**ESAIM: Control, Optimisation and Calculus of Variations** November 2000, Vol. 5, 579–590
URL: `http://www.emath.fr/cocv/`

# LINKS BETWEEN YOUNG MEASURES ASSOCIATED TO CONSTRAINED SEQUENCES

Anca-Maria Toader[1]

**Abstract**. We give necessary and sufficient conditions which characterize the Young measures associated to two oscillating sequences of functions, $u_n$ on $\omega_1 \times \omega_2$ and $v_n$ on $\omega_2$ satisfying the constraint $v_n(y) = \frac{1}{|\omega_1|}\int_{\omega_1} u_n(x,y)dx$. Our study is motivated by nonlinear effects induced by homogenization. Techniques based on equimeasurability and rearrangements are employed.

**AMS Subject Classification.** 28, 49.

Received February 8, 2000. Revised July, 2000.

## 1. Introduction

Given two sequences of functions, $u_n$ on $\omega_1 \times \omega_2$ and $v_n$ on $\omega_2$ such that the constraint $v_n(y) = \frac{1}{|\omega_1|}\displaystyle\int_{\omega_1} u_n(x,y)dx$ holds, one wants to characterize the relationship between their associated Young measures.

Our main result (Th. 3.1) gives necessary and sufficient conditions for the above problem in terms of distribution measures and decreasing rearrangements. It can be interpreted as follows: Young measures capture something of the oscillating behaviour of the sequence $(u_n)$ and the integration $\displaystyle\int_{\omega_1} u_n(x,y)dx$ destroyes part of the oscillations of $u_n$. We use Young measure techniques introduced by Ball, Tartar, Balder, Valadier. Notions and results from Probability Theory are employed; particularly, the order relation $\prec$ on the set of positive measures due to Choquet and Loomis, and related results (see Cartier *et al.* [4] and Meyer [6]). The corresponding preorder relation $\prec$ on the set of nonnegative $L^1$ functions introduced by Hardy *et al.* is used, as well as properties of doubly stochastic operators proved by Ryff.

Our study was motivated by more complex questions in relation with nonlocal effects induced by homogenization.

Consider the degenerate elliptic equation studied by Amirat *et al.* in [1]:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(a_n(x,y)\dfrac{\partial u_n}{\partial x}(x,y)\right) = f(x,y) \text{ in } ]0,1[\times]0,1[, \\ \qquad\qquad u_n(0,y) = u_n(1,y) = 0 \text{ on } ]0,1[. \end{cases} \tag{1.1}$$

By homogenization a nonlocal effect appears expressed in terms of a kernel. We introduce a parameter $\gamma$ (following an idea of Tartar [10]) by setting $a_n(x,y) := a_-(x,y)/(1+\gamma b_n(x,y))$ where $1/a_n \rightharpoonup 1/a_-$ for the

*Keywords and phrases:* Young measures, rearrangements, doubly stochastic operators.

[1] Faculdade de Ciências, CMAF, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal;
e-mail: `amtan@lmc.fc.ul.pt`

weak $*$ topology and $b_n := a_-/a_n - 1$. Then the above problem corresponds to the case $\gamma = 1$. The solution $u_n$ writes in series of powers of $\gamma$ like $u_{n,\gamma}(x,y) = v_n^0(x,y) + v_n^1(x,y)\gamma + \cdots + v_n^k(x,y)\gamma^k + \ldots$ and the analysis would be based on analyticity properties in the parameter $\gamma$. The kernel describing the nonlocal effect is the weak limit of

$$K_{n,\gamma}(x,\xi,y) = \frac{b_n(\xi,y)}{1+\gamma\int_0^1 b_n(x,y)dx}\left(\int_0^1 b_n(x,y)dx - b_n(x,y)\right). \tag{1.2}$$

One would like to characterize the nature of kernels that may appear as weak limits of $K_{n,\gamma}$ in (1.2), in order to understand the constitutive laws of the homogenized materials. This is still an open problem.

We inscribe our contribution in the effort to characterize such kernels by solving the symplified problem presented in the beginning.

In Section 2 we make a brief recall on notions to be employed: distribution measures, equimeasurability, rearrangements, the relation $\prec$, Young measures.

Section 3 is dedicated to our main result: we state it and give its proof by making use of auxiliary results contained in Section 4.

## 2. Preliminary notions

To describe precisely the type of questions we are considering here, we begin by recalling a few facts about distribution measures and equimeasurable functions. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, denote by $\mathcal{L}_N$ the Lebesgue measure on $\mathbb{R}^N$ and by $\mathcal{B}(\Omega)$ the Borel $\sigma$-field of $\Omega$. Let $f:\Omega\to\mathbb{R}_+$ be a nonnegative, measurable function; we represent by $\mu_f$ its *distribution measure* defined by

$$\mu_f(B) := \mathcal{L}_N(f^{-1}(B)), \quad \forall B\in\mathcal{B}(\mathbb{R}_+).$$

We denote by $f^*$ the *decreasing rearrangement* of $f$ given by:

$$f^*(s) := \sup\{t>0 : \mu_f(]t,\infty[) > s\}\cdot$$

It is easily checked that $f^*$ is the unique (up to modifications on Lebesgue zero-measure sets) nonincreasing function on $[0,\mathcal{L}_N(\Omega)[$ such that $f^*$ and $f$ have the same distribution measure $\mu_f$. Finally, we will say that $f,g\geq 0$ are equimeasurable or, equivalently, that $f$ is a rearrangement of $g$ if they have the same distribution measure. We will denote this equivalence relation by $f\sim g$.

Hardy *et al.* introduced (see [5]) the following preorder relation on the set of nonnegative functions in $L^1(\Omega)$: for $f$ and $g$ in $L^1(\Omega)$

$$g\prec f \text{ iff } \int_\Omega F(g(t))dt \leq \int_\Omega F(f(t))dt\,,$$

for all convex, continuous functions $F:\mathbb{R}_+\to\mathbb{R}$. Observe that $f\prec g$ and $g\prec f$ is equivalent to $f\sim g$. We recall from [5] the following property of the preorder relation between functions in $L^\infty(\Omega)$, in terms of decreasing rearrangements

$$g\prec f \text{ iff } \begin{cases} \displaystyle\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds \quad \forall t\in[0,\mathcal{L}_N(\Omega)[,\\ \displaystyle\int_0^{\mathcal{L}_N(\Omega)} g^*(s)ds = \int_0^{\mathcal{L}_N(\Omega)} f^*(s)ds. \end{cases} \tag{2.1}$$

Actually we shall only use functions which take values in a compact interval.

On the set of positive finite measures, the corresponding order relation was introduced by Choquet and, in a different framework, by Loomis (see [4] and [6]).

**Definition 2.1.** *Two positive finite measures on* $([\alpha, \beta], \mathcal{B}([\alpha, \beta]))$, $\nu$ *and* $\mu$, *having the same total mass* $(\mu([\alpha, \beta]) = \nu([\alpha, \beta]))$ *satisfy* $\mu \prec \nu$ *if and only if*

$$\int_{\alpha}^{\beta} \phi(z) d\mu(z) \leq \int_{\alpha}^{\beta} \phi(z) d\nu(z)\,,$$

*for all continuous and convex functions* $\phi : [\alpha, \beta] \to \mathbb{R}$.

**Remark 2.2.** Given two positive functions such that $g \prec f$, their distribution measures $\mu_g$ and $\mu_f$ satisfy $\mu_g \prec \mu_f$. Conversely, given two measures that satisfy $\mu \prec \nu$, if $g$ and $f$ have the distribution measures $\mu$ and $\nu$ respectively, then $g \prec f$.

The "old" tool of Young measures has proven crucial for applications in asymptotic analysis. Young measures techniques were developed by Tartar [9], Ball [3], Balder [2], Valadier [11], etc.

In the sequel we make a brief recall on Young measures following the notations and framework from [11].

We call *Young measure* any positive measure $\mu$ on $\Omega \times S$ ($S$ is a metrizable space) whose projection on $\Omega$ is $\mathcal{L}_N$. Let $\mathcal{Y}(\Omega \times S)$ be the set of all Young measures on $\Omega \times S$.

We will not distinguish $\mu$ from its disintegration $(\mu_x)_{x\in\Omega}$ which is a measurable family of probabilities on $S$ such that for any $\psi : \Omega \times S \to \mathbb{R}$, $\mu$-integrable,

$$\int_{\Omega\times S} \psi \, d\mu = \int_{\Omega} \int_{S} \psi\,(x, \xi)\, d\mu_x(\xi) d\mathcal{L}_N(x).$$

For each measurable function $a : \Omega \to S$ we associate a Young measure $\mu_a$, with support in the graph of $a$, defined by

$$\langle \mu_a, \phi \rangle = \int_{\Omega\times S} \phi(x, \lambda) d\mu_a(x, \lambda) = \int_{\Omega} \phi(x, a(x)) dx,$$

for all positive Carathéodory integrands $\phi : \Omega \times S \to \mathbb{R}$.

On $\mathcal{Y}(\Omega \times S)$ we consider the *narrow* topology, *i.e.*, the weakest topology that makes continuous the maps

$$\mu \mapsto \int_{\Omega\times S} \phi(x, \lambda) d\mu(x, \lambda),$$

for all bounded Carathéodory integrands $\phi$.

**Remark 2.3.** 1) The set of Young measures associated to a sequence of functions uniformly bounded in $L^1(\Omega; \mathbb{R}^d)$ is relatively compact in $\mathcal{Y}(\Omega \times \mathbb{R}^d)$.

2) If $S$ is a metrizable compact space, then every set of Young measures associated to a sequence of measurable functions $a_n : \Omega \to S$ is relatively compact in $\mathcal{Y}(\Omega \times S)$.

**Remark 2.4.** 1) Given a uniformly bounded sequence $(a_n)$ in $L^1(\Omega; \mathbb{R}^d)$, using Remark 2.3 1), we may assume that, up to a subsequence of $(a_n)$, the sequence of their associated Young measures, narrow converges to some $\mu = (\mu_x)_{x\in\Omega} \in \mathcal{Y}(\Omega \times \mathbb{R}^d)$.

2) For a sequence of functions $a_n : \Omega \to S$ we say that $\mu$ is the Young measure associated to the sequence $(a_n)$, or, that $(a_n)$ gives rise to the Young measure $\mu$, if the Young measures associated to $(a_n)$ narrow converge to $\mu$, *i.e.*, for all bounded Carathéodory integrand $\phi$,

$$\int_{\Omega} \phi(x, a_n(x))\; dx \to \int_{\Omega\times S} \phi(x, \lambda)\; d\mu(x, \lambda).$$

## 3. Main result

Theorem 3.1 below was conjectured by Tartar in some discussions we had together, during his visits to Lisbon, in December 1998.

Consider two domains $\omega_1 \subset \mathbb{R}^N$ and $\omega_2 \subset \mathbb{R}^M$ and a compact interval $[\alpha, \beta]$ in $\mathbb{R}$. We may assume without losing generality that $|\omega_1| = |\omega_2| = 1$.

**Theorem 3.1.** *Given are the sequences $u_n : \omega_1 \times \omega_2 \to [\alpha, \beta]$ and $v_n : \omega_2 \to [\alpha, \beta]$ such that $v_n(y) = \int_{\omega_1} u_n(x, y)dx$. Assume that the sequence $(u_n)$ gives rise to the Young measure $\nu = (\nu_{x,y})_{(x,y)\in\omega_1\times\omega_2}$ and that the sequence $(v_n)$ gives rise to the Young measure $\mu = (\mu_y)_{y\in\omega_2}$. One defines a function $f : \omega_1 \times \omega_2 \times (0, 1) \to [\alpha, \beta]$ such that $f(x, y, \cdot)$ is nonincreasing and its distribution measure is $\nu_{x,y}$, and similarly, a function $g : \omega_2 \times (0, 1) \to [\alpha, \beta]$ such that $g(y, \cdot)$ is nonincreasing and its distribution measure is $\mu_y$. Then one has*

$$\int_0^t g(y, s)ds \le \int_{\omega_1} \int_0^t f(x, y, s)dsdx, \tag{3.1}$$

*with equality for $t = 1$.*

*Conversely, if $f$ and $g$ satisfy the above inequality for all $t \in [0, 1)$ and the corresponding equality for $t = 1$, then, denoting by $\nu_{x,y}$ and $\mu_y$ the distribution measures of $f(x, y, \cdot)$ and $g(y, \cdot)$, respectively, there exists a sequence $u_n : \omega_1 \times \omega_2 \to [\alpha, \beta]$ that gives rise to the Young measure $\nu$ and the sequence defined by $v_n := \int_{\omega_1} u_n(x, y)dx$ gives rise to the Young measure $\mu$.*

The author presented a particular case of the above result (the Young measure $\nu = (\nu_{x,y})_{(x,y)\in\omega_1\times\omega_2}$ did not depend on $x$: $\nu_{x,y} = \nu_y$) at the Equadiff99 conference held in Berlin.

*Proof.* For the direct implication let $t \in [0, 1]$ arbitrarily fixed. Then by Lemma 4.5 there exists a subsequence $v_{n_k}$ of $v_n$ and a sequence of characteristic functions $\chi_k^v : \omega_2 \to \{0, 1\}$, $\chi_k^v \rightharpoonup t$ such that

$$\chi_k^v v_{n_k} \rightharpoonup \int_0^t g(y, s)ds.$$

On the other hand, applying Lemma 4.5 with the corresponding subsequence $(u_{n_k})$ and with the same number $t$, we obtain that for any sequence of characteristic functions $\chi_k : \omega_1 \times \omega_2 \to \{0, 1\}$ such that $\chi_k \rightharpoonup t$, up to a subsequence, we have that

$$\text{weak } \lim_k \chi_k u_{n_k} \le \int_0^t f(x, y, s)ds, \tag{3.2}$$

and in particular the above inequality holds for $\chi_k^v$. Then

$$\begin{aligned}\int_0^t g(y, s)ds &= \text{weak } \lim_k \chi_k^v v_{n_k} = \text{weak } \lim_k \chi_k^v \int_{\omega_1} u_{n_k}(x, y)dx \\ &= \text{weak } \lim_k \int_{\omega_1} \chi_k^v(y) u_{n_k}(x, y)dx \le \int_{\omega_1} \int_0^t f(x, y, s)dsdx,\end{aligned}$$

and the direct implication turns out, for $t \in [0, 1[$. For $t = 1$, $\chi_k^v \to 1$ strongly and (3.2) holds with equality which yields equality in (3.1).

Conversely, suppose that $f$ and $g$ satisfy (3.1) for $t \in [0, 1)$ and the corresponding equality for $t = 1$. We construct a sequence $u_n$ giving rise to the Young measure $\nu$ and such that the sequence $v_n := \int_{\omega_1} u_n(x, y)dx$ gives rise to the Young measure $\mu$ as follows: by property (2.1), (3.1) is equivalent to $g(y, \cdot) \prec \int_{\omega_1} f(x, y, \cdot)dx$. Applying Lemma 4.4 it turns out that there exists a positive measure $\theta$ on $[0, 1] \times [0, 1]$ whose both projections are the Lebesgue measure and such that

$$g(y,t) = \int_0^1 \int_{\omega_1} f(x,y,s)dxd\theta_t(s).$$

Here $\theta_t$ is the disintegration of $\theta$ with respect to the Lebesgue measure. Define $u : \omega_1 \times \omega_2 \times [0,1] \times [0,1] \to [\alpha, \beta]$ by $u(x,y,s,t) := f(x,y,\Phi_t(s))$, where $\Phi_t : [0,1] \to [0,1]$ has the distribution measure $\theta_t$. The existence of $\Phi_t$ is ensured by Remark 4.1. Then $g$ writes

$$g(y,t) = \int_0^1 \int_{\omega_1} f(x,y,\Phi_t(s))dxds = \int_0^1 \int_{\omega_1} u(x,y,s,t)dxds. \tag{3.3}$$

One takes $u_n(x,y) := u(x,y,[nx_1],[ny_1])$ (here $[z]$, with $z \in \mathbb{R}$, represents the fractional part of $z$ and, by $x_1$, we represent the first component of $x$). By Riemann-Lebesgue theorem it turns out that $u_n$ gives rise to the Young measure $\nu$. By a similar argument, the sequence $w_n$ defined by $w_n(y) := g(y,[ny_1])$ gives rise to the Young measure $\mu$. Having in mind the definition of the sequence $v_n(y) := \int_{\omega_1} u_n(x,y)dx$, it turns out that $v_n - w_n$ converges uniformly to 0, and since $w_n$ gives rise to the Young measure $\mu$, so does the sequence $v_n$, and the proof is complete. □

## 4. Auxiliary results

**Remark 4.1.** Given a finite positive measure $\mu$ on $[\alpha,\beta]$, $\mu : \mathcal{B}([\alpha,\beta]) \to \mathbb{R}_+$, there exists a measurable function $f : [0, \mu([\alpha,\beta])] \to [\alpha,\beta]$ that transports the Lebesgue measure on $[0,\mu([\alpha,\beta])]$ into $\mu$, *i.e.* $\mu(B) = \mathcal{L}_1(f^{-1}(B))$ for all $B$ in $\mathcal{B}([\alpha,\beta])$. We can take for instance the nonincreasing function that is given by

$$f(x) := \sup\{t \in [\alpha,\beta] : \mu([t,\beta]) > x\}\cdot$$

The following two properties are to be used in the proof of Lemma 4.4.

**Property 4.2.** *Consider a finite positive measure $\theta$ on $\Omega \times \Omega'$ with $proj_{\Omega'}\theta = \nu$. If the measure $\nu$ is absolutely continuous with respect to a positive measure $l$, $d\nu(y) = \tau(y)dl(y)$ for a function $\tau \in L^1_l(\Omega')$, then $\theta$ may be disintegrated with respect to $l$ according to the formula*

$$d\theta(x,y) = \theta^l_y(x)dl(y),$$

*where $\theta^l_y(x) = \tau(y)\theta_y(x)$ and $(\theta_y)_{y\in\Omega'}$ is the disintegration of $\theta$ with respect to $\nu$.*

Let $(\Omega, m)$ and $(\Omega', l)$ be two measure spaces with the same total mass ($m(\Omega) = l(\Omega') < +\infty$), $m$ and $l$ being positive measures.

**Property 4.3.** *If $f : \Omega \to \mathbb{R}$ and $g : \Omega' \to \mathbb{R}$ are integrable functions such that $f(x) < g(y)$ for all $x \in \Omega$ and $y \in \Omega'$, then $\displaystyle\int_\Omega f(x)dm(x) < \int_{\Omega'} g(y)dl(y)$.*

**Lemma 4.4.** *Given two measurable functions $u : \Omega \to [\alpha,\beta]$ and $v : \Omega' \to [\alpha,\beta]$ such that for all continuous and convex functions $\Phi : [\alpha,\beta] \to \mathbb{R}$, $\displaystyle\int_{\Omega'} \Phi(v(y))dl(y) \le \int_\Omega \Phi(u(x))dm(x)$ holds. There exists then a positive measure $\theta$, on $\Omega\times\Omega'$, whose projections on $\Omega$ and $\Omega'$ are $m$ and $l$, respectively, and such that $l$-almost everywhere in $\Omega'$*

$$v(y) = \int_\Omega u(x)d\theta_y(x),$$

*where $(\theta_y)_{y\in\Omega'}$ is the disintegration of $\theta$ with respect to $l$.*

This result is a consequence of the results obtained by Ryff in [7] and [8] regarding equivalent characterizations, in terms of doubly stochastic operators, for the preorder relation $\prec$ between functions. Lemma 4.4 is also very close to Cartier's theorems stated in terms of dilatations (see Ths. 1 and 2 in [4] and Ths. 35 and 36 in [6], p. 288). We shall give a direct (sketched) proof.

The proof is to be done in 3 steps.

**Step 1.**
Denote by $\Theta$ the set of positive measures $\theta$ on $\Omega \times \Omega'$ such that the projections of $\theta$ on $\Omega$ and $\Omega'$ are $m$ and $l$, respectively. $\Theta$ is a subset of the linear space of measures on $\Omega \times \Omega'$. The set $\Theta$ is compact with respect to the weak topology of measures.

Let us define for each $\theta \in \Theta$ the function $w(y) = \int_\Omega u(x) d\theta_y(x)$ where $(\theta_y)_{y\in\Omega'}$ is the disintegration of $\theta$ with respect to $l$. Note that the dependency of $w$ on $\theta$ is linear. Let us introduce the functional $J : \Theta \to \mathbb{R}_+$,

$$J(\theta) = \int_{\Omega'} (v - w)^2 dl(y).$$

$J$ depends on $\theta$ through $w$, and, due to the lower semicontinuity of the norm in $L^2_l(\Omega')$, it turns out that $J$ is lower semicontinuous. Since $\Theta$ is compact, $J$ attaines its minimum, *i.e.* there exists $\theta^0 \in \Theta$ such that $J(\theta^0) = \inf_{\theta\in\Theta} J(\theta)$. Denote by $w_0(y) := \int_\Omega u(x) d{\theta^0}_y(x)$ the function corresponding to $\theta^0$. We shall prove that $w_0 = v$.

**Step 2.**
$\theta^0$ has the following property: if $A_1$ and $A_2$ are subsets of $\Omega$ such that

$$u(x_1) < u(x_2) \text{ for all } x_1 \in A_1, x_2 \in A_2$$

and if $B_1$ and $B_2$ are subsets of $\Omega'$ such that

$$v(y_1) - w_0(y_1) < v(y_2) - w_0(y_2) \text{ for all } y_1 \in B_1, y_2 \in B_2,$$

then, either $\theta^0(A_1 \times B_2) = 0$ or $\theta^0(A_2 \times B_1) = 0$. Otherwise, if $\theta^0(A_1 \times B_2) > 0$ and $\theta^0(A_2 \times B_1) > 0$ one can increase $\theta^0$ in the sets $A_1 \times B_1$ and $A_2 \times B_2$ and decrease $\theta^0$ in the sets $A_1 \times B_2$ and $A_2 \times B_1$ by maintaining the same projections $m$ and $l$ on $\Omega$ and $\Omega'$, respectively, and obtain a smaller value for $J$. Indeed, assuming that $\theta^0(A_1 \times B_2) > 0$ and $\theta^0(A_2 \times B_1) > 0$ we can choose two measures on $\Omega \times \Omega'$, $\theta^{12}$ and $\theta^{21}$ such that $\text{supp}\theta^{12} \subset A_1 \times B_2$, $\theta^{12} \le \theta^0 \mid_{A_1\times B_2}$, $\text{supp}\theta^{21} \subset A_2 \times B_1$, $\theta^{21} \le \theta^0 \mid_{A_2\times B_1}$ and $\theta^{12}(A_1 \times B_2) = \theta^{21}(A_2 \times B_1) = \gamma > 0$.

Denote $\mu_1 := \text{proj}_\Omega\theta^{12}$ ($\text{supp}\mu_1 \subset A_1$), $\nu_2 := \text{proj}_{\Omega'}\theta^{12}$ ($\text{supp}\nu_2 \subset B_2$), $\mu_2 := \text{proj}_\Omega\theta^{21}$ ($\text{supp}\mu_2 \subset A_2$), $\nu_1 := \text{proj}_{\Omega'}\theta^{21}$ ($\text{supp}\nu_1 \subset B_1$). Note that $\mu_1(A_1) = \nu_2(B_2) = \mu_2(A_2) = \nu_1(B_1) = \gamma$.

Let us define $\theta^{11} := \frac{1}{\gamma}\mu_1 \otimes \nu_1$, $\theta^{22} := \frac{1}{\gamma}\mu_2 \otimes \nu_2$ and note that $\text{supp}\theta^{11} \subset A_1 \times B_1$ and $\text{supp}\theta^{22} \subset A_2 \times B_2$.

Consider $\theta^t := \theta^0 + t\left(\theta^{11} - \theta^{12} - \theta^{21} + \theta^{22}\right)$. For $t \in [0, 1]$ the measure $\theta^t$ is positive since $\theta^{12} \le \theta^0$ and $\theta^{21} \le \theta^0$ on the disjoint sets $A_1 \times B_2$ and, respectively, $A_2 \times B_1$. Calculating the projection of $\theta^t$ on $\Omega$ we have for every $A \in \mathcal{B}(\Omega)$ that $\theta^t(A \times \Omega') = \theta^0(A \times \Omega') + t\left(\theta^{11}(A \times \Omega') - \theta^{12}(A \times \Omega') - \theta^{21}(A \times \Omega') + \theta^{22}(A \times \Omega')\right) = m(A) + t\left(\frac{1}{\gamma}\mu_1(A)\nu_1(\Omega') - \theta^{12}(A \times B_2) - \theta^{21}(A \times B_1) + \frac{1}{\gamma}\mu_2(A)\nu_2(\Omega')\right) = m(A) + t\left(\frac{1}{\gamma}\mu_1(A)\nu_1(B_1) - \mu_1(A) - \mu_2(A) + \frac{1}{\gamma}\mu_2(A)\nu_2(B_2)\right) = m(A)$, that is $\text{proj}_\Omega\theta^t = m$ and by similar arguments we obtain also that $\text{proj}_{\Omega'}\theta^t = l$. Therefore $\theta^t \in \Theta$ for $t \in [0, 1]$.

Let us calculate $J(\theta^t)$ in order to evaluate the derivative $\frac{d}{dt}J(\theta^t)\mid_{t=0}$. Let us make first some remarks useful to the computations. $\theta^{21} \le \theta^0$ on $A_2 \times B_1$, consequently, their projections satisfy $\nu_1 \le l$ and therefore $\nu_1$ is absolutely continuous with respect to $l$. According to Property 4.2 we can disintegrate $\theta^{11}$ and $\theta^{21}$ with respect to $l$. Similar arguments employed with the measure $\nu_2$ permit us to conclude that $\theta^{12}$ and $\theta^{22}$ may be disintegrated with respect to $l$. In the following, by $(\bar\theta^{11}_y)$, $(\bar\theta^{12}_y)$, $(\bar\theta^{21}_y)$ and $(\bar\theta^{22}_y)$, we denote the disintegrations with respect to $l$ of $\theta^{11}$, $\theta^{12}$, $\theta^{21}$ and $\theta^{22}$, respectively.

$$\begin{aligned}J(\theta^t) &= \int_{\Omega'} (v(y) - \int_{\Omega} u(x)d\theta^t_y(x))^2 dl(y)\\ &= \int_{\Omega'} \left(v(y) - w_0(y) - t\left(\int_{\Omega} u(x)d\bar{\theta}^{11}_y(x) - \int_{\Omega} u(x)d\bar{\theta}^{12}_y(x)\right.\right.\\ &\quad \left.\left. - \int_{\Omega} u(x)d\bar{\theta}^{21}_y(x) + \int_{\Omega} u(x)d\bar{\theta}^{22}_y(x)\right)\right)^2 dl(y).\end{aligned}$$

Since in the above expression the integrand is a polynomial of degree 2 in $t$, we obtain that

$$\begin{aligned}\frac{d}{dt}J(\theta^t)\mid_{t=0} &= 2\int_{\Omega'} (v(y) - w_0(y))\left(\int_{\Omega} u(x)d\bar{\theta}^{11}_y(x) - \int_{\Omega} u(x)d\bar{\theta}^{12}_y(x)\right.\\ &\quad \left. - \int_{\Omega} u(x)d\bar{\theta}^{21}_y(x) + \int_{\Omega} u(x)d\bar{\theta}^{22}_y(x)\right) dl(y)\\ &= 2\left(\int_{\Omega\times\Omega'} u(x)\delta v(y)d\bar{\theta}^{11}(x,y) - \int_{\Omega\times\Omega'} u(x)\delta v(y)d\bar{\theta}^{12}(x,y)\right.\\ &\quad \left. - \int_{\Omega\times\Omega'} u(x)\delta v(y)d\bar{\theta}^{21}(x,y) + \int_{\Omega\times\Omega'} u(x)\delta v(y)d\bar{\theta}^{22}(x,y)\right),\end{aligned}$$

where $\delta v(y) := v(y) - w_0(y)$ and we had in mind the disintegration formulae for $\theta^{11}$, $\theta^{12}$, $\theta^{21}$ and $\theta^{22}$ with respect to $l$.

Let $(\theta^{11}_y)$ and $(\theta^{21}_y)$ be the disintegrations of $\theta^{11}$ and $\theta^{21}$ with respect to $\nu_1$, which is the projection of both on $\Omega'$. Similarly, let $(\theta^{12}_y)$ and $(\theta^{22}_y)$ be the disintegrations of $\theta^{12}$ and $\theta^{22}$ with respect to $\nu_2$. Then

$$\begin{aligned}\frac{d}{dt}J(\theta^t)\mid_{t=0} &= 2\int_{B_1} \delta v(y)\left(\int_{A_1} \frac{1}{\gamma}u(x)d\mu_1(x) - \int_{A_2} u(x)d\theta^{21}_y(x)\right) d\nu_1(y)\\ &\quad + 2\int_{B_2} \delta v(y)\left(\int_{A_2} \frac{1}{\gamma}u(x)d\mu_2(x) - \int_{A_2} u(x)d\theta^{12}_y(x)\right) d\nu_2(y).\end{aligned}$$

Applying Property 4.3 in each point $y \in B_1$ with the function $u$ on $A_1$ and $A_2$, and the probability measures $\frac{1}{\gamma}\mu_1$ on $A_1$ and $\theta^{21}_y$ on $A_2$, we obtain that $\alpha(y) := \int_{A_1} \frac{1}{\gamma}u(x)d\mu_1(x) - \int_{A_2} u(x)d\theta^{21}_y(x) < 0$. Similar arguments lead to $\beta(y) := \int_{A_2} \frac{1}{\gamma}u(x)d\mu_2(x) - \int_{A_2} u(x)d\theta^{12}_y(x) > 0$. With the above notations we have

$$\frac{d}{dt}J(\theta^t)\mid_{t=0} = 2\left(\int_{B_1} \delta v(y)\alpha(y)d\nu_1(y) + \int_{B_2} \delta v(y)\beta(y)d\nu_2(y)\right). \tag{4.1}$$

Note that the measures $-\alpha(y)d\nu_1$ and $\beta(y)d\nu_2$ have the same mass. Indeed

$$\begin{aligned}-\int_{B_1} \alpha(y)d\nu_1(y) &= \int_{B_1}\int_{A_2} u(x)d\theta^{21}_y(x)d\nu_1(y) - \int_{B_1}\int_{A_1} \frac{1}{\gamma}u(x)d\mu_1(x)d\nu_1(y)\\ &= \int_{A_2\times B_1} u(x)d\theta^{21}(x,y) - \int_{A_1\times B_1} u(x)d\theta^{11}(x,y) = \int_{A_2} u(x)d\mu_2(x) - \int_{A_1} u(x)d\mu_1(x)\end{aligned}$$

and

$$\int_{B_2} \beta(y)d\nu_2(y) = \int_{A_2\times B_2} \frac{1}{\gamma}u(x)d\mu_2(x)d\nu_2(y) - \int_{A_1\times B_2} u(x)d\theta^{12}(x,y)$$
$$= \int_{A_2} u(x)d\mu_2(x) - \int_{A_1} u(x)d\mu_1(x).$$

Now we can apply Property 4.3 with the function $\delta v$ on $B_1$ and $B_2$, and with the measures $-\alpha(y)d\nu_1$ on $B_1$ and $\beta(y)d\nu_2$ on $B_2$, respectively, and from (4.1) we obtain that

$$\frac{d}{dt}J(\theta^t)\mid_{t=0}< 0$$

which contradicts the fact that $\theta^0$ is a minimum for $J$.

**Step 3.**
Suppose that $v(y) - w_0(y) \neq 0$ on a set of positive $l$ measure. We show then that there exists a real number $r$ that satisfies:

$$\int_{\Omega'} (v(y) - r)_+ dl(y) > \int_{\Omega} (u(x) - r)_+ dm(x),$$

which is in contradiction with the hypothesis since $\Phi(\lambda) := (\lambda - r)_+$ is a continuous convex function (by $f_+$ we represent the positive part of the function $f$). The number $r$ is constructed from the following considerations:

Define for all $p \in \mathbb{R}$ the set $C_p := \{x \in \Omega \mid u(x) > p\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}$. Note that if for some $p_1 \in \mathbb{R}$, $\theta^0(C_{p_1}) = 0$ then $\theta^0(C_p) = 0$ for every $p > p_1$, since $C_p \subset C_{p_1}$. Let $p_0 := \inf\{p \in \mathbb{R} \mid \theta^0(C_p) = 0\}$. Then $\theta^0(C_{p_0}) = 0$ since $C_{p_0}$ may be written as $C_{p_0} := \bigcup_n C_{p_n}$ for some sequence $p_n \searrow p_0$ and having in mind that $\theta^0(C_{p_n}) = 0$ for each $n$. For all $p < p_0$, $\theta^0(C_p) > 0$ and by Step 2 it turns out that $\theta^0(\{x \in \Omega \mid u(x) \leq p\} \times \{y \in \Omega' \mid v(y) - w_0(y) \geq 0\}) = 0$. Taking a sequence $p_n \nearrow p_0$ we get by employing the same arguments as above, that $\theta^0(\{x \in \Omega \mid u(x) < p_0, v(y) - w_0(y) \geq 0\}) = 0$.

Analogously, define for all $q \in \mathbb{R}$ the set $D_q := \{x \in \Omega \mid u(x) < q\} \times \{y \in \Omega' \mid v(y) - w_0(y) > 0\}$ and note that if for some $q_1 \in \mathbb{R}$, $\theta^0(D_{q_1}) = 0$, then $\theta^0(D_q) = 0$ for every $q < q_1$ since $D_q \subset D_{q_1}$. Let $q_0 := \sup\{q \in \mathbb{R} \mid \theta^0(D_q) = 0\}$. By similar arguments to the ones used above with $C_{p_0}$, $\theta^0(D_{q_0}) = 0$. Moreover, using a sequence $q_n \searrow q_0$ we obtain also that $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}) = 0$.

Note that $p_0 \leq q_0$ otherwise, consider $s$ such that $q_0 < s < p_0$. Then $\theta^0(C_s) > 0$ and $\theta^0(D_s) > 0$ that is, $\theta^0(\{x \in \Omega \mid u(x) > s\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}) > 0$ and $\theta^0(\{x \in \Omega \mid u(x) < s\} \times \{y \in \Omega' \mid v(y) - w_0(y) > 0\}) > 0$ which contradicts Step 2.

Consider now $r = q_0$ (any number between $p_0$ and $q_0$ may be taken). Let us evaluate

$$\int_{\Omega} (u(x) - q_0)_+ dm(x) = \int_{\Omega\times\Omega'} (u(x) - q_0)_+ d\theta^0(x,y) = \int_{\substack{\{x|u>q_0\}\times \\ \{y|v-w_0>0\}}} (u(x) - q_0) d\theta^0(x,y), \tag{4.2}$$

where for the last equality we had in mind the above deduced relation $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}) = 0$.

In order to evaluate $\int_{\Omega'} (v(y) - q_0)_+ dl(y)$ let us first make the following analysis:

In the set $\{y \in \Omega' \mid v(y) - w_0(y) < 0\}$, since $\theta^0(C_{p_0}) = 0$ *i.e.* $\theta^0(\{x \in \Omega \mid u(x) > p_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) < 0\}) = 0$, and having in mind the definition $w_0 := \int_\Omega u(x)d\theta^0_y(x)$, it turns out that $w_0(y) \leq p_0$ and therefore $v(y) < w_0(y) \leq p_0 \leq q_0$.

In the set $\{y \in \Omega' \mid v(y) - w_0(y) = 0\}$, since $\theta^0(\{x \in \Omega \mid u(x) < p_0\} \times \{y \in \Omega' \mid v(y) = w_0(y)\}) = 0$ and $\theta^0(\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) = w_0(y)\}) = 0$ as subsets of $\{x \in \Omega \mid u(x) < p_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \geq 0\}$ and respectively, $\{x \in \Omega \mid u(x) > q_0\} \times \{y \in \Omega' \mid v(y) - w_0(y) \leq 0\}$, we have that $w_0 \in [p_0, q_0]$ and in particular $v(y) \leq q_0$.

In the set $\{y \in \Omega' \mid v(y) - w_0(y) > 0\}$, since $\theta^0(D_{q_0}) = 0$ we obtain that $w_0(y) \geq q_0$ and hence $v(y) > q_0$.

So the only points where $(v - q_0)_+$ does not vanish are $\{y \in \Omega' \mid v(y) - w_0(y) > 0\}$ and in this set we have $v(y) > w_0(y) \geq q_0$.

Now we can evaluate $\int_{\Omega'}(v(y) - q_0)_+ dl(y)$ as follows, where the first inequality occurs applying Property 4.3 with the functions $v$ and $w_0$ on $\Omega'$ and with the same measure $l$:

$$\begin{aligned}\int_{\Omega'}(v(y) - q_0)_+ dl(y) &= \int_{\{y|v-w_0>0\}}(v(y) - q_0)dl(y) > \int_{\{y|v-w_0>0\}}(w_0(y) - q_0)dl(y)\\ &= \int_{\{y|v-w_0>0\}}\left(\int_\Omega u(x)d\theta^0_y(x) - q_0\right)dl(y)\\ &= \int_{\Omega\times\{y|v-w_0>0\}}(u(x) - q_0)d\theta^0(x,y) \geq \int_{\substack{\{x|u>q_0\}\times\\ \{y|v-w_0>0\}}}(u(x) - q_0)d\theta^0(x,y). \end{aligned} \tag{4.3}$$

Then (4.2) and (4.3) yield the contradiction. □

The following lemma is the most important ingredient in the proof of Theorem 3.1. Tartar suggested the statement below and the main idea of its proof as well.

The domain $\omega_2$ in the sequel is as in Theorem 3.1.

**Lemma 4.5.** *Given a sequence $v_n : \omega_2 \to [\alpha, \beta]$, assume that it gives rise to the Young measure $\mu = (\mu_y)$. One defines a function $g : \omega_2 \times (0,1) \to [\alpha,\beta]$ such that $g(y,\cdot)$ is nonincreasing and its distribution measure is $\mu_y$. Then, given $\theta \in [0,1]$, for all sequences $\chi_n : \omega_2 \to \{0,1\}$ such that $\chi_n \rightharpoonup \theta$, up to a subsequence, we have that*

$$\text{weak}\lim_n \chi_n v_n \leq \int_0^\theta g(y,s)ds. \tag{4.4}$$

*Moreover, there exists a sequence $\bar\chi_n : \omega_2 \to \{0,1\}$, $\bar\chi_n \rightharpoonup \theta$ such that, up to a subsequence*

$$\bar\chi_n v_n \rightharpoonup \int_0^\theta g(y,s)ds. \tag{4.5}$$

*Proof.* Consider a real number $\theta \in [0,1]$ arbitrarily fixed and consider a sequence of characteristic functions $\chi_n \rightharpoonup \theta$. Denote by $\pi$ the Young measure associated to the pair $(v_n, \chi_n)$. So $\pi = (\pi_y)_{y\in\omega_2}$ is a measure on $[\alpha,\beta] \times \{0,1\}$, where $\pi_y = \pi^1_y \otimes \delta_{\chi=0} + \pi^2_y \otimes \delta_{\chi=1}$. The projection of $\pi_y$ on $\{0,1\}$ is $(1-\theta)\delta_0 + \theta\delta_1$ and the projection of $\pi_y$ on $[\alpha,\beta]$ is $\mu_y$. Therefore $\mu_y = \pi^1_y + \pi^2_y$ and since $\pi^1_y$ and $\pi^2_y$ are positive measures, it turns out that $\pi^2_y$ is absolutely continuous with respect to $\mu_y$. Then by Radon-Nikodym theorem we have that $d\pi^2_y(v) = \eta_y d\mu_y(v)$ where $\eta_y$ is a positive function in $L^1_{\mu_y}([\alpha,\beta])$ such that $0 \leq \eta_y \leq 1$ $\mu_y$-almost everywhere in $[\alpha,\beta]$.

The weak limit of $\chi_n$ is $\theta$ so $\pi^2_y([\alpha,\beta]) = \theta$. The weak limit of $\chi_n v_n$ is calculated as follows:

$$\chi_n v_n \rightharpoonup \int \chi v d\pi_y(\chi,v) = \int \chi v d(\pi^1_y \otimes \delta_{\chi=0}) + \int \chi v d(\pi^2_y \otimes \delta_{\chi=1}) = \int v d\pi^2_y(v) = \int_0^\theta h(y,s)ds, \tag{4.6}$$

where $h : \omega_2 \times (0,\theta) \to [\alpha,\beta]$ is a nonincreasing function whose distribution measure is $\pi^2_y$. Consider $h$, for instance, given by

$$h(y,s) := \sup\{x \in [\alpha,\beta] : \pi^2_y(]x,\beta]) > s\}\cdot$$

Since for $g$ one can take the following similar definition

$$g(y,s) := \sup\{x \in [\alpha,\beta] : \mu_y(]x,\beta]) > s\},$$

having in mind that $\pi_y^2 \leq \mu_y$, one obtains that for all $y \in \omega_2$, $h(y,s) \leq g(y,s)$, for $\mathcal{L}_1$ almost every $s \in ]0,\theta[$. Thus $\int_0^\theta h(y,s)ds \leq \int_0^\theta g(y,s)ds$ and then (4.6) implies (4.4).

We prove in the sequel that the equality is reached, up to a subsequence, that is, there exists a sequence $\bar{\chi}_k \to \theta$ such that $\bar{\chi}_k v_{n_k} \to \int_0^\theta g(y,s)ds$, for a subsequence $(v_{n_k})_k$ of $(v_n)_n$. Note that there exists $c \in [\alpha,\beta]$ such that $\mu_y(]c,\beta]) \leq \theta \leq \mu_y([c,\beta])$. Indeed, having in mind the definition of $g$, it is sufficient to take $c = g(y,\theta)$. Let us define

$$\bar{\eta}_y(v) = \begin{cases} 1, & \text{if } v > c, \\ \gamma, & \text{if } v = c, \\ 0, & \text{if } v < c, \end{cases}$$

where $\gamma = \dfrac{\theta - \mu_y(]c,\beta])}{\mu_y(\{c\})}$ if $\mu_y(\{c\}) \neq 0$ and $\gamma$ may be any number between 0 and 1 if $\mu_y(\{c\}) = 0$. Then the following expression writes:

$$\int v\bar{\eta}_y(v)d\mu_y(v) = \int_{]c,\beta]} v d\mu_y(v) + c\gamma\mu_y(\{c\}).$$

Denote by $I$ the first term and by $II$ the second term in the above sum. Let us calculate the $\mathcal{L}_1$-measure of the set $\{s : g(y,s) > c\}$:

$$\mathcal{L}_1(\{s : g(y,s) > c\}) = \int_0^1 \chi_{\{g(y,s)>c\}}(s)ds = \int_{[\alpha,\beta]} \chi_{\{v>c\}}(v)d\mu_y(v) = \mu_y(]c,\beta]).$$

Since $g(y,\cdot)$ is nonincreasing it turns out that $\{s : g(y,s) > c\}$ is the interval with extremities 0 and $\mu_y(]c,\beta])$. Hence for all $s \in ]\mu_y(]c,\beta]),\theta]$, we have that $g(y,s) = c$ and then the first term $I$ yields

$$I = \int_{]c,\beta]} v d\mu_y(v) = \int_0^1 g(y,s)\chi_{\{s:g(y,s)>c\}}(s)ds = \int_0^{\mu_y(]c,\beta])} g(y,s)ds.$$

If $\mu_y(\{c\}) \neq 0$, the second term gives:

$$II = c(\theta - \mu_y(]c,\beta])) = c\int_{\mu_y(]c,\beta)}^\theta ds = \int_{\mu_y(]c,\beta])}^\theta g(y,s)ds,$$

and consequently

$$I + II = \int_0^\theta g(y,s)ds.$$

If $\mu_y(\{c\}) = 0$ then $\mu_y(]c,\beta]) = \theta = \mu_y([c,\beta])$ and $II = 0$ while

$$I = \int_{]c,\beta]} v d\mu_y(v) = \int_0^{\mu_y(]c,\beta])} g(y,s)ds = \int_0^\theta g(y,s)ds.$$

It remains now to show that there exists a sequence $\bar{\chi}_n$ such that the pair $(v_n, \bar{\chi}_n)$ gives rise to the Young measure $\pi_y = \pi_y^1 \otimes \delta_0 + \pi_y^2 \otimes \delta_1$ with $d\pi_y^2 = \bar{\eta}_y d\mu_y$ and $d\pi_y^1 = (1 - \bar{\eta}_y)d\mu_y$. This existence is ensured by the following lemma applied with $K = \{0,1\}$. □

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$.

**Lemma 4.6.** *Given $v_n$ a sequence $v_n : \Omega \to [\alpha,\beta]$ giving rise to the Young measure $\mu$ and given a family of probability measures $(\pi_{x,v})_{(x,v)\in\Omega\times[\alpha,\beta]}$ with support in $\overline{K}$ ($K$ is a bounded subset of $\mathbb{R}^p$), there exists a sequence*

*$(\chi_n)$, $\chi_n : \Omega \to K$ such that for all continuous functions $F : [\alpha, \beta] \times \overline{K} \to \mathbb{R}$ we have, up to a subsequence:*

$$F(v_n, \chi_n) \rightharpoonup \int_{[\alpha,\beta]\times\overline{K}} F(v, u) d\pi_{x,v}(u) d\mu_x(v),$$

*that is the sequence $(v_n, \chi_n)$ gives rise to the Young measure $\pi$.*

The above result generalizes Theorem 5 in [9] (p. 147), and the proof uses analogous arguments.

*Proof.* Consider the following two sets:

$$M_1 = \{\pi \text{ measure on } \Omega \times [\alpha, \beta] \times \overline{K} \text{ such that it is a narrow limit of Young measures associated to } (v_n, \chi) \text{ where } \chi : \Omega \to K \text{ is some measurable function}\}$$

and

$$M_2 = \{\pi \text{ measure on } \Omega \times [\alpha, \beta] \times \overline{K}, \pi \geq 0, \mathrm{supp}\pi \subset \Omega \times [\alpha, \beta] \times \overline{K}, \mathrm{proj}_{\Omega\times[\alpha,\beta]}\pi = \mu\}\cdot$$

We prove that equality $\overline{M}_1 = M_2$ holds.

First step. We prove that $\overline{M}_1$ is convex.

Consider $\pi_1, \pi_2, \ldots, \pi_q \in M_1$. Then each $\pi_i$ is the narrow limit of Young measures associated to pairs $(v_n, \chi_i)$ that is, for all bounded Carathéodory integrands $\phi$,

$$\int_{[\alpha,\beta]\times\overline{K}} \phi(x, v, \lambda) d\pi_i(v, \lambda) = \lim_m \phi(x, v_m(x), \chi_i(x)).$$

We shall show that $\pi := \sum_{i=1}^q \pi_i\theta_i$ belongs to $\overline{M}_1$, where $\theta_i$ are real nonnegative numbers such that $\sum_{i=1}^q \theta_i = 1$. By Theorem 3 in [9], there exist the following sequences of characteristic functions $\psi_{i,n} : \Omega \to \{0, 1\}$ such that $\sum_{i=1}^q \psi_{i,n} = 1$ and $\psi_{i,n} \rightharpoonup \theta_i$. Consider $\chi_n := \sum_{i=1}^q \psi_{i,n}(x)\chi_i(x)$. Then $\phi(x, v_m(x), \chi_n(x)) = \sum_{i=1}^q \phi(x, v_m(x), \chi_i(x))\psi_{i,n}(x)$ and passing to the limit first in $m$ and then in $n$, one obtains that, for all bounded Carathéodory integrands $\phi$,

$$\lim_n \lim_m \phi(x, v_m(x), \chi_n(x)) = \int_{[\alpha,\beta]\times\overline{K}} \phi(x, v, \lambda) d\pi(v, \lambda)$$

and consequently $\pi \in \overline{M}_1$.

Second step. We prove that $\overline{\mathrm{conv}}(\overline{M}_1) = M_2$. We know that in order to find the closed convex hull of $M_1$ we need only to consider the affine continuous functions which are positive on $M_1$ (by Hahn-Banach theorem). But an affine continuous function on the space of measures on $\Omega \times [\alpha, \beta] \times \overline{K}$ has the following form

$$\pi \to \langle \pi, \Phi_0(x, v, \lambda)\rangle + \delta$$

where $\Phi_0$ is continuous bounded and $\delta \in \mathbb{R}$. Such affine continuous function is positive on $M_1$ if

$$\langle \pi, \Phi_0(x, v, \lambda)\rangle + \delta \geq 0$$

for all $\pi \in M_1$. This is equivalent to

$$\lim_m \int_\Omega \Phi_0(x, v_m(x), \chi(x)) dx + \delta \geq 0 \tag{4.7}$$

for all $\chi : \Omega \to K$. Define $\Psi_0(x,v) := \inf_{\lambda\in\overline{K}}\{\Phi_0(x,v,\lambda)\}$. Then $\Psi_0$ is continuous bounded and therefore (4.7) is equivalent to

$$\lim_m \int_\Omega \Psi_0(x, v_m(x))dx + \delta \geq 0.$$

Let $\Phi_0(x,v,\lambda) = \Psi_0(x,v) + \chi_0(x,v,\lambda)$, with $\chi_0(x,v,\lambda) \geq 0$. So, we have that

$$\begin{aligned}\overline{\mathrm{conv}}(\overline{M}_1) = &\{\pi \text{ measure that satisfies } \langle\pi,\Phi_0\rangle + \delta \geq 0 \text{ for all } \Phi_0 \text{ and } \delta \text{ satisfying}\\ &\Phi_0 = \Psi_0 + \chi_0, \text{ for some function } \chi_0 \geq 0, \text{ and } \langle\Psi_0,\mu\rangle + \delta \geq 0\} \cdot\end{aligned}$$

Now it remains to prove that the above conditions characterize $M_2$.

1) Take $\Psi_0 = 0$ and $\delta = 0$. Then $\langle\pi,\chi_0\rangle \geq 0$ for all $\chi_0 \geq 0$. Therefore $\pi \geq 0$.
2) If $\chi_0 = 0$ in $\Omega \times [\alpha,\beta] \times \overline{K}$ then $\chi_0 \geq 0$ and $-\chi_0 \geq 0$. Then $\langle\pi,\chi_0\rangle \geq 0$ and $\langle\pi,-\chi_0\rangle \geq 0$ and hence $\langle\pi,\chi_0\rangle = 0$, therefore supp$\pi \subset \Omega \times [\alpha,\beta] \times \overline{K}$.
3) Consider $\delta = -\langle\Psi_0,\mu\rangle$ and $\chi_0 = 0$. Then $\langle\pi,\Psi_0\rangle + \delta \geq 0$ for all continuous bounded function $\Psi_0$ such that $\delta = -\langle\Psi_0,\mu\rangle$. So, $\langle\pi,\Psi_0\rangle \geq \langle\mu,\Psi_0\rangle$. On the other hand $\langle\pi,-\Psi_0\rangle - \delta \geq 0$. Then $\langle\pi,\Psi_0\rangle \geq \langle\mu,\Psi_0\rangle$. Hence $\langle\pi,\Phi_0\rangle = \langle\mu,\Psi_0\rangle$, and consequently $\mathrm{proj}_{\Omega\times[\alpha,\beta]}\pi = \mu$.

From 1), 2) and 3) above it turns out that $\overline{\mathrm{conv}}(\overline{M}_1) \subset M_2$.

Conversely, let us prove that $M_2 \subset \overline{\mathrm{conv}}(\overline{M}_1)$. Consider $\pi \in M_2$. Then $\langle\pi,\Phi_0\rangle + \delta \geq 0$ since $\langle\pi,\Phi_0\rangle = \langle\pi,\chi_0\rangle + \langle\pi,\Psi_0\rangle$, $\langle\pi,\chi_0\rangle \geq 0$ and $\langle\pi,\Psi_0\rangle = \langle\mu,\Psi_0\rangle \geq 0$. Hence $\pi \in \overline{\mathrm{conv}}(\overline{M}_1)$.

Thus we obtain the existence of a sequence $\bar{\chi}_m : \Omega \to K$ that satisfies

$$\text{weak} \lim_m \text{ weak} \lim_n \phi(x, v_n(x), \bar{\chi}_m(x)) = \int_{[\alpha,\beta]\times\overline{K}} \phi(x,v,\lambda)d\pi_x(v,\lambda),$$

for all bounded Carathéodory integrands $\phi$. One can extract then a diagonal subsequence $n_k$ such that:

$$\text{weak} \lim_k \phi(x, v_{n_k}(x), \bar{\chi}_{n_k}(x)) = \int_{[\alpha,\beta]\times\overline{K}} \phi(x,v,\lambda)d\pi_x(v,\lambda).$$

□

I warmly thank Professor Guy Bouchitté for suggesting the study of problem (1.1). I am very grateful to Professor Luc Tartar, who conjectured the main result of this paper, for many stimulating conversations we have had together. This research work was supported by FEDER-PRAXIS/2/2.1/MAT/125/94, PRAXIS-FEDER/3/3.1/CTM/10/94, Universidade de Lisboa, JNICT-MENESR 97/166.

## References


[1] Y. Amirat, K. Hamdache and A. Ziani, Some Results on Homogenization of Convection-Diffusion Equations. *Arch. Rational Mech. Anal.* **114** (1991) 155–178.

[2] E.J. Balder, *Lectures on Young measures.* Preprint No. 9517, CEREMADE. Université Paris IX - Dauphine, France (1995).

[3] J.M. Ball, A version of the fundamental theorem for Young measures, *Partial Differential Equations and Continuum Models of Phase Transitions*, edited by M. Rascle, M. Slemrod and D. Serre. Springer Verlag, Berlin (1989) 207–215.

[4] P. Cartier, J.M.G. Fell and P.A. Meyer, Comparaison des mesures portées par un ensemble convexe compact. *Bull. Soc. Math. France* **92** (1964) 435–445.

[5] G.H. Hardy, J.E. Littlewood and G.E. Pólya, *Inequalities.* Cambridge (1952).

[6] P.A. Meyer, *Probabilités et potentiel.* Hermann, Paris (1966).

[7] J.V. Ryff, On the representation of doubly stochastic operators. *Pacific J. Math.* **13** (1963) 1379–1386.

[8] J.V. Ryff, Orbits of $L^1$-functions under doubly stochastic transformations. *Trans. Amer. Math. Soc.* **117** (1965) 92–100.

[9] L. Tartar, Compensated compactness and applications to partial differential equations. *Nonlinear Analysis and Mechanics: Heriot Watt Symposium, Vol. IV. Res. Notes in Math. Pitman* (1979) 136–212.

[10] L. Tartar, Memory effects and homogenization. *Arch. Rational Mech. Anal.* **111** (1990) 121–133.

[11] M. Valadier, Young measures. *Methods of Nonconvex Analysis,* edited by A. Cellina. Springer Verlag, *Lecture Notes in Math.* **1446** (1990) 152.