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SIMULTANEOUS CONTROLLABILITY IN SHARP TIME FOR TWO ELASTIC STRINGS

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Abstract. We study the simultaneously reachable subspace for two strings controlled from a common endpoint. We give necessary and sufficient conditions for simultaneous spectral and approximate controllability. Moreover we prove the lack of simultaneous exact controllability and we study the space of simultaneously reachable states as a function of the position of the joint. For each type of controllability result we give the sharp controllability time.

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1. Introduction

In recent years boundary controllability of elastic systems has been intensively studied (see, for instance [2,10,12] and the references therein). In the present paper we focus on a particular case of the following general question: if we consider two exactly controllable systems, find the assumptions allowing the control of both systems by using the same input function. This property is called *simultaneous controllability.* Simultaneous exact controllability was first considered by Russell in [16] and it is the subject of Chapter 5 in Lions [12]. The case in which one of the systems is finite dimensional was studied in [17].

The problem we tackle is the one dimensional version of an open question raised in [12], and it considers the simultaneous controllability of two strings. More precisely, for $\xi \in (0,1)$ we consider the problems

$$\begin{cases} \ddot{w}_1(x,t) - \dfrac{\partial^2 w_1}{\partial x^2}(x,t) = 0 & \forall\, x \in (0,\xi), & \forall\, t \in (0,\infty), \\ w_1(0,t) = 0, & w_1(\xi,t) = u(t) & \forall\, t \in (0,\infty), \\ w_1(x,0) = 0, & \dot{w}_1(x,0) = 0 & \forall\, x \in (0,\xi) \end{cases} \tag{1.1}$$

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and

$$\begin{cases} \ddot{w}_2(x,t) - \dfrac{\partial^2 w_2}{\partial x^2}(x,t) = 0, & \forall\, x \in (\xi,1), & \forall\, t \in (0,\infty), \\ w_2(1,t) = 0, & w_2(\xi,t) = u(t) & \forall\, t \in (0,\infty), \\ w_2(x,0) = 0, & \dot{w}_2(x,0) = 0 & \forall\, x \in (\xi,1). \end{cases} \tag{1.2}$$

The systems above model the vibrations of two strings joined at a common endpoint at $x = \xi$, the common input being the displacement of this common point.

It is well known that, if $u \in L^2(0,T)$ the system (1.1) (respectively (1.2)) is well posed in $L^2(0,\xi) \times H^{-1}(0,\xi)$ (respectively in $L^2(\xi,1) \times H^{-1}(\xi,1)$) (see for instance [12]). This allows us to define the linear bounded operator

$$W^T : L^2(0,T) \to L^2(0,\xi) \times H^{-1}(0,\xi) \times L^2(\xi,1) \times H^{-1}(\xi,1),$$

$$W^T u = (w_1(\cdot,T), \dot{w}_1(\cdot,T), w_2(\cdot,T), \dot{w}_2(\cdot,T)).$$

The space $\mathcal{R}^T$ of the states which are simultaneously reachable at instant T is simply defined as the range $W^T(L^2(0,T))$ of the operator W^T. According to the properties of the space of simultaneously reachable states we can define several types of simultaneous controllability.

Definition 1.1.

1. The systems (1.1, 1.2) are called simultaneously approximately controllable in time T if $\mathcal{R}^T$ is dense in $L^2(0,\xi) \times H^{-1}(0,\xi) \times L^2(\xi,1) \times H^{-1}(\xi,1)$.
2. The systems (1.1, 1.2) are called simultaneously spectrally controllable in time T if, for all $n \geq 1$, the states
$$\left(\sin\left(\frac{n\pi x}{\xi}\right), 0, 0, 0\right), \left(0, \sin\left(\frac{n\pi x}{\xi}\right), 0, 0\right),$$
$$\left(0, 0, \sin\left(\frac{n\pi(1-x)}{1-\xi}\right), 0\right), \left(0, 0, 0, \sin\left(\frac{n\pi(1-x)}{1-\xi}\right)\right),$$
are reachable at time T, *i.e.* they belong to $\mathcal{R}^T$.
3. The systems (1.1, 1.2) are called simultaneously exactly controllable in time T if $\mathcal{R}^T = L^2(0,\xi) \times H^{-1}(0,\xi) \times L^2(\xi,1) \times H^{-1}(\xi,1)$.

One can easily notice that the simultaneous exact controllability implies the simultaneous spectral controllability, which implies the simultaneous approximate controllability.

The main results in this paper are the following three theorems concerning, respectively, simultaneous spectral controllability, simultaneous approximate controllability and characterization of the simultaneously reachable space. The first one concerns the simultaneous spectral controllability.

Theorem 1.2. *For any irrational ξ the systems (1.1, 1.2) are simultaneously spectrally controllable in time $T \geq 2$.*

The result above implies, in particular, that the system (1.1, 1.2) is simultaneously approximately controllable for any irrational ξ and any $T \geq 2$. Concerning the approximate controllability we will first check the following simple result.

Proposition 1.3. *For any rational ξ the systems (1.1, 1.2) are not simultaneously approximately controllable for any $T > 0$.*

Moreover we will prove the following result:

Theorem 1.4. *For any $\xi \in (0,1)$ the systems (1.1, 1.2) are not approximately controllable in time $T < 2$. In particular, for any $\xi \in (0,1)$, the system (1.1, 1.2) is not spectrally controllable in time $T < 2$.*

For $s > -\frac{1}{2}$, we introduce the space $\mathcal{W}_s$ of quadruples of functions $(w_1^0, w_1^1, w_2^0, w_2^1)$ satisfying

$$(w_1^0, w_1^1, w_2^0, w_2^1) \in H^{s+1}(0,\xi) \times H^s(0,\xi) \times H^{s+1}(\xi,1) \times H^s(\xi,1),$$

$$w_1^0(0) = 0,\ w_2^0(1) = 0,\ w_1^0(\xi) = w_2^0(\xi).$$

Denote by $\mathbb{Q}$ the set of all rational numbers. Let us also denote by $\mathcal{S}$ the set of all numbers $\rho \in (0,1)$ such that $\rho \notin \mathbb{Q}$ and if $[0, a_1, \ldots, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximation (*cf.* [5], p. 120), its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are "badly" approximable by rational numbers. In particular, by Euler-Lagrange theorem (*cf.* [11], p. 57) $\mathcal{S}$ contains all $\xi \in (0,1)$ such that ξ is an irrational quadratic number (*i.e.* satisfying a second degree equation with rational coefficients). According to a classical result (see for instance [11], p. 24) $\xi \in \mathcal{S}$ if and only if there exists a constant $C_\xi > 0$ such that

$$\left|\xi - \frac{p}{q}\right| \geq \frac{C_\xi}{q^2}, \qquad \forall\, q \geq 1. \tag{1.3}$$

We can now state our main result concerning the lack of simultaneous exact controllability and giving the characterization of the simultaneous reachable space as a function of ξ.

Theorem 1.5. *Suppose that $T \geq 2$. Then the following holds:*

(a) *The inclusion $\mathcal{R}^T \supset \mathcal{W}_0$ holds true if and only if $\xi \in \mathcal{S}$.*
(b) *For almost all $\xi \in [0,1]$ and for all $s > 0$, we have that $\mathcal{R}^T \supset \mathcal{W}_s$.*
(c) *The results above are sharp in the sense that, for any $\xi \in (0,1)$ and $s < 0$ we can find a state in $\mathcal{W}_s$ which is not reachable by means of an input $u \in L^2(0,T)$. In particular, for any $T > 0$, the systems (1.1, 1.2) are not simultaneously exactly controllable in time T.*

Remark 1.6. The proof of Theorem 1.5 is partially based on the observability inequality in Lemma 4.7 (see below). This inequality has been proved in [3] for a.e. ξ when $T > \max\{4\xi, 4(1-\xi)\}$. This result was generalized in [4], which was published after the submission of the present paper. In this note the authors announced general results implying in particular that the inequality holds for $T > 2$. A similar generalization, which holds in the case $T \geq 2$, was then announced in [1]. The inequality proved in [3] implies a particular case of assertion (b) in Theorem 1.5, namely the fact that, for almost all ξ, the states in $\mathcal{W}_s$, $s > 0$, which **vanish** at $x = \xi$, can be reached in time $T > \max\{4\xi, 4(1-\xi)\}$. The reachability of **all** the states in $\mathcal{W}_s$ in time $T > \max\{4\xi, 4(1-\xi)\}$ was first proved in [17].

2. Some background on exponential families and on Riesz basis theory

In this section we gather, for easy reference, some results on minimality and the basis property of exponential families in $L^2(0,T)$, with $0 < T \leq \infty$ together with some results on families of simple fractions in the Hardy space $H^2(\Pi_+)$, where Π_+ is the upper half-plane in $\mathbb{C}$ (see for instance [2], Sect. 1.1.1 for the definition and the properties of $H^2(\Pi_+)$). The results in this section are particular cases of some theorems of Paley–Wiener and of Vasyunin. For further details and related questions we refere to [2] and the references therein.

Let H be a Hilbert space and $(e_k)_{k\in\mathbb{Z}} \subset H$. We first recall the notion of minimality of a family of vectors in H.

Definition 2.1. The family (e_k) is said to be minimal if each vector in the set lies outside the closed subspace spanned by the others.

Consider a family of exponentials

$$\mathcal{E}_T = \left\{\mathrm{e}^{i\lambda_n t}\right\}_{n\in\mathbb{Z}} \tag{2.4}$$

in the space $L^2(0,T)$, where the sequence (λ_n) satisfies the conditions:

$$\sup_{n\in\mathbb{Z}} |\operatorname{Im}\lambda_n| < \infty, \quad \lambda_n \neq \lambda_m \text{ if } n \neq m, \quad \operatorname{Re}\lambda_n \leq \operatorname{Re}\lambda_{n+1} \quad \forall n \in \mathbb{Z}. \tag{2.5}$$

In the case of exponential functions, the property of minimality, defined above, can be characterized by a classical result of Paley and Wiener (see for instance Th. II.4.1 in [2], p. 99 or Problem 1 in [18], p. 130).

Proposition 2.2. *The family $\mathcal{E}_T$ defined in* (2.4) *is minimal in $L^2(0,T)$ if and only if there exists an entire function F of exponential type not greater than $T/2$ such that*

$$F(\lambda_n) = 0 \qquad \forall\, n \in \mathbb{Z}, \tag{2.6}$$

$$\int_{\mathbb{R}} \frac{|F(x)|^2}{1+x^2} \mathrm{d}x < \infty. \tag{2.7}$$

We will also need the notions of Riesz basis and of Riesz basis from subspaces. For the convenience of the reader we recall the definitions below.

Definition 2.3. The family $(e_k) \subset H$ is said to form a Riesz basis in H if for every $f \in H$ there exists a unique sequence $(a_k) \subset l^2(\mathbb{C})$ such that

$$f = \sum_{k\in\mathbb{Z}} a_k e_k \ \text{ in } H,$$

$$C_1\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |a_k|^2 \leq C_2\|f\|^2 \qquad \forall\, f \in H,$$

where the constants $C_1,\ C_2 > 0$ are independent on $f \in H$.

A family (X_k) of subspaces of H is called a Riesz basis of subspaces of H if for any $g \in H$ there exists the unique sequence of elements $e_k \in X_k$ such that

$$g = \sum_{k\in\mathbb{Z}} e_k,$$

and

$$C_1\|g\|^2 \leq \sum_{k\in\mathbb{Z}} \|e_k\|^2 \leq C_2\|g\|^2,$$

where the constants $C_1,\ C_2 > 0$ are independent on $g \in H$.

In order to state the results concerning the Riesz basis property we first introduce some notations. Let $\Theta = (\theta_n)$ be a countable set in $\mathbb{C}$ satisfying, for all $n \in \mathbb{Z}$, the conditions $C_1 \leq \text{Im } \theta_n \leq C_2$ with $C_1, C_2 > 0$ and $\text{Re } \theta_n \leq \text{Re } \theta_{n+1}$. For $r > 0$ we put

$$G(r) = \bigcup_{n\in\mathbb{Z}} B(\theta_n; r),$$

where $B(\theta_n; r)$ is the disk of center θ_n and of radius r and denote by $G_m(r)$, $m = 1, 2, \ldots$ the connected components of $G(r)$. Moreover let us set $\Theta_m(r) = \Theta \cap G_m(r)$ and write $L_m(r)$ for the closed linear space spanned in the Hardy space $H^2(\Pi_+)$ by the rational functions $z \to (z - \overline{\theta})^{-1}$ (the so-called simple fractions) with $\theta \in \Theta_m(r)$. Finally we denote by $\mathcal{L}$ the closed linear space in $H^2(\Pi_+)$ spanned by the simple fractions $z \to (z - \overline{\theta})^{-1}$, with $\theta \in \Theta$. The following result will be essentially used in Section 4.

Proposition 2.4. *Suppose that Θ satisfies* (2.5) *and that Θ is the union of two separate sets (i.e. in which the distance between any two different points is bounded from below). Then, for all $r > 0$, the family $(L_m(r))$ forms a Riesz basis from subspaces in $\mathcal{L}$.*

Proof. The proposition is a simple consequence of a theorem of Vasyunin (see for instance [2], Prop. II.2.11). In order to apply this result to our case we first remark that a separate set contained is a bounded strip parallel to the real axis is Carlesonian (see [2], p. 53), so the set Θ is the union of two Carlesonian sets. Moreover in a bounded strip parallel to the real axis the hyperbolic metric used by Vasyunin (see [2], p. 68) is clearly equivalent to the standard euclidian metric. Therefore a direct application of Proposition II.2.11 in [2], with $N = 2$, yields the conclusion of the proposition. $\square$

3. Proof of Theorem 1.2

Let us consider the operators $A_i,\ i = 1, 2$ defined by

$$\mathcal{D}(A_1) = H^2(0,\xi) \cap H_0^1(0,\xi),\ A_1 : \mathcal{D}(A_1) \to L^2(0,\xi),\ A_1 h = \frac{\mathrm{d}^2 h}{\mathrm{d}x^2},$$

where the derivative $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}$ is calculated in $\mathcal{D}'(0,\xi)$ and

$$\mathcal{D}(A_2) = H^2(\xi,1) \cap H_0^1(\xi,1),\ A_2 : \mathcal{D}(A_2) \to L^2(\xi,1),\ A_2 h = \frac{\mathrm{d}^2 h}{\mathrm{d}x^2},$$

where the derivative $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}$ is this time calculated in $\mathcal{D}'(\xi,1)$. We notice that $A_i,\ i = 1, 2$, are selfadjoint and negative. Moreover the eigenfunctions (e_n) of A_1 given by

$$e_n(x) = \sqrt{\frac{2}{\xi}} \sin\left(\frac{n\pi x}{\xi}\right), \qquad \forall\, n \geq 1,$$

form an orthonormal basis in $L^2(0,\xi)$. In the same way the eigenfunctions (f_n) of A_2 given by

$$f_n(x) = \sqrt{\frac{2}{1-\xi}} \sin\left(\frac{n\pi(1-x)}{1-\xi}\right), \qquad \forall\, n \geq 1,$$

form an orthonormal basis in $L^2(\xi,1)$.

Proof of Proposition 1.3. Let

$$w_1(x,t) = \sum_{n\geq 1} \beta_n(t) e_n(x),\ \text{ in } L^2(0,\xi),$$
$$w_2(x,t) = \sum_{n\geq 1} \gamma_n(t) f_n(x),\ \text{ in } L^2(\xi,1),$$

be the expansions of $w_1,\ w_2$ in the bases (e_n) and (f_n) defined above. Standard calculations show that the coefficients $\beta_n(\cdot)$ and $\gamma_n(\cdot)$ satisfy the equalities

$$\beta_n(t) = (-1)^{n+1} \sqrt{\frac{2}{\xi}} \int_0^t \sin\left[\frac{n\pi(t-s)}{\xi}\right] u(s)\mathrm{d}s, \tag{3.8}$$

$$\dot\beta_n(t) = (-1)^{n+1} \frac{n\pi}{\xi} \sqrt{\frac{2}{\xi}} \int_0^t \cos\left[\frac{n\pi(t-s)}{\xi}\right] u(s)\mathrm{d}s, \tag{3.9}$$

$$\gamma_n(t) = (-1)^{n+1} \sqrt{\frac{2}{1-\xi}} \int_0^t \sin\left[\frac{n\pi(t-s)}{1-\xi}\right] u(s)\mathrm{d}s, \tag{3.10}$$

$$\dot{\gamma}_n(t) = (-1)^{n+1}\frac{n\pi}{1-\xi}\sqrt{\frac{2}{1-\xi}}\int_0^t \cos\left[\frac{n\pi(t-s)}{1-\xi}\right]u(s)\mathrm{d}s. \tag{3.11}$$

Relations (3.8–3.11) and a simple calculation imply that if $\frac{1-\xi}{\xi} = \frac{p}{q}$ with $p, q \in \mathbb{N}$, then the state

$$\left((-1)^{mp-mq-1}\sin\frac{mq\pi x}{\xi}, 0, \sin\frac{mp\pi(1-x)}{1-\xi}, 0\right)$$

is orthogonal (in $L^2(0,\xi)\times H^{-1}(0,\xi)\times L^2(\xi,1)\times H^{-1}(\xi,1)$) to the simultaneously reachable space of (1.1) and (1.2), for any $T > 0$ and for all $m \in \mathbb{N}$. This fact clearly implies the conclusion of Proposition 1.3. □

From (3.8–3.11) we can easily deduce that the simultaneous spectral controllability of (1.1, 1.2) can be characterized as follows:

Lemma 3.1. *The systems* (1.1) *and* (1.2) *are simultaneously spectrally controllable in time T if and only if the family of functions*

$$\mathcal{F}_T = \left\{\sin\left(\frac{n\pi t}{\xi}\right)\right\}_{n\in\mathbb{N}} \bigcup \left\{\cos\left(\frac{n\pi t}{\xi}\right)\right\}_{n\in\mathbb{N}}$$
$$\bigcup \left\{\sin\left(\frac{n\pi t}{1-\xi}\right)\right\}_{n\in\mathbb{N}} \bigcup \left\{\cos\left(\frac{n\pi t}{1-\xi}\right)\right\}_{n\in\mathbb{N}}$$

admits a biorthogonal family in $L^2(0,T)$.

Proof of Theorem 1.2. The Hahn–Banach theorem shows that for a given family $\mathcal{F}_T$ the existence of a biorthogonal family is equivalent to the fact that the family $\mathcal{F}_T$ is minimal in $L^2(0,T)$.

The minimality of $\mathcal{F}_T$ is clearly equivalent to the minimality of the family of functions

$$\mathcal{E}_T = \left\{\exp\left(\frac{in\pi t}{\xi}\right)\right\}_{n\in\mathbb{Z}^*} \bigcup \left\{\exp\left(\frac{in\pi t}{1-\xi}\right)\right\}_{n\in\mathbb{Z}^*}, \ \mathbb{Z}^* := \mathbb{Z}\setminus 0\,.$$

Since ξ is irrational, the sequences $\left(\frac{in\pi t}{\xi}\right)_{n\in\mathbb{Z}^*}$ and $\left(\frac{in\pi t}{1-\xi}\right)_{n\in\mathbb{Z}^*}$ have no common element.

The function

$$F(z) = \sin(\xi z)\sin[(1-\xi)z], \quad z\in\mathbb{C}, \tag{3.12}$$

obviously satisfies (2.6, 2.7), and it is of exponential type

$$\xi + (1-\xi) = 1 \le T/2,$$

if $T \ge 2$. Hence the family $\mathcal{E}_T$ is minimal in $L^2(0,T)$ for any irrational ξ. □

More detailed information about connections of controllability types with properties of corresponding exponential families can be found in [2] (Sect. III.3).

4. Proof of Theorem 1.5

Suppose that ξ is irrational and denote by $(\lambda_k)_{k\in\mathbb{Z}^*}$ the strictly increasing sequence formed by the elements of the set

$$\Lambda = \left[\cup_{n\in\mathbb{Z}^*}\left\{\frac{n\pi}{\xi}\right\}\right] \bigcup \left[\cup_{n\in\mathbb{Z}^*}\left\{\frac{n\pi}{1-\xi}\right\}\right].$$

It is clear that

$$\lambda_{k+2} - \lambda_k \geq \delta \qquad \forall\, k \in \mathbb{Z}^*, \tag{4.1}$$

where $\delta = \min\left\{\frac{\pi}{\xi}, \frac{\pi}{1-\xi}\right\}$. We denote

$$A = \left\{ k \in \mathbb{Z}^* \text{ s.t. } \lambda_{k+1} - \lambda_k < \frac{\delta}{2} \right\} \cdot$$

Moreover we define the set

$$B = \{ k \in \mathbb{Z}^* \text{ s.t. } k \notin A \text{ and } k-1 \notin A \} \cdot$$

The main tool used in the proof of Theorem 1.5 is the following result:

Theorem 4.1. *The family*

$$\mathcal{E} = \left\{ \mathrm{e}^{i\lambda_k t} \right\}_{k \in \mathbb{Z}^* \setminus A} \bigcup \left\{ \frac{\mathrm{e}^{i\lambda_{k+1} t} - \mathrm{e}^{i\lambda_k t}}{\lambda_{k+1} - \lambda_k} \right\}_{k \in A} \bigcup \{1, \mathrm{e}^{-t}\}$$

forms a Riesz basis in $L^2(0,2)$.

In order to prove Theorem 4.1, let us split Λ in two disjoint sets: $\Lambda = \Lambda_B \bigcup \Lambda_{\bar{B}}$, where

$$\Lambda_B = \{\lambda_k : k \in B\},\ \Lambda_{\bar{B}} = \{\lambda_k : k \in \mathbb{Z}^* \setminus B\} \cdot$$

Notice that

$$B = \{ k \in \mathbb{Z}^* : \min\{\lambda_k - \lambda_{k-1},\ \lambda_{k+1} - \lambda_k\} \geq \delta/2\},$$
$$\mathbb{Z}^* \setminus B = \{ k \in \mathbb{Z}^* : \min\{\lambda_k - \lambda_{k-1},\ \lambda_{k+1} - \lambda_k\} < \delta/2\}$$

and $\Lambda_{\bar{B}}$ is the union of the pairs of "close" points:

$$\Lambda_{\bar{B}} = \bigcup\nolimits_{k \in A} \{\lambda_k, \lambda_{k+1}\} \cdot$$

Denote by $\mathcal{E}_k, k \in B$, the one-dimensional subspace in $L^2(0,2)$ spanned by $\mathrm{e}^{i\lambda_k t}$, by $\mathcal{G}_k, k \in A$, – the two-dimensional subspace spanned by $\mathrm{e}^{i\lambda_k t}$ and $\mathrm{e}^{i\lambda_{k+1} t}$, and by $\mathcal{G}_0$ – the two-dimensional subspace spanned by 1 and e^{-t}. The proof of Theorem 4.1 is essentially based on the result below.

Proposition 4.2. *The family*

$$\{\mathcal{E}_k\}_{k \in B} \bigcup \{\mathcal{G}_k\}_{k \in A} \bigcup \mathcal{G}_0 \tag{4.2}$$

forms a Riesz basis from subspaces in $L^2(0,2)$.

In order to prove Proposition 4.2 we first consider the shifted sets

$$\Lambda'_A = \{\lambda'_k = \lambda_k + i | k \in A\}\,, \quad \Lambda'_B = \{\lambda'_k = \lambda_k + i | k \in B\} \cdot$$

Moreover denote by $\mathcal{E}'_k, k \in B$, the one dimensional space spanned in $L^2(0,2)$ by $\mathrm{e}^{i\lambda'_k t}$ and by $\mathcal{G}'_k, k \in A$, (respectively by $\mathcal{G}'_0$) the two dimensional space spanned in $L^2(0,2)$ by $\mathrm{e}^{i\lambda'_k t}$ and $\mathrm{e}^{i\lambda'_{k+1} t}$ (respectively by e^{-t} and e^{-2t}). Since $\mathrm{e}^{-t}\mathrm{e}^{i\lambda t} \in L^2(0,\infty)$, for all $\lambda \in \Lambda$ we can denote by $\widetilde{\mathcal{E}}'_k, k \in B$, the one dimensional space spanned in $L^2(0,\infty)$ by $\mathrm{e}^{i\lambda'_k t}$ and by $\widetilde{\mathcal{G}}'_k, k \in A$, (respectively by $\mathcal{G}'_0$) the two dimensional space spanned in $L^2(0,\infty)$ by $\mathrm{e}^{i\lambda'_k t}$ and $\mathrm{e}^{i\lambda'_{k+1} t}$ (respectively by e^{-t} and e^{-2t}).

We will next prove the following result:

Proposition 4.3. *The family*

$$\left\{\tilde{\mathcal{E}}'_k\right\}_{k\in B}\bigcup\left\{\tilde{\mathcal{G}}'_k\right\}_{k\in A}\bigcup\tilde{\mathcal{G}}'_0 \tag{4.3}$$

forms a Riesz basis from subspaces in $\mathcal{L}$, where $\mathcal{L}$ is the closure of the linear span of the family (4.3) *in* $L^2(0,\infty)$.

Proof. If $f\in L^2(0,\infty)$ we extend f by zero outside $[0,\infty)$ and we denote by $\mathcal{F}(f)$ the complex Fourier transform of f defined by

$$[\mathcal{F}(f)](z)=\sqrt{\frac{1}{2\pi}}\int_0^\infty f(t)\mathrm{e}^{itz}\mathrm{d}t,\ \mathcal{I}m\ z\geq 0.$$

According to the Paley–Wiener theorem (see for instance [2], pp. 40-41 or [15], pp. 372), $\mathcal{F}$ is an isometric isomorphism from $L^2(0,\infty)$ to the Hardy space $H^2(\Pi_+)$. The facts above imply that the family (4.3) is a Riesz basis of subspaces in $L^2(0,\infty)$ if and only if the family

$$\left\{\mathcal{F}(\tilde{\mathcal{E}}'_k)\right\}_{k\in B}\bigcup\left\{\mathcal{F}(\tilde{\mathcal{G}}'_k)\right\}_{k\in A}\bigcup\mathcal{F}(\tilde{\mathcal{G}}'_0) \tag{4.4}$$

forms a Riesz basis from subspaces in $\widetilde{\mathcal{L}}$, where $\widetilde{\mathcal{L}}$ is the closure of the linear span of the family (4.4) in $H^2(\Pi_+)$. This last assertion can be now checked by a simple application of Proposition 2.4.

More precisely, in order to apply this result to our case we notice first that the Fourier transform of $\mathrm{e}^{i\lambda'_k t}$ is $h_k(z)=i\sqrt{\frac{1}{2\pi}}\frac{1}{\lambda'_k+z}$ and that $\mathcal{F}(\widetilde{\mathcal{E}}'_k), k\in B$, is the one dimensional space spanned in $H^2(\Pi_+)$ by h_k, that $\widetilde{\mathcal{G}}'_k, k\in A$, is the two dimensional space spanned in $H^2(\Pi_+)$ by h_k and h_{k+1} and that $\mathcal{F}(\mathcal{G}'_0)$ is the two dimensional space spanned in $H^2(\Pi_+)$ by $\frac{1}{z+i}$ and $\frac{1}{z+2i}$. By applying Proposition 2.4, with $r=\frac{\delta}{2}$ we obtain that the family (4.4) forms a Riesz basis from subspaces in $\widetilde{\mathcal{L}}$. Since, as we noticed above, the family (4.4) is obtained from the family (4.3) by a boundedly invertible continuous operator, this implies that the family defined in (4.3) forms a Riesz basis in $\mathcal{L}$, which is the conlusion of the proposition. □

Proof of Proposition 4.2. Since multiplication by e^{-t} is the isomorphism in $L^2(0,2)$ the family (4.2) forms a Riesz basis from subspaces in $L^2(0,2)$ if and only if the family 4.3 does. From Proposition 4.3 it follows that the family 4.3 forms a Riesz basis from subspaces in $L^2(0,2)$ if and only if the orthoprojector P from $L^2(0,\infty)$ onto $L^2(0,2)$ is an isomorphism from $\mathcal{L}$ to $L^2(0,2)$. The necessary and sufficient conditions for P to be an isomorphism were obtained by Pavlov [14] (see also Th. II.3.14 a) and Prop. II.3.17 b) in [2]): there exists an entire function $G(z)$ of exponential type 1 with zero set

$$\{\Lambda+i\}\cup\{i\}\cup\{2i\} \tag{4.5}$$

such that

$$\sup\left(\frac{1}{|I|}\int_I|G(x)|^2\,\mathrm{d}x\,\frac{1}{|I|}\int_I|G(x)|^{-2}\,\mathrm{d}x\right)<\infty\,, \tag{4.6}$$

where sup is taken over all the intervals $I=(\alpha,\beta)\subset\mathbb{R}$ (the so called Muckenhoupt (A_2) condition). It is easy to see that the function

$$G(z)=\frac{z-2i}{z-i}\sin\left[\xi(z-i)\right]\sin\left[(1-\xi)(z-i)\right]$$

is of exponential type 1 in the both upper and lower half planes and has the set of zeros (4.5). Since

$$0 < \inf_{x\in\mathbb{R}} |G(x)| \le \sup_{x\in\mathbb{R}} |G(x)| < \infty\,,$$

the condition (4.6) is obviously satisfied. Lemma 4.2 is proved. □

Proof of Theorem 4.1. By using Proposition 4.2 it clearly suffices to show that the angle ϕ_k formed (in $L^2(0,2)$) by

$$\mathrm{e}^{i\lambda_k t} \quad \text{and} \quad \frac{\mathrm{e}^{i\lambda_{k+1}t} - \mathrm{e}^{i\lambda_k t}}{\lambda_{k+1} - \lambda_k}$$

is bounded away from zero uniformly with respect to $k \in A$. Elementary calculations show that

$$\cos^2(\phi_k) = \rho(\lambda_{k+1} - \lambda_k) \tag{4.7}$$

where the function $\rho(\nu)$ is defined by

$$\rho(\nu) = \frac{\left(1 - \frac{\sin(2\nu)}{2\nu}\right)^2 + \frac{\sin^4(\nu)}{\nu^2}}{2\left(1 - \frac{\sin(2\nu)}{2\nu}\right)}\cdot$$

Since, by the definition of the set A, $\lambda_{k+1} - \lambda_k$ lies in $(0, \delta/2)$ it suffices to show that $\rho(\nu)$ is bounded away from 1 for $\nu \in (0, \delta/2)$. Moreover, since ρ can be extended to a continuous function on the closed interval $[0, \delta/2]$ (in fact we have $\lim_{\nu\to 0} \rho(\nu) = \frac{3}{4}$), it suffices to prove that $\rho(\nu) \ne 1$, for all $\nu \in [0, \delta/2]$. This follows directly from the fact that, by (4.7), $\sqrt{\rho(\nu)}$ is the cosine of the angle formed by the functions 1 and $\dfrac{\mathrm{e}^{\nu t} - 1}{\nu}$ which is obviously different of 1, for all $\nu \in [0, \delta/2]$. This ends up the proof of Theorem 4.1. □

As an immediate consequence of Theorem 4.1 we obtain:

Corollary 4.4. *Suppose that $T \ge 2$. Then there exist positive constants C_1 and C_2 such that for any $(a_n) \subset l^2(\mathbb{C})$*

$$C_1 \left\{ \sum_{n\in A} \left[\left(|a_n|^2 + |a_{n+1}|^2\right) |\lambda_{n+1} - \lambda_n|^2 + |a_n + a_{n+1}|^2 \right] + \sum_{n\in B} |a_n|^2 \right\} \le \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\lambda_n t} \right|^2 \mathrm{d}t$$
$$\le C_2 \left\{ \sum_{n\in A} \left[\left(|a_n|^2 + |a_{n+1}|^2\right) |\lambda_{n+1} - \lambda_n|^2 + |a_n + a_{n+1}|^2 \right] + \sum_{n\in B} |a_n|^2 \right\}\cdot$$

Remark 4.5. Constants C_1 and C_2 may depend on ξ and T. Using some additional arguments we can prove that C_1 is an absolute constant and C_2 has the form of $C_3 T$ where C_3 is also an absolute constant.

The corollary above improves the similar results obtained in [8] for $T > \dfrac{12\sqrt{6}}{\delta}$ and in [3] for $T > \frac{4\pi}{\delta}$.

Let us now consider the initial and boundary value problems

$$\ddot{\phi}_1(x,t) - \frac{\partial^2 \phi_1}{\partial x^2}(x,t) = 0 \quad \forall x \in (0,\xi), \quad \forall t \in (0,\infty), \tag{4.8}$$

$$\phi_1(0,t) = \phi_1(\xi,t) = 0 \quad \forall t \in (0,\infty), \tag{4.9}$$

$$\phi_1(x,0) = \phi_1^0(x), \quad \dot{\phi}_1(x,0) = \phi_1^1(x) \quad \forall x \in (0,\xi) \tag{4.10}$$

and

$$\ddot{\phi}_2(x,t) - \frac{\partial^2 \phi_2}{\partial x^2}(x,t) = 0 \quad \forall x \in (\xi,1), \quad \forall t \in (0,\infty), \tag{4.11}$$

$$\phi_2(1,t) = \phi_2(\xi,t) = 0 \quad \forall t \in (0,\infty), \tag{4.12}$$

$$\phi_2(x,0) = \phi_2^0(x), \quad \dot{\phi}_2(x,0) = \phi_2^1(x) \quad \forall x \in (\xi,1). \tag{4.13}$$

As a tool in our proof, for all $s > -\frac{1}{2}$ we introduce the spaces

$$\mathcal{V}_s = H_0^{s+1}(0,\xi) \times H^s(0,\xi) \times H_0^{s+1}(\xi,1) \times H^s(\xi,1).$$

It is clear that $\mathcal{V}_s$ is a subspace of W_s (with finite codimension). In order to prove Theorem 1.5, we notice first that for $s < \frac{1}{2}$, the reachability of $\mathcal{W}_s$ is equivalent to the reachability of its subspace $\mathcal{V}_s$. More precisely, we have the following result, which is proved in [17] (Sect. 5):

Lemma 4.6. *Let $s \in (-\frac{1}{2}, \frac{1}{2})$. Then $\mathcal{R}^T \supset \mathcal{W}_s$ if and only if $\mathcal{R}^T \supset \mathcal{V}_s$.*

We will use the following duality result, which follows from Theorem 2.1 in Dolecki and Russell [6] or from the HUM method of Lions (see [12]).

Lemma 4.7. *The space of the states of (1.1, 1.2) which can be reached by means of the same input $u \in L^2(0,T)$ contains the space $\mathcal{V}_s$, $s \in (-\frac{1}{2}, \frac{1}{2})$, if and only if there exist $C, T > 0$ such that the solutions ϕ_1, ϕ_2 of (4.8–4.13) satisfy*

$$\int_0^T \left| \frac{\partial \phi_2}{\partial x}(\xi,t) - \frac{\partial \phi_1}{\partial x}(\xi,t) \right|^2 \mathrm{d}t \geq C \left(\|\phi_1^0\|^2_{H^{-s}(0,\xi)} + \|\phi_1^1\|^2_{H^{-1-s}(0,\xi)} + \|\phi_2^0\|^2_{H^{-s}(\xi,1)} + \|\phi_2^1\|^2_{H^{-1-s}(\xi,1)} \right)$$

$$\forall (\phi_1^0, \phi_1^1, \phi_2^0, \phi_2^1) \in \left(H^2(0,\xi) \cap H_0^1(0,\xi)\right) \times H_0^1(0,\xi) \times \left(H^2(\xi,1) \cap H_0^1(\xi,1)\right) \times H_0^1(\xi,1).$$

Remark 4.8. The reachability of all the elements in $\mathcal{W}_s$ is equivalent to the inequality

$$\int_0^T \left| \frac{\partial \phi_2}{\partial x}(\xi,t) - \frac{\partial \phi_1}{\partial x}(\xi,t) \right|^2 \mathrm{d}t \geq C \left(\|\phi^0\|^2_{H^{-s}(0,1)} + \|\phi^1\|^2_{H^{-1-s}(0,1)} \right),$$

$$\forall (\phi_1^0, \phi_1^1, \phi_2^0, \phi_2^1) \in \left(H^2(0,\xi) \cap H_0^1(0,\xi)\right) \times H_0^1(0,\xi) \times \left(H^2(\xi,1) \cap H_0^1(\xi,1)\right) \times H_0^1(\xi,1),$$

where

$$\phi^i(x) = \begin{cases} \phi_1^i(x) & \text{if} \quad 0 < x < \xi \\ \phi_2^i(x) & \text{if} \quad \xi < x < 1 \end{cases}, \quad i \in \{0,1\}.$$

Unlike it was claimed in [3] the inequality above doesn't follow from the inequality in Lemma 4.7 or from a direct application of Ingham type results. This is why the inequality in Lemma 4.7 implies only the reachability of elements in $\mathcal{V}_s$. For the reachability of the elements of $\mathcal{W}_s$ having non vanishing trace at $x = \xi$ we use a different argument, which is given in Lemma 4.6 above.

Proof of Theorem 1.5. It is known that we have the expansions

$$\left.\begin{array}{rl}\phi_1^0(x) = & \sum_{n\geq 1} c_n \sin\left(\dfrac{n\pi x}{\xi}\right)\\ \phi_1^1(x) = & \frac{\pi}{\xi}\sum_{n\geq 1} n d_n \sin\left(\dfrac{n\pi x}{\xi}\right)\end{array}\right\} \; x\in(0,\xi),$$

$$\left.\begin{array}{rl}\phi_2^0(x) = & \sum_{n\geq 1} e_n \sin\left(\dfrac{n\pi(1-x)}{1-\xi}\right)\\ \phi_2^1(x) = & \dfrac{\pi}{1-\xi}\sum_{n\geq 1} n f_n \sin\left(\dfrac{n\pi(1-x)}{1-\xi}\right)\end{array}\right\} \; x\in(\xi,1),$$

where the sequences (c_n), (d_n), (e_n) and (f_n) are in l^2. A standard calculation shows that the solutions ϕ_1, ϕ_2 of (4.8–4.13) are given by

$$\phi_1(x,t) = \sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\frac{n\pi}{\xi}t}\sin\left(\frac{|n|\pi x}{\xi}\right), \quad x\in(0,\xi), \tag{4.14}$$

$$\phi_2(x,t) = \sum_{n\in\mathbb{Z}^*} b_n \mathrm{e}^{i\frac{n\pi}{1-\xi}t}\sin\left(\frac{|n|\pi(1-x)}{1-\xi}\right), \quad x\in(\xi,1), \tag{4.15}$$

where

$$a_n = \begin{cases} \dfrac{c_n - id_n}{2} & \text{for } \; n\geq 1,\\ \dfrac{c_{-n}+id_{-n}}{2} & \text{for } \; n\leq -1,\end{cases} \tag{4.16}$$

$$b_n = \begin{cases} \dfrac{e_n - if_n}{2} & \text{for } \; n\geq 1,\\ \dfrac{e_{-n}+if_{-n}}{2} & \text{for } \; n\leq -1.\end{cases} \tag{4.17}$$

In order to prove the first assertion of the theorem we notice that from (1.3) it easily follows (see [7] for details) that, for any $\xi\in\mathcal{S}$, there exists a constant $C_\xi>0$ with

$$\lambda_{n+1}-\lambda_n \geq \frac{C_\xi}{|\lambda_n|} \qquad \forall\, n\in\mathbb{Z}^* \;. \tag{4.18}$$

Moreover (4.14, 4.15) imply

$$\frac{\partial\phi_2}{\partial x}(\xi,t)-\frac{\partial\phi_1}{\partial x}(\xi,t) = \sum_{n\in\mathbb{Z}^*}(-1)^{|n|+1}|n|\pi\left(\frac{a_n}{\xi}\mathrm{e}^{i\frac{n\pi t}{\xi}}+\frac{b_n}{1-\xi}\mathrm{e}^{i\frac{n\pi t}{1-\xi}}\right), \tag{4.19}$$

which yields

$$\frac{\partial\phi_2}{\partial x}(\xi,t)-\frac{\partial\phi_1}{\partial x}(\xi,t) = \sum_{n\in\mathbb{Z}^*}\alpha_n\lambda_n\mathrm{e}^{i\lambda_n t} \tag{4.20}$$

with

$$\sum_{n\in\mathbb{Z}^*} |\alpha_n|^2 = \sum_{n\in\mathbb{Z}^*} (|a_n|^2 + |b_n|^2). \tag{4.21}$$

Relations (4.18, 4.20, 4.21) and the first inequality in Corollary 4.4 imply that there exists a constant $K_\xi > 0$ such that

$$\int_0^T \left| \frac{\partial \phi_2}{\partial x}(\xi,t) - \frac{\partial \phi_1}{\partial x}(\xi,t) \right|^2 \mathrm{d}t \geq K_\xi \sum_{n\in\mathbb{Z}^*} \left(|a_n|^2 + |b_n|^2\right) \tag{4.22}$$

for all $\xi \in \mathcal{S}$ and for all $T \geq 2$. Inequality (4.22) combined with Lemma 4.7 implies that the elements in $\mathcal{V}_0$ are reachable by means of an input in $L^2(0,T)$. By using Lemma 4.6 we obtain the first implication in the assertion (a) of Theorem 1.5. Conversely, if we suppose that $\xi \notin \mathcal{S}$, by the definition of the set $\mathcal{S}$ we can construct a sequence $(p(n)) \in \mathbb{N}$ such that

$$\lim_{n\to\infty} p(n) \left[\lambda_{p(n)+1} - \lambda_{p(n)}\right] = 0.$$

Let us now consider the sequence of solutions (ϕ_{1n}) (respectively (ϕ_{2n})) of (4.8–4.10) (respectively of (4.11–4.13)) having initial data $\left(\frac{\xi}{p(n)\pi}\sin\left(\frac{p(n)\pi}{\xi}x\right), 0\right)$ (respectively $\left(\frac{1-\xi}{(p(n)+1)\pi}\sin\left(\frac{(p(n)+1)\pi}{1-\xi}(1-x)\right), 0\right)$. A simple calculation (see the proof of the assertion (c) below for details) yields

$$\lim_{n\to\infty} \frac{\displaystyle\int_0^T \left| \frac{\partial \phi_{2n}}{\partial x}(\xi,t) - \frac{\partial \phi_{1n}}{\partial x}(\xi,t) \right|^2 \mathrm{d}t}{\|\phi_{1n}(0)\|^2_{L^2(0,\xi)} + \|\phi_{2n}(0)\|^2_{L^2(\xi,1)}} = 0.$$

This ends up the proof of the assertion (a) of the theorem.

In order to prove assertion (b) we notice that, according to Lemma 7.3 in [7], for any $\varepsilon > 0$ there exists a set $B_\varepsilon \subset (0,1)$, of Lebesgue measure equal to 1, such that for any $\xi \in B_\varepsilon$, there exists a constant $C_\xi > 0$ with

$$\lambda_{n+1} - \lambda_n \geq \frac{C_\xi}{|\lambda_n|^{1+\varepsilon}} \qquad \forall\, n \in \mathbb{Z}^*. \tag{4.23}$$

Relations (4.20, 4.21, 4.23) and Corollary 4.4 imply that there exists a constant $K_\xi > 0$ such that

$$\int_0^T \left| \frac{\partial \phi_2}{\partial x}(\xi,t) - \frac{\partial \phi_1}{\partial x}(\xi,t) \right|^2 \mathrm{d}t \geq K_\xi \sum_{n\in\mathbb{Z}^*} \left(\frac{|a_n|^2 + |b_n|^2}{|\lambda_n|^{2\varepsilon}} \right) \tag{4.24}$$

for all $\xi \in B_\varepsilon$ and for all $T \geq 2$. By applying again Lemma 4.7 and Lemma 4.6 we get assertion (b) of Theorem 1.5.

In order to prove assertion (c) we notice that, for any $\xi \in (0,1)$, we can use the continued fractions expansion of $\frac{1-\xi}{\xi}$ to construct a sequence $(p(n))$ with values in $\mathbb{N}$, with $\lim_{n\to\infty} p(n) = \infty$, such that

$$\lambda_{p(n)+1} - \lambda_{p(n)} \leq \frac{C}{p(n)} \qquad \forall\, n \in \mathbb{N}. \tag{4.25}$$

Let us denote by (ϕ_{1n}) (respectively by (ϕ_{2n})) the sequence of solutions of (4.8–4.10) (respectively of (4.11–4.13)) having initial data $\left(\frac{\xi}{p(n)\pi}\sin\left(\frac{p(n)\pi}{\xi}x\right), 0\right)$ (respectively $\left(\frac{1-\xi}{(p(n)+1)\pi}\sin\left(\frac{(p(n)+1)\pi}{1-\xi}(1-x)\right), 0\right)$.

A simple calculation shows that

$$\phi_{1n}(x,t) = \frac{\xi}{p(n)\pi}\cos\left(\frac{p(n)\pi}{\xi}t\right)\sin\left(\frac{p(n)\pi x}{\xi}\right), \qquad \forall\, x \in (0,\xi),$$

$$\phi_{2n}(x,t) = \frac{1-\xi}{(p(n)+1)\pi}\cos\left(\frac{(p(n)+1)\pi}{1-\xi}t\right)\sin\left(\frac{(p(n)+1)\pi(1-x)}{1-\xi}\right), \qquad \forall\, x \in (\xi,1).$$

Relations above and (4.25) imply that

$$\lim_{n\to\infty} \frac{\displaystyle\int_0^T \left|\frac{\partial\phi_{2n}}{\partial x}(\xi,t) - \frac{\partial\phi_{1n}}{\partial x}(\xi,t)\right|^2 \mathrm{d}t}{\|\phi_{1n}(0)\|^2_{H^{-s}(0,\xi)} + \|\phi_{2n}(0)\|^2_{H^{-s}(\xi,1)}} = 0$$

for all $s < 0$. Using again Lemma 4.7 we conclude that (c) also holds. □

5. Proof of Theorem 1.4

The main ingredient of the proof of Theorem 1.4 is the following result:

Lemma 5.1. *Let (λ_n) be the sequence introduced in Section 3, $\xi \in \mathbb{R}\setminus\mathbb{Q}$ and $T < 2$. Then the family $\mathcal{E}_T = (\mathrm{e}^{i\lambda_n t})_{n\in\mathbb{Z}^*}$ contains a subfamily $\mathcal{G}_T$, with $\mathcal{G}_T \neq \mathcal{E}_T$ such that $\mathcal{G}_T$ is a Riesz basis in $L^2(0,T)$.*

Proof. For $\varepsilon \in (0,1/2)$ we define the set

$$\mathbb{Z}_\varepsilon := \left\{k \in \mathbb{Z}^* : \inf_{n\in\mathbb{Z}^*}\left|\frac{k\pi}{\xi} - \frac{n\pi}{1-\xi}\right| < \frac{\varepsilon\pi}{1-\xi}\right\} = \left\{k \in \mathbb{Z}^* : \inf_{n\in\mathbb{Z}^*}\left|k\frac{1-\xi}{\xi} - n\right| < \varepsilon\right\}.$$

For $\theta \in \mathbb{R}\setminus\mathbb{Q}$, it is known that the sequence $(k\theta)\,, k \in \mathbb{Z}^*$ is well-distributed mod 1 (see *e.g.* [9], pp. 40-42 for definition and properties). It follows that

$$\frac{\#\{k \in \mathbb{Z}_\varepsilon : x \le k < x+r\}}{r} \to 2\varepsilon \tag{5.1}$$

as $r \to \infty$ uniformly relative to $x \in \mathbb{R}$. Here we denote by $\#\mathcal{A}$ the number of elements in the set $\mathcal{A}$. It follows from (5.1) that

$$\frac{1}{r}\left[\#\left\{\frac{k\pi}{\xi} : k \in \mathbb{Z}^*\setminus\mathbb{Z}_\varepsilon,\ x \le \frac{k\pi}{\xi} < x+r\right\} + \#\left\{\frac{k\pi}{1-\xi} : k \in \mathbb{Z}^*,\ x \le \frac{k\pi}{1-\xi} < x+r\right\}\right] \to \frac{1-2\varepsilon\xi}{\pi} \tag{5.2}$$

as $r \to \infty$ uniformly in $x \in \mathbb{R}$. Let us suppose additionally that $\varepsilon < \frac{2-T}{4\xi}$. Then the right hand side of (5.2) is greater than $T/2\pi$. In this case, by the force of Theorem II.4.18 of [2], the family

$$\left\{\mathrm{e}^{i\frac{k\pi}{\xi}t}\right\}_{k\in\mathbb{Z}^*\setminus\mathbb{Z}_\varepsilon} \bigcup \left\{\mathrm{e}^{i\frac{k\pi}{1-\xi}t}\right\}_{k\in\mathbb{Z}^*}$$

contains a subfamily $\mathcal{G}_T$ which forms a Riesz basis in $L^2(0,T)$. Lemma is proved. □

Remark 5.2. The algorithm proposed in Theorem II.4.18 of [2] allows us to construct a subfamily $\mathcal{G}_T$ with symmetric spectrum: inclusion $\mathrm{e}^{i\lambda t} \in \mathcal{G}_T$ implies $\mathrm{e}^{-i\lambda t} \in \mathcal{G}_T$. We use this fact below in the proof of Theorem 1.4.

Consider again the problems (4.8–4.10) and (4.11–4.13). It is well-known (see for instance [12]) that (4.8–4.10) (respectively (4.11–4.13)) is well-posed in $H_0^1(0,\xi)\times L^2(0,\xi)$ (respectively in $H_0^1(\xi,1)\times L^2(\xi,1)$) and that the solution ϕ_1 (respectively ϕ_2) has the hidden regularity property $\frac{\partial\phi_1}{\partial x}(\xi,\cdot)\in L^2(0,T)$ (respectively $\frac{\partial\phi_2}{\partial x}(\xi,\cdot)\in L^2(0,T)$). It is by now well-known that approximate controllability is equivalent to a unique continuation result for the solutions of the dual problem. In our case, simultaneous approximate controllability is characterized by the result below, which we state here without proof.

Lemma 5.3. *The systems* (1.1) *and* (1.2) *are simultaneously approximately controllable in time* $T>0$ *if and only if the only solutions*

$$\phi_1\in C(0,T;H_0^1(0,\xi))\cap C^1(0,T;L^2(0,\xi)), \tag{5.3}$$

$$\phi_2\in C(0,T;H_0^1(\xi,1))\cap C^1(0,T;L^2(\xi,1)) \tag{5.4}$$

of (4.8–4.13) satisfying the condition

$$\frac{\partial\phi_2}{\partial x}(\xi,\cdot)-\frac{\partial\phi_1}{\partial x}(\xi,\cdot)=0 \quad \textit{in } L^2(0,T),$$

are $\phi_1\equiv 0$ *and* $\phi_2\equiv 0$.

Proof of Theorem 1.4. Suppose that $0<T<T'<2$. Moreover let $f\in L^2(0,2)$ be a real valued function such that $f(t)=0$ if $t\le T$ and $||f||_{L^2(0,T')}\neq 0$. In order to prove the lack of simultaneous approximate controllability, by Lemma 5.3, it suffices to show that there exist ϕ_1,ϕ_2 not identically zero satisfying (4.8–4.13, 5.3) and (5.4) such that

$$\frac{\partial\phi_2}{\partial x}(\xi,\cdot)-\frac{\partial\phi_1}{\partial x}(\xi,\cdot)=f(t) \quad \text{in } L^2(0,T'). \tag{5.5}$$

Let us denote

$$P=\left\{n\in\mathbb{Z}^*|\mathrm{e}^{i\frac{n\pi t}{\xi}}\in\mathcal{G}_{T'}\right\},$$
$$Q=\left\{n\in\mathbb{Z}^*|\mathrm{e}^{i\frac{n\pi t}{1-\xi}}\in\mathcal{G}_{T'}\right\},$$

where $\mathcal{G}_{T'}$ is the set introduced in Lemma 5.1. According to Lemma 5.1 there exist the sequences (k_n), $(l_n)\subset l^2(\mathbb{C})$ such that

$$f(t)=\sum_{n\in P}k_n\mathrm{e}^{i\frac{n\pi t}{\xi}}+\sum_{n\in Q}l_n\mathrm{e}^{i\frac{n\pi t}{1-\xi}} \quad \text{in } L^2(0,T'), \tag{5.6}$$

$$0<\sum_{n\in P}|k_n|^2+\sum_{n\in Q}|l_n|^2<\infty. \tag{5.7}$$

By using (4.14, 4.15, 4.19) and (5.6) we get that $f(t)$ satisfies (5.5), with ϕ_1,ϕ_2 solutions of (4.8–4.13) with initial data

$$\phi_1^0(x)=2\xi\sum_{n\in P,n\ge1}\frac{(-1)^{n+1}}{n\pi}\mathcal{R}e(k_n)\sin\left(\frac{n\pi x}{\xi}\right),$$
$$\phi_1^1(x)=2\sum_{n\in P,n\ge1}(-1)^n\mathcal{I}m(k_n)\sin\left(\frac{n\pi x}{\xi}\right),$$

$$\phi_2^0(x) = 2(1-\xi) \sum_{n\in Q, n\geq 1} \frac{(-1)^{n+1}}{n\pi} \mathcal{R}e(l_n) \sin\left(\frac{n\pi(1-x)}{1-\xi}\right),$$

$$\phi_2^1(x) = 2(1-\xi) \sum_{n\in Q, n\geq 1} (-1)^n \mathcal{I}m(l_n) \sin\left(\frac{n\pi(1-x)}{1-\xi}\right).$$

Relations above and (5.7) imply that ϕ_1, ϕ_2 are not identically zero on $[0,T]$ and they satisfy (4.8–4.13, 5.3, 5.4) and (5.5), so we have

$$\frac{\partial\phi_2}{\partial x}(\xi,\cdot) - \frac{\partial\phi_1}{\partial x}(\xi,\cdot) = 0 \quad \text{ in } L^2(0,T).$$

This ends up the proof of the theorem. □

Remark 5.4. A different proof of Theorem 1.4 can be obtained by applying Theorem III.3.10 of [2]. According to this result the systems (1.1, 1.2) are simultaneously approximately controllable in time T if and only if the family

$$\mathcal{E}_T = \left\{\exp\left(\frac{in\pi t}{\xi}\right)\right\}_{n\in\mathbb{Z}^*} \bigcup \left\{\exp\left(\frac{in\pi t}{1-\xi}\right)\right\}_{n\in\mathbb{Z}^*}, \ \mathbb{Z}^* := \mathbb{Z}\setminus 0$$

is weakly linear independent (see Definition I.1.11 in [2]) in $L^2(0,T)$. By Lemma 5.1 it is not the case when $T < 2$.

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