal{O}_s \} \cdot$$

Also we denote by $H^1_0(\mathcal{O})$ the set $H^1_{\partial\mathcal{O}}(\mathcal{O})$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1_0(\mathcal{O})$ and $H^{-1}(\mathcal{O})$.

A weak formulation of the pure radiation problem can be stated as follows.

Let Ω be an open bounded set with boundary $\partial\Omega$ and Γ be a non-empty subset of the boundary. Given $f \in L^2(\Gamma)$, find $u \in L^2(\Gamma)$ satisfying

$$\int_\Gamma u(s)\,\phi(s)\,ds - \int_\Gamma \phi(s) \int_\Gamma K(s,s')\,u(s')\,ds'ds = \int_\Gamma f(s)\,\phi(s)\,ds \qquad \forall \phi \in L^2(\Gamma), \tag{2.8}$$

where

$$K(s,s') = \frac{\big(\vec{n}(s')\cdot(\vec{r}-\vec{r}\,')\big)\big(\vec{n}(s)\cdot(\vec{r}\,'-\vec{r})\big)}{2\|\vec{r}\,'-\vec{r}\|^3}\;\Xi(s,s'). \tag{2.9}$$

The vectors $\vec{r}$ and $\vec{r}\,'$ indicate the position of s and s' and $\vec{n}$ denotes the normal to the boundary Γ. The function $\Xi(s,s')$ is defined by

$$\Xi(s,s') = \begin{cases} 0 & \text{if} \quad \overline{ss'} \cap \Gamma \neq 0 \\ 1 & \text{if} \quad \overline{ss'} \cap \Gamma = 0, \end{cases} \tag{2.10}$$

where $\overline{ss'}$ denotes the segment

$$\overline{ss'} = \left\{ \vec{x} \in \mathbb{R}^2 \,|\, \vec{x} = \vec{r}(s)\,t + (1-t)\vec{r}(s'), t \in (0,1),\; s,s' \in \Gamma \right\}\cdot$$

In the case of a convex set Ω we have $\Xi(s,s') = 1$ for all $s,s' \in \Gamma$. The equivalence of the weak formulation in (2.8) with the classical problem in (1.1) is clear since the kernel $K(\cdot,\cdot)$ has finite norm $\|K\|$ in $L^2(\Gamma)\times L^2(\Gamma)$.

The equation in (2.8) can be written in operator form. The operator $\hat{K} : L^2(\Gamma) \to L^2(\Gamma)$ is defined by

$$\langle \hat{K}u, \phi\rangle = \int_\Gamma \phi(s) \int_\Gamma K(s,s')\,u(s')\,ds'ds \quad \forall \phi \in L^2(\Gamma),$$

and (2.8) becomes

$$u - \hat{K}(u) = f \tag{2.11}$$

with $u \in L^2(\Gamma)$ if $f \in L^2(\Gamma)$.

Existence and uniqueness of solutions of the above problems has been studied in [22] under the assumption that the source term is known on the boundary Γ. If Γ is piecewise $C^{1,1}$ and $f \in L^2(\Gamma)$ the problem in (2.8) is well posed and has unique solution in $L^2(\Gamma)$. Also some useful properties of the operator $\hat{K}$ can be found in [22] and briefly summarised as follows:

i) $\hat{K}$ defined on a closed curve is well defined and its norm $\|\widehat{K}\|$ is equal to 1;
ii) the operator $\hat{K}$ is nonnegative;
iii) $\hat{K}$ maps $L^p(\Gamma)$ into itself compactly and $\|\hat{K}\| \leq 1$.

In order to solve the boundary equation (2.8) we need to specify the curve Γ and rewrite (2.8) in the cartesian coordinate system. Let Ω be an open simply connected set in $\mathbb{R}^2$ with boundary $\partial\Omega$ and η be the curvilinear coordinate on $\partial\Omega$ which is piecewise $C^{1,1}$. We consider a smooth one-to-one mapping $\vec{r}(\eta) = (x,z)$ from $[0,1]$ into $\Gamma \subset \mathbb{R}^2$. The tangent direction is defined by

$$\bar{e}_1 = \frac{\partial \vec{r}}{\partial \eta}$$

and the corresponding covariant matrix by the element

$$g(\eta) = \bar{e}_1 \cdot \bar{e}_1 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 \cdot \tag{2.12}$$

The unit normal vector $\vec{n}(\eta)$ at η is given by $\vec{n} = \bar{e}_2/|g|$ where

$$\bar{e}_2 = \left(-\frac{\partial z}{\partial \eta}, \frac{\partial x}{\partial \eta}\right)^T. \tag{2.13}$$

We can define the unit tangent vector as $\vec{s} = \bar{e}_1/|g|$ and the covariant basis in $\mathbb{R}^2$ for the curve Γ as the pair $(\vec{s}, \vec{n})$. Therefore we have two fundamental coordinate systems: the global system, defined by (x, z), and the local system or covariant system defined by (s, n).

In this representation the integral of the function $y(s)$ over Γ reads

$$\int_\Gamma y(s)\, ds = \int_I y(s(\eta))\, \sqrt{g(\eta)}\, d\eta. \tag{2.14}$$

By introducing the kernel of the integral equation in the form

$$K(s, s') = \frac{\big(\vec{n}(s') \cdot (\vec{r} - \vec{r}\,')\big)\big(\vec{n}(s) \cdot (\vec{r}\,' - \vec{r})\big)}{2\|\vec{r}\,' - \vec{r}\|^3}\, \Xi(x, x') \tag{2.15}$$

then (2.8) becomes

$$\int_I u(\eta)\, \phi(\eta)\, \sqrt{g(\eta)} d\eta - \int_I \phi(\eta) \int_I K(\eta, \eta') u(\eta') d\eta' d\eta = \int_I \phi(\eta) f(\eta) \sqrt{g(\eta)} d\eta, \tag{2.16}$$

where

$$K(\eta, \eta') = \frac{(\bar{e}_2 \cdot (\vec{r}\,' - \vec{r}))(\bar{e}_2\,' \cdot (\vec{r} - \vec{r}\,'))}{2\,\|\vec{r} - \vec{r}\,'\|^3}\, \Xi(\eta, \eta'). \tag{2.17}$$

Numerical solutions of the boundary equation (2.16) can be found by using a wide range of schemes (see for example [16, 23] and references therein). Also an abstract *"a priori"* error estimate for the finite dimensional approximations of (2.8) and (2.6) can be found in [23].

3. Optimal control problem for intensity and location of sources

3.1. Model

Let Ω be an open bounded domain with boundary $\partial\Gamma = \Sigma_1 \cup \Gamma_1 \cup \Sigma_2 \cup \Gamma_2$. We consider a geometric configuration with isotropic non-conductive grey surfaces and radiating sources S_i, $i = 1, 2 \ldots, n$. The radiating boundary Γ is supposed to be an isotropic non-conductive grey surface and consists of the top surface Γ_1 and the bottom surface Γ_2. The rest of the boundary, which is a non-radiating surface, is denoted by $\Sigma = \Sigma_1 \cup \Sigma_2$ (see Fig. 1).

Under these assumptions, we can model the heat transfer by the function $u = \sigma T^4$, where u is the solution of

$$\int_\Gamma u\, \phi\, ds - \int_\Gamma \phi(s) \int_\Gamma K(s, s')\, u(s')\, ds' ds = \int_\Gamma f(s, \vec{q})\, \phi\, ds \quad \forall \phi \in L^2(\Gamma) \tag{3.1}$$

and $f(s, \vec{q}) \in L^2(\Gamma)$ is the source term which is Lipschitz continuous with respect to the control $\vec{q} = \vec{q}(T_b)$.

There are several analytical and empirical models available from the manufacturing industry but here we consider a simple model. Let Γ_{S_b} be a convex surface emitting radiation at the temperature T_b. In this case we can write the source term in (3.1) as integral over the known radiating surface Γ_{S_b}, *i.e.*

$$f(s, \cdot) = \int_{\Gamma_{S_b}} K(s, s')\, u_b(s')\, ds',$$

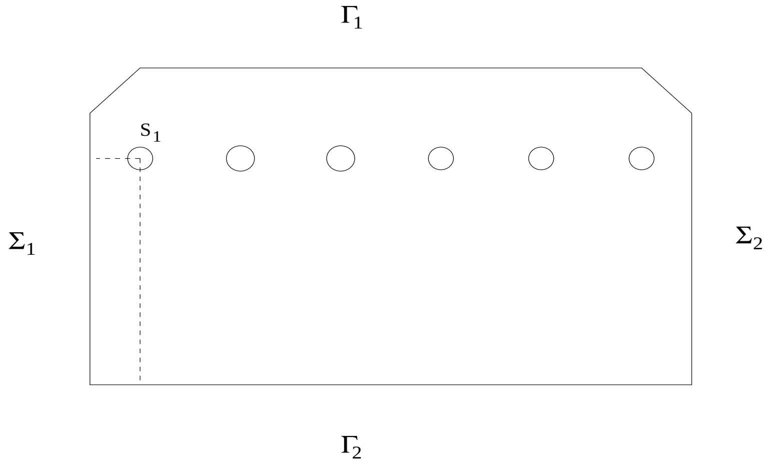

FIGURE 1. Setup for the optimal control problem of the sources. S_i denotes the radiating sources and Γ_i the radiating surfaces.

where u_b is the solution of the equation

$$\int_{\Gamma_{S_b}} u_b(s')\,\phi(s')\,ds' = \int_{\Gamma_{S_b}} \phi(s') \int_{\Gamma} K(s,s')\,u(s)\,ds\,ds' + \int_{\Gamma_{S_b}} Q(s')\,\phi(s')\,ds' \qquad \forall \phi \in L^2(\Gamma) \tag{3.2}$$

and $Q(\cdot) = Q(T_b,\cdot)$ is the known function modelling the surface intensity of the emitting source at temperature T_b. By solving the system in (3.1) and (3.2) we have the exact solution of the problem. However the source temperature is much higher than the surface temperature so that the first integral on the right hand side can be neglected, namely u_b satisfies

$$\int_{\Gamma_{S_b}} u_b(s')\,\phi(s')\,ds' = \int_{\Gamma_{S_b}} Q(s')\,\phi(s')\,ds' \quad \forall \phi \in L^2(\Gamma).$$

We set $\phi(s') = K(s',s)$ in the above equation then we have

$$f(s,\cdot) = \int_{\Gamma_{S_b}} Q(s')\,K(s',s)\;ds'$$

for all $s \in \Gamma$. The integral can be computed if we consider a two-dimensional cylindrical point source with small radius r_0 when the radius tends asymptotically to zero. In fact in this case the source, located at the point $\vec{x}_b$, takes the form [16]

$$f(s,\cdot) = \frac{Q_b}{\pi} W(s,\vec{x}_b) = \frac{Q_b}{2\pi} \frac{\vec{n}(s)\cdot(\vec{x}_b - \vec{r}(s))}{|\vec{r}(s) - \vec{x}_b|^2}, \tag{3.3}$$

where $Q_b = 2\pi r_0 \sigma T_b^4$ is the total energy emitted by the source, the vector $\vec{r}(s)$ indicates the position s on the boundary and $\vec{n}$ is the normal to Γ at $\vec{r}(s)$. Also we note that, if $|\vec{r}(s) - \vec{x}_b| \geq r_0$, then

$$\sup_{s\in\Gamma} |f| \leq \frac{Q_b}{2\pi r_0}, \tag{3.4}$$

which states the conservation of energy on Γ.

In the rest of the paper we refer to (3.3) as our source model which is characterised by few parameters: the source centre of mass $\vec{x}_b$ and the total source intensity Q_b which is a function of its average temperature T_b and radius r_0. If the temperature of the boundary is close to the source temperature this model is no longer valid. For example in a steady situation where a source is located in a closed domain and no radiation can escape the temperature of the boundary is the same as the temperature of the source and the correct source model defined by (3.2) must be used.

For n two-dimensional point sources emitting energy Q_{bi} we write

$$f(s,\vec{q}) = \sum_{i=1}^{n} \frac{Q_{bi}}{\pi} W(s,\vec{x}_{bi}), \tag{3.5}$$

where $\vec{x}_{bi} = (x_{bi}, z_{bi})$ is the position of the i-th point source and $W(\cdot,\cdot)$ defined by (3.3). If we desire to control the intensity we write $\vec{q} = (q_1, q_2, \dots, q_n)$, $Q_{bi} = q_i$ for $i = 1, 2, \dots, n$ and

$$f(s,\vec{q}) = \sum_{i=1}^{n} \frac{q_i}{\pi} W(s,\vec{x}_{bi}). \tag{3.6}$$

If we desire to control the vertical position of the sources we write $z_{bi} = q_i$ for $i = 1, 2, \dots, n$ or

$$f(s,\vec{q}) = \sum_{i=1}^{n} \frac{Q_{bi}}{\pi} W(s,(x_{bi}, q_i)). \tag{3.7}$$

In this paper the control $\vec{q}$ is a vector which lies in a n-dimensional space $\mathbb{R}^n$ but a generalisation to infinite dimensional spaces is straightforward. In particular we consider n-dimensional controls bounded by the following constraints

$$\chi \le q_i \le \omega \quad \forall i = 1, \dots, n, \tag{3.8}$$

where χ and ω are the given limits for the control. We note that in (3.7) a constraint is necessary to keep the vertical coordinate far from the boundary Γ, namely the function f in $L^2(\Gamma)$. Given $\chi < \omega$, we define by $\mathbf{L}_{\chi\omega}$ the set of all vectors in $\mathbb{R}^n$ bounded by (3.8). It is easy to see that $\mathbf{L}_{\chi\omega}$ is a closed convex set.

We can think the source term f as an operator from $\mathbb{R}^n$ to $L^2(\Gamma)$ and the function $f(\vec{q})$ is always assumed known and Lipschitz continuous with respect to $\vec{q}$ inside the range of interest. The source term defined in (3.6) is clearly Lipschitz continuous but the source term in (3.7) is Lipschitz continuous only if specified over an appropriate range. From (3.7) we have

$$\frac{\partial f}{\partial \vec{q}} = \sum_{i=1}^{n} \frac{Q_{bi}}{\pi} \frac{\partial W}{\partial q_i}(s,(\vec{x}_{bi}, q_i)) = \sum_{i=1}^{n} \frac{I_{bi}}{2\pi} \left[\frac{-n_z}{|\vec{r}_i - \vec{x}_{bi}|^2} + 2 \frac{(\vec{n} \cdot (\vec{r}_i - \vec{x}_{bi}))(z - q_i)}{|\vec{r}_i - \vec{x}_{bi}|^4} \right],$$

where $\vec{n}$ is the normal to Γ and n_z its z-component. If $\chi > 0$ and $\omega < \min\{z : (x,z) \in \Gamma \subset \mathbb{R}^2\}$ then clearly $df/d\vec{q}$ is bounded and f is Lipschitz continuous with respect to $\vec{q}$.

The objective of the optimisation is to have a desired temperature profile on the bottom surface Γ_2, which is equivalent to having a desired profile of the function u since $\sigma T^4 = u$ on Γ_2. This optimisation can be reached through the minimisation of the following functional

$$\mathcal{J}(u,q) = \int_{\Gamma_2} (u - U)^2 d\vec{s}, \tag{3.9}$$

where U is the desired distribution of radiosity. *We say that U is in the set of admissible targets U_{ad} if $U \in L^2(\Gamma_2)$.* The function U is equal to σT_0^4 if T_0 is the desired profile of temperature on Γ_2.

The optimal control problem can be stated as follows:

Find $(u, \vec{q}) \in (L^2(\Gamma), \mathbf{L}_{\chi\omega})$ *such that* $(u, \vec{q})$ *satisfies (3.1) and minimises (3.9) under the constraint (3.8).*

By using standard techniques, it is possible to show that this problem has at least one solution.

Theorem 3.1. *If* $\Gamma \in C^{1,1}$ *and* f *is Lipschitz continuous with respect to the control* $\vec{q}$ *then the optimal control problem has at least one solution* $(u, \vec{q}) \in L^2(\Gamma) \times \mathbb{R}^n$*.*

Proof. We briefly sketch the proof (for details see [11, 12]). Since the set of admissible solutions is not empty there exist minimising sequences u_n and $\vec{q}_n$. Since these minimising sequences are uniformly bounded then there exist a pair $(u, \vec{q})$ and subsequences u_k, q_k converging weakly to $(u, \vec{q})$. This limit is solution of the problem. In fact the lower semicontinuity of the functional and the Lipschitz continuous condition of the source term allows us to pass to the limit in the functional and in the equation. □

3.2. The first order necessity condition

We shall show that the optimal control solution must satisfy a first-order necessity condition, which leads to a variational inequality. By studying this variational inequality, a possible candidate for the optimal control solution can be found. Now we need to prove differentiability.

Theorem 3.2. *Let* $\vec{q}$ *and* $\widetilde{q}$ *be in* $\mathbf{L}_{\chi\omega}$*. The mapping* $u = u(\vec{q})$ *from* $\mathbf{L}_{\chi\omega}$ *to* $L^2(\Gamma)$*, defined as the solution of (3.1), has a Gateaux derivative* $du/d\vec{q} \cdot \vec{h}$ *in the direction* $\vec{h} = \widetilde{q} - \vec{q}$*. Furthermore,* $\widetilde{w}(h) = (du/d\vec{q}) \cdot \vec{h}$ *is the solution of the problem*

$$\int_\Gamma \widetilde{w}(s)\, \phi(s) ds - \int_\Gamma \phi(s) \int_\Gamma K(s,s')\widetilde{w}(s')\, ds'\, ds = \int_\Gamma \phi(s) \left(\frac{df}{d\vec{q}} \cdot \vec{h} \right) ds \quad \forall \phi \in L^2(\Gamma), \tag{3.10}$$

Proof. This follows from the linearity of the operator $\hat{K}$. For details see for example [24]. □

From the definition of optimal solution $(u, \vec{q})$, we have

$$\mathcal{J}(u, \vec{q}) - \mathcal{J}(u, \overline{q}) \leq 0 \quad \forall \overline{q} \in \mathbf{L}_{\chi\omega}.$$

As $\mathbf{L}_{\chi\omega}$ is convex, then we can set $\overline{q} = \widetilde{q}t + (1-t)\vec{q}$ for all $t \in [0,1]$ and for all $\widetilde{q} \in \mathbf{L}_{\chi\omega}$. Hence

$$\mathcal{J}(u, \vec{q} - t(\vec{q} - \widetilde{q})) - \mathcal{J}(u, \vec{q}) \geq 0 \quad \forall t \in [0,1],$$

which implies the following lemma for t tending to zero.

Lemma 3.3. *If* $(u, \vec{q}) \in L^2(\Gamma) \times \mathbf{L}_{\chi\omega}$ *is an optimal solution then the variational inequality*

$$\mathcal{J}'(u, \vec{q}) \cdot (\widetilde{q} - \vec{q}) \geq 0 \quad \forall \widetilde{q} \in \mathbf{L}_{\chi\omega} \tag{3.11}$$

must hold.

Fortunately, in order to minimise the functional we need only an integral over all the directions and this can more easily be obtained through the solution of a single adjoint equation. To see this, the following result is needed.

Lemma 3.4. *Let* $0 < \chi < \omega$*,* $U \in U_{ad}$*,* $\vec{h} \in \mathbf{L}_{\chi\omega}$ *and* $\widetilde{w}(\vec{h})$ *be defined through* (3.10)*. For every* $h_2 \in L^2(\Gamma)$ *we have*

$$\int_\Gamma h_2\, \widetilde{w}(h)\, ds = -\int_\Gamma \lambda(s) \left(\frac{df}{d\vec{q}} \cdot \vec{h} \right) ds, \tag{3.12}$$

where λ is the solution of the adjoint problem

$$\int_\Gamma \lambda(s)\phi(s)ds - \int_\Gamma \phi(s) \int_\Gamma K(s,s')\lambda(s')\,ds'\,ds = -\int_\Gamma h_2(s)\,\phi(s)\,ds \quad \forall\phi\in L^2(\Gamma). \tag{3.13}$$

Proof. This lemma follows from the fact that the operator $\hat{K}$ is self-adjoint. In fact

$$\int_\Gamma h_2\,\widetilde{w}(h)\,ds = -\int_\Gamma \lambda(s)\,\widetilde{w}(s)\,ds + \int_\Gamma \widetilde{w}(s)\int_\Gamma K(s,s')\lambda(s')\,ds'\,ds = -\int_\Gamma \lambda(s)\left(\frac{df}{d\vec{q}}\cdot\vec{h}\right)ds.$$

□

It is now easy to show that a solution of the optimal problem implies a variational inequality involving the adjoint variable.

Theorem 3.5. *Let $0<\chi<\omega$ and $U\in U_{ad}$. If $(u,\vec{q})\in L^2(\Gamma)\times\mathbf{L}_{\chi\omega}$ is an optimal pair, then we have*

$$\mathcal{J}'(u,\vec{q})\cdot(\widetilde{q}-\vec{q}) = \int_{\Gamma_2}\lambda(s)\left(\frac{df}{d\vec{q}}\cdot(\widetilde{q}-\vec{q})\right)ds \ge 0 \quad \forall\widetilde{q}\in\mathbf{L}_{\chi\omega}, \tag{3.14}$$

where λ is the solution of the adjoint problem

$$\int_\Gamma \lambda(s)\,\phi(s)\,ds - \int_\Gamma \phi(s)\int_\Gamma K(s',s)\lambda(s')ds'ds = -\int_\Gamma (u(s)-U(s))\,\phi(s)\,ds \qquad \forall\phi\in L^2(\Gamma) \tag{3.15}$$

and u the solution of (3.1).

Proof. From Theorem 3.2 and Lemma 3.4 we have

$$\mathcal{J}'(u,\vec{q})\cdot\vec{h} = \int_{\Gamma_2}\widetilde{w}(s)\,(u(s)-U(s))\,ds = \int_{\Gamma_2}\lambda(s)\left(\frac{df}{d\vec{q}}\cdot\vec{h}\right)ds.$$

The theorem follows from Lemma 3.3. □

3.3. The optimality system

The adjoint method used here requires the solution of the pure radiative equation

$$\int_\Gamma u(s)\,\phi(s)\,ds - \int_\Gamma \phi(s)\int_\Gamma K(s,s')\,u(s')\,ds'ds = \int_\Gamma f(s,\vec{q})\,\phi(s)\,ds \qquad \forall\phi\in L^2(\Gamma), \tag{3.16}$$

the adjoint equation

$$\int_\Gamma \lambda(s)\phi(s)\,ds - \int_\Gamma \phi(s)\int_\Gamma K(s,s')\lambda(s')\,ds'\,ds = -\int_\Gamma (u(s)-U(s))\,\phi(s)\,ds \qquad \forall\phi\in L^2(\Gamma) \tag{3.17}$$

and the variational inequality

$$\mathcal{J}'(u,\vec{q})\cdot(\widetilde{q}-\vec{q}) = \int_{\Gamma_2}\lambda(s)\left(\frac{df}{d\vec{q}}\cdot(\widetilde{q}-\vec{q})\right)ds \ge 0 \quad \forall\widetilde{q}\in\mathbf{L}_{\chi\omega}, \tag{3.18}$$

where f is defined by (3.5). The numerical solution of the optimality system must be found iteratively. In order to solve the above system we propose a projected gradient algorithm [14]. Set $\mathcal{J}(k)=\mathcal{J}(u,q(k))$, where $\mathcal{J}(\cdot)$ is given by (3.9) and k is the iteration counter of the gradient algorithm. In the algorithm, τ will denote a prescribed tolerance used to test for the convergence of the functional. The *gradient algorithm* proceeds as follows.

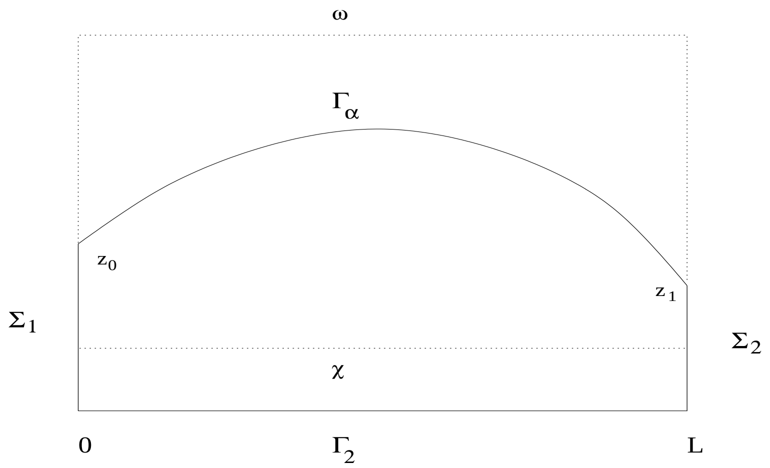

FIGURE 2. Setup of the optimal shape design problem. The dot line marks the limits on the variation of the surface Γ_α.

a) initialisation:
  i) choose τ and $\vec{q}(0)$; set $k = 0$ and $\rho_0 = 1$;
  ii) solve for the starting field $u(0)$ from (3.16) with $\vec{q} = \vec{q}(0)$;
  iii) evaluate $\mathcal{J}(0)$;
b) main loop:
  iv) set $k = k + 1$;
  v) solve for $\lambda(k)$ from (3.17) with $u = u(k-1)$;
  vi) set $\vec{q}(k) = P_L\left(\vec{q}(k-1) - \rho_k \int_{\Gamma_2} \lambda(k-1)\, \frac{df}{d\vec{q}}(k-1)\, ds\right)$;
  vii) solve for $u(k)$ from (3.16) with $\vec{q} = \vec{q}(k)$;
  viii) evaluate $\mathcal{J}(k)$;
  ix) if $\mathcal{J}(k) \geq \mathcal{J}(k-1)$, set $\rho_{k-1} = .5\rho_{k-1}$ and go to vi); otherwise, continue;
  x) if $|\mathcal{J}(k) - \mathcal{J}(k-1)|/|\mathcal{J}(k)| > \tau$, set $\rho_k = 1.3\rho_{k-1}$ and go to iv); otherwise, stop.

The operator P_L is the projection operator over the set $\mathbf{L}_{\chi\omega}$. A discussion of this algorithm can be found in [7] or [1]. Other, more sophisticated, strategies and implementation can be devised to speed up convergence. For example the trust region algorithm can be used [9].

4. Optimal shape design problem

4.1. **Preliminaries and description of the problem**

In this section we shall study the shape optimisation of the pure radiation problem which leads to a predefined bottom temperature distribution. The shape of the domain Ω can be defined by four surfaces (see Fig. 2): the top surface Γ_1, the bottom surface Γ_2, which are the radiating boundaries and two non-radiating surfaces Σ_1 and Σ_2. We assume, without loss of generality, that the bottom is flat and fixed, namely $\Gamma_2 = \{(x, z) \in \mathbb{R}^2 : z = 0, x \in (0, L)\}$. We take the top surface $\Gamma_\alpha = \{(x, z) \in \mathbb{R}^2 \,|\, x \in (0, L), z = \alpha(x)\} = \Gamma_1$ as control, which is assumed to be bounded and regular. In the rest of the paper we denote $(0, L)$ simply by I and the radiating surface by $\Gamma(\alpha) = \Gamma_\alpha \cup \Gamma_2$.

We can define a set of possible shapes in the following way. Let χ and ω be two positive constants and z_0, z_1 be the boundary points of the controlled surface Γ_α. The set of all convex $\alpha \in H^2(I)$ with $\chi \leq \alpha \leq \omega$ and $\alpha(0) = z_0$, $\alpha(L) = z_1$ and $d^2\alpha/dx^2(0) = d^2\alpha/dx^2(L) = 0$ may be a suitable set of admissible controls α. In fact from the Sobolev imbedding theorem we have that the boundary Γ_α is in $H^2(I) \subset C^1(\bar{I}) \subset \mathcal{C}^{0,1}(\bar{I})$, which is regular enough to suppress excessive oscillations of the boundary (see for example [18]). However the regularity of $\alpha \in H^2(I)$ may not be enough to write explicitly the first order necessity condition. From the nature of the integral equation we must assume $d^2\alpha/dx^2$ positive and bounded by M so that α is in $C^{1,1}(I)$. We note that the kernel $K(s,s')$ in (4.4) is defined through the function Ξ (see (2.10)), which may introduce many relative minimal points. Also the function Ξ may not be differentiable and therefore the first order necessity condition may not be achievable in explicit form. In order to avoid these unfortunate situations we assume convexity for α. The convexity of Γ_α and the shape of Ω guarantee that the factor Ξ is always equal to 1. Furthermore the convexity of α implies that $\alpha \geq \chi$ if $\chi = \min\{z_0, z_1\}$ and the continuity ensures that there exists a real number ω such that $\alpha \leq \omega$.

Hence the set of *admissible controls* α is defined by the following convex closed set

$$\mathcal{Q}_{ad}(I) = \left\{ \alpha \in H^2(I) \text{ such that } -M \leq \frac{d^2\alpha}{dx^2} \leq 0 \right\},$$

with M real positive constant. The objective of the optimisation is to have a desired profile of temperature on the bottom surface Γ_2, which is equivalent to having a desired profile of the function u since $\sigma T^4 = u$. This optimisation can be reached through the minimisation of the following functional

$$\mathcal{J}(u,\alpha) = \frac{1}{2}\int_{\Gamma_2}(u-U)^2 ds + \gamma\frac{1}{2}\int_I \left(\frac{d^2\alpha}{dx^2}\right)^2 dx, \tag{4.1}$$

where $U \in U_{ad}$ is the desired distribution of radiosity and γ is a positive real constant. The penalty term $\gamma(d^2\alpha/dx^2)^2$, with $\gamma > 0$, is necessary to have α in $H^2(I)$. However this does not imply $\alpha \in C^{1,1}$ and therefore we need to include the constraint $\alpha \in \mathcal{Q}_{ad}(I)$ in order to have a well posed problem.

We introduce the set of admissible controls α and solutions u as

$$\mathcal{A}_{ad} = \{(u,\alpha) \in L^2(\Gamma(\alpha)) \times \mathcal{Q}_{ad} \text{ such that } \mathcal{J}(u,\alpha) < \infty, \text{where } (u,\alpha) \text{ is a solution of (2.8)}\} \cdot \tag{4.2}$$

The optimal shape design problem can be stated as follows:

Find $\alpha \in \mathcal{Q}_{ad}$ such that

$$\mathcal{J}(u,\alpha) \leq \mathcal{J}(u,\widetilde{\alpha}) \quad \forall\widetilde{\alpha} \in \mathcal{Q}_{ad}(I), \tag{4.3}$$

with $u \in L^2(\Gamma(\alpha))$ solution of

$$\int_{\Gamma(\alpha)} u(s)\,\phi(s)\,ds - \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s,s')\,u(s')ds'ds = \int_{\Gamma(\alpha)} f(s)\,\phi(s)ds \qquad \forall\phi \in L^2(\Gamma(\alpha)) \tag{4.4}$$

and $f \in L^2(\Gamma(\alpha))$.

In the rest of the paper we shall assume the source term of the form (3.3) which can be easily extended to n-point sources. Now we shall show the existence of at least one solution of the optimal shape design problem formulated in (4.3–4.4).

Theorem 4.1. *Let $U \in U_{ad}$ be given, then there exists at least one solution $(u,\alpha) \in \mathcal{A}_{ad}(I)$ of the optimal control problem.*

Proof. In order to prove this theorem we have to show that there exists a convergent minimising sequence and the corresponding limit satisfies the optimal shape control problem.

The admissible set of solutions $\mathcal{A}_{ad}$ is bounded and not empty which implies the existence of a minimising sequence $\{\alpha_m\} \in \mathcal{Q}_{ad}(I)$ and a corresponding sequence $\{u_m\}$. In fact the boundness of $\mathcal{A}_{ad}$ is obvious from the definition of the admissible set and the non-emptiness follows from the fact that the linear shape α between the points $(0, z_0)$ and (L, z_1) is in the set $\mathcal{Q}_{ad}$ and the corresponding solution $u(\alpha)$ in $L^2(\Gamma(\alpha))$.

Now we have to show that these sequences are uniformly bounded. The minimising sequence $\{\alpha_m\} \in \mathcal{Q}_{ad}(I)$ is clearly uniformly bounded in $H^2(I)$ when $\gamma > 0$. From (4.4) the norm of $\vec{u}$ can be estimated as

$$\|\vec{u}\|_{\Gamma(\alpha)} \leq C_1 \|\vec{f}\|_{\Gamma(\alpha)} \leq C_2 Q_{\rm tot}$$

for some constant C_1 and C_2 and any $\alpha \in \mathcal{Q}_{ad}$ where $Q_{\rm tot}$ is the total source intensity. Let J be an open bounded set in $\mathbb{R}$ and the extension of u be the function which is equal to u on J and zero otherwise [2, 13]. In the rest of the proof we use the same notation for the function and its extension. If the sequence $\{\alpha_n\}$ is in $C^{1,1}$ then the corresponding sequence of extended functions $\{u_n\}$ is uniformly bounded in $L^2(\mathbb{R})$.

As Hilbert spaces are reflexive, every ball is weakly compact and thus there exist a pair (u, α) and a subsequence (u_k, α_k) such that

$$\begin{array}{cccl} u_k & \longrightarrow & u & \text{weakly in } L^2(\mathbb{R}) \\ \alpha_k & \longrightarrow & \alpha & \text{weakly in } H^2(I). \end{array}$$

Since $\mathcal{Q}_{ad}$ is a convex closed set then $\mathcal{Q}_{ad}$ is also a weakly closed set and the limits α and u are in $\mathcal{Q}_{ad}$ and $L^2(\Gamma(\alpha))$, respectively (see [20]).

Now we have to show that the pair (u, α) solves (4.4) and minimises the functional $\mathcal{J}(u, \alpha)$. In order to show that (u, α) satisfies (4.4) we have to pass (u_k, α_k) to the limit in the equation. Let $\Gamma(\alpha)$ and u be the weak limits of the sequence $\Gamma(\alpha_k)$ and u_k respectively. We need to prove that

$$\lim_{k\to\infty} \Bigg(\int_{\Gamma(\alpha_k)} \varphi(s_k) \Big(u_k(s_k) - \int_{\Gamma(\alpha_k)} K(s_k, s'_k)\, u_k(s'_k)\, ds'_k - f(s_k) \Big) ds_k \\ - \int_{\Gamma(\alpha)} \varphi(s) \Big(u(s) - \int_{\Gamma(\alpha)} K(s, s')\, u(s')\, ds' - f(s) \Big) ds \Bigg) = 0.$$

We shall show that the limit is true for all $\varphi \in C_0^\infty(\mathbb{R})$ and then we claim the theorem by a continuity argument as the set $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ [20].

For the first term we have to prove that

$$\lim_{k\to\infty} \left(\int_{\Gamma(\alpha_k)} \varphi(s_k)\, u_k(s_k) ds_k - \int_{\Gamma(\alpha)} \varphi(s)\, u(s)\, ds \right) = 0.$$

Since the limit is clearly zero on the fix boundary $\Gamma(\alpha) - \Gamma_{\alpha_k}$ we can restrict the domain to Γ_{α_k}. By using the covariant coordinates of $\Gamma(\alpha)$ to describe $\Gamma(\alpha_k)$ we have

$$\left(\int_{\Gamma_{\alpha_k}} \varphi(s_k)\, u_k(s_k)\, ds_k - \int_{\Gamma_\alpha} \varphi(s) u(s)\, ds \right) = \left(\int_{\Gamma_{\alpha_k}} \varphi(s_k)\, u_k(s_k)\, ds_k - \int_{\Gamma_\alpha} \varphi(s)\, u_k(s)\, ds \right) \\ + \int_{\Gamma_\alpha} \varphi(s)\, (u_k(s) - u(s))\, ds.$$

Since $\alpha_k - \alpha$ converges in C^1 the first term of the right hand side converges to zero. From the weakly convergence of the extended u_k in $L^2(\mathbb{R})$ the second term on the right hand side converges to zero when $\alpha_k \to \alpha$. The theorem follows by treating all the other terms in a similar way.

Now it remains to show that (u, α) minimises the functional $\mathcal{J}(u, \alpha)$. In fact, by the convexity of $\mathcal{Q}_{ad}$ we have lower semi-continuity of the functional in (4.1) [20] and hence

$$\mathcal{J}(u, \alpha) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, \alpha_n).$$

□

4.2. Optimal shape design

In this subsection we shall find the first order necessity conditions for the optimal shape design in the case of pure radiation. Let Ω be an open bounded set in $\mathbb{R}^2$ with boundary Γ. As the function α defines the shape of Γ we write $\Gamma(\alpha)$. After deformation, the domain $\Omega(\alpha)$ takes a new shape $\Omega(\widetilde{\alpha})$ with boundary $\Gamma(\widetilde{\alpha})$ corresponding to the function $\widetilde{\alpha}$. The field, defined on $\Gamma(\alpha)$, transforming $\Gamma(\alpha)$ into $\Gamma(\widetilde{\alpha})$ is indicated with $\vec{V}$ and the corresponding variation with $\delta\alpha = \widetilde{\alpha} - \alpha$. In our specific case the vector $\vec{V}$ may be taken equal to $(0, \delta\alpha)$ in Γ_α and zero in the remaining part of the boundary. In this way we generate a family of boundaries parameterised by t as

$$\Gamma_{\alpha + t\delta\alpha} = \{\vec{x}_\alpha + t\,\vec{V}(x_\alpha) \,|\, \vec{x}_\alpha \in \Gamma_\alpha\}$$

for all $t \in [0, 1]$ and the corresponding family $\Gamma(\alpha_t) = \Gamma(\alpha + t\delta\alpha)$. Let X and Y be two Banach spaces then $u(\alpha) : B \subset X \to Y$ is said to have Gateaux derivative at $\alpha \in B$ if there exists a function $\widetilde{w} \in Y$ such that [18]

$$\lim_{t \to 0+} \frac{\|u_{\alpha_t}(\Gamma(\alpha_t)) - u_\alpha(\Gamma(\alpha)) - t\widetilde{w}(\Gamma(\alpha))\|_{Y(\Gamma(\alpha))}}{t} = 0. \tag{4.5}$$

Before proving Gateaux differentiability we need to prove the following fundamental introductory lemma. This lemma has been proved for our particular optimal control situation but a more general framework can be found in [21].

Lemma 4.2. *Let $\alpha, \widetilde{\alpha} \in \mathcal{Q}_{ad}(I)$, $\delta\alpha = \widetilde{\alpha} - \alpha$, $\alpha_t = \alpha + t\delta\alpha$ and $y(s)$ be in $L^2(\mathbb{R})$. Then we have*

$$\lim_{t \to 0+} \frac{1}{t} \left(\int_{\Gamma(\alpha_t)} y(s_t)\, ds_t - \int_{\Gamma(\alpha)} y(s)\, ds - t \int_{\Gamma_\alpha} y(s)\, \kappa(s)\, (\vec{V} \cdot \vec{n}(s))\, ds \right) = 0, \tag{4.6}$$

where $\vec{V}$ is the vector $(0, \delta\alpha)$, κ is the curvature and $\vec{n}$ is the unitary vector normal to Γ_α.

Proof. The result is obvious on the fix part of the boundary so that we can limit our considerations to the controlled boundary $\Gamma_\alpha = \{\vec{x}_\alpha = (x, z) \in \mathbb{R}^2 \,|\, z = \alpha(x)\}$. Since the set $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ we can prove the proposition for all $y \in C_0^\infty(\mathbb{R})$ and claim the theorem by a continuity argument.

In order to compute the integral in (4.6) we use a parameterisation over the fix domain I. As discussed in previous sections we have the global coordinate system (x, z) and the local or covariant coordinate system (s, n) on Γ_α. From the global coordinate system we can evaluate the components in the tangent and the normal directions by

$$\left| \begin{array}{c} s \\ n \end{array} \right| = \frac{1}{\sqrt{1 + \alpha'^2}} \left| \begin{array}{cc} 1 & \alpha' \\ \alpha' & -1 \end{array} \right| \left| \begin{array}{c} x \\ z \end{array} \right|.$$

For a variation $\delta\alpha$ we can write the family $\Gamma_{\alpha+t\delta\alpha}$ as $\{(x_t, z_t) \in \mathbb{R}^2 | x_t = x,\ z_t = \alpha(x) + t\delta\alpha(x)\}$. In order to prove (4.6) we have to show the following limit

$$\lim_{t\to 0} \frac{1}{t}\left(\int_{\Gamma_{\alpha_t}} y(s_t)\, ds_t - \int_{\Gamma_\alpha} y(s)\, ds\right) = \lim_{t\to 0} \frac{1}{t}\left(\int_{\Gamma_{\alpha_t}} y(s_t)\, ds_t - \int_{\Gamma_\alpha} y(s_t(s))\, ds\right) + \lim_{t\to 0} \frac{1}{t}\int_{\Gamma_\alpha} (y(s_t(s)) - y(s))\, ds$$
$$= \int_{\Gamma_\alpha} y(s)\,\kappa(s)\,(\vec{V}\cdot\vec{n})\, ds.$$

The determinant of the covariant matrix for the transformation from Γ_α to Γ_{α_t} takes the form $1 + (\alpha' + t\delta\alpha')^2$ and therefore the first term can be computed as

$$\lim_{t\to 0+} \frac{1}{t}\left(\int_{\Gamma_{\alpha_t}} y(s_t)\, ds_t - \int_{\Gamma_\alpha} y(s_t(s))\, ds\right) = \int_I y(s) \frac{\alpha'}{\sqrt{1+\alpha'^2}}\,\delta\alpha'\, dx. \tag{4.7}$$

If we note that $s = (x + \alpha' z)/\sqrt{1+\alpha'^2})$ so that $ds/dz = \alpha'/\sqrt{1+\alpha'^2})$ the second term can be written as

$$\lim_{t\to 0+} \frac{1}{t}\left(\int_{\Gamma_\alpha} y(s_t(s)) - y(s) ds\right) = \int_I \frac{\partial y(s)}{\partial s}\frac{\alpha'}{\sqrt{1+\alpha'^2}}\delta\alpha dx = \int_I -\left(y(s)\frac{\alpha''}{(1+\alpha'^2)^{3/2}}\delta\alpha + y(s)\frac{\alpha'}{\sqrt{1+\alpha'^2}}\delta\alpha'\right)dx$$
$$= -\int_I y(s)\frac{\alpha'}{\sqrt{1+\alpha'^2}}\delta\alpha' dx + \int_{\Gamma_\alpha} y(s)\kappa(s)(\vec{V}\cdot\vec{n})ds. \tag{4.8}$$

The theorem follows by adding (4.8) and (4.7). □

We shall show that the optimal control solution must satisfy a first-order necessity condition, which leads to a variational inequality. By studying this variational inequality, a possible candidate for the optimal control solution can be found. In order to find this necessity condition we need to prove differentiability.

Theorem 4.3. *Let α and $\widetilde{\alpha}$ be in $\mathcal{Q}_{ad}(I)$. The mapping*

$$u(\alpha) : \mathcal{Q}_{ad}(I) \to L^2(\Gamma(\alpha))$$

defined as solution of (4.4)*, has a Gateaux derivative $\widetilde{w} = (Du/D\alpha)\cdot\delta\alpha$ at α in the direction $\delta\alpha = \widetilde{\alpha} - \alpha$. Furthermore, $\widetilde{w} = (Du/D\alpha)\cdot\delta\alpha$ is the solution of the problem*

$$\int_{\Gamma(\alpha)} \widetilde{w}(s)\phi(s)\, ds - \int_{\Gamma(\alpha)} \phi(s)\int_{\Gamma(\alpha)} K(s,s')\,\widetilde{w}(s')\, ds' ds$$
$$= -\int_{\Gamma(\alpha)} \phi(s)\int_{\Gamma_\alpha} \kappa(s')\,(\vec{V}(\delta\alpha')\cdot\vec{n}(s'))\,K(s,s')\,u(s')\, ds'\, ds \qquad \forall\phi \in L^2(\Gamma), \tag{4.9}$$

where $\vec{V}(\delta\alpha)$ is the vector $(0,\delta\alpha)$, κ is the curvature and $\vec{n}$ the normal to Γ_α.

Proof. In this proof we use the same notation for a function and its canonical extension to $\mathbb{R}$ as already discussed in the previous section. Let α and $\widetilde{\alpha}$ be in $H^2(I)$ and $\alpha_t = \alpha + t\,\delta\alpha$. We need to prove the following result

$$\lim_{t\to 0+}\left(\frac{\|u_t - u - t\widetilde{w}\|_{L^2(\Gamma(\alpha))}}{t}\right) = 0. \tag{4.10}$$

The test functions can be defined on $\mathbb{R}$ and considered constant during the deformation of the domain. The function u is the solution of

$$\int_{\Gamma(\alpha)} u(s)\,\phi(s)\,ds - \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s,s')\,u(s')\,ds'\,ds - \int_{\Gamma(\alpha)} f(s)\phi(s)\,ds = 0$$

and $u_t(s_t)$ is solution of

$$\int_{\Gamma(\alpha_t)} u_t(s_t)\,\phi(s_t)\,ds_t - \int_{\Gamma(\alpha_t)} \phi(s_t) \int_{\Gamma(\alpha_t)} K(s_t,s'_t)\,u_t(s'_t)\,ds'_t ds_t - \int_{\Gamma(\alpha_t)} f(s_t)\phi(s_t)\,ds_t = 0.$$

We set $\widetilde{u} = (u_t - u - t\widetilde{w})/t$ so that $\widetilde{u}$ is the solution of the equation

$$\begin{aligned}
\int_{\Gamma(\alpha)} \widetilde{u}(s)\phi(s)\,ds \qquad & - \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s,s')\,\widetilde{u}(s')\,ds' ds \qquad \forall \phi \in L^2(\Gamma) \\
= \quad & \frac{1}{t}\left(\int_{\Gamma(\alpha_t)} \phi(s_t) \left(u_t(s_t) - \int_{\Gamma(\alpha_t)} K(s_t,s'_t)\,u_t(s'_t)\,ds'_t - f(s_t) \right) ds_t \right. \\
& - \int_{\Gamma(\alpha)} \phi(s) \left(u_t(s) - \int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t)\,ds'_t - f(s) \right) ds \\
& \left. - \int_{\Gamma(\alpha)} \phi(s) \left(\int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t)\,ds'_t - \int_{\Gamma(\alpha)} K(s,s')\,u_t(s')\,ds' \right) ds \right) \\
& + \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma_\alpha} \kappa(s')\,(\vec{V}(\delta\alpha') \cdot \vec{n}(s')) K(s,s')\,u(s')\,ds' ds.
\end{aligned}$$

In order to evaluate the right hand side we use the Lemma 4.2 and the estimate of the norm $\|u_t - u\|$ which is obtained by using similar techniques. Since the function

$$\phi(s) \left(u_t(s) - \int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t) ds'_t - f(s) \right)$$

is in $L^2(\Gamma(\alpha))$ then for every $\epsilon/2 > 0$ there exists a $\delta_1 > 0$ such that if $|t| < \delta_1$ we have

$$\begin{aligned}
& \left| \frac{1}{t}\left(\int_{\Gamma(\alpha_t)} \phi(s_t) \left(u_t(s_t) - \int_{\Gamma(\alpha_t)} K(s_t,s'_t)\,u_t(s'_t)\,ds'_t - f(s_t) \right) ds_t \right. \right. \\
& \qquad \left. - \int_{\Gamma(\alpha)} \phi(s) \left(u_t(s) - \int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t)\,ds'_t - f(s) \right) ds \right) \\
& \qquad \left. - \int_{\Gamma(\alpha)} \phi(s)\kappa(s)(\vec{V} \cdot \vec{n}(s)) \left(u_t(s) - \int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t)\,ds'_t - f(s) \right) ds \right| \le \frac{\epsilon}{2}
\end{aligned}$$

which implies

$$\begin{aligned}
& \left| \frac{1}{t} \int_{\Gamma(\alpha_t)} \phi(s_t) \left(u_t(s_t) - \int_{\Gamma(\alpha_t)} K(s_t,s'_t)\,u_t(s'_t)\,ds'_t - f(s_t) \right) ds_t \right. \\
& \qquad\qquad \left. - \frac{1}{t} \int_{\Gamma(\alpha)} \phi(s) \left(u_t(s) - \int_{\Gamma(\alpha_t)} K(s,s'_t)\,u_t(s'_t)\,ds'_t - f(s) \right) ds \right| \le \frac{\epsilon}{2}
\end{aligned}$$

since

$$\lim_{t\to 0}\int_{\Gamma(\alpha)}\phi(s)\kappa(s)(\vec{V}\cdot\vec{n}(s))\left(u_t(s)-\int_{\Gamma(\alpha_t)}K(s,s_t')\,u_t(s_t')\,ds_t'-f(s)\right)ds=0.$$

In a similar way for every $\epsilon/2>0$ there exists a $\delta_2>0$ such that, if $|t|<\delta_2$, we have

$$\left|\int_{\Gamma(\alpha)}\phi(s)\left(\int_{\Gamma(\alpha_t)}K(s,s_t')\,u_t(s_t')\,ds_t'-\int_{\Gamma(\alpha)}K(s,s')\,u_t(s')\,ds'\right)ds\right.$$
$$\left.-\int_{\Gamma_\alpha}\phi(s)\int_{\Gamma_\alpha}\kappa(s')\,(\vec{V}'\cdot\vec{n}(s'))K(s,s')u(s')\,ds'ds\right|\le\frac{\epsilon}{2}\cdot$$

This means that for $|t|<\delta=\min\{\delta_1,\delta_2\}$ and $\phi=\widetilde{u}$ we can write

$$\|\widetilde{u}\|^2\le\left|\int_{\Gamma(\alpha)}\widetilde{u}(s)\int_{\Gamma(\alpha)}K(s,s')\,\widetilde{u}(s')\,ds'\,ds\right|+\epsilon$$

and therefore

$$\left(1-\|K\|^2_{L^2(\Gamma(\alpha))\times L^2(\Gamma(\alpha))}\right)\|\widetilde{u}\|^2\le\epsilon.$$

The theorem follows as the above inequality is true for $\|K\|_{L^2(\Gamma(\alpha))\times L^2(\Gamma(\alpha))}<1$ and all $\epsilon>0$. □

Under the hypotheses of the Theorem 4.3 we have the existence of the Gateaux derivative of the map $u=u(\alpha)$. Now we can use the Gateaux derivative to compute the derivative of the functional $\mathcal{J}(u,\alpha)$.

Theorem 4.4. *The functional in (4.1) defines a mapping*

$$\mathcal{J}(u,\alpha):\mathcal{A}_{ad}(I)\to\mathbb{R},\tag{4.11}$$

which is Gateaux differentiable in the direction $\delta\alpha=\alpha-\widetilde{\alpha}$ *for all* $\widetilde{\alpha}\in\mathcal{Q}_{ad}$. *Furthermore we have*

$$\frac{D\mathcal{J}(u,\alpha)}{D\alpha}\cdot\delta\alpha=\int_{\Gamma_2}\widetilde{w}\,(u-U)\,ds+\gamma\int_I\frac{d^2\alpha}{dx^2}\,\frac{d^2\delta\alpha}{dx^2}\,dx.$$

Proof. The proof is obvious as the functional is defined on a fix boundary $\Gamma_2=\Gamma(\alpha)-\Gamma_\alpha$. □

We have shown that the functional is Gateaux differentiable and can be written as a function of $\widetilde{w}$. Now we can show that the optimal control problem implies a first-order necessary condition. From the definition of optimal solution and convexity of $\mathcal{Q}_{ad}(I)$ we have that

$$\mathcal{J}(u,\alpha+\lambda\delta\alpha)\ge\mathcal{J}(u,\alpha)$$

for every $\delta\alpha=\widetilde{\alpha}-\alpha$ such that $\widetilde{\alpha}\in\mathcal{Q}_{ad}(I)$, and for every $\lambda\in\mathbb{R}^+$. The above inequality implies

$$\frac{\mathcal{J}(u,\alpha+\lambda\delta\widetilde{\alpha})-\mathcal{J}(u,\alpha)}{\lambda}\ge 0\quad if\quad\lambda\ge 0.$$

The limit must be positive when λ tends to zero and this leads to the following first-order necessary condition.

Theorem 4.5. *If (u, α) is an optimal pair for the problem in (4.3–4.4) then the following variational inequality*

$$\mathcal{J}'(u, \widetilde{\alpha} - \alpha) \geq 0 \quad \forall \widetilde{\alpha} \in \mathcal{Q}_{ad} \tag{4.12}$$

must hold.

We would like to write the first-order necessary condition in a more explicit form. In order to do this we need this interesting preliminary result.

Lemma 4.6. *Given α and $\widetilde{\alpha}$ in $\mathcal{Q}_{ad}$ and $\delta\alpha = \widetilde{\alpha} - \alpha$. Let $\widetilde{w}$ be defined by* (4.9) *then, for every $\widetilde{h}_2$ in $L^2(\Gamma(\alpha))$, we have*

$$\int_{\Gamma(\alpha)} \widetilde{h}_2 \, \widetilde{w}(\delta\alpha) \, ds = \int_{\Gamma(\alpha)} w(s) \int_{\Gamma_\alpha} \kappa(s') \, (\vec{V}(\delta\alpha') \cdot \vec{n}(s')) \, K(s, s') \, u(s') \, ds' \, ds,$$

where $\vec{V} = (0, \delta\alpha)$ on Γ_α and w is the solution of the following adjoint linear problem

$$\int_{\Gamma(\alpha)} w(s) \, \phi(s) \, ds - \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s, s') \, w(s') \, ds' \, ds = - \int_{\Gamma(\alpha)} \widetilde{h}_2(s) \, \phi(s) \, ds \qquad \forall \phi \in L^2(\Gamma(\alpha)). \tag{4.13}$$

Proof. By using (4.13) with $\phi = \widetilde{w}$ we have

$$\int_{\Gamma(\alpha)} \widetilde{h}_2 \, \widetilde{w} \, ds = - \int_{\Gamma(\alpha)} w \, \widetilde{w} \, ds + \int_{\Gamma(\alpha)} \widetilde{w}(s) \int_{\Gamma(\alpha)} K(s, s') \, w(s') \, ds' \, ds. \tag{4.14}$$

The result follows easily from (4.9) with $\phi = w$ as

$$- \int_{\Gamma(\alpha)} w\widetilde{w} ds + \int_{\Gamma(\alpha)} \widetilde{w}(s) \int_{\Gamma(\alpha)} K(s, s') w(s') ds' ds = \int_{\Gamma(\alpha)} w \left(\int_{\Gamma_\alpha} \kappa(s') (\vec{V}(\delta\alpha') \cdot \vec{n}(s')) K(s, s') u(s') ds' \right) ds.$$

□

In the next theorem we write explicitly the first order necessity condition as a function of the adjoint variable.

Theorem 4.7. *If $(u, \alpha) \in \mathcal{A}_{ad}$ is optimal for the problem in* (4.3)*, then (u, α) is solution of*

$$\mathcal{J}'(u, \widetilde{\alpha} - \alpha)(w) \geq 0 \quad \forall \widetilde{\alpha} \in \mathcal{Q}_{ad}, \tag{4.15}$$

where the function $\mathcal{J}'(u, v)(w)$ is defined by

$$\mathcal{J}'(u, v)(w) = \gamma \int_I \frac{d^2\alpha}{dx^2} \frac{d^2 v}{dx^2} \, dx + \int_{\Gamma(\alpha)} w \int_{\Gamma_\alpha} \kappa(s') \, (\vec{V}(v) \cdot \vec{n}(s')) \, K(s, s') \, u'(s') \, ds' \, ds, \tag{4.16}$$

with boundary conditions $\alpha(0) = z_0$, $\alpha(L) = z_1$ and $\alpha''(0) = \alpha''(L) = 0$. The function w is the solution of the adjoint problem

$$\int_{\Gamma(\alpha)} w(s) \, \phi(s) \, ds - \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s, s') \, w(s') \, ds' ds = - \int_{\Gamma_2} (u(s) - U(s)) \phi(s) \, ds \qquad \forall \phi \in L^2(\Gamma(\alpha)) \tag{4.17}$$

and u solution of the radiation equation

$$\int_{\Gamma(\alpha)} u(s) \, \phi(s) \, ds = \int_{\Gamma(\alpha)} \phi(s) \int_{\Gamma(\alpha)} K(s, s') \, u(s') \, ds' ds + \int_{\Gamma(\alpha)} f(s) \, \phi(s) \, ds \qquad \forall \phi \in L^2(\Gamma(\alpha)). \tag{4.18}$$

Proof. Let (u, α) be an optimal solution of the problem (4.3–4.4). We compute the Gateaux derivative of the functional $\mathcal{J}(u, \alpha)$ in the direction $\delta\alpha$ from Theorem 4.4 and then Lemma 4.6 completes the proof. We have

$$\frac{D\mathcal{J}(u,\alpha)}{D\alpha} \cdot \delta\alpha = \int_{\Gamma_2} \widetilde{w}\,(u - U)\,ds + \gamma \int_I \frac{d^2\alpha}{dx^2}\frac{d^2\delta\alpha}{dx^2}\,dx.$$

Now, by using Lemma 4.6, we obtain

$$\mathcal{J}'(u,\alpha)\cdot\delta\alpha = \frac{D\mathcal{J}(u,\alpha)}{D\alpha}\cdot\delta\alpha = \gamma\int_I \frac{d^2\alpha}{dx^2}\frac{d^2\delta\alpha}{dx^2}\,dx + \int_{\Gamma(\alpha)} w(s)\int_{\Gamma_\alpha} \kappa(s')\,(\vec{V}(\delta\alpha')\cdot\vec{n}(s'))K(s,s')\,u'(s')\,ds'ds, \tag{4.19}$$

where w is the solution of (4.17). Now, the theorem follows from the Theorem 4.5. □

The system (4.15–4.18) is not linear and it must be solved iteratively. It is convenient to write (4.15–4.18) in a different form. We define two new variables q and μ by

$$q = \frac{d^2\alpha}{dx^2}$$

and

$$\int_I \mu\frac{d^2v}{dx^2}\,dx = \int_{\Gamma(\alpha)} w(s)\int_{\Gamma_\alpha}\kappa(s')\,(\vec{V}(v')\cdot\vec{n}(s'))K(s,s')\,u(s')\,ds'\,ds \quad \forall\, v\in H^1_0(I)\cap H^2(I)$$

with Dirichlet boundary conditions. With these variables in order to solve the optimal shape problem we have to solve, for (u, w, α, μ, q), the pure radiation equation

$$\int_{\Gamma(\alpha)} u(s)\,\phi(s)\,ds - \int_{\Gamma(\alpha)}\phi(s)\int_{\Gamma(\alpha)} K(s,s')\,u(s')\,ds'\,ds = \int_{\Gamma(\alpha)} f(s)\,\phi(s)ds \qquad \forall\phi\in L^2(\Gamma), \tag{4.20}$$

the adjoint equation

$$\int_{\Gamma(\alpha)} w(s)\,\phi(s)\,ds - \int_{\Gamma(\alpha)}\phi(s)\int_{\Gamma(\alpha)} K(s,s')\,w(s')\,ds'\,ds = -\int_{\Gamma_2}(u(s)-U(s))\,\phi(s)\,ds \qquad \forall\phi\in L^2(\Gamma), \tag{4.21}$$

the shape equation

$$-\int_I \frac{d\alpha}{dx}\frac{dv}{dx}\,dx = \int_I q\,v\,dx \quad \forall\, v\in H^1_0(I) \tag{4.22}$$

with $\alpha(0) = 0$ and $\alpha(L) = 0$, the adjoint shape equation

$$-\int_I \frac{d\mu}{dx}\frac{dv}{dx}\,dx = \int_{\Gamma(\alpha)} w(s)\int_{\Gamma_\alpha}\kappa(s')\,(\vec{V}(v)\cdot\vec{n}(s'))\,K(s,s')\,u(s')\,ds'\,ds, \tag{4.23}$$

for all $v \in H_0^1(I) \cap H^2(I)$ with Dirichlet boundary conditions and the variational equation

$$\int_I (\gamma q + \mu)\,(\widetilde{q} - q)dx \geq 0 \qquad \forall\,\widetilde{\alpha} \in \mathcal{Q}_{ad}(I), \tag{4.24}$$

where $\widetilde{q} = \widetilde{\alpha}''$.

From (4.22) we have that if $q \in L^2(I)$ then $\alpha \in H^2(I)$ and $\alpha \in C^1(\bar{I})$. Furthermore $\alpha \in \mathcal{Q}_{ad}$ and therefore we can conclude that $\alpha \in C^{1,1}(\bar{I})$. Also Γ_2 is supposed to be in $C^{1,1}(\bar{I})$ and thus $\Gamma \in C^{1,1}(\bar{I})$. This regularity is sufficient enough to guarantee $u \in L^2(\Gamma(\alpha))$. The formulation in (4.20–4.24) has some advantages from the numerical point of view. For example this mixed variable formulation suggests a straightforward projected gradient algorithm and the test functions in (4.23) have not to be necessarily in $H^2(I)$ (if the problem is regular enough) leading to a very simple numerical solution for α and μ.

Numerical solutions of the pure radiative boundary equation can be found by using a wide range of schemes [16, 23] and abstract *"a priori"* error estimates for finite elements approximations are available in literature (see for example [23]). A finite element approximation and some industrial applications based on the system (4.20–4.24) can be found in [14]. Since the curvature κ is not a smooth function then the integrand in (4.23) is not smooth and the numerical integration of the kernel may not be accurate. In order to have accurate results the integration over the basis functions (piecewice constant) may be done analytically. This presents many problems as the curvature and the normal to the curve α change during the optimisation and are not known in analytical form. In [14] we discretise the solution, perform the analytical integration of the kernel and then find the first order necessity conditions directly on the discrete system.

Also in [14] we propose the following projected gradient algorithm for the solution of (4.20–4.24). Let $\mathcal{J}(k)$ be $\mathcal{J}(u, q(k))$, where $\mathcal{J}(\cdot)$ is given by (4.1) and k be the iteration counter of the gradient algorithm. In the algorithm τ will denote a prescribed tolerance used to test for the convergence of the functional. The *projected gradient algorithm* proceeds as follows:

a) initialisation:
   i) choose τ and $\alpha(0)$; set $k = 0$ and $\rho_0 = 1$;
   ii) solve for the starting field $u(0)$ from (4.20) with $\alpha = \alpha(0)$;
   iii) evaluate $\mathcal{J}(0)$;
b) main loop:
   iv) set $k = k + 1$;
   v) solve for $w(k)$ from (4.21) with $u = u(k-1)$ and $\alpha = \alpha(k-1)$;
   vi) solve for $\mu(k)$ from (4.23) with $u = u(k-1)$, $w = w(k)$ and $\alpha = \alpha(k-1)$;
   vii) set $q(k) = P_L\Big(q(k-1) + \rho_k \mu(k)/\gamma\Big)$ and solve for $\alpha(k)$ from (4.22);
   viii) solve for $u(k)$ from (4.20) with $\alpha = \alpha(k)$;
   ix) evaluate $\mathcal{J}(k)$;
   x) if $\mathcal{J}(k) \geq \mathcal{J}(k-1)$, set $\rho_{k-1} = .5\rho_{k-1}$ and go to vii); otherwise, continue;
   xi) if $|\mathcal{J}(k) - \mathcal{J}(k-1)|/|\mathcal{J}(k)| > \tau$, set $\rho_k = 1.3\rho_{k-1}$ and go to v); otherwise, stop.

In the above algorithm the operator P_L is the projection operator over the set $\mathcal{Q}_{ad}$. This is a very straightforward approach and other more sophisticated methods can be devised for the solution of the system (4.20–4.23). One can use also sensitivity methods. The sensitivity of the map $u = u(\alpha)$ is given by (4.9) and can be used to know how the first derivative of the functional behaves in some particular directions. As result of numerical computations in [14] we can summarise that the prescribed algorithm converges fast and produces good results depending on initial data like source power, maximum admissible curvature and initial surface design.

5. Conclusions

In this paper we have presented some optimal control problems based on the heat radiative transfer equation. The problems are formulated in such a way that the existence of solutions can be proved and consistent first order necessary conditions can be found in form of variational inequalities. The problems described in this paper are basic and fundamental in the design of a desired environment where radiation is the predominant form of heat exchange. The optimal control theory is a very flexible tool for industrial applications. Constraints arising from real situations can be easily included in the formulation and different objective functionals can be used.

The authors would like to thank Prof. H. Neunzert and Dr. N. Siedow for their help and support.

References

[1] F. Abergel and R. Temam, On some control problems in fluid mechanics. *Theoret. Computational Fluid Dynamics* **1** (1990) 303-326.
[2] R. Adams, *Sobolev Spaces*. Academic Press, New York (1975).
[3] V. Alekseev, V. Tikhomirov and S. Fomin, *Optimal Control*. Consultants Bureau, New York (1987).
[4] I. Babuska, The finite element method with Lagrangian multipliers. *Numer. Math.* **16** (1973) 179-192.
[5] D.M. Bedivan and G.J. Fix, An extension theorem for the space $H_{\rm div}$. *Appl. Math. Lett.* (to appear).
[6] N. Di Cesare, O. Pironneau and E. Polak, *Consistent approximations for an optimal design problem*. Report 98005 Labotatoire d'analyse numérique, Paris, France (1998).
[7] P. Ciarlet, *Introduction to Numerical Linear Algebra and Optimization*. Cambridge University, Cambridge (1989).
[8] P. Ciarlet, *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam (1978).
[9] J.E. Dennis and R.B. Schnabel, *Numerical methods for unconstrained optimisation and non-linear equations*. Prentice-Hall Inc., New Jersey (1983).
[10] V. Girault and P. Raviart, *The Finite Element Method for Navier-Stokes Equations: Theory and Algorithms*. Springer-Verlag, New York (1986).
[11] M. Gunzburger and S. Manservisi, *Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with distributed control. SIAM J. Numer. Anal.* (to appear).
[12] M. Gunzburger and S. Manservisi, The velocity tracking problem for for Navier-Stokes flows with bounded distributed control. *SIAM J. Control Optim.* (to appear).
[13] J. Haslinger and P. Neittaanmäki, *Finite Element Approximation for Optimal Shape Design*. Wiley, Chichester (1988).
[14] K. Heusermann and S. Manservisi, Optimal design for heat radiative transfer systems. *Comput. Methods Appl. Mech. Engrg.* (to appear).
[15] F.P. Incropera and D.P. DeWitt, *Fundamentals of Heat and Mass Transfer*. Wiley, New York (1990).
[16] M. Modest, *Radiative heat transfer*. McGraw-Hill, New York (1993).
[17] O. Pironneau, *Optimal shape design in fluid mechanics*. Thesis, University of Paris (1976).
[18] O. Pironneau, *On optimal design in fluid mechanics. J. Fluid. Mech.* **64** (1974) 97-110.
[19] O. Pironneau, *Optimal shape design for elliptic systems*. Springer, Berlin (1984).
[20] R.E. Showalter, Hilbert Space Methods for Partial Differential Equations. *Electron. J. Differential Equations* (1994) `http://ejde.math.swt.edu/mono-toc.html`
[21] J. Sokolowski and J. Zolesio, *Introduction to shape optimisation: Shape sensitivity analysis*. Springer, Berlin (1992).
[22] T. Tiihonen, Stefan-Boltzmann radiation on Non-convex Surfaces. *Math. Methods Appl. Sci.* **20** (1997) 47-57.
[23] T. Tiihonen, *Finite Element Approximations for a Heat Radiation Problem*. Report 7/1995, Dept. of Mathematics, University of Jyväskylä (1995).
[24] V. Tikhomirov, *Fundamental Principles of the Theory of Extremal Problems*. Wiley, Chichester (1986).