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A-QUASICONVEXITY: RELAXATION AND HOMOGENIZATION

Andrea Braides[1], Irene Fonseca[2] and Giovanni Leoni [3]

Abstract. Integral representation of relaxed energies and of Γ-limits of functionals

$$(u,v) \mapsto \int_\Omega f(x,u(x),v(x))\,dx$$

are obtained when sequences of fields v may develop oscillations and are constrained to satisfy a system of first order linear partial differential equations. This framework includes the treatement of divergence-free fields, Maxwell's equations in micromagnetics, and curl-free fields. In the latter case classical relaxation theorems in $W^{1,p}$ are recovered.

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1. Introduction

In a recent paper Fonseca and Müller [22] have proved that $\mathcal{A}$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$(u,v) \mapsto \int_\Omega f(x,u(x),v(x))\,dx,$$

whenever $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0,\infty)$ is a Carathéodory integrand satisfying

$$0 \le f(x,u,v) \le a(x,u)\left(1+|v|^q\right),$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^m \times \mathbb{R}^d$, where $1 \le q < \infty$, $a \in L^\infty_{\rm loc}(\Omega \times \mathbb{R}; [0,\infty))$, $\Omega \subset \mathbb{R}^N$ is open, bounded, $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^q(\Omega;\mathbb{R}^d)$ and $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega;\mathbb{R}^l)$ (see also [14]). Here, and in what follows, following [32]

$$\mathcal{A} : L^q(\Omega;\mathbb{R}^d) \to W^{-1,q}(\Omega;\mathbb{R}^l), \quad \mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i},$$

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[1] SISSA, Trieste, Italy; e-mail: `braides@sissa.it`

[2] Department of MathematicalSciences, Carnegie-Mellon University, Pittsburgh, PA, U.S.A.; e-mail: `fonseca@andrew.cmu.edu`

[3] Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Alessandria, Italy;
e-mail: `leoni@al.unipmn.it`

is a constant–rank, first order linear partial differential operator, with $A^{(i)} : \mathbb{R}^d \to \mathbb{R}^l$ linear transformations, $i = 1, \dots, N$. We recall that $\mathcal{A}$ satisfies the *constant-rank* property if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}w = r \quad \text{for all } w \in S^{N-1}, \tag{1.1}$$

where

$$\mathbb{A}w := \sum_{i=1}^{N} w_i\, A^{(i)}, \qquad w \in \mathbb{R}^N.$$

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(v) \leq \int_Q f(v + w(y))\, dy$$

for all $v \in \mathbb{R}^d$ and all $w \in C^\infty_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$. Here Q denotes the unit cube in $\mathbb{R}^N$, and the space $C^\infty_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^d)$ is introduced in Section 2.

The relevance of this general framework, as emphasized by Tartar (see [32, 34–39]), lies on the fact that in continuum mechanics and electromagnetism PDEs other than $\operatorname{curl} v = 0$ arise naturally, and this calls for a relaxation theory which encompasses PDE constraints of the type $\mathcal{A}v = 0$. Some important examples included in this general setting are given by:

(a) [Unconstrained Fields]

$$\mathcal{A}v \equiv 0.$$

Here, due to Jensen's inequality A-quasiconvexity reduces to convexity.

(b) [Divergence Free Fields]

$$\mathcal{A}v = 0 \quad \text{if and only if } \operatorname{div} v = 0,$$

where $v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ (see [33]).

(c) [Magnetostatics Equations]

$$\mathcal{A}\begin{pmatrix} m \\ h \end{pmatrix} := \begin{pmatrix} \operatorname{div}(m+h) \\ \operatorname{curl} h \end{pmatrix} = 0,$$

where $m : \mathbb{R}^3 \to \mathbb{R}^3$ is the *magnetization* and $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the *induced magnetic field* (see [17,38]); often these are also called Maxwell's Equations in the micromagnetics literature.

(d) [Gradients]

$$\mathcal{A}v = 0 \quad \text{if and only if } \operatorname{curl} v = 0.$$

Note that $w \in C^\infty_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^d)$ is such that curl $w = 0$ and $\int_Q w(y)\, dy = 0$ if and only if there exists $\varphi \in C^\infty_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^n)$ such that $\nabla\varphi = v$, where $d = n \times N$. Thus in this case we recover the well-known notion of *quasiconvexity* introduced by Morrey [30].

(e) [Higher Order Gradients]

Replacing the target space $\mathbb{R}^d$ by an appropriate finite dimensional vector space E^n_s, it is possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in L^p(\Omega; E^n_s)$ and $\mathcal{A}v = 0$ if and only if there exists $\varphi \in W^{s,q}(\Omega; \mathbb{R}^n)$ such that $v = \nabla^s \varphi$ (see Th. 1.3).

This paper is divided into two parts. In the first part we give an integral representation formula for the relaxed energy in the context of $\mathcal{A}$–quasiconvexity. Precisely, let $1 \leq p < \infty$ and $1 < q < \infty$, and consider the functional

$$F : L^p(\Omega; \mathbb{R}^m) \times L^q(\Omega; \mathbb{R}^d) \times \mathcal{O}(\Omega) \to [0, \infty)$$

defined by

$$F((u,v); D) := \int_D f(x, u(x), v(x))\, dx,$$

where $\mathcal{O}(\Omega)$ is the collection of all open subsets of Ω, and the density f satisfies the following hypothesis:

(H) $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty)$ is Carathéodory function satisfying

$$0 \le f(x, u, v) \le C\,(1 + |u|^p + |v|^q)$$

for a.e. $x \in \Omega$ and all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^d$, and where $C > 0$.

For $D \in \mathcal{O}(\Omega)$ and $(u, v) \in L^p(\Omega; \mathbb{R}^m) \times \big(L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}\big)$ define

$$\begin{aligned}\mathcal{F}((u, v); D) := \inf\Big\{ &\liminf_{n\to\infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d),\\ &u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d), \quad \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,q}(D; \mathbb{R}^l) \Big\}.\end{aligned} \tag{1.2}$$

It turns out that the condition $\mathcal{A}v_n \to 0$ imposed in (1.2) may be replaced by requiring that v_n do satisfy the homogeneous PDE $\mathcal{A}v = 0$. Precisely, and in view of Lemma 3.1 and Corollary 3.2 below, it can be shown that

$$\mathcal{F}((u, v); D) = \inf\left\{ \liminf_{n\to\infty} F((u, v_n); D) : v_n \in L^q(D; \mathbb{R}^d), v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d), \mathcal{A}v_n = 0 \right\},$$

and thus

$$\begin{aligned}\mathcal{F}((u, v); D) = \inf\Big\{ &\liminf_{n\to\infty} F((u_n, v_n); D) : (u_n, v_n) \in L^p(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d),\\ &u_n \to u \text{ in } L^p(D; \mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d), \quad \mathcal{A}v_n = 0 \Big\} =: \mathcal{F}_0((u, v); D).\end{aligned} \tag{1.3}$$

The first main result of the paper is given by the following theorem:

Theorem 1.1. *Under condition (H) and the constant-rank hypothesis* (1.1)*, for all $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega; \mathbb{R}^m)$, and $v \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$, we have*

$$\mathcal{F}((u, v); D) = \int_D \mathcal{Q}_{\mathcal{A}} f(x, u(x), v(x))\, dx$$

where, for each fixed $(x, u) \in \Omega \times \mathbb{R}^m$, the function $\mathcal{Q}_{\mathcal{A}} f(x, u, \cdot)$ is the $\mathcal{A}$-quasiconvexification of $f(x, u, \cdot)$, namely

$$\mathcal{Q}_{\mathcal{A}} f(x, u, v) := \inf\left\{ \int_Q f(x, u, v + w(y))\, dy : w \in C^\infty_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A},\ \int_Q w(y)\, dy = 0 \right\}$$

for all $v \in \mathbb{R}^d$.

Remarks 1.2. (i) Note that in the degenerate case where $\mathcal{A} = 0$, $\mathcal{A}$-quasiconvex functions are convex and Theorem 1.1 together with condition (1.4) yield a convex relaxation result with respect to $L^p \times L^q$(weak) convergence. See the monograph of Buttazzo [12] for related results in this context.

(ii) If the function f also satisfies a growth condition of order q from below in the variable v, that is

$$f(x, u, v) \ge \frac{1}{C}|v|^q - C \tag{1.4}$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^m \times \mathbb{R}^d$, then a simple diagonalization argument shows that $(u,v) \mapsto \mathcal{F}((u,v);D)$ is $L^p \times (L^q\text{-weak})$ lower semicontinuous, *i.e.*,

$$\int_D \mathcal{Q}_{\mathcal{A}} f(x,u(x),v(x))\,dx \leq \liminf_{n\to\infty} \int_D \mathcal{Q}_{\mathcal{A}} f(x,u_n(x),v_n(x))\,dx \tag{1.5}$$

whenever $u_n \in L^p(\Omega;\mathbb{R}^m)$, $v_n \in L^q(\Omega;\mathbb{R}^d) \cap \ker\mathcal{A}$, $u_n \to u$ in $L^p(D;\mathbb{R}^m)$, $v_n \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$. In particular $\mathcal{Q}_{\mathcal{A}} f$ is $\mathcal{A}$-quasiconvex if f is continuous and

$$\frac{1}{C}|v|^q - C \leq f(v) \leq C(1+|v|^q)$$

for some $C>0$, and all $v \in \mathbb{R}^d$ (see the proof of Cor. 5.7).

The lower semicontinuity result (1.5) is not covered by Theorem 3.7 in [22], where it is assumed that the integrand be $\mathcal{A}$-quasiconvex and *continuous* in the v variable. However, as remarked in [22], in the realm of general $\mathcal{A}$-quasiconvexity the function $\mathcal{Q}_{\mathcal{A}} f(x,u,\cdot)$ may not be continuous, even if $f(x,u,\cdot)$ is. Indeed in the degenerate case $\ker\mathcal{A} = \{0\}$ all functions are $\mathcal{A}$-quasiconvex. Also, when $N=1$, $d=2$, and $v=(v_1,v_2)$, consider

$$\mathcal{A}v := \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}.$$

Then for $w \in \mathbb{R}$

$$\mathbb{A}w = \begin{pmatrix} 0 & w \end{pmatrix}$$

and thus when $|w|=1$ the matrix $\mathbb{A}w$ has constant rank 1. For any given function $f(v)$ the $\mathcal{A}$-quasiconvex envelope of f is obtained by convexification in the first component, so that by considering *e.g.* (*cf.* [22,28])

$$f_1(v) := e^{-|v_1|\,v_2^2}, \qquad f_2(v) := (1+|v_1|)^{|v_2|},$$

one gets

$$\mathcal{Q}_{\mathcal{A}} f_1(v) = \begin{cases} 0 & \text{if } v_2 \neq 0 \\ 1 & \text{if } v_2 = 0 \end{cases}, \qquad \mathcal{Q}_{\mathcal{A}} f_1(v) = \begin{cases} (1+|v_1|)^{|v_2|} & \text{if } |v_2| \geq 1 \\ 1 & \text{if } |v_2| < 1. \end{cases}$$

(iii) The continuity of f with respect to v is essential to ensure the representation of $\mathcal{F}$ provided in Theorem 1.1, in contrast with the case where $\mathcal{A}v = 0$ if and only if $\operatorname{curl} v = 0$. In fact, if $f : \mathbb{R}^{n\times N} \to [0,\infty)$ is a Borel function satisfying the growth condition

$$0 \leq f(v) \leq C(1+|v|^q)$$

for $C>0$, $1 \leq q < \infty$, $v \in \mathbb{R}^{n\times N}$, then it can be shown easily that

$$\mathcal{F}(w;D) = \int_D \mathcal{Q}f(\nabla w(x))\,dx \tag{1.6}$$

for all $D \in \mathcal{O}(\Omega)$, $w \in W^{1,q}(\Omega;\mathbb{R}^n)$, where $\mathcal{Q}f$ is the quasiconvex envelope of f. Indeed, Qf is a (continuous) quasiconvex function satisfying (H) (see [18], [8] Th. 4.3); therefore by Theorem 1.1

$$w \mapsto \int_D Qf(\nabla w(x))\,dx$$

is $W^{1,q}$-sequentially weakly lower semicontinuous, and so

$$\int_D Qf(\nabla w(x))\,dx \leq \mathcal{F}(w;D).$$

Conversely, under hypothesis (H) it is known that $\mathcal{F}(v;\cdot)$ admits an integral representation (see Th. 9.1 in [10], Th. 20.1 in [15])

$$\mathcal{F}(w;D)=\int_D \varphi(\nabla w(x))\,dx,$$

where φ is a quasiconvex function, and $\varphi(v)\leq f(v)$ for all $v\in\mathbb{R}^{n\times N}$. Hence $\varphi\leq Qf$ and we conclude that (1.6) holds.

For general constant-rank operators $\mathcal{A}$, and if f is not continuous with respect to v, it may happen that $\mathcal{F}_0((u,v);\cdot)$ is not even the trace of a Radon measure in $\mathcal{O}(\Omega)$ and thus (1.3) fails. As an example, consider $d=2$, $N=1$, $\Omega:=(0,1)$, $v=(v_1,v_2)$, and let $\mathcal{A}(v)=0$ if and only if $v_2'=0$ as in (ii) above. Let

$$f(v):=\begin{cases}(v_1-1)^2+v_2^2, & \text{if } v_2\in\mathbb{Q}\\ (v_1+1)^2+v_2^2, & \text{if } v_2\notin\mathbb{Q}.\end{cases}$$

Although f satisfies a quadratic growth condition of the type (H), and (A_3) holds with $q=2$, it is easy to see that for all intervals $(a,b)\subset(0,1)$,

$$\mathcal{F}_0((u,v);(a,b))=\mathcal{F}_0(v;(a,b))=\min\left\{\int_a^b((v_1-1)^2+v_2^2)dx,\int_a^b((v_1+1)^2+v_2^2)dx\right\}$$

which is not the trace of a Radon measure on $\mathcal{O}(\Omega)$. On the other hand, it may be shown that (see the Appendix below for a proof)

$$\mathcal{F}((u,v);(a,b))=\mathcal{F}(v;(a,b))=\int_a^b(\psi^{**}(v_1)+v_2^2)\,dx,$$

where $\psi^{**}(v_1)$ is the convex envelope of

$$\psi(v_1):=\min\left\{(v_1-1)^2,(v_1+1)^2\right\}\cdot$$

(iv) Using the growth condition (H), a mollification argument, and the linearity of $\mathcal{A}$, it can be shown that (see Rem. 3.3 in [22])

$$\mathcal{Q}_{\mathcal{A}}f(x,u,v)=\inf\left\{\int_Q f(x,u,v+w(y))\,dy : w\in L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d)\cap\ker\mathcal{A},\quad \int_Q w(y)\,dy=0\right\}\cdot$$

We write $w\in L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d)\cap\ker\mathcal{A}$ when $w\in L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d)$ and $\mathcal{A}w=0$ in $W^{-1,q}(Q;\mathbb{R}^l)$.
(v) We may also treat the cases $q=1$, ∞ and $p=\infty$. See Theorem 3.6 below.

The proof of Theorem 1.1 relies heavily on the use of Young measures (see [5, 40]). However, instead of applying directly the arguments of Fonseca and Müller [22] (based on Balder's [4] and Kristensen's [26] approach in the curl–free case), we use these together with the blow-up method introduced by Fonseca and Müller in [20].

Although in Theorem 1.1 the functions u and v are not related to each other, the arguments of the proof work equally well when u and v are not independent. Indeed as a corollary, we can prove the following two theorems:

Theorem 1.3. *Let $1\leq p\leq\infty$, $s\in\mathbb{N}$, and suppose that $f:\Omega\times E^n_{[s-1]}\times E^n_s\to[0,\infty)$ is a Carathéodory function satisfying*

$$0\leq f(x,\mathbf{u},v)\leq C\left(1+|\mathbf{u}|^p+|v|^p\right),\qquad 1\leq p<\infty,$$

for a.e. $x\in\Omega$ and all $(\mathbf{u},v)\in E^n_{[s-1]}\times E^n_s$, where $C>0$, and

$$\chi_\Omega f\in L^\infty_{\rm loc}(\mathbb{R}^N\times E^n_{[s-1]}\times E^n_s;[0,\infty))\qquad \textit{if } p=\infty.$$

Then for every $u \in W^{s,p}(\Omega;\mathbb{R}^n)$ *we have*

$$\int_\Omega \mathcal{Q}^s f(x,u,\dots,\nabla^s u)\,dx = \inf\left\{ \liminf_{k\to\infty}\int_\Omega f(x,u_k,\dots,\nabla^s u_k)\,dx : \{u_k\}\subset W^{s,p}(\Omega;\mathbb{R}^n),\right.$$
$$\left. u_k \rightharpoonup u \text{ in } W^{s,p}(\Omega;\mathbb{R}^n) \quad (\overset{\star}{\rightharpoonup} \text{ if } p=\infty)\right\},$$

where, for a.e. $x\in\Omega$ *and all* $(\mathbf{u},v)\in E^n_{[s-1]}\times E^n_s$,

$$\mathcal{Q}^s f(x,\mathbf{u},v) := \inf\left\{ \int_Q f(x,\mathbf{u},v+\nabla^s w(y))\,dy : w\in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^n)\right\}.$$

Remarks 1.4. (i) Here E^n_s stands for the space of n-tuples of symmetric s-linear maps on $\mathbb{R}^{\mathbb{N}}$,

$$E^n_{[s-1]} := \mathbb{R}^n\times E^n_1\times\cdots\times E^n_{s-1},$$

and

$$\nabla^l u := \left(\frac{\partial^l u}{\partial x_1^{\alpha_1}\dots\partial x_N^{\alpha_N}}\right)_{\alpha_1+\ldots+\alpha_N=l}, \quad l\geq 1.$$

(ii) When $s=1$ we recover classical relaxation results (see *e.g.* the work of Acerbi and Fusco [1], Dacorogna [13], Marcellini and Sbordone [28] and the references contained therein).

When $s>1$ lower semicontinuity results related to Theorem 1.3 are due to Meyers [29], Fusco [23] and Guidorzi and Poggiolini [25], while we are not aware of any integral representation formula for the relaxed energy, when the integrand depends on the full set of variables, that is $f=f(x,u,\dots,\nabla^s u)$. This is due to the fact that classical truncation methods for $s=1$ cannot be extended in a simple way to truncate higher order derivatives. The results of Fonseca and Müller (see the proof of Lem. 2.15 in [22]), where the truncation is only on the highest order derivative $\nabla^s u$, and Corollary 3.2 below, allows us to overcome this difficulty. Note however that this technique relies heavily on p-equi-integrability, and thus cannot work in the case $p=1$, if one replaces weak convergence in $W^{s,1}(\Omega;\mathbb{R}^n)$ with the natural convergence, which is strong convergence in $W^{s-1,1}(\Omega;\mathbb{R}^n)$. In this context, a relaxation result has been given by Amar and De Cicco [2], but only when $f=f(\nabla^s u)$, so that truncation is not needed. The general case where f depends also on lower order derivatives has been addressed by Fonseca *et al.* [19].

Theorem 1.5. *Let* $1\leq p\leq\infty$, *let* $\Omega\subset\mathbb{R}^N$ *be an open, bounded, connected set, and suppose that* $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{N^2}\to[0,\infty)$ *is a Carathéodory function satisfying*

$$0\leq f(x,u,v)\leq C\left(1+|u|^p+|v|^p\right),\qquad 1\leq p<\infty,$$

for a.e. $x\in\Omega$ *and all* $(u,v)\in\mathbb{R}^N\times\mathbb{R}^{N^2}$, *where* $C>0$, *and*

$$\chi_\Omega f\in L^\infty_{\text{loc}}(\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{N^2};[0,\infty))\qquad \text{if } p=\infty.$$

Then for every $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ *such that* $\operatorname{div} u=0$, *we have*

$$\int_\Omega \bar f(x,u(x),\nabla u(x))\,dx = \inf\left\{\liminf_{n\to\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx : \{u_n\}\subset W^{1,p}(\Omega;\mathbb{R}^N),\right.$$
$$\left.\operatorname{div} u_n=0,\quad u_n\rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)\quad(\overset{\star}{\rightharpoonup} \text{ if } p=\infty)\right\}, \tag{1.7}$$

where, for a.e. $x \in \Omega$ *and all* $(u, v) \in \mathbb{R}^N \times \mathbb{R}^{N^2}$,

$$\bar{f}(x, u, v) := \inf \left\{ \int_Q f(x, u, v + \nabla w(y))\, dy : w \in C^\infty_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^N),\ \operatorname{div} w = 0 \right\}.$$

Remark 1.6. To the authors' knowledge, this result is new in this generality (for a different proof, with additional smoothness assumptions, see [9]). A related problem was addressed by Dal Maso *et al.* in [16], where it was shown that the Γ-limit of a family of functionals of the type (1.7) may be non local if (H) is violated.

In the second part of the paper we present (Γ-convergence) homogenization results for periodic integrands in the context of $\mathcal{A}$-quasiconvexity. Let $\varepsilon > 0$ and $1 < q < \infty$, and consider a family of functionals

$$\mathcal{F}_\varepsilon : \left(L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}\right) \times \mathcal{O}(\Omega) \to [0, \infty)$$

defined by

$$\mathcal{F}_\varepsilon(v; D) := \int_D f\left(\frac{x}{\varepsilon}, v(x)\right) dx,$$

where the density f satisfies the following hypotheses:

(A_1) $f : \mathbb{R}^N \times \mathbb{R}^d \to [0, \infty)$ is a continuous function, Q-periodic in the first argument, that is $f(x+e_i, v) = f(x, v)$ for every $i = 1, \ldots, N$, where e_i are the elements of the canonical basis of $\mathbb{R}^N$;

(A_2) there exists $C > 0$ such that

$$0 \leq f(x, v) \leq C(1 + |v|^q)$$

for all $(x, v) \in \mathbb{R}^N \times \mathbb{R}^d$;

(A_3) there exists $C > 0$ such that

$$f(x, v) \geq \frac{1}{C}|v|^q - C$$

for all $(x, v) \in \mathbb{R}^N \times \mathbb{R}^d$.

Let $\varepsilon_n \to 0^+$. We say that a functional

$$\mathcal{J} : \left(L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}\right) \times \mathcal{O}(\Omega) \to [0, +\infty]$$

is the $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) of the sequence of functionals $\{\mathcal{F}_{\varepsilon_n}\}$ with respect to the weak convergence in $L^q(\Omega; \mathbb{R}^d)$ if for every $v \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$

$$\mathcal{J}(v; \Omega) = \inf \Big\{ \liminf_{n\to\infty} \text{ (resp. } \limsup_{n\to\infty}) \, \mathcal{F}_{\varepsilon_n}(v_n; \Omega) : \ v_n \in L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}, \quad v_n \rightharpoonup v \text{ in } L^q(\Omega; \mathbb{R}^d) \Big\}, \tag{1.8}$$

and we write

$$\mathcal{J} = \Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n} \left(\text{resp. } \mathcal{J} = \Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n} \right).$$

When finite energy sequences are L^q-equibounded then the infimum in the definition of $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) is attained. We say that the sequence $\{\mathcal{F}_{\varepsilon_n}\}$ Γ-*converges* to $\mathcal{J}$ if the $\Gamma - \liminf$ and $\Gamma - \limsup$ coincide, and we write

$$\mathcal{J} = \Gamma - \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}.$$

The functional $\mathcal{J}$ is said to be the $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) of the *family* of functionals $\{\mathcal{F}_\varepsilon\}$ with respect to the weak convergence in $L^q(\Omega;\mathbb{R}^d)$ if for *every* sequence $\varepsilon_n \to 0^+$ we have that

$$\mathcal{J} = \Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n} \quad \left(\text{resp. } \mathcal{J} = \Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}\right),$$

and we write

$$\mathcal{J} = \Gamma - \liminf_{\varepsilon\to 0} \mathcal{F}_{\varepsilon} \quad \left(\text{resp. } \mathcal{J} = \Gamma - \limsup_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}\right).$$

Finally, we say that $\mathcal{J}$ is the is the Γ-*limit* of the *family* of functionals $\{\mathcal{F}_\varepsilon\}$, and we write

$$\mathcal{J} = \Gamma - \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n},$$

if $\Gamma - \liminf$ and $\Gamma - \limsup$ coincide.

In the sequel we will also consider functionals $\mathcal{J}$ given by (1.8) where we replace the weak convergence $v_n \rightharpoonup v$ with the convergence $v_n \to v$ with respect to some metric d. In order to highlight this dependence on the metric d these functionals will be denoted as

$$\mathcal{J} = \Gamma(d) - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n} \quad \left(\text{resp. } \mathcal{J} = \Gamma(d) - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}\right),$$

as it is customary (see [10, 15]).

Theorem 1.7. *Under hypotheses* $(A_1) - (A_2)$ *and the constant-rank hypothesis* (1.1),

$$\mathcal{F}_{\rm hom} = \Gamma - \liminf_{\varepsilon\to 0} \mathcal{F}_{\varepsilon},$$

where

$$\mathcal{F}_{\rm hom}(v; D) := \int_D f_{\rm hom}(v)\, dx$$

for all $v \in L^q(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ *and* $D \in \mathcal{O}(\Omega)$, *and*

$$f_{\rm hom}(v) := \inf_{k\in\mathbb{N}} \frac{1}{k^N} \inf \left\{ \int_{kQ} f(x, v + w(x))\, dx : \quad w \in L^q_{k\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}, \quad \int_{kQ} w(x)\, dx = 0 \right\} \tag{1.9}$$

for all $v \in \mathbb{R}^d$. *Moreover, if* (A_3) *holds then*

$$\mathcal{F}_{\rm hom} = \Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon}.$$

For the definition of the space $L^q_{k\text{-per}}(\mathbb{R}^N;\mathbb{R}^d)$, we direct the reader to Section 2.

Remarks 1.8. (i) Using the growth condition (A_2), a mollification argument, and the linearity of $\mathcal{A}$, it can be shown that

$$f_{\rm hom}(v) = \inf_{k\in\mathbb{N}} \frac{1}{k^N} \inf \left\{ \int_{kQ} f(x, v + w(x))\, dx : \quad w \in L^\infty_{k\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}, \quad \int_{kQ} w(x)\, dx = 0 \right\}.$$

See also Corollary 5.7 below.

(ii) When f satisfies the q-Lipschitz condition

$$|f(x, v_1) - f(x, v_2)| \le C(|v_1|^{q-1} + |v_2|^{q-1} + 1)|v_1 - v_2| \tag{1.10}$$

for all $x \in \mathbb{R}^N$, v_1, $v_2 \in \mathbb{R}^d$, and for some $C > 0$, then the continuity of $f(\cdot, v)$ can be weakened to measurability, namely f can be assumed to be simply Carathéodory. Note that (1.10) is not restrictive when $\mathcal{A} = \mathrm{curl}$, that is when $v = \nabla u$ for some $u \in W^{1,q}(\Omega; \mathbb{R}^m)$, $d = N \times m$. Indeed, in this case in the definition of Γ-convergence we may replace the weak convergence of the gradients in $L^q(\Omega; \mathbb{R}^d)$ with the strong convergence in $L^q(\Omega; \mathbb{R}^m)$ of the potentials normalized to have zero average over Ω, and thus

$$\begin{aligned}\Gamma - \lim_{\varepsilon \to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, v(x)\right)\, dx &= \Gamma(L^q(\Omega; \mathbb{R}^m)) - \lim_{\varepsilon \to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\, dx \\ &= \Gamma(L^q(\Omega; \mathbb{R}^m)) - \lim_{\varepsilon \to 0} \int_\Omega \mathcal{Q}f\left(\frac{x}{\varepsilon}, \nabla u(x)\right)\, dx,\end{aligned}$$

by Proposition 7.13 in [10]. As shown in [27], if $f(x, v)$ is a Borel function which satisfies the growth condition (A_2) then its quasiconvex envelope $\mathcal{Q}f$ satisfies (1.10).

A similar argument fails for general $\mathcal{A}$-quasiconvexity, since the function $\mathcal{Q}_{\mathcal{A}} f(x, \cdot)$ may not even be continuous, see Remark 1.2(i) above.

In Section 2 we collect preliminary results on Young measures and Γ-convergence. The general relaxation results (see Th. 1.1 and its exstension Th. 3.6) are proved in Section 3, and Section 4 is devoted to the applications of the general relaxation principle to Theorems 1.3 and 1.5. Finally, in Section 5 we address homogenization of functionals of $\mathcal{A}$-constrained vector fields.

2. Preliminaries

We start with some notation. Here Ω is an open, bounded subset of $\mathbb{R}^N$, $\mathcal{L}^N$ is the N dimensional Lebesgue measure, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere, and $Q := (-1/2, 1/2)^N$ the unit cube centered at the origin. We set $Q(x_0, \varepsilon) := x_0 + \varepsilon Q$ for $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$. A function $w \in L^q_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ is said to be *Q–periodic* if $w(x + e_i) = w(x)$ for a.e. all $x \in \mathbb{R}^N$ and every $i = 1, \ldots, N$, where $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$. We write $w \in L^q_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^d)$. More generally, $w \in L^q_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ is said to be *kQ–periodic*, $k \in \mathbb{N}$, if $w(k\cdot)$ is Q–periodic. We write $w \in L^q_{k\text{-per}}(\mathbb{R}^N; \mathbb{R}^d)$. Also $C^\infty_{\text{1-per}}(\mathbb{R}^N; \mathbb{R}^d)$ will stands for the space of Q–periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$.

We recall briefly some facts about Young measures which will be useful in the sequel (see *e.g.* [5, 33]). If D is an open set (not necessarily bounded), we denote by $C_c(D; \mathbb{R}^d)$ the set of continuous functions with compact support in D, endowed with the supremum norm. The dual of the closure of $C_c(D; \mathbb{R}^d)$ may be identified with the set of $\mathbb{R}^d$-valued Radon measures with finite mass $\mathcal{M}(D; \mathbb{R}^d)$, through the duality

$$\langle \nu, f \rangle := \int_D f(y)\, d\nu(y), \qquad \nu \in \mathcal{M}(D; \mathbb{R}^d), \quad f \in C_c(D; \mathbb{R}^d).$$

A map $\nu : \Omega \to \mathcal{M}(D; \mathbb{R}^d)$ is said to be *weak-∗ measurable* if $x \mapsto \langle \nu_x, f \rangle$ are measurable for all $f \in C_c(D; \mathbb{R}^d)$.

The following result is a corollary of the Fundamental Theorem on Young Measures (see [5, 7, 34])

Theorem 2.1. *Let $z_n : \Omega \to \mathbb{R}^d$ be measurable functions such that*

$$\sup_{n \in \mathbb{N}} \int_\Omega |z_n|^q\, dx < \infty,$$

for some $q > 0$. Then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a weak-∗ measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ such that

(i) $\nu_x \geq 0$, $||\nu_x||_{\mathcal{M}} = \int_{\mathbb{R}^d} d\nu_x = 1$ *for a.e.* $x \in \Omega$*;*

(ii) *if* $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *is a normal function bounded from below then*

$$\liminf_{k\to\infty} \int_\Omega f(x, z_{n_k}(x))\, dx \geq \int_\Omega \overline{f}(x) dx < \infty,$$

where

$$\overline{f}(x) := \langle \nu_x, f(x,\cdot)\rangle = \int_{\mathbb{R}^d} f(x,y)\, d\nu_x(y);$$

(iii) *for any Carathéodory function* $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *bounded from below one has*

$$\lim_{k\to\infty} \int_\Omega f(x, z_{n_k}(x)\, dx = \int_\Omega \overline{f}(x) dx < \infty$$

if and only if $\{f(\cdot, z_{n_k}(\cdot))\}$ *is equi-integrable.*

The map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ is called the *Young measure* generated by the sequence $\{z_{n_k}\}$.

Proposition 2.2. *If* $\{z_n\}$ *generates a Young measure* ν *and* $v_n \to 0$ *in measure, then* $\{z_n + v_n\}$ *still generates the Young measure* ν*.*

If $1 < q \leq \infty$ then $W^{-1,q}(\Omega;\mathbb{R}^l)$ is the dual of $W_0^{1,q'}(\Omega;\mathbb{R}^l)$, where q' is the Hölder conjugate exponent of q, that is $1/q + 1/q' = 1$. It is well known that $F \in W^{-1,q}(\Omega;\mathbb{R}^l)$ if and only if there exist $g_1, \dots, g_N \in L^q(\Omega;\mathbb{R}^l)$ such that

$$\langle F, w\rangle = \sum_{i=1}^N \int_\Omega g_i \cdot \frac{\partial w}{\partial x_i}\, dx \qquad \text{for all } w \in W_0^{1,q}(\Omega;\mathbb{R}^l).$$

Consider a collection of linear operators $A^{(i)} : \mathbb{R}^d \to \mathbb{R}^l$, $i = 1, \dots, N$, and define the differential operator

$$\begin{aligned} \mathcal{A} : L^q(\Omega;\mathbb{R}^d) &\longrightarrow W^{-1,q}(\Omega;\mathbb{R}^l) \\ v &\longmapsto \mathcal{A}v \end{aligned}$$

as follows:

$$\langle \mathcal{A}v, w\rangle := \left\langle \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, w \right\rangle = -\sum_{i=1}^N \int_\Omega A^{(i)} v\, \frac{\partial w}{\partial x_i}\, dx \qquad \text{for all } w \in W_0^{1,q}(\Omega;\mathbb{R}^l).$$

Even though the operator $\mathcal{A}$ so defined depends on Ω, we will omit reference to the underlying domain whenever it is clear from the context. In particular, if $v \in L^q_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^d)$ then we will say that $v \in \ker \mathcal{A}$ if $\mathcal{A}v = 0$ in $W^{-1,q}(Q;\mathbb{R}^l)$.

Throughout the paper we assume that $\mathcal{A}$ satisfies the constant-rank property (1.1).

The following proposition is due to Fonseca and Müller [22].

Proposition 2.3. (i) $(1 < q < +\infty)$ *Let* $1 < q < +\infty$*, let* $\{V_n\}$ *be a bounded sequence in* $L^q(\Omega;\mathbb{R}^d)$ *such that* $\mathcal{A}V_n \to 0$ *in* $W^{-1,q}(\Omega;\mathbb{R}^l)$*,* $V_n \rightharpoonup V$ *in* $L^q(\Omega;\mathbb{R}^d)$*, and assume that* $\{V_n\}$ *generates a Young measure* ν*. Then there exists a* q*-equi-integrable sequence* $\{v_n\} \subset L^q(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ *such that*

$$\int_\Omega v_n\, dx = \int_\Omega V\, dx, \quad ||v_n - V_n||_{L^s(\Omega)} \to 0 \quad \textit{for all}\ 1 \leq s < q,$$

and, in particular, $\{v_n\}$ *still generates* ν*. Moreover, if* $\Omega = Q$ *then* $v_n - V \in L^q_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$*.*

(ii) ($q = 1$) *Let* $\{V_n\}$ *be a sequence converging weakly in* $L^1(\Omega;\mathbb{R}^d)$ *to a function* V, $\mathcal{A}V_n \to 0$ *in* $W^{-1,r}(\Omega;\mathbb{R}^l)$ *for some* $r \in (1, N/(N-1))$, *and assume that* $\{V_n\}$ *generates a Young measure* ν. *Then there exists an equi-integrable sequence* $\{v_n\} \subset L^1(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ *such that*

$$\int_\Omega v_n\,dx = \int_\Omega V\,dx, \quad ||v_n - V_n||_{L^1(\Omega)} \to 0,$$

and, in particular, $\{v_n\}$ *still generates* ν. *Moreover, if* $\Omega = Q$ *then* $v_n - V \in L^1_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$.

(iii) ($q = +\infty$) *Let* $\{V_n\}$ *be a sequence that satisfies* $V_n \overset{\star}{\rightharpoonup} V$ *in* $L^\infty(\Omega;\mathbb{R}^d)$, $\mathcal{A}V_n \rightharpoonup 0$ *in* $L^r(\Omega)$ *for some* $r > N$, *and assume that* $\{V_n\}$ *generates a Young measure* ν. *Then there exists a sequence* $\{v_n\} \subset L^\infty(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$ *such that*

$$\int_\Omega v_n\,dx = \int_\Omega V\,dx, \quad ||v_n - V_n||_{L^\infty(\Omega)} \to 0,$$

and, in particular, $\{v_n\}$ *still generates* ν. *Moreover, if* $\Omega = Q$ *then* $v_n - V \in L^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$.

In the second part of the paper we will need the following classical results from Γ-convergence. For a proof see [10].

Proposition 2.4. *Let* (X, d) *be a separable metric space and let* $f_n : X \to [-\infty, \infty]$. *Then*

(i) *there exists an increasing sequence of integers* $\{n_k\}$ *such that*

$$\Gamma(d) - \lim_{k\to\infty} f_{n_k}(x) \qquad \text{exists for all } x \in X.$$

(ii) *Moreover*

$$f_\infty = \Gamma(d) - \lim_{n\to\infty} f_n$$

if and only if for every subsequence $\{f_{n_k}\}$ *there exists a further subsequence* $\{f_{n_{k_j}}\}$ *which* $\Gamma(d)$*-converges to* f_∞.

3. Relaxation

In this section we prove Theorem 1.1 and its generalization to the case where $q \in \{1, \infty\}$ and $p = \infty$ (see Th. 3.6).

Lemma 3.1. *Let* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, \infty)$ *be a Carathéodory function satisfying* (*H*), *with* $1 \le p < \infty$ *and* $1 < q < \infty$. *Let* $(u, v) \in L^p(D;\mathbb{R}^m) \times \big(L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}\big)$, *where* $D \in \mathcal{O}(\Omega)$, *and consider a sequence of functions* $\{(u_k, \hat{v}_k)\} \subset L^p(D;\mathbb{R}^m) \times L^q(D;\mathbb{R}^d)$ *such that*

$$\begin{aligned} u_k \to u \quad in \quad L^p(D;\mathbb{R}^m), \quad \hat{v}_k \rightharpoonup v \quad in \quad L^q(D;\mathbb{R}^d) \\ \mathcal{A}\hat{v}_k \to 0 \quad in\ W^{-1,q}(D;\mathbb{R}^l). \end{aligned} \tag{3.1}$$

Then we can find a q-equi-integrable sequence $\{v_k\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ *such that*

$$v_k \rightharpoonup v \quad in \quad L^q(D;\mathbb{R}^d), \qquad \int_D v_k\,dx = \int_D v\,dx,$$

and

$$\liminf_{k\to\infty} \int_D f(x, u(x), v_k(x))\,dx \le \liminf_{k\to\infty} \int_D f(x, u_k(x), \hat{v}_k(x))\,dx.$$

Proof of Lemma 3.1. Consider a subsequence $\{(u_n, \hat{v}_n)\}$ of $\{(u_k, \hat{v}_k)\}$ such that

$$\lim_{n\to\infty} \int_D f(x, u_n(x), \hat{v}_n(x))\, dx = \liminf_{k\to\infty} \int_D f(x, u_k(x), \hat{v}_k(x))\, dx$$

and $\{(u_n, \hat{v}_n)\}$ generates the Young measure $\{\delta_{u(x)} \otimes \nu_x\}_{x\in D}$. For $i \in \mathbb{N}$ let

$$F_i := \left\{ x \in D : \operatorname{dist}(x, \partial D) < \frac{1}{i} \right\},$$

and consider cut-off functions θ_i with compact support in D and such that $\theta_i \equiv 1$ in $D \setminus F_i$. Set $w_{i,n} := \theta_i(\hat{v}_n - v) \in L^q(D; \mathbb{R}^d)$ and fix $\varphi \in L^{q'}(D; \mathbb{R}^d)$, where q' is the Hölder conjugate exponent of q. Then

$$\lim_{i\to\infty} \lim_{n\to\infty} \int_D \varphi(x) w_{i,n}(x)\, dx = \lim_{i\to\infty} \lim_{n\to\infty} \int_D \varphi(x)\theta_i(x)(\hat{v}_n(x) - v(x)) dx = 0, \tag{3.2}$$

where we have used the fact that $\hat{v}_k \rightharpoonup v$ in $L^q(D; \mathbb{R}^d)$. Hence $w_{i,n} \rightharpoonup 0$ in $L^q(D; \mathbb{R}^d)$ as $n \to \infty$ and $i \to \infty$. Moreover, in view of the compact embedding

$$L^q(D; \mathbb{R}^l) \hookrightarrow W^{-1,q}(D; \mathbb{R}^l)$$

and the assumption that $\mathcal{A}\hat{v}_k \to 0$ in $W^{-1,q}(D; \mathbb{R}^l)$, we have that

$$\lim_{i\to\infty} \lim_{n\to\infty} \mathcal{A} w_{i,n} = 0 \quad \text{in} \quad W^{-1,q}(D; \mathbb{R}^l).$$

Let $\mathbb{G}$ be a countable dense subset of $L^q(D; \mathbb{R}^d)$. By means of a diagonalization process we obtain subsequences $\{u_i := u_{n_i}\}$ and $\{\hat{w}_i := w_{i,n_i} = \theta_i(\hat{v}_{n_i} - v)\}$ such that $||u_i - u||_{L^p} \to 0$, (3.2) holds for each $\varphi \in \mathbb{G}$, and

$$\mathcal{A}\hat{w}_i \to 0 \quad \text{in} \quad W^{-1,q}(D; \mathbb{R}^l).$$

Hence $\hat{w}_i \rightharpoonup 0$ in $L^q(D; \mathbb{R}^d)$, by the density of $\mathbb{G}$ in $L^q(D; \mathbb{R}^d)$. By Proposition 2.3(i) there exists a q-equi-integrable sequence $\{w_i\} \subset L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $w_i \rightharpoonup 0$ in $L^q(D; \mathbb{R}^d)$, and

$$\int_D w_i\, dx = 0, \quad ||\hat{w}_i - w_i||_{L^s(D)} \to 0 \quad \text{for all } 1 \leq s < q. \tag{3.3}$$

Set $v_i := v + w_i$. Then $\int_D v_i\, dx = \int_D v\, dx$, $v_i \rightharpoonup v$ in $L^q(D; \mathbb{R}^d)$. By Hölder's inequality and by (3.3), for $1 \leq s < q$

$$\begin{aligned} ||\hat{v}_{n_i} - v_i||_{L^s(D)} &\leq ||\hat{v}_{n_i} - v - \hat{w}_i||_{L^s(D)} + ||\hat{w}_i - w_i||_{L^s(D)} \\ &\leq ||(1-\theta_i)(\hat{v}_{n_i} - v)||_{L^s(D)} + ||\hat{w}_i - w_i||_{L^s(D)} \\ &\leq ||\hat{v}_{n_i} - v||_{L^q(D)} |F_i|^r + ||\hat{w}_i - w_i||_{L^s(D)} \to 0 \end{aligned} \tag{3.4}$$

as $i \to \infty$ and where $r := (q-s)/sq$. By (3.4) and Proposition 2.2, the two sequences

$$\{(u(x), v_i(x))\} \qquad \text{and} \qquad \{(u_i(x), \hat{v}_i(x))\}$$

generate the same Young measure $\{\delta_{u(x)} \otimes \nu_x\}_{x\in D}$. Hence by Theorems 2.1(ii) and (iii)

$$\begin{aligned} \lim_{i\to\infty} \int_D f(x, u(x), v_i(x))\, dx &= \int_D \int_{\mathbb{R}^d} f(x, u(x), V) d\nu_x(V) dx \leq \liminf_{i\to\infty} \int_D f(x, u_i(x), \hat{v}_i(x))\, dx \\ &= \liminf_{k\to\infty} \int_D f(x, u_k(x), \hat{v}_k(x))\, dx, \end{aligned} \tag{3.5}$$

where we have used the fact that $\{f(x, u(x), v_i(x))\}$ is equi-integrable over D, which follows from (H) and the q-equi-integrability of $\{v_i\}$ over D. □

It follows immediately from Lemma 3.1 that under its assumptions on f it holds:

Corollary 3.2. *For $D \in \mathcal{O}(\Omega)$ and $(u, v) \in L^p(\Omega; \mathbb{R}^m) \times \big(L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}\big)$*

$$\mathcal{F}((u,v); D) = \inf\left\{\liminf_{n\to\infty} \int_D g(x, v_n(x))\, dx : \{v_n\} \subset L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A} \quad \text{is } q\text{-equi-integrable} \right.$$
$$\left. \text{and} \quad v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d)\right\},$$

where g is the Carathéodory function defined by

$$g(x, v) := f(x, u(x), v).$$

Note that, by (H), the function g satisfies the growth condition

$$0 \le g(x, v) \le C\,(1 + |u(x)|^p + |v|^q) \tag{3.6}$$

for a.e. $x \in \Omega$ and all $v \in \mathbb{R}^d$. Moreover, since g is a Carathéodory function, by the Scorza-Dragoni theorem for each $j \in \mathbb{N}$ there exists a compact set $K_j \subset \Omega$, with $|\Omega \setminus K_j| \le 1/j$, such that $g : K_j \times \mathbb{R}^d \to [0, \infty)$ is continuous. Let K_j^* be the set of Lebesgue points of χ_{K_j}, and set

$$\omega := \bigcup_j^{\infty} (K_j \cap K_j^*) \cap L(u, v), \tag{3.7}$$

where $L(u, v)$ is the set of Lebesgue points of (u, v). Then

$$|\Omega \setminus \omega| \le |\Omega \setminus K_j| \le \frac{1}{j} \to 0 \quad \text{as } j \to \infty.$$

Corollary 3.3. *Assume that $x_0 \in \omega$, let $v \in L^q(Q; \mathbb{R}^d) \cap \ker \mathcal{A}$, and consider $r_k \to 0^+$ and a sequence of functions*

$$\{\hat{v}_k\} \subset L^q(Q; \mathbb{R}^d) \cap \ker \mathcal{A}$$

such that

$$\hat{v}_k \rightharpoonup v \quad \text{in} \quad L^q(Q; \mathbb{R}^d).$$

Then we can find a q-equi-integrable sequence $\{w_k\} \subset L^q_{1\text{-per}}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$w_k \rightharpoonup 0 \quad \text{in} \quad L^q(Q; \mathbb{R}^d), \qquad \int_Q w_k\, dx = 0,$$

and

$$\liminf_{k\to\infty} \int_Q g(x_0, v(y) + w_k(y))\, dy \le \liminf_{k\to\infty} \int_Q g(x_0 + r_k y, \hat{v}_k(y))\, dy.$$

Proof of Corollary 3.3. We proceed as in the proof of Lemma 3.1 up to (3.4). Since the sequence $\{v_i\}$ is q-equi-integrable, for any $\eta > 0$ there exists $\delta > 0$ such that

$$\sup_i \int_D C(1 + |u(x_0)|^p + |v_i(y)|^q)\, dy < \eta \tag{3.8}$$

for any measurable set $D \subset Q$, with $|D| < \delta$, and where C is the constant given in (H). Fix $\eta > 0$ and let $\delta > 0$ be given according to (3.8). By the Biting Lemma (see [6]) we may find a further subsequence $\{\hat{v}_{n_j}\} \subset \{\hat{v}_{n_i}\}$ and a set $E \subset Q$ such that $|Q \setminus E| < \delta$ and $\{\hat{v}_{n_j}\}$ is q-equi-integrable over E. Hence there exists $0 < \delta_1 < \delta$ such that

$$\sup_j \int_D C(1 + |u(x_0)|^p + |\hat{v}_{n_j}(y)|^q)\, dy < \eta \tag{3.9}$$

for any measurable set $D \subset E$, with $|D| < \delta_1$. Moreover, as $\{\hat{v}_{n_j}\}$, $\{v_j\}$ are bounded in $L^q(Q;\mathbb{R}^d)$, we may find $L > 0$ such that

$$|E \setminus E_j| \le \delta_1, \quad \text{where} \quad E_j := \{y \in E : |\hat{v}_{n_j}(y)| \le L,\, |v_j(y)| \le L\} \cdot \tag{3.10}$$

Note that by construction of v_i and by Proposition 2.3, $v_i = v + w_i$ where $w_i \in L^q_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$. From the definition of the set ω there exists an integer j_0 such that $x_0 \in K_{j_0} \cap K^*_{j_0}$. Since

$$g : K_{j_0} \times \overline{B_d(0,L)} \to [0,\infty)$$

is uniformly continuous, there exists $\rho > 0$ such that

$$|g(x,v) - g(x_1,v)| \le \eta \tag{3.11}$$

for all (x,v), $(x_1,v) \in K_{j_0} \times \overline{B_d(0,L)}$, with $|x - x_1| \le \rho$. By (3.10) and (3.11)

$$\begin{aligned}
\lim_{i\to\infty} \int_Q g(x_0 + r_{n_i} y, \hat{v}_{n_i}(y))\, dy &\ge \liminf_{j\to\infty} \int_{E_j} g(x_0 + r_{n_j} y, \hat{v}_{n_j}(y))\, dy \\
&\ge \liminf_{j\to\infty} \frac{1}{r_{n_j}^N} \int_{(x_0 + r_{n_j} E_j)\cap K_{j_0}} g(x, \hat{v}_{n_j}((x - x_0)/r_{n_j}))\, dx \\
&\ge -\eta + \liminf_{j\to\infty} \frac{1}{r_{n_j}^N} \int_{(x_0 + r_{n_j} E_j)\cap K_{j_0}} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j}))\, dx.
\end{aligned} \tag{3.12}$$

Using, once again, the fact that $|\hat{v}_{n_j}(y)| \le L$ for $y \in E_j$, by (3.6) we have that

$$\frac{1}{r_{n_j}^N} \int_{(x_0 + r_{n_j} E_j)\setminus K_{j_0}} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j}))\, dx \le C(1 + |u(x_0)|^p + L^q) \frac{|Q(x_0, r_{n_j}) \setminus K_{j_0}|}{r_{n_j}^N} \to 0$$

as $j \to \infty$, because x_0 is a Lebesgue point of $\chi_{K_{j_0}}$. Consequently, from (3.12) we get

$$\begin{aligned}
\lim_{i\to\infty} \int_Q g(x_0 + r_{n_i} y, \hat{v}_{n_i}(y))\, dy &\ge -\eta + \liminf_{j\to\infty} \frac{1}{r_{n_j}^N} \int_{x_0 + r_{n_j} E_j} g(x_0, \hat{v}_{n_j}((x - x_0)/r_{n_j}))\, dx \\
&= -\eta + \liminf_{j\to\infty} \int_{E_j} g(x_0, \hat{v}_{n_j}(y))\, dy \\
&\ge -2\eta + \liminf_{j\to\infty} \int_E g(x_0, \hat{v}_{n_j}(y))\, dy,
\end{aligned}$$

where we have used (3.6, 3.9) and the fact that $|E \setminus E_j| \leq \delta_1$. We may now proceed as in the previous lemma, using the Carathéory function $h(x,v) := \chi_E(x) g(x_0, v)$, to obtain

$$\lim_{i\to\infty} \int_Q g(x_0 + r_{n_i} y, \hat{v}_{n_i}(y))\, dy \geq -2\eta + \liminf_{j\to\infty} \int_E g(x_0, v_j(y))\, dy \geq -3\eta + \liminf_{j\to\infty} \int_Q g(x_0, v_j(y))\, dy$$

by (3.8). It now suffices to let $\eta \to 0^+$. □

Theorem 1.1 follows from Lemmas 3.4 and 3.5 below. We will use the notation $\mu\lfloor A$ to denote the restriction of a Radon measure μ to the Borel set A, *i.e.*, $\mu\lfloor A(X) := \mu(X \cap A)$ where X is an arbitrary Borel set in the domain of μ.

Lemma 3.4. *$\mathcal{F}((u,v);\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N\lfloor\Omega$.*

Proof of Lemma 3.4. As it is usual, it suffices to prove subadditivity (see *e.g.* [3, 21]), *i.e.*

$$\mathcal{F}((u,v);D) \leq \mathcal{F}((u,v);D \setminus \overline{B}) + \mathcal{F}((u,v);C)$$

if $B \subset\subset C \subset\subset D$. Fix $\eta > 0$. By Corollary 3.2 there exist two q-equi-integrable sequences

$$\{v_k\} \subset L^q(D \setminus \overline{B}; \mathbb{R}^d) \cap \ker \mathcal{A}, \qquad \{w_k\} \subset L^q(C; \mathbb{R}^d) \cap \ker \mathcal{A},$$

such that

$$v_k \rightharpoonup v \quad \text{in} \quad L^q(D \setminus \overline{B}; \mathbb{R}^d), \qquad w_k \rightharpoonup v \quad \text{in} \quad L^q(C; \mathbb{R}^d),$$

and

$$\lim_{k\to\infty} \int_{D\setminus\overline{B}} g(x, v_k(x))\, dx \leq \mathcal{F}((u,v); D \setminus \overline{B}) + \eta,$$
$$\lim_{k\to\infty} \int_C g(x, w_k(x))\, dx \leq \mathcal{F}((u,v); C) + \eta.$$

Let θ_j be smooth cut-off functions, $\theta_j \in C_c^\infty(C;[0,1])$, $\theta_j(x) = 1$ for all $x \in B$, and $|\{0 < \theta_j < 1\}| \to 0$ as $j \to \infty$. Set

$$\hat{V}_{j,k} := (1 - \theta_j) v_k + \theta_j w_k.$$

Then, for j fixed,

$$\mathcal{A}\hat{V}_{j,k} = (1-\theta_j)\mathcal{A}v_k + \theta_j \mathcal{A}w_k - \sum_{i=1}^N A^{(i)} v_k \frac{\partial \theta_j}{\partial x_i} + \sum_{i=1}^N A^{(i)} w_k \frac{\partial \theta_j}{\partial x_i} \to 0$$

as $k \to \infty$ in $W^{-1,q}(D;\mathbb{R}^l)$ strong. Using a diagonalization procedure such as that adopted in the proof of Lemma 3.1, we get

$$\hat{V}_j \rightharpoonup v \quad \text{in} \quad L^q(D;\mathbb{R}^d), \qquad \mathcal{A}\hat{V}_j \to 0 \quad \text{in} \quad W^{-1,q}(D;\mathbb{R}^l),$$

where $\hat{V}_j := \hat{V}_{j,k_j}$. By Lemma 3.1 we can find a q-equi-integrable sequence $\{V_j\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $V_j \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$ and

$$\liminf_{j\to\infty} \int_D g(x, V_j(x))\, dx \leq \liminf_{j\to\infty} \int_D g(x, \hat{V}_j(x))\, dx.$$

Consequently, in view of Corollary 3.2

$$\begin{aligned}\mathcal{F}((u,v);D) &\leq \liminf_{j\to\infty}\int_D g(x,V_j(x))\,dx \leq \liminf_{j\to\infty}\int_D g(x,\hat{V}_j(x))\,dx\\ &\leq \limsup_{j\to\infty}\int_{\{\theta_j=0\}} g(x,v_{k_j}(x))\,dx + \limsup_{j\to\infty}\int_{\{\theta_j=1\}} g(x,w_{k_j}(x))\,dx\\ &\quad + \limsup_{j\to\infty}\int_{\{0<\theta_j<1\}} C(1+|u(x)|^p+|w_{k_j}(x)|^q+|v_{k_j}(x)|^q)\,dx\\ &\leq 2\eta + \mathcal{F}((u,v);D\setminus\overline{B}) + \mathcal{F}((u,v);C).\end{aligned}$$

It suffices to let $\eta \to 0^+$. Finally, note that by (H) we have that

$$\mathcal{F}((u,v),\cdot) \leq C(1+|u|^p+|v|^q)\mathcal{L}^N\lfloor\Omega.$$

□

Lemma 3.5. *For $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ we have*

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) = \mathcal{Q}_{\mathcal{A}}f(x_0,u(x_0),v(x_0)).$$

Proof of Lemma 3.5. Fix $x_0 \in \omega$, where ω is defined as in (3.7), and such that

$$\lim_{r\to 0^+}\frac{1}{r^N}\int_{Q(x_0,r)}|u(x)-u(x_0)|^p\,dx = \lim_{r\to 0^+}\frac{1}{r^N}\int_{Q(x_0,r)}|v(x)-v(x_0)|^q\,dx = 0 \tag{3.13}$$

and

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\frac{\mathcal{F}((u,v);Q(x_0,r))}{r^N} < \infty,$$

where, by virtue of Lemma 3.4, we have chosen the radii $r \to 0^+$ such that

$$\mathcal{F}((u,v);\partial(Q(x_0,r))) = 0.$$

By Corollary 3.2 and for $r > 0$ fixed, let $\{v_{n,r}\} \subset L^q(Q(x_0,r);\mathbb{R}^d) \cap \ker\mathcal{A}$ be such that $v_{n,r} \rightharpoonup v$ in $L^q(Q(x_0,r);\mathbb{R}^d)$ as $n \to \infty$ and

$$\lim_{n\to\infty}\int_{Q(x_0,r)} g(x,v_{n,r}(x))\,dx \leq \mathcal{F}((u,v);Q(x_0,r)) + r^{N+1}.$$

Then

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{r\to 0^+}\lim_{n\to\infty}\frac{1}{r^N}\int_{Q(x_0,r)} g(x,v_{n.r}(x))\,dx = \liminf_{r\to 0^+}\lim_{n\to\infty}\int_Q g(x_0+ry,v(x_0)+w_{n,r}(y))\,dy$$

where $w_{n,r}(y) := v_{n,r}(x_0+ry) - v(x_0)$. We claim that $w_{n,r} \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ if we first let $n \to \infty$ and then $r \to 0^+$. Indeed let $\varphi \in L^{q'}(Q;\mathbb{R}^d)$, where q' is the Hölder conjugate exponent of q. Using Hölder's inequality and then making a change of variables, we get

$$\begin{aligned}\left|\int_Q \varphi(y)\, w_{r,n}(y)\, dy\right| &\le \left|\int_Q \varphi(y)(v_{n,r}(x_0+ry)-v(x_0+ry))\, dy\right| + \left|\int_Q \varphi(y)(v(x_0+ry)-v(x_0))\, dy\right| \\ &\le \left|\frac{1}{r^N}\int_{Q(x_0,r)} \varphi((x-x_0)/r)(v_{n,r}(x)-v(x))\, dx\right| \\ &\quad + ||\varphi||_{L^{q'}(Q)} \left(\frac{1}{r^N}\int_{Q(x_0,r)} |v(x)-v(x_0)|^q\, dx\right)^{1/q}.\end{aligned}$$

If we now let $n \to \infty$ the first integral tends to zero, since $v_{n,r} \rightharpoonup v$ in $L^q(Q(x_0,r);\mathbb{R}^d)$. The claim then follows by letting $r \to 0^+$ and by using (3.13). Diagonalize to get $\hat{w}_k \in L^q(Q;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $\hat{w}_k \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ and

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) \ge \lim_{k\to\infty}\int_Q g(x_0 + r_k\, y, v(x_0) + \hat{w}_k(y))\, dy$$

where $r_k \to 0$. By Corollary 3.3 there is a q-equi-integrable sequence $\{w_k\} \subset L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$w_k \rightharpoonup 0 \quad \text{in} \quad L^q(Q;\mathbb{R}^d), \qquad \int_Q w_k\, dy = 0,$$

and

$$\begin{aligned}\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) &\ge \lim_{k\to\infty}\int_Q g(x_0 + r_k\, y, v(x_0) + \hat{w}_k(y))\, dy \\ &\ge \liminf_{k\to\infty}\int_Q f(x_0, u(x_0), v(x_0) + w_k(y))\, dy \ge \mathcal{Q}_{\mathcal{A}} f(x_0, u(x_0), v(x_0)).\end{aligned}$$

To conclude the proof of the lemma it remains to show that

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) \le \mathcal{Q}_{\mathcal{A}} f(x_0, u(x_0), v(x_0)) \quad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega.$$

Fix $\eta > 0$ and let $w \in C^\infty_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$ be such that $\int_Q w\, dy = 0$ and

$$\int_Q f(x_0, u(x_0), v(x_0) + w(y))\, dy \le \mathcal{Q}_{\mathcal{A}} f(x_0, u(x_0), v(x_0)) + \eta. \tag{3.14}$$

For any fixed $r > 0$ set $w_{n,r}(x) := w(n(x-x_0)/r)$. Then $w_{n,r} \rightharpoonup 0$ in $L^q(Q(x_0,r);\mathbb{R}^d)$ as $n \to \infty$. Hence, by Corollary 3.2,

$$\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\frac{\mathcal{F}((u,v);Q(x_0,r))}{r^N} \le \liminf_{r\to 0^+}\liminf_{n\to\infty}\frac{1}{r^N}\int_{Q(x_0,r)} g(x, v(x) + w_{n,r}(x))\, dx. \tag{3.15}$$

Fix $L > |v(x_0)| + ||w||_{L^\infty} + 1$, and let j be such that $x_0 \in K_j \cap K_j^* \cap L(u,v)$, where we are using the notation introduced in (3.7). Since

$$g : K_j \times \overline{B_d(0,L)} \to [0,\infty)$$

is uniformly continuous, there exists $0 < \rho < 1$ such that

$$|g(x,v) - g(x_1,v_1)| \le \eta \tag{3.16}$$

for all (x,v), $(x_1,v_1) \in K_j \times \overline{B_d(v(x_0),L)}$, with $|x-x_1| \le \rho$ and $|v-v_1| \le \rho$. Let

$$E_{r,\rho} := \{x \in Q(x_0,r) : |v(x)-v(x_0)| \le \rho\}\cdot$$

We claim that

$$\limsup_{r\to 0^+} \frac{1}{r^N} \int_{Q(x_0,r)\setminus(E_{r,\rho}\cap K_j)} C(1+|u(x)|^p+|v(x)|^q+||w||^q_{L^\infty})dx = 0. \tag{3.17}$$

Since $|v(x)-v(x_0)| \ge \rho$ for $x \in Q(x_0,r)\setminus E_{r,\rho}$, we have

$$\frac{1}{r^N}\int_{Q(x_0,r)\setminus(E_{r,\rho}\cap K_j)} C(1+|u(x)|^p+|v(x)|^q+||w||^q_{L^\infty})dx \le C\,\frac{|Q(x_0,r)\setminus(E_{r,\rho}\cap K_j)|}{r^N} + \frac{C}{r^N}\int_{Q(x_0,r)} \left(|u(x)-u(x_0)|^p+|v(x)-v(x_0)|^q\right)\,dx$$

and

$$\frac{|Q(x_0,r)\setminus(E_{r,\rho}\cap K_j)|}{r^N} \le \frac{|Q(x_0,r)\setminus K_j|}{r^N} + \frac{|Q(x_0,r)\setminus E_{r,\rho}|}{r^N} \le \frac{|Q(x_0,r)\setminus K_j|}{r^N} + \frac{1}{\rho^q}\frac{C}{r^N}\int_{Q(x_0,r)} |v(x)-v(x_0)|^q\,dx \to 0 \quad \text{as } r\to 0^+,$$

where we have used (3.13) and the fact that x_0 is a Lebesgue point of χ_{K_j}. Then by (3.6, 3.15–3.17) and (3.14),

$$\begin{aligned}
\frac{d\mathcal{F}((u,v);\cdot)}{d\mathcal{L}^N}(x_0) &\le \liminf_{r\to 0^+}\liminf_{n\to\infty} \frac{1}{r^N}\int_{E_{r,\rho}\cap K_j} g(x,v(x)+w_{n,r}(x))\,dx \\
&\quad + \limsup_{r\to 0^+}\frac{1}{r^N}\int_{Q(x_0,r)\setminus(E_{r,\rho}\cap K_j)} C(1+|u(x)|^p+|v(x)|^q+||w||^q_{L^\infty})dx \\
&\le \eta + \liminf_{r\to 0^+}\liminf_{n\to\infty}\frac{1}{r^N}\int_{Q(x_0,r)} g(x_0,v(x_0)+w_{n,r}(x))\,dx \\
&= \eta + \liminf_{n\to\infty}\int_Q g(x_0,v(x_0)+w(ny))\,dy \\
&= \eta + \int_Q g(x_0,v(x_0)+w(y))\,dy \le 2\eta + \mathcal{Q}_{\mathcal{A}}f(x_0,u(x_0),v(x_0)),
\end{aligned}$$

by virtue of the equality

$$\liminf_{n\to\infty}\int_Q g(x_0,v(x_0)+w(ny))\,dx = \int_Q g(x_0,v(x_0)+w(y))\,dy,$$

which follows from the Q-periodicity of the function $g(x_0,v(x_0)+w(\cdot))$. It now suffices to let $\eta\to 0^+$. □

As mentioned in the introduction, Theorem 1.1 continues to hold when $q\in\{1,\infty\}$ and $p=\infty$. Indeed, let $1\le p,\, q\le\infty$ and assume that

(A_4) $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^d\to[0,\infty)$ is a Carathéodory function satisfying the following growth conditions for a.e. $x\in\Omega$ and all $(u,v)\in\mathbb{R}^m\times\mathbb{R}^d$:

$$0\le f(x,u,v)\le C\,(1+|u|^p+|v|^q) \qquad \text{if } 1\le p,\, q<\infty,$$

where $C > 0$;

$$0 \le f(x,u,v) \le a(x,u)\,(1+|v|^q) \qquad \text{if } p=\infty \text{ and } 1 \le q < \infty, \tag{3.18}$$

where $\chi_\Omega a \in L^\infty_{\rm loc}(\mathbb{R}^N \times \mathbb{R}^d;[0,\infty))$;

$$0 \le f(x,u,v) \le b(x,v)\,(1+|u|^p) \qquad \text{if } 1 \le p < \infty \text{ and } q=\infty,$$

where $\chi_\Omega b \in L^\infty_{\rm loc}(\mathbb{R}^N \times \mathbb{R}^m;[0,\infty))$;

$$\chi_\Omega f \in L^\infty_{\rm loc}(\mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^d;[0,\infty)) \qquad \text{if } p=q=\infty.$$

For $D \in \mathcal{O}(\Omega)$ and $(u,v) \in L^p(\Omega;\mathbb{R}^m) \times \big(L^q(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}\big)$ define

$$\mathcal{F}((u,v);D) := \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) : \ (u_n,v_n) \in L^p(D;\mathbb{R}^m) \times L^1(D;\mathbb{R}^d), \\ u_n \to u \text{ in } L^p(D;\mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^1(D;\mathbb{R}^d), \quad \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,r}(D;\mathbb{R}^l)\Big\}$$

if $q=1$ and for some $r \in (1, N/(N-1))$; as in (1.2), we set

$$\mathcal{F}((u,v);D) := \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) : \ (u_n,v_n) \in L^p(D;\mathbb{R}^m) \times L^q(D;\mathbb{R}^d) \\ u_n \to u \text{ in } L^p(D;\mathbb{R}^m), \quad v_n \rightharpoonup v \text{ in } L^q(D;\mathbb{R}^d), \quad \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,q}(D;\mathbb{R}^l)\Big\}$$

if $1 < q < \infty$;

$$\mathcal{F}((u,v);D) := \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) : \ (u_n,v_n) \in L^p(D;\mathbb{R}^m) \times L^\infty(D;\mathbb{R}^d), \\ u_n \to u \text{ in } L^p(D;\mathbb{R}^m), \quad v_n \overset{\star}{\rightharpoonup} v \text{ in } L^\infty(D;\mathbb{R}^d), \quad \mathcal{A}v_n \rightharpoonup 0 \quad \text{in } L^r(D;\mathbb{R}^l)\Big\}$$

if $q=\infty$ and for some $r > N$.

We can prove the following theorem:

Theorem 3.6. *Under condition* (A_4) *and the constant-rank hypothesis* (1.1), *for all* $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega;\mathbb{R}^m)$ *and* $v \in L^q(\Omega;\mathbb{R}^d) \cap \ker \mathcal{A}$, *we have*

$$\mathcal{F}((u,v);D) = \int_D \mathcal{Q}_\mathcal{A} f(x,u(x),v(x))\,dx.$$

Proof of Theorem 3.6.

Step 1. Assume first that $1 \le p < \infty$ and $q=1$. The proof is similar to the one of Theorem 1.1, with the exceptions that in Lemma 3.1 condition (3.1) should be replaced by

$$u_k \to u \quad \text{in} \quad L^p(D;\mathbb{R}^m), \quad \hat{v}_k \rightharpoonup v \quad \text{in} \quad L^1(D;\mathbb{R}^d), \\ \mathcal{A}\hat{v}_n \to 0 \quad \text{in } W^{-1,r}(D;\mathbb{R}^l) \text{ for some } r \in (1, N/(N-1)),$$

that we use the compact embedding

$$L^1(D;\mathbb{R}^l) \hookrightarrow W^{-1,r}(D;\mathbb{R}^l), \qquad r \in (0, N/(N-1),$$

to diagonalize $\{w_{i,n}\}$, and (3.3, 3.4) are replaced, respectively, by

$$\int_D w_i\,dx = 0, \quad ||\hat{w}_i - w_i||_{L^1(D)} \to 0,$$

$$\begin{aligned} ||\hat{v}_{n_i} - v_i||_{L^1(D)} &\le ||\hat{v}_{n_i} - v - \hat{w}_i||_{L^1(D)} + ||\hat{w}_i - w_i||_{L^1(D)} \le ||(1-\theta_i)(\hat{v}_{n_i} - v)||_{L^1(D)} + ||\hat{w}_i - w_i||_{L^1(D)} \\ &\le ||\hat{v}_{n_i} - v||_{L^1(F_i)} + ||\hat{w}_i - w_i||_{L^1(D)} \to 0, \end{aligned}$$

where we have used the fact that $||\hat{v}_{n_i} - v||_{L^1(F_i)} \to 0$ as $i \to \infty$, which is due to the equi-integrability of the original sequence $\{\hat{v}_k - v\}$ and the fact that $|F_i| \to 0$.

Step 2. If $p = \infty$ and $1 \le q < \infty$ then in Lemma 3.1 the only change needed is in deriving (3.5), which now follows from the fact that, by (3),

$$0 \le f(x, u(x), v_i(x)) \le A_\infty \left(1 + |v_i(x)|^q\right),$$

where $A_\infty := \operatorname{esssup}\{a(x,u) : x \in \Omega, \quad |u| \le ||u||_\infty\} < \infty$, and thus equi-integrability of $\{f(x,u,v_i)\}$ follows from the q-equi-integrability of $\{v_i\}$ over D. Moreover in the remaining of the proof of Theorem 1.1, the growth condition (3.6) should be replaced by

$$0 \le g(x,v) \le A_\infty \left(1 + |v|^q\right) \tag{3.19}$$

for a.e. $x \in \Omega$ and all $v \in \mathbb{R}^d$.

Step 3. If $1 \le p \le \infty$ and $q = \infty$ then in Lemma 3.1 the hypothesis (3.1) should be replaced by

$$\begin{gathered} u_k \to u \quad \text{in} \quad L^p(D;\mathbb{R}^m), \quad \hat{v}_k \overset{\star}{\rightharpoonup} v \quad \text{in} \quad L^\infty(D;\mathbb{R}^d), \\ A\hat{v}_n \rightharpoonup 0 \quad \text{in } L^r(D;\mathbb{R}^l) \text{ for some } r > N, \end{gathered}$$

the growth condition should be replaced by (3.19) if $1 \le p < \infty$, $q = \infty$, and by $\chi_\Omega g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^d;[0,\infty))$ if $p = q = \infty$, and we can proceed similarly to the proof of Lemma 3.1 to show that $w_{i,n} \overset{\star}{\rightharpoonup} 0$ in $L^\infty(D;\mathbb{R}^d)$ and $\mathcal{A}w_{i,n} \rightharpoonup 0$ in $L^r(D;\mathbb{R}^l)$, and use Proposition 2.3(iii) to get

$$||v_i - \hat{v}_{n_i}||_\infty \to 0.$$

We omit the details. □

4. Proofs of Theorems 1.3 and 1.5

Proof of Theorem 1.3. We present the proof for $1 \le p < \infty$, the case $p = \infty$ being very similar. Fix $u \in W^{s,p}(\Omega;\mathbb{R}^n)$, and for $D \in \mathcal{O}(\Omega)$ define

$$\mathcal{F}(u;D) := \inf\left\{ \liminf_{k\to\infty} \int_D f(x, u_k, \dots, \nabla^s u_k)\,dx : \{u_k\} \subset W^{s,p}(D;\mathbb{R}^n), \right. \\ \left. u_k \rightharpoonup u \text{ in } W^{s,p}(D;\mathbb{R}^n) \right\},$$

and let g be the Carathéodory function

$$g(x,v) := f(x,u(x),\dots,\nabla^{s-1}u(x),v).$$

Reasoning as in Lemma 3.4, it is easy to show that $\mathcal{F}(u;\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^N\lfloor\Omega$.

For any function $v \in L^p(\Omega;E_s^n)$ set

$$\mathcal{G}(v;D) := \inf\Big\{\liminf_{k\to\infty}\int_D g(x,V_k(x))\,dx : \{V_k\}\subset L^p(D;E_s^n)\cap\ker\mathcal{A} \quad \text{is } p\text{-equi-integrable}, \quad \text{and} \quad V_k \rightharpoonup v \text{ in } L^p(D;E_s^n)\Big\},$$

where the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}v := \left(\frac{\partial}{\partial x_i}v_{i_1\dots i_h j i_{h+2}\dots i_s} - \frac{\partial}{\partial x_j}v_{i_1\dots i_h i i_{h+2}\dots i_s}\right)_{0\le h\le s-1,\,1\le i,j,i_1\dots i_s\le N}.$$

Here $h=0$ and $h=s-1$ correspond to the multi-indeces $ji_2\dots i_s$ and $i_1\dots i_{s-1}j$. By Theorem 3.6 (and Cor. 3.2), and where the target space $\mathbb{R}^d$ is being replaced by the finite dimensional vector space E_s^n, for any $D\in\mathcal{O}(\Omega)$

$$\mathcal{G}(v;D) = \int_D \mathcal{Q}_{\mathcal{A}}g(x,v(x))\,dx,$$

where for a.e. $x\in\Omega$ and for all $v\in E_s^n$,

$$\mathcal{Q}_{\mathcal{A}}g(x,v) := \inf\left\{\int_Q g(x,v+w(y))\,dy : \; w\in C^\infty_{\text{1-per}}(\mathbb{R}^N;E_s^n)\cap\ker\mathcal{A},\; \int_Q w(y)\,dy = 0\right\}\cdot$$

As shown in [22],

$$\left\{w\in C^\infty_{\text{1-per}}(\mathbb{R}^N;E_s^n) : \;\mathcal{A}w=0, \quad \int_Q w\,dx = 0\right\} = \left\{\nabla^s\varphi : \;\varphi\in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^n)\right\}\cdot \tag{4.1}$$

Hence

$$\mathcal{Q}_{\mathcal{A}}g(x,v) = \inf\left\{\int_Q g(x,v+\nabla^s\varphi(y))\,dy : \varphi\in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^N)\right\}\cdot$$

In particular

$$\mathcal{G}(\nabla^s u;D) = \int_D \mathcal{Q}^s f(x,u,\dots,\nabla^s u)\,dx. \tag{4.2}$$

Let $\{u_k\}\subset W^{s,p}(\Omega;\mathbb{R}^n)$ be any sequence such that $u_k \rightharpoonup u$ in $W^{s,p}(\Omega;\mathbb{R}^n)$. Extracting a subsequence, if necessary, we may assume that

$$\mathbf{u}_k := (u_k,\dots,\nabla^{s-1}u_k) \to \mathbf{u} := (u,\dots,\nabla^{s-1}u) \quad \text{in } L^p(D;E^n_{[s-1]}).$$

Since $\nabla^s u_k \rightharpoonup \nabla^s u$ in $L^p(D; E_s^n)$ and $\mathcal{A}\nabla^s u_k = 0$, by Lemma 3.1 there exists a p-equi-integrable sequence $\{V_k\} \subset L^p(D; E_s^n) \cap \ker \mathcal{A}$ such that $V_k \rightharpoonup \nabla^s u$ in $L^p(D; E_s^n)$ and

$$\liminf_{k\to\infty} \int_D g(x, V_k(x))\, dx \le \liminf_{k\to\infty} \int_\Omega f(x, u_k, \dots, \nabla^s u_k)\, dx.$$

Thus

$$\mathcal{G}(\nabla^s u; D) \le \mathcal{F}(u; D). \tag{4.3}$$

To prove the converse inequality, fix $x_0 \in \Omega$ and $r > 0$, and consider any p-equi-integrable sequence $\{V_k\} \subset L^p(B(x_0; r); E_s^n) \cap \ker \mathcal{A}$ such that $V_k \rightharpoonup \nabla^s u$ in $L^p(B(x_0; r); E_s^n)$. An induction argument, similar to the one used in [22] to prove (4.1) above, shows that $\mathcal{A}V_k = 0$ if and only if there exists $\varphi_k \in W^{s,p}(B(x_0; r); \mathbb{R}^n)$ such that $\nabla^s \varphi_k = V_k$. By Lemmas 1.1–1.3 in [24], for any $\varphi \in W^{s,p}(B(x_0; r); \mathbb{R}^n)$ we may find a unique function $P \in C^\infty(\mathbb{R}^N; \mathbb{R}^n)$ whose components are polynomials of degree $s - 1$ such that

$$\int_{B(x_0,r)} \nabla^l(\varphi - P)\, dx = 0 \qquad 0 \le l \le s-1, \tag{4.4}$$

and a constant $C(n, N, s, p, r) > 0$ such that the following Poincaré type inequality holds

$$||\varphi - P||_{W^{s,p}(B(x_0;r);\mathbb{R}^n)} \le C||\nabla^s \varphi||_{L^p(B(x_0;r);E_s^n)}. \tag{4.5}$$

Let P_k and P be the functions associated to φ_k and u, respectively, and satisfying (4.4, 4.5). Since $\nabla^s \varphi_k \rightharpoonup \nabla^s u$ in $L^p(B(x_0; r); E_s^n)$, we have that

$$\varphi_k - P_k \rightharpoonup u - P \text{ in } W^{s,p}(B(x_0; r); \mathbb{R}^n),$$

so

$$u_k := \varphi_k - P_k + P \rightharpoonup u \text{ in } W^{s,p}(B(x_0; r); \mathbb{R}^N).$$

Consider a subsequence of $\{V_k\}$ (not relabelled) such that the two sequences

$$\{(u_k, \dots, \nabla^{s-1} u_k, V_k)\} \quad \text{and} \quad \{(u, \dots, \nabla^{s-1} u, V_k)\}$$

generate the Young measure $\{\delta_{(u(x),\dots,\nabla^{s-1}u)} \otimes \nu_x\}_{x\in B(x_0,r)}$, and

$$(u_k, \dots, \nabla^{s-1} u_k) \to (u, \dots, \nabla^{s-1} u)$$

pointwise and in $L^p(B(x_0; r); E_{s-1}^n)$. Since $\{V_k\}$ is p-equi-integrable and u_k converge to u strongly in $W^{s-1,p}(\Omega; \mathbb{R}^N)$, it follows from Theorem 2.1 and the growth condition on f that

$$\lim_{k\to\infty} \int_{B(x_0,r)} f(u_k, \dots, \nabla^{s-1} u_k, V_k)\, dx = \lim_{k\to\infty} \int_{B(x_0,r)} g(x, V_k(x))\, dx.$$

Thus

$$\mathcal{G}(\nabla^s u; B(x_0, r)) \ge \mathcal{F}(u; B(x_0, r)),$$

which, together with (4.3), yields

$$\mathcal{G}(\nabla^s u; B(x_0, r)) = \mathcal{F}(u; B(x_0, r)). \tag{4.6}$$

Since $\mathcal{F}(u;\cdot)$ and $\mathcal{G}(\nabla^s u;\cdot)$ are both traces of a Radon measures absolutely continuous with respect to $\mathcal{L}^N\lfloor\Omega$, by (4.2) and (4.6) we immediately obtain that

$$\mathcal{F}(u;D) = \mathcal{G}(\nabla^s u;D) = \int_D \mathcal{Q}f(x,u,\dots,\nabla^s u)\,dx.$$

□

Proof of Theorem 1.5. We only proof Theorem 1.5 for $1 \le p < \infty$, the case $p = \infty$ being very similar.

For $v \in \mathbb{R}^{N^2-1}$ let

$$v = (v^{(1)},\dots,v^{(N)}), \qquad \text{where} \quad v^{(i)} \in \mathbb{R}^N, \quad i = 1,\dots,N-1, \quad v^{(N)} \in \mathbb{R}^{N-1}.$$

Given a function $v \in L^p(\Omega;\mathbb{R}^{N^2-1})$ define the differential operator $\mathcal{A}$ as follows

$$\mathcal{A}v := \begin{pmatrix} \operatorname{curl} v^{(1)} \\ \vdots \\ \operatorname{curl} v^{(N-1)} \\ \operatorname{curl}\,(v^{(N)}, -v_1^{(1)} - \dots - v_{N-1}^{(N-1)}) \end{pmatrix}.$$

A straightforward calculation shows that $\mathcal{A}$ satisfies the *constant-rank* property (1.1).

Given a Carathéodory function $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{N^2} \to [0,\infty)$, we define $\hat{f}(x,u,v)$, for $(x,u,v) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N^2-1}$, as

$$\hat{f}(x,u,v) = f\left(x,u,\left(v^{(1)},\dots,v^{(N-1)},\begin{pmatrix} v^{(N)} \\ -v_1^{(1)} - \dots - v_{N-1}^{(N-1)} \end{pmatrix}\right)\right).$$

Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, with $\operatorname{div} u = 0$, and let $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^N)$ be such that $\operatorname{div} u_n = 0$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^N)$. By Lemma 3.1 there exists a p-equi-integrable sequence $\{V_n\} \subset L^p(\Omega;\mathbb{R}^{N^2-1}) \cap \ker\mathcal{A}$ such that $V_n \rightharpoonup v$ in $L^p(D;\mathbb{R}^{N^2-1})$ and

$$\liminf_{n\to\infty} \int_\Omega \hat{f}(x,u,V_n)\,dx \le \liminf_{n\to\infty} \int_\Omega \hat{f}(x,u_n,v_n)\,dx = \liminf_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)\,dx, \tag{4.7}$$

where

$$v_n := \left(\nabla u_n^{(1)},\dots,\nabla u_n^{(N-1)},\begin{pmatrix} \frac{\partial u_n^{(N)}}{\partial x_1} \\ \vdots \\ \frac{\partial u_n^{(N)}}{\partial x_{N-1}} \end{pmatrix}\right), \quad v := \left(\nabla u^{(1)},\dots,\nabla u^{(N-1)},\begin{pmatrix} \frac{\partial u^{(N)}}{\partial x_1} \\ \vdots \\ \frac{\partial u^{(N)}}{\partial x_{N-1}} \end{pmatrix}\right). \tag{4.8}$$

Define

$$\mathcal{G}(v;D) := \inf\Big\{\liminf_{n\to\infty} \int_D \hat{g}(x,V_n(x))\,dx : \{V_n\} \subset L^p(D;\mathbb{R}^{N^2-1}) \cap \ker\mathcal{A} \quad \text{is } p\text{-equi-integrable,}$$

$$\text{and} \quad V_n \rightharpoonup v \text{ in } L^p(D;\mathbb{R}^{N^2-1})\Big\},$$

where $\hat{g}$ is the Carathéodory function defined by $\hat{g}(x,v) := \hat{f}(x,u(x),v)$. By Theorem 3.6 (and Cor. 3.2)

$$\mathcal{G}(v;\Omega) = \int_\Omega \mathcal{Q}_\mathcal{A}\hat{g}(x,v)\,dx, \tag{4.9}$$

where

$$\mathcal{Q}_\mathcal{A}\hat{g}(x,v(x)) := \inf\left\{ \int_Q \hat{f}(x,u(x),v(x)+w(y))\,dy : \; w \in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^{N^2-1}) \cap \ker\mathcal{A}, \quad \int_Q w(y)\,dy = 0 \right\}.$$

Now

$$w \in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^{N^2-1}) \cap \ker\mathcal{A} \quad \text{and} \quad \int_Q w(y)\,dy = 0$$

if and only if there exists $\varphi \in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$w = \left(\nabla\varphi^{(1)}, \dots, \nabla\varphi^{(N-1)}, \begin{pmatrix} \frac{\partial\varphi^{(N)}}{\partial x_1} \\ \vdots \\ \frac{\partial\varphi^{(N)}}{\partial x_{N-1}} \end{pmatrix} \right)$$

and $\frac{\partial\varphi^{(N)}}{\partial x_N} = -\frac{\partial\varphi^{(1)}}{\partial x_1} - \dots - \frac{\partial\varphi^{(N-1)}}{\partial x_{N-1}}$. Hence

$$\begin{aligned} \mathcal{Q}_\mathcal{A}\hat{g}(x,v(x)) &= \inf\left\{ \int_Q f(x,u(x),\nabla u(x)+\nabla\varphi(y))\,dy : \varphi \in C^\infty_{\text{1-per}}(\mathbb{R}^N;\mathbb{R}^N), \quad \operatorname{div}\varphi = 0 \right\} \\ &= \bar{f}(x,u(x),\nabla u(x)). \end{aligned} \tag{4.10}$$

Thus, by (4.7, 4.9), and (4.10),

$$\int_\Omega \bar{f}(x,u(x),\nabla u(x))\,dx = \mathcal{G}(v;\Omega) \le \liminf_{n\to\infty} \int_\Omega \hat{f}(x,u,V_n)\,dx \le \liminf_{n\to\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx,$$

and, in turn,

$$\int_\Omega \bar{f}(x,u(x),\nabla u(x))\,dx \le \inf\left\{ \liminf_{n\to\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx : \; \{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^N), \right.$$
$$\left. \operatorname{div} u_n = 0, \quad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N) \right\}.$$

To prove the converse inequality, fix $\varepsilon > 0$. By the definition of $\mathcal{G}(v;\Omega)$, there exists a p-equi-integrable sequence $\{V_n\} \subset L^p(D;\mathbb{R}^{N^2-1}) \cap \ker\mathcal{A}$ such that $V_n \rightharpoonup v$ in $L^p(D;\mathbb{R}^{N^2-1})$ and

$$\int_\Omega \bar{f}(x,u(x),\nabla u(x))\,dx + \varepsilon > \liminf_{n\to\infty} \int_D \hat{g}(x,V_n(x))\,dx = \liminf_{n\to\infty} \int_D \hat{f}(x,u(x),V_n(x))\,dx, \tag{4.11}$$

where we used for v the notation introduced in (4.8). Now $\mathcal{A}V_n = 0$ if and only if there exists $\varphi_n \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$V_n = \left(\nabla\varphi_n^{(1)}, \dots, \nabla\varphi_n^{(N-1)}, \begin{pmatrix} \frac{\partial\varphi_n^{(N)}}{\partial x_1} \\ \vdots \\ \frac{\partial\varphi_n^{(N)}}{\partial x_{N-1}} \end{pmatrix} \right)$$

and $\frac{\partial \varphi_n^{(N)}}{\partial x_N} = -\frac{\partial \varphi_n^{(1)}}{\partial x_1} - \ldots - \frac{\partial \varphi_n^{(N-1)}}{\partial x_{N-1}}$. Since $\nabla\varphi_n \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^{N^2})$, we have that

$$\varphi_n - \frac{1}{|\Omega|}\int_\Omega \varphi_n(x)\,dx \rightharpoonup U \text{ in } W^{1,p}(\Omega;\mathbb{R}^N),$$

where $U = u + c$ for some constant $c \in \mathbb{R}^N$. So

$$u_n := \varphi_n - \frac{1}{|\Omega|}\int_\Omega \varphi_n(x)\,dx - c \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N),$$

and div $u_n = 0$. Consider a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\lim_{k\to\infty}\int_\Omega \hat{f}(x,u(x),V_{n_k}(x))\,dx = \liminf_{n\to\infty}\int_\Omega \hat{f}(x,u(x),V_n(x))\,dx$$

and $\{(u_{n_k},V_{n_k})\}$ and $\{(u,V_{n_k})\}$ generates the Young measure $\{\delta_{u(x)}\otimes\nu_x\}_{x\in\Omega}$. Since $\{V_{n_k}\}$ is p-equi-integrable and u_{n_k} converge to u strongly in $L^p(\Omega;\mathbb{R}^N)$, it follows from Theorem 2.1 and the growth condition on f that

$$\lim_{k\to\infty}\int_\Omega \hat{f}(x,u(x),V_{n_k}(x))\,dx = \lim_{k\to\infty}\int_\Omega \hat{f}(x,u_{n_k}(x),V_{n_k}(x))\,dx.$$

By (4.11)

$$\begin{aligned}\int_\Omega \bar{f}(x,u(x),\nabla u(x))\,dx + \varepsilon &> \lim_{k\to\infty}\int_\Omega \hat{f}(x,u_{n_k}(x),V_{n_k}(x))\,dx = \lim_{k\to\infty}\int_\Omega f(x,u_{n_k}(x),\nabla u_{n_k}(x))\,dx\\ &\geq \inf\Bigg\{\liminf_{n\to\infty}\int_\Omega f(x,u_n(x),\nabla u_n(x))\,dx : \{u_n\}\subset W^{1,p}(\Omega;\mathbb{R}^N),\\ &\qquad \operatorname{div} u_n = 0, \quad u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^N)\Bigg\}.\end{aligned}$$

It now suffices to let $\varepsilon \to 0^+$. ☐

5. Homogenization

In this section we will limit our analysis to the case where $1 < q < \infty$.

Lemma 5.1. *Let $f : \mathbb{R}^N\times\mathbb{R}^d \to [0,\infty)$ be a continuous function satisfying (A_1)-(A_2). Let $v \in L^q(D;\mathbb{R}^d)\cap\ker\mathcal{A}$, where $D \in \mathcal{O}(\Omega)$, $\varepsilon_k \to 0^+$, and let $\{\hat{v}_k\}\subset L^q(D_1;\mathbb{R}^d)$ be a sequence of functions such that*

$$\hat{v}_k \rightharpoonup v \quad in \quad L^q(D_1;\mathbb{R}^d), \qquad \mathcal{A}\hat{v}_k \to 0 \quad in \quad W^{-1,q}(D_1;\mathbb{R}^l),$$

for some $D_1 \in \mathcal{O}(\Omega)$, with $D_1 \subset D$. Then we can find a q-equi-integrable sequence $\{v_k\}\subset L^q(D;\mathbb{R}^d)\cap\ker\mathcal{A}$ such that $\int_D v_k\,dx = \int_D v\,dx$,

$$v_k \rightharpoonup v \quad in \quad L^q(D;\mathbb{R}^d), \qquad ||\hat{v}_k - v_k||_{L^s(D_1)} \to 0 \quad for\ all\ 1 \leq s < q \tag{5.1}$$

and

$$\begin{aligned}\liminf_{k\to\infty}\int_{D_1} f\left(x/\varepsilon_k, v_k(x)\right)\,dx &\le \liminf_{k\to\infty}\int_{D_1} f\left(x/\varepsilon_k, \hat{v}_k(x)\right)\,dx,\\ \limsup_{k\to\infty}\int_{D\setminus D_1}|v_k(x)|^q\,dx &\le \int_{D\setminus D_1}|v(x)|^q\,dx.\end{aligned}\tag{5.2}$$

Moreover, if $D=Q$*, then* $v_k=v+w_k$*, with* $w_k\in L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d)\cap\ker\mathcal{A}$.

Remark 5.2. Lemma 5.1 implies, in particular, that for every $v\in L^q(\Omega;\mathbb{R}^d)\cap\ker\mathcal{A}$

$$\Gamma-\liminf_{n\to\infty}\mathcal{F}_{\varepsilon_n}(v;D)=\inf\Big\{\liminf_{n\to\infty}\mathcal{F}_{\varepsilon_n}(v_n;D):\ v_n\in L^q(D;\mathbb{R}^d)\cap\ker\mathcal{A},\quad v_n\rightharpoonup v \text{ in } L^q(D;\mathbb{R}^d),\quad \int_D v_n\,dx=\int_D v\,dx\Big\},$$

and if $D=Q$ then

$$\Gamma-\liminf_{n\to\infty}\mathcal{F}_{\varepsilon_n}(v;Q)=\inf\Big\{\liminf_{n\to\infty}\mathcal{F}_{\varepsilon_n}(v+w_n;Q):\ w_n\in L^q_{1-\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)\cap\ker\mathcal{A}\quad w_n\rightharpoonup 0 \text{ in } L^q(Q;\mathbb{R}^d),\quad \int_Q w_n\,dx=0\Big\}.$$

Proof of Lemma 5.1. Let $g(x):=x$ in Q and extend it periodically to $\mathbb{R}^N$ with period 1. Set $g_k(x):=g(x/\varepsilon_k)$. Since $\{g_k\}$ is bounded in L^∞ and $\hat{v}_k\rightharpoonup v$ in $L^q(D_1;\mathbb{R}^d)$, by Theorem 2.1 there exists a subsequence $\{\varepsilon_n\}$ of $\{\varepsilon_k\}$ such that

$$\{(g_n(x),\hat{v}_n(x))\}\quad\text{generates a Young measure }\{\nu_x\}$$

and

$$\lim_{n\to\infty}\int_{D_1} f\left(x/\varepsilon_n,\hat{v}_n(x)\right)\,dx=\liminf_{k\to\infty}\int_{D_1} f\left(x/\varepsilon_k,\hat{v}_k(x)\right)\,dx.$$

For $i\in\mathbb{N}$ let

$$F_i:=\left\{x\in D_1:\ \mathrm{dist}(x,\partial D_1)<\frac{1}{i}\right\}$$

and consider cut-off functions θ_i with compact support in D_1 and such that $\theta_i\equiv 1$ in $D_1\setminus F_i$. Set $w_{i,n}:=\theta_i(\hat{v}_n-v)\in L^q(Q;\mathbb{R}^d)$. Then we can proceed as in the proof of Lemma 3.1 to find a q-equi-integrable sequence $\{v_i:=v+w_i\}$, where $\{w_i\}$ satisfies (3.3, 5.1) holds, and the two sequences

$$\{(g_{n_i}(x),v_i(x))\}\qquad\text{and}\qquad\{(g_{n_i}(x),\hat{v}_{n_i}(x))\}$$

generate the same Young measure $\{\nu_x\}$. Hence by Theorem 2.1

$$\begin{aligned}\lim_{i\to\infty}\int_{D_1} f\left(x/\varepsilon_{n_i},v_i(x)\right)\,dx&=\int_{D_1}\left(\int_{\mathbb{R}^N\times\mathbb{R}^d} f(X,V)d\nu_x(X,V)\right)dx\le\lim_{i\to\infty}\int_{D_1} f\left(x/\varepsilon_{n_i},\hat{v}_{n_i}(x)\right)\,dx\\&=\liminf_{k\to\infty}\int_{D_1} f\left(x/\varepsilon_k,\hat{v}_k(x)\right)\,dx,\end{aligned}$$

where we have used (A_2), and the facts that $\{v_i(x)\}$ is q-equi-integrable over D_1, and that f is a continuous function.

To prove the second inequality in (5.2), we remark that by (3.3) and the fact that $\hat{w}_i = \theta_i(\hat{v}_{n_i} - v) \equiv 0$ outside D_1, we have for all $1 \leq s < q$

$$||v_i - v||_{L^s(D\setminus D_1)} = ||\hat{w}_i - w_i||_{L^s(D\setminus D_1)} \to 0.$$

Hence $\{v_i(x)\}$ generates the Young measure $\{\mu_x = \delta_{v(x)}\}$ on $D \setminus D_1$, and since $\{v_i\}$ is q-equi-integrable we have that

$$\limsup_{i\to\infty} \int_{D\setminus D_1} |v_i(x)|^q \, dx = \int_{D\setminus D_1} |Y|^q d\mu_x(Y) dx = \int_{D\setminus D_1} |v(x)|^q \, dx.$$

To complete the proof it suffices to define $v_k := v_{n_i}$ for each $n_i \leq k < n_{i+1}$. Clearly

$$\liminf_{k\to\infty} \int_{D_1} f\left(x/\varepsilon_k, v_k(x)\right) \, dx \leq \liminf_{i\to\infty} \int_{D_1} f\left(x/\varepsilon_{n_i}, v_i(x)\right) \, dx.$$

□

Lemma 5.3. *Let $\varepsilon_n \to 0^+$ and let $\mathcal{R}(\Omega)$ be the family of all finite unions of open cubes contained in Ω and with vertices in $\mathbb{Q}^N$. Then there exists a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that the Γ–limit*

$$\Gamma - \lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(v; R)$$

exists for all $v \in L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$ and for all $R \in \mathcal{R}(\Omega)$.

Proof of Lemma 5.3. Fix $R \in \mathcal{R}(\Omega)$. For simplicity set $\mathcal{F}_n := \mathcal{F}_{\varepsilon_n}$ and let $\mathcal{B}$ denote the closed unit ball of $L^q(R;\mathbb{R}^d)$. For each $l \in \mathbb{N}$ consider

$$l\,\mathcal{B} := \{v \in L^q(R;\mathbb{R}^d) : \, ||v||_{L^q} \leq l\}\cdot$$

Since $q > 1$ the dual of $L^q(R;\mathbb{R}^d)$ is separable, and hence the space $l\,\mathcal{B}$ endowed with the weak topology is metrizable. Let d_l be any metric which generates the L^q-weak topology. Consider $l = 1$ and apply Proposition 2.4 to the sequence of functionals $\{\mathcal{F}_n(\cdot;R)\}$ restricted to $(\mathcal{B} \cap \ker \mathcal{A}, d_1)$. Then we can find an increasing sequence of integers $\{n_j^1\}$ such that

$$\Gamma(d_1) - \lim_{j\to\infty} \mathcal{F}_{n_j^1}(v; R)$$

exists for all $v \in \mathcal{B} \cap \ker \mathcal{A}$. We now proceed recursively, so that given $l \in \mathbb{N}$ we apply Proposition 2.4 to the sequence of functionals $\{\mathcal{F}_{n_j^{l-1}}(\cdot;R)\}$ restricted to $(l\,\mathcal{B} \cap \ker \mathcal{A}, d_l)$ to obtain a subsequence $\{n_j^l\}$ of $\{n_j^{l-1}\}$ such that

$$\Gamma(d_l) - \lim_{j\to\infty} \mathcal{F}_{n_j^l}(v; R)$$

exists for all $v \in l\,\mathcal{B} \cap \ker \mathcal{A}$. Let $n_k := n_k^k$. Since $\{n_k\}$ is a subsequence of all $\{n_j^l\}$ we have that for each $l \in \mathbb{N}$

$$\Gamma(d_l) - \lim_{k\to\infty} \mathcal{F}_{n_k}(v; R)$$

exists for all $v \in l\,\mathcal{B} \cap \ker \mathcal{A}$.

We claim that the Γ–limit

$$\Gamma - \lim_{k\to\infty} \mathcal{F}_{n_k}(v; R) \tag{5.3}$$

exists for all $v \in L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$. Indeed assume by contradiction that this is not the case. Then there exists $v \in L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$ for which

$$\mathcal{F}^-(v;R) := \Gamma - \liminf_{k\to\infty} \mathcal{F}_{n_k}(v;R) < \mathcal{F}^+(v;R) := \Gamma - \limsup_{k\to\infty} \mathcal{F}_{n_k}(v;R).$$

Let $v_k \in L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$ be such that $v_k \rightharpoonup v$ in $L^q(R;\mathbb{R}^d)$ and

$$\liminf_{k\to\infty} \mathcal{F}_{n_k}(v_k;R) = \mathcal{F}^-(v;R).$$

Since $v_k \rightharpoonup v$ in $L^q(R;\mathbb{R}^d)$, we may find an integer l_0 such that $v_k,\, v \in l_0\,\mathcal{B} \cap \ker \mathcal{A}$ for all $k \in \mathbb{N}$. Consequently

$$d_{l_0}(v_k, v) \to 0 \qquad \text{as } k \to \infty,$$

and thus

$$\Gamma(d_{l_0}) - \liminf_{k\to\infty} \mathcal{F}_{n_k}(v;R) \le \liminf_{k\to\infty} \mathcal{F}_{n_k}(v_k;R) = \mathcal{F}^-(v;R) < \mathcal{F}^+(v;R) \le \Gamma(d_{l_0}) - \limsup_{k\to\infty} \mathcal{F}_{n_k}(v;R),$$

which contradicts the existence of the Γ-limit $\Gamma(d_{l_0}) - \lim_{k\to\infty} \mathcal{F}_{n_k}(v;R)$, and where we have used the fact that

$$\begin{aligned}
\mathcal{F}^+(v;R) &= \inf\left\{ \limsup_{k\to\infty} \mathcal{F}_{n_k}(z_k;R) :\ z_k \in L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}, \quad z_k \rightharpoonup v \text{ in } L^q(R;\mathbb{R}^d) \right\} \\
&\le \Gamma(d_{l_0}) - \limsup_{k\to\infty} \mathcal{F}_{n_k}(v;R) \\
&= \inf\left\{ \limsup_{k\to\infty} \mathcal{F}_{n_k}(z_k;R) :\ z_k \in l_0\,\mathcal{B} \cap \ker \mathcal{A}, \quad z_k \rightharpoonup v \text{ in } L^q(R;\mathbb{R}^d) \right\}\cdot
\end{aligned}$$

Hence (5.3) holds. To conclude the proof of the lemma it suffices to observe that since the family $\mathcal{R}(\Omega)$ is countable, with a diagonal process it is possible to extract a further subsequence for which (5.3) holds for all $R \in \mathcal{R}(\Omega)$.

□

Remark 5.4. The previous proof asserts that for any given $D \in \mathcal{O}(\Omega)$ and $\varepsilon_n \to 0^+$ there exists a subsequence $\{\varepsilon_{n_k}\}$ (depending on the particular set D) of $\{\varepsilon_n\}$ such that such that the Γ–limit

$$\Gamma - \lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(v;D)$$

exists for all $v \in L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$.

Lemma 5.5. *Assume that conditions* (A_1)-(A_2) *hold. Given* $\varepsilon_n \to 0^+$, *let* $\{\varepsilon_{n_k}\}$ *be as in Lemma* 5.3, *and for any* $D \in \mathcal{O}(\Omega)$ *set*

$$\mathcal{F}^-(\cdot;D) := \Gamma - \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(\cdot;D).$$

Then $\mathcal{F}^-(v;\cdot)$ *is the trace of a Radon measure.*

Proof of Lemma 5.5. We start by establishing inner regularity. Precisely, we claim that for any $v \in L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ and $D \in \mathcal{O}(\Omega)$

$$\mathcal{F}^-(v;D) = \sup\left\{ \mathcal{F}^-(v;R) :\ R \in \mathcal{R}(\Omega),\ R \subset D \right\} = \lim_{R \nearrow D} \mathcal{F}^-(v;R), \tag{5.4}$$

where the limit is taken over all finite unions of cubes $R \in \mathcal{R}(\Omega)$ with $R \subset D$. For fixed $\eta > 0$ there exists $\delta > 0$ such that

$$\int_{D_0} C(1 + |v(x)|^q)\, dx < \eta \tag{5.5}$$

for any measurable set $D_0 \subset D$, with $|D_0| < \delta$, and where C is the constant given in (A_2). Let $R \in \mathcal{R}(\Omega)$, with $R \subset D$ and $|D \setminus R| < \delta$, and, in light of Lemma 5.3, consider a sequence $\{\hat{v}_k\} \subset L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$, with $\hat{v}_k \rightharpoonup v$ in $L^q(R;\mathbb{R}^d)$, and such that

$$\lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(\hat{v}_k; R) = \mathcal{F}^-(v;R).$$

By Lemma 5.1 there exists a q-equi-integrable sequence $\{v_k\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$v_k \rightharpoonup v \quad \text{in} \quad L^q(D;\mathbb{R}^d), \qquad \int_D v_k\,dx = \int_D v\,dx,$$

and

$$\begin{aligned} \liminf_{k\to\infty} \int_R f\left(x/\varepsilon_{n_k}, v_k(x)\right)\,dx &\le \lim_{k\to\infty} \int_R f\left(x/\varepsilon_{n_k}, \hat{v}_k(x)\right)\,dx, \\ \limsup_{k\to\infty} \int_{D\setminus R} |v_k(x)|^q\,dx &\le \int_{D\setminus R} |v(x)|^q\,dx. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{F}^-(v;D) &\le \liminf_{k\to\infty} \int_D f\left(x/\varepsilon_{n_k}, v_k(x)\right)\,dx \le \lim_{k\to\infty} \int_R f\left(x/\varepsilon_{n_k}, \hat{v}_k(x)\right)\,dx + \limsup_{k\to\infty} \int_{D\setminus R} C(1+|v_k(x)|^q)\,dx \\ &\le \mathcal{F}^-(v;R) + \int_{D\setminus R} C(1+|v(x)|^q)\,dx \le \mathcal{F}^-(v;R) + \eta, \end{aligned}$$

where we have used (A_2) and (5.5). Consequently

$$\mathcal{F}^-(v;D) \le \sup\left\{\mathcal{F}^-(v;R):\ R \in \mathcal{R}(\Omega),\ R \subset D\right\} + \eta,$$

and letting $\eta \to 0^+$ we obtain one inequality in (5.4). To show the opposite inequality, note that if $\{v_k\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$, with $v_k \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$, then the restriction of v_k to R belongs to $L^q(R;\mathbb{R}^d) \cap \ker \mathcal{A}$, and $v_k \rightharpoonup v$ in $L^q(R;\mathbb{R}^d)$. Therefore

$$\mathcal{F}^-(v;R) \le \liminf_{k\to\infty} \int_R f\left(x/\varepsilon_{n_k}, v_k(x)\right)\,dx \le \liminf_{k\to\infty} \int_D f\left(x/\varepsilon_{n_k}, v_k(x)\right)\,dx$$

and by taking the infimum over all such sequences we get that

$$\mathcal{F}^-(v;R) \le \mathcal{F}^-(v;D), \tag{5.6}$$

and in turn (5.4) holds.

In order to prove that $\mathcal{F}^-(v;\cdot)$ is the trace of a Radon measure, as it is usual it suffices to prove subadditivity for nested sets (see [3, 21]). Let $B \subset\subset C \subset\subset D$. By (5.4) for fixed $\eta > 0$ we find $R \in \mathcal{R}(\Omega)$ such that $R \subset D$ and

$$\mathcal{F}^-(v;D) \le \eta + \mathcal{F}^-(v;R).$$

Construct $R_1,\ R_2 \in \mathcal{R}(\Omega)$ with

$$R \subset R_1 \cup R_2, \qquad R_1 \subset D \setminus \overline{B} \quad \text{and} \quad R_2 \subset C.$$

By (5.6) we have

$$\mathcal{F}^-(v;D) \le \eta + \mathcal{F}^-(v;R) \le \eta + \mathcal{F}^-(v;R_1 \cup R_2). \tag{5.7}$$

By the definition of Γ-convergence and Lemma 5.1 there exist $v_k \in L^q(R_1;\mathbb{R}^d) \cap \ker \mathcal{A}$ and $w_k \in L^q(R_2;\mathbb{R}^d) \cap \ker \mathcal{A}$, with $v_k \rightharpoonup v$ in $L^q(R_1;\mathbb{R}^d)$ and $w_k \rightharpoonup v$ in $L^q(R_2;\mathbb{R}^d)$, such that

$$\mathcal{F}^-(v;R_1) = \lim_{k\to\infty} \mathcal{F}_{\tilde{\varepsilon}_{n_k}}(v_k;R_1), \qquad \mathcal{F}^-(v;R_2) = \lim_{k\to\infty} \mathcal{F}_{\tilde{\varepsilon}_{n_k}}(w_k;R_2), \tag{5.8}$$

where $\{\tilde{\varepsilon}_{n_k}\}$ is a subsequence of $\{\varepsilon_{n_k}\}$ and $\{v_k\}$, $\{w_k\}$ are q-equi-integrable over R_1 and R_2, respectively. Let θ_j be smooth cut-off functions which are equal to 1 on B and 0 on $D \setminus \overline{C}$, and such that $|\{0 < \theta_j < 1\}| \to 0$ as $j \to \infty$. Set

$$\hat{V}_{j,k} := (1-\theta_j)v_k + \theta_j w_k.$$

For j fixed

$$\mathcal{A}\hat{V}_{j,k} = (1-\theta_j)\mathcal{A}v_k + \theta_j \mathcal{A}w_k - \sum_{i=1}^N A^{(i)} v_k \frac{\partial \theta_j}{\partial x_i} + \sum_{i=1}^N A^{(i)} w_k \frac{\partial \theta_j}{\partial x_i} \to 0$$

in $W^{-1,q}(R_1 \cup R_2;\mathbb{R}^l)$ strong, because $||v_k - w_k||_{W^{-1,q}(B\setminus\overline{C};\mathbb{R}^l)} \to 0$ as $k \to \infty$. Diagonalize to get $\hat{V}_j := \hat{V}_{j,k_j}$ such that

$$\hat{V}_j \rightharpoonup v \quad \text{in} \quad L^q(R_1 \cup R_2;\mathbb{R}^d), \qquad \mathcal{A}\hat{V}_j \to 0 \quad \text{in} \quad W^{-1,q}(R_1 \cup R_2;\mathbb{R}^l).$$

By Lemma 5.1 we can find $V_j \in L^q(R_1 \cup R_2;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $V_j \rightharpoonup v$ in $L^q(R_1 \cup R_2;\mathbb{R}^d)$ and

$$\liminf_{j\to\infty} \int_{R_1\cup R_2} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, V_j(x)\right)\,dx \le \liminf_{j\to\infty} \int_{R_1\cup R_2} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, \hat{V}_j(x)\right)\,dx.$$

Consequently, by (5.7)

$$\begin{aligned}
\mathcal{F}^-(v;D) &\le \eta + \mathcal{F}^-(R_1 \cup R_2;D) \le \eta + \liminf_{j\to\infty} \int_{R_1\cup R_2} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, V_j(x)\right)\,dx \\
&\le \eta + \liminf_{j\to\infty} \int_{R_1\cup R_2} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, \hat{V}_j(x)\right)\,dx \le \eta + \limsup_{j\to\infty} \int_{R_1} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, v_{k_j}(x)\right)\,dx \\
&\quad + \limsup_{j\to\infty} \int_{R_2} f\left(x/\tilde{\varepsilon}_{n_{k_j}}, w_{k_j}(x)\right)\,dx + \limsup_{j\to\infty} \int_{\{0<\theta_j<1\}} C(1 + |w_{k_j}(x)|^q + |v_{k_j}(x)|^q) \\
&\le \eta + \mathcal{F}^-(v;R_1) + \mathcal{F}^-(v;R_2) \le \eta + \mathcal{F}^-(v;D \setminus \overline{B}) + \mathcal{F}^-(v;C),
\end{aligned}$$

where we have used (5.6) and the fact that in (5.8) inferior limits are actually limits. It now suffices to let $\eta \to 0^+$.

□

Lemma 5.6. *Under conditions* (A_1)-(A_2), *for* $\mathcal{L}^N$ *a.e.* $x_0 \in \Omega$ *we have*

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) = f_{\text{hom}}(v(x_0)).$$

Proof of Lemma 5.6. We divide the proof in three steps.

Given $\varepsilon_n \to 0^+$, let $\{\varepsilon_{n_k}\}$ be as in Lemma 5.3. In order to simplify the notations, in the proof of this lemma *we will represent* $\{\varepsilon_{n_k}\}$ *simply by* $\{\varepsilon\}$.

Step 1. We claim that

$$\mathcal{F}^-(v(\cdot - x_0);D + x_0) = \mathcal{F}^-(v;D).$$

The proof is similar to the one of Lemma 3.9 in [11]. We present it here for the convenience of the reader. Let $v_\varepsilon \in L^q(D;\mathbb{R}^d)\cap \ker\mathcal{A}$ be such that $v_\varepsilon \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$ and

$$\mathcal{F}^-(v;D) = \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(v_\varepsilon;D). \tag{5.9}$$

Consider the sequence $z_\varepsilon := [x_0/\varepsilon] \in \mathbb{Z}^N$, so that $x_\varepsilon := z_\varepsilon\,\varepsilon$ converges to x_0. Here $[z] := ([z_1],\dots[z_N])$, with $[z_i]$ denoting the integer part of $z_i \in \mathbb{R}$. By the periodicity of f,

$$\mathcal{F}_\varepsilon(v_\varepsilon;D) = \int_D f\left(\frac{x+x_\varepsilon}{\varepsilon}, v_\varepsilon(x)\right)dx = \int_{D+x_\varepsilon} f\left(\frac{y}{\varepsilon}, v_\varepsilon(y-x_\varepsilon)\right)dy.$$

Let $B \subset\subset D$. For ε sufficiently small we have that $D + x_\varepsilon \supset B + x_0$, and thus

$$\mathcal{F}_\varepsilon(v_\varepsilon;D) \geq \int_{B+x_0} f\left(\frac{y}{\varepsilon}, v_\varepsilon(y-x_\varepsilon)\right)dy. \tag{5.10}$$

Since $v_\varepsilon(\cdot - x_\varepsilon) \rightharpoonup v(\cdot - x_0)$ in $L^q(B+x_0;\mathbb{R}^d)$, and $v_\varepsilon(\cdot - x_\varepsilon) \in L^q(B+x_0;\mathbb{R}^d)\cap\ker\mathcal{A}$, by (5.9, 5.10), we obtain

$$\mathcal{F}^-(v;D) \geq \mathcal{F}^-(v(\cdot - x_0); B+x_0).$$

By letting $R \nearrow D + x_0$, $R \in \mathcal{R}(\Omega)$, setting $B := R - x_0$ above, we obtain by (5.4)

$$\mathcal{F}^-(v;D) \geq \mathcal{F}^-(v(\cdot - x_0); D+x_0).$$

The converse inequality follows in a similar way.

Step 2. Next, we show that

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \geq f_{\text{hom}}(v(x_0)) \quad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega.$$

Fix $x_0 \in \Omega$ such that

$$\lim_{r\to 0^+} \frac{1}{r^N}\int_{Q(x_0,r)} |v(x)-v(x_0)|^q\,dx = 0 \tag{5.11}$$

and

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\liminf_{\varepsilon\to 0^+} \frac{1}{r^N}\int_{rQ} f\left(\frac{x}{\varepsilon}, v_{\varepsilon,r}(x)\right)dx < \infty,$$

where we have used Step 1 and Lemma 5.5, and where we have chosen the radii $r \to 0^+$ such that $\mathcal{F}^-(v(\cdot + x_0);\partial(rQ)) = 0$. Here $v_{\varepsilon,r} \in L^q(rQ;\mathbb{R}^d)\cap\ker\mathcal{A}$ and $v_{\varepsilon,r} \rightharpoonup v(\cdot + x_0)$ in $L^q(rQ;\mathbb{R}^d)$ as $\varepsilon \to 0^+$. Then

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+}\liminf_{\varepsilon\to 0^+} \int_Q f\left(\frac{r}{\varepsilon}\,y, v(x_0)+w_{\varepsilon,r}(y)\right)dy$$

where $w_{\varepsilon,r}(y) := v_{\varepsilon,r}(ry) - v(x_0)$. As in the proof of Lemma 3.5, we have that $w_{\varepsilon,r} \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ if we first let $\varepsilon \to 0$ and then $r \to 0^+$. Diagonalize to get $\hat{w}_k \in L^q(Q;\mathbb{R}^d)\cap\ker\mathcal{A}$ such that $\hat{w}_k \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$,

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty}\int_Q f(s_k\,y, v(x_0)+\hat{w}_k(y))\,dy,$$

and where $s_k := 1/\varepsilon_k \to \infty$. By Lemma 5.1, applied to the Carathéodory function $h(x,v) := f(x, v(x_0)+v)$, there exists a q-equi-integrable sequence $\{w_k\} \subset L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$w_k \rightharpoonup 0 \quad \text{in} \quad L^q(Q;\mathbb{R}^d), \qquad \int_Q w_k\, dy = 0,$$

and

$$\liminf_{k\to\infty} \int_Q f(s_k\, y, v(x_0) + w_k(y))\, dy \le \lim_{k\to\infty} \int_Q f(s_k\, y, v(x_0) + \hat{w}_k(y))\, dy.$$

Consequently

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \ge \liminf_{k\to\infty} \int_Q f(s_k\, y, v(x_0) + w_k(y))\, dy \ge \liminf_{i\to\infty} \liminf_{k\to\infty} \int_Q f(s_k\, y, v(x_0) + \theta_i(y) w_k(y))\, dy,$$

where $0 \le \theta_i \le 1$ are smooth cut-off functions with compact support in Q such that $\theta_i \equiv 1$ in $(1-1/i)Q$, and where we used the q-equi-integrability of $\{w_k\}$ and (A_2). Then $\theta_i\, w_k \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ as $k \to \infty$ and $i \to \infty$, in this order, and

$$\lim_{i\to\infty} \lim_{k\to\infty} \mathcal{A}(\theta_i\, w_k) = 0 \quad \text{in} \quad W^{-1,q}(Q;\mathbb{R}^l).$$

Diagonalize to get $U_i := \theta_i\, w_{k_i}$ extended by zero outside Q, such that $U_i \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$, $\mathcal{A}U_i \to 0$ in $W^{-1,q}(Q;\mathbb{R}^l)$ as $i \to \infty$, and

$$\liminf_{i\to\infty} \liminf_{k\to\infty} \int_Q f(s_k\, y, v(x_0) + \theta_i(y) w_k(y))\, dy = \liminf_{i\to\infty} \int_Q f(s_{k_i}, v(x_0) + U_i(y))\, dy.$$

Thus

$$\begin{aligned}\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \ge& \liminf_{i\to\infty} \frac{1}{s_{k_i}^N} \int_{s_{k_i} Q} f\left(x, v(x_0) + U_i\left(\frac{x}{s_{k_i}}\right)\right) dx \\ \ge& \liminf_{i\to\infty} \frac{1}{s_{k_i}^N} \int_{([s_{k_i}]+1)Q} f\left(x, v(x_0) + U_i\left(\frac{x}{s_{k_i}}\right)\right) dx \\ &- \limsup_{i\to\infty} \frac{1}{s_{k_i}^N} \int_{([s_{k_i}]+1)Q\setminus s_{k_i} Q} f\left(x, v(x_0) + U_i\left(\frac{x}{s_{k_i}}\right)\right) dx,\end{aligned}$$

where $[s_{k_i}]$ denotes the integer part of s_{k_i}. We claim that the last limit is zero. Indeed

$$\frac{1}{s_{k_i}^N} \int_{([s_{k_i}]+1)Q\setminus s_{k_i} Q} f\left(x, v(x_0) + U_i\left(\frac{x}{s_{k_i}}\right)\right) dx = \int_{\frac{([s_{k_i}]+1)}{s_{k_i}}Q\setminus Q} f(s_{k_i}\, y, v(x_0) + U_i(y))\, dy.$$

Since $([s_{k_i}]+1)/s_{k_i} \to 1$, we have that

$$\left| \frac{([s_{k_i}]+1)}{s_{k_i}} Q \setminus Q \right| = \left(\frac{([s_{k_i}]+1)}{s_{k_i}} \right)^N - 1 \to 0,$$

and thus the claim follows from the q-equi-integrability of $\{U_i\}$ and (A_2). Hence, setting

$$m_i := 1/s_{k_i}, \qquad n_i := [s_{k_i}] + 1 \in \mathbb{N},$$

we obtain

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \ge \liminf_{i\to\infty} \frac{1}{n_i^N} \int_{n_i Q} f(x, v(x_0) + U_i(m_i\, x))\, dx = \liminf_{i\to\infty} \int_Q f(n_i\, y, v(x_0) + U_i(n_i\, m_i y))\, dy.$$

We claim that

$$U_i(n_i\, m_i\, \cdot) \rightharpoonup 0 \quad \text{in } L^q(Q;\mathbb{R}^d), \qquad \mathcal{A}U_i(n_i\, m_i\, \cdot) \to 0 \quad \text{in } W^{-1,q}(Q;\mathbb{R}^l) \tag{5.12}$$

as $i \to \infty$. Assuming that the claim holds, by Lemma 5.1 there exists a q-equi-integrable sequence $\{V_i\} \subset L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$V_i \rightharpoonup 0 \quad \text{in} \quad L^q(Q;\mathbb{R}^d), \qquad \int_Q V_i\, dy = 0,$$

and

$$\begin{aligned}\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) &\geq \liminf_{i\to\infty} \int_Q f\left(n_i\, y, v(x_0) + U_i(n_i\, m_i y)\right) dy \geq \liminf_{i\to\infty} \int_Q f\left(n_i\, y, v(x_0) + V_i(y)\right) dy \\ &= \liminf_{i\to\infty} \frac{1}{n_i^N} \int_{n_i Q} f\left(x, v(x_0) + V_i\left(\frac{x}{n_i^N}\right)\right) dx \geq f_{\text{hom}}(v(x_0)),\end{aligned}$$

and where we have used the facts that

$$V_i\left(\frac{1}{n_i^N}\,\cdot\right) \in L^q_{n_i\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}, \qquad \int_{n_i Q} V_i\left(\frac{1}{n_i^N}\, y\right) dy = 0.$$

Thus it remains to show (5.12). If $\varphi \in C_c^\infty(Q)$ then

$$\begin{aligned}\int_Q U_i(n_i\, m_i\, y)\, \varphi(y)\, dy &= \frac{1}{(n_i\, m_i)^N} \int_{n_i\, m_i\, Q} U_i(x)\, \varphi\left(\frac{x}{n_i\, m_i}\right) dx \\ &= \frac{1}{(n_i\, m_i)^N}\left(\int_Q U_i(x)\, \varphi(x)\, dx + \int_Q U_i(x)\left(\varphi\left(\frac{x}{n_i\, m_i}\right) - \varphi(x)\right) dx\right),\end{aligned}$$

where we have used the fact that $U_i(x) \equiv 0$ in $n_i\, m_i\, Q \setminus Q$. Since $U_i \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ and $n_i\, m_i \to 1$ the first integral on the right hand side of the previous inequality tends to zero as $i \to \infty$. By Hölder's inequality

$$\left|\int_Q U_i(x)\left(\varphi\left(\frac{x}{n_i\, m_i}\right) - \varphi(x)\right) dx\right| \leq \left(\sup_l ||U_l||_{L^q(Q)}\right)\left(\int_Q \left|\varphi\left(\frac{x}{n_i\, m_i}\right) - \varphi(x)\right|^{q'} dx\right)^{1/q'}.$$

Since φ is bounded we can apply Lebesgue Dominated Convergence Theorem to conclude that the right hand side approaches zero as $i \to \infty$. In a similar way we can show that

$$\mathcal{A}U_i(n_i\, m_i \cdot) \to 0 \quad \text{in } W^{-1,q}(Q;\mathbb{R}^l) \text{ as } i \to \infty.$$

We omit the details.

Step 3. To conclude the proof of the lemma it remains to show that

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \leq f_{\text{hom}}(v(x_0)) \quad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega.$$

By Remark 1.8(i), for any fixed $\eta > 0$ we may find $k \in \mathbb{N}$, $w \in L^\infty_{k\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $\int_{kQ} w\, dx = 0$ and

$$\frac{1}{k^N} \int_{kQ} f(x, v(x_0) + w(x))\, dx \leq f_{\text{hom}}(v(x_0)) + \eta. \tag{5.13}$$

For any fixed $r > 0$ and for any $n \in \mathbb{N}$, let $u_{n,r}(x) := w(xnk/r)$. Then $u_{n,r} \in L^\infty(rQ;\mathbb{R}^d) \cap \ker A$, $u_{n,r} \rightharpoonup 0$ in $L^q(rQ;\mathbb{R}^d)$ as $n \to \infty$, and by Step 1

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+} \frac{\mathcal{F}^-(v(\cdot + x_0); rQ)}{r^N} \le \liminf_{r\to 0^+} \liminf_{n\to\infty} \frac{1}{r^N} \int_{rQ} f\left(\frac{x}{\varepsilon_{n,r}}, v(x_0 + x) + u_\varepsilon(x)\right) dx$$
$$= \liminf_{r\to 0^+} \liminf_{n\to\infty} \int_Q f\left(nk\, y, v(x_0 + r\, y) + w\left(nk\, y\right)\right) dy,$$

where $\varepsilon_{n,r} := r/nk$. Since $f(\cdot, v)$ is Q-periodic, there exists $\delta > 0$ such that if $|v - v(x_0)| < \delta$ then

$$\sup_{s\in\mathbb{R}^N} |f(s, v + w(s)) - f(s, v(x_0) + w(s))| < \eta.$$

Setting $E_{r,\delta} := \{y \in Q : |v(x_0 + ry) - v(x_0)| \ge \delta\}$, we deduce that

$$\frac{d\mathcal{F}^-(v;\cdot)}{d\mathcal{L}^N}(x_0) \le \eta + \limsup_{r\to 0^+} \limsup_{n\to\infty} \int_Q f\left(nk\, y, v(x_0) + w\left(nk\, y\right)\right) dy$$
$$+ \limsup_{r\to 0^+} \limsup_{n\to\infty} \int_{E_{r,\delta}} C\left(1 + |v(x_0 + r\, y)|^q + |w\left(nk\, y\right)|^q\right) dy$$
$$= \eta + \frac{1}{k^N} \int_{kQ} f(y, v(x_0) + w(y))\, dy \le f_{\mathrm{hom}}(v(x_0)) + 2\eta,$$

where we have used (5.13), the kQ-periodicity of the function $h(y) := f(y, v(x_0) + w(y))$, the equi-integrability of $\{|u_\varepsilon|^q\}$, and the fact that (5.11) entails

$$\lim_{r\to 0^+} |\{y \in Q : |v(x_0 + r\, y) - v(x_0)| \ge \delta\}| = 0.$$

It suffices to let $\eta \to 0^+$.

□

Proof of Theorem 1.7. We claim that for any $\varepsilon_n \to 0^+$

$$\mathcal{F}_{\mathrm{hom}}(\cdot; D) = \Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(\cdot; D).$$

By Lemmas 5.5 and 5.6 we always have

$$\mathcal{F}_{\mathrm{hom}}(\cdot; D) \ge \Gamma - \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}} \ge \Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(\cdot; D).$$

Thus assume for contradiction that that there exists $\varepsilon_n \to 0^+$ and $v \in L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\mathcal{F}_{\mathrm{hom}}(v; D) > \Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v; D).$$

Let $\{v_n\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ be such that $v_n \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$ and

$$\mathcal{F}_{\mathrm{hom}}(v; D) > \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v_n; D),$$

and choose a subsequence $\{\varepsilon_{n_k}\}$ such that

$$\mathcal{F}_{\mathrm{hom}}(v; D) > \lim_{k\to\infty} \mathcal{F}_{\varepsilon_{n_k}}(v_{n_k}; D).$$

Then, by the previous lemmas, we can extract a further subsequence $\{\varepsilon_{n_{k_j}}\}$ such that

$$\mathcal{F}_{\rm hom}(v;D) = \Gamma - \liminf_{j\to\infty} \mathcal{F}_{\varepsilon_{n_{k_j}}}(v;D) \le \lim_{j\to\infty} \mathcal{F}_{\varepsilon_{n_{k_j}}}(v_{n_{k_j}};D) < \mathcal{F}_{\rm hom}(v;D),$$

which is a contradiction and proves the claim.

Hence it remains to show that, when (A_3) holds, for any $\varepsilon_n \to 0^+$ and $v \in L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$

$$\Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v;D) \le \mathcal{F}_{\rm hom}(v;D).$$

By taking $w_n \equiv v$ and using (A_2) we get

$$\Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v;D) \le \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(w_n;D) \le C \int_D (|v(x)|^q + 1)dx.$$

Hence for *any* sequence $\{v_n\} \subset L^q(D;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $v_n \rightharpoonup v$ in $L^q(D;\mathbb{R}^d)$ and

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v_n;D) \le C \int_D (|v(x)|^q + 1)dx,$$

by (A_3) we get that $\sup_n ||v_n||_{L^q(D)} \le L < \infty$, where the constant L depends only on the constants in (A_2), (A_3), and on $||v||_{L^q(\Omega;\mathbb{R}^d)}$. Using the notation introduced in the proof of Lemma 5.3, we conclude that

$$\Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v;D) = \Gamma(d_L) - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v;D). \tag{5.14}$$

By Remark 5.4 and by Lemma 5.6, for any subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ there exists a subsequence $\{\varepsilon_{n_{k_j}}\}$ (depending on D) such that

$$\Gamma - \lim_{j\to\infty} \mathcal{F}_{\varepsilon_{n_{k_j}}}(v;D) = \mathcal{F}_{\rm hom}(v;D).$$

By (5.14) this implies that

$$\Gamma(d_L) - \lim_{j\to\infty} \mathcal{F}_{\varepsilon_{n_{k_j}}}(v;D) = \mathcal{F}_{\rm hom}(v;D).$$

We can now apply the second part of Proposition 2.4 in the metric space $(L\mathcal{B}, d_L)$ to conclude that

$$\Gamma(d_L) - \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v;D) = \mathcal{F}_{\rm hom}(v;D).$$

□

Corollary 5.7. *Under hypotheses (A_1)-(A_3) the function $f_{\rm hom}$ is $\mathcal{A}$-quasiconvex and the following asymptotic formula holds*

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty} \frac{1}{T^N} \inf\left\{ \int_{TQ} f(x, \xi + v(x))\, dx : \ v \in L^q_{T\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}, \int_{TQ} v(x)\, dx = 0 \right\} \cdot \tag{5.15}$$

Proof of Corollary 5.7. It may be shown easily, via a diagonalization procedure and in view of the coercivity condition (A_3), that $\mathcal{F}_{\rm hom}(\cdot;Q)$ is L^q-sequentially weakly lower semicontinuous in $\ker \mathcal{A}$. In particular, this entails $\mathcal{A}$-quasiconvexity for $f_{\rm hom}$. Indeed, fix $v \in \mathbb{R}^d$ and $w \in L^q_{1\text{-per}}(\mathbb{R}^N;\mathbb{R}^d) \cap \ker \mathcal{A}$, with $\int_Q w(y)\, dy = 0$, and define $w_n(x) := w(nx)$. Then $w_n \in L^q(Q;\mathbb{R}^d) \cap \ker \mathcal{A}$, $w_n \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$, and so

$$f_{\rm hom}(v) = \mathcal{F}_{\rm hom}(v;Q) \le \liminf_{n\to\infty} \mathcal{F}_{\rm hom}(v + w_n;Q) = \liminf_{n\to\infty} \int_Q f_{\rm hom}(v + w(nx))\, dx = \int_Q f_{\rm hom}(v + w(x))\, dx.$$

Finally, using Theorem 1.7 (A_3), and recalling Remark 5.2, we conclude that

$$f_{\rm hom}(v) = \min\left\{\mathcal{F}_{\rm hom}(v+w;Q) :\ w \in L^q_{1-{\rm per}}(\mathbb{R}^N;\mathbb{R}^d)\cap \ker \mathcal{A},\ \int_Q w(x)\,dx = 0\right\}$$
$$= \lim_{\varepsilon\to 0} \inf\left\{\mathcal{F}_\varepsilon(v+w;Q) :\ w \in L^q_{1-{\rm per}}(\mathbb{R}^N;\mathbb{R}^d)\cap \ker \mathcal{A},\quad \int_Q w(x)\,dx = 0\right\},$$

and (5.15) follows by setting $T = 1/\varepsilon$ and changing variables in the last expression. □

As a corollary of Theorem 1.7, we obtain the following result via the same choice of the underlying operator $\mathcal{A}$ as in the proof of Theorem 1.5.

Theorem 5.8 (Homogenization with constraint on the divergence)**.** *Assume that conditions $(A_1)-(A_3)$ hold, with $d = N^2$, and let F_ε be defined by*

$$F_\varepsilon(u;D) := \int_D f\Big(\frac{x}{\varepsilon},\nabla u\Big)\,dx$$

on functions $u \in W^{1,q}(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} u = 0$*. Then the Γ-limit*

$$F(u;D) := \Gamma(L^q) - \lim_{\varepsilon\to 0} F_\varepsilon(u;D) = \int_D f_{\rm hom}(\nabla u)dx$$

exists on functions $u \in W^{1,q}(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} u = 0$*, where*

$$f_{\rm hom}(v) = \inf_{k\in\mathbb{N}} \frac{1}{k^N} \inf\left\{\int_{kQ} f(x, v + \nabla w(x))\,dx :\quad w \in W^{1,q}_{k\text{-per}}(\mathbb{R}^N;\mathbb{R}^N),\ \operatorname{div} w = 0\right\}$$

for all $v \in \mathbb{R}^{N^2}$.

Appendix

We prove that in Remark 1.2(iii) in the introduction

$$\mathcal{F}(v;(a,b)) = \mathcal{F}((v_1,v_2);(a,b)) = \int_a^b (\psi^{**}(v_1) + v_2^2)\,dx, \tag{5.16}$$

where $\psi^{**}(v_1)$ is the convex envelope of

$$\psi(v_1) := \min\left\{(v_1-1)^2,(v_1+1)^2\right\} = \begin{cases}(v_1+1)^2 & \text{if} \quad v_1 \geq 0,\\ (v_1-1)^2 & \text{if} \quad v_1 < 0.\end{cases}$$

Indeed, if $v_1^n \rightharpoonup v_1$ in $L^2(a,b)$, $v_2^n \rightharpoonup v_2$ in $L^2(a,b)$ and $(v_2^n)' \to 0$ in $H^{-1}(a,b)$ then the function v_2 is constant and Jensen's inequality yields

$$\liminf_{n\to\infty} \int_a^b f(v_1^n,v_2^n)\,dx \geq \liminf_{n\to\infty}\int_a^b [\psi(v_1^n) + (v_2^n)^2]\,dx \geq \liminf_{n\to\infty}\int_a^b \psi(v_1^n)\,dx + \liminf_{n\to\infty}\int_a^b (v_2^n)^2\,dx$$
$$\geq \int_a^b \psi^{**}(v_1)\,dx + (v_2)^2(b-a).$$

The arbitrariness of the sequence $\{(v_1^n, v_2^n)\}$ allows us to conclude that

$$\mathcal{F}(v;(a,b)) \geq \int_a^b \psi^{**}(v_1)\,dx + (v_2)^2(b-a).$$

Conversely, suppose that if v_1 is smooth, $v_1 \in L^2(a,b)$, and $|\{x \in (a,b) : v_1(x) = 0\}| = 0$, $v_2 \in \mathbb{R}$, then

$$\mathcal{F}(v;(a,b)) \leq \int_a^b \psi(v_1)\,dx + (v_2)^2(b-a). \tag{5.17}$$

Then this inequality remains true for $v_1 \in L^2(a,b)$, $v_2 \in \mathbb{R}$ arbitrary, because we may approximate v_1 in L^2 strong by a sequence $\{v_1^n\} \subset L^2(a,b) \cap C^\infty(a,b)$, $|\{x \in (a,b) : v_1^n(x) = 0\}| = 0$, and

$$v_1 \mapsto \mathcal{F}((v_1,v_2);(a,b)), \qquad v_1 \mapsto \int_a^b \psi(v_1)\,dx$$

are, respectively, L^2-weak lower semicontinuous and L^2-strong continuous. Once we establish (5.17) for $(v_1,v_2) \in L^2(a,b) \times \mathbb{R}$ then (5.16) follows because, once again, $\mathcal{F}((\cdot,v_2);(a,b))$ is L^2-weak lower semicontinuous.

Fix now $v_1 \in L^2(a,b) \cap C^\infty(a,b)$, with $|\{x \in (a,b) : v_1(x) = 0\}| = 0$, and let $v_2 \in \mathbb{R}$, $\delta > 0$ be fixed. Set

$$A_\delta^- := \{x \in (a,b) : (v_1(x)-1)^2 < (v_1(x)+1)^2 - \delta\},$$
$$A_\delta^+ := \{x \in (a,b) : (v_1(x)-1)^2 > (v_1(x)+1)^2 + \delta\}\cdot$$

If for all $\delta > 0$ $A_\delta^- = \emptyset$ then $(v_1(x)-1)^2 \geq (v_1(x)+1)^2$ for every $x \in (a,b)$ and we choose $w_n \notin \mathbb{Q}$ with $w_n \to v_2$. Then

$$\mathcal{F}((v_1,v_2);(a,b)) \leq \liminf_{n\to\infty} \int_a^b f(v,w_n)\,dx = \liminf_{n\to\infty} \int_a^b [(v_1(x)+1)^2 + (w_n)^2]\,dx$$
$$\leq \int_a^b \psi(v_1)\,dx + (v_2)^2(b-a).$$

Similary (5.17) holds if for all $\delta > 0$ $A_\delta^+ = \emptyset$. Thus assume that for $\delta > 0$ sufficiently small $A_\delta^- \neq \emptyset \neq A_\delta^+$. Choose a cut-off function $\varphi_\delta \in C^\infty((a,b);[0,1])$ such that $\varphi_\delta \equiv 1$ in A_δ^-, $\varphi_\delta \equiv 0$ in A_δ^+, and let $z_n \in \mathbb{Q}$, $w_n \notin \mathbb{Q}$, be such that with w_n, $z_n \to v_2$. Define

$$v_2^{n,\delta}(x) := \varphi_\delta(x)\,z_n + (1-\varphi_\delta(x))\,w_n.$$

Since $(v_2^{n,\delta})' = \varphi_\delta'(x)(z_n - w_n)$, it is clear that

$$\lim_{\delta\to 0^+}\lim_{n\to\infty} ||v_2^{n,\delta} - v_2||_{L^2} = \lim_{\delta\to 0^+}\lim_{n\to\infty} ||(v_2^{n,\delta})'||_{H^{-1}} = 0.$$

We have

$$
\begin{aligned}
\mathcal{F}((v_1,v_2);(a,b)) &\le \liminf_{\delta\to 0^+}\liminf_{n\to\infty}\int_a^b f(v_1, v_2^{n,\delta})\,dx \\
&= \liminf_{\delta\to 0^+}\liminf_{n\to\infty}\left\{\int_{A_\delta^-} f(v_1,z_n)\,dx + \int_{A_\delta^+} f(v_1,w_n)\,dx + \int_{(a,b)\setminus(A_\delta^-\cup A_\delta^+)} f(v_1^n, v_2^{n,\delta})\,dx\right\} \\
&\le \liminf_{\delta\to 0^+}\liminf_{n\to\infty}\left\{\int_{A_\delta^-}[\psi(v_1)+(z_n)^2]\,dx + \int_{A_\delta^+}[\psi(v_1)+(w_n)^2]\,dx + C\left|(a,b)\setminus(A_\delta^-\cup A_\delta^+)\right|\right\} \\
&= \liminf_{\delta\to 0^+}\left\{\int_{A_\delta^-\cup A_\delta^+}[\psi(v_1)+(v_2)^2]\,dx + C\left|(a,b)\setminus(A_\delta^-\cup A_\delta^+)\right|\right\} \\
&= \int_a^b \psi(v_1)\,dx + (v_2)^2(b-a),
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\left|(a,b)\setminus(A_\delta^-\cup A_\delta^+)\right| &= \left|\{x\in(a,b):\ \left|(v_1-1)^2-(v_1+1)^2\right|<\delta\}\right| = |\{x\in(a,b):\ |v_1(x)|<\delta/2\}| \\
&\to |\{x\in(a,b):\ v_1(x)=0\}| = 0
\end{aligned}
$$

as $\delta\to 0^+$.

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