ESAIM: Control, Optimisation and Calculus of Variations March 2001, Vol. 6, 333–360
URL: http://www.emath.fr/cocv/

CONTROLLABILITY OF A SLOWLY ROTATING TIMOSHENKO BEAM

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Abstract. Consider a Timoshenko beam that is clamped to an axis perpendicular to the axis of the beam. We study the problem to move the beam from a given initial state to a position of rest, where the movement is controlled by the angular acceleration of the axis to which the beam is clamped. We show that this problem of controllability is solvable if the time of rotation is long enough and a certain parameter that describes the material of the beam is a rational number that has an even numerator and an odd denominator or *vice versa*.

Mathematics Subject Classification. 93C20, 93B05, 93B60.

Received October, 2000. Revised January, 2001.

1. Introduction

In [2], a model for an Euler–Bernoulli beam rotating in a plane has been derived. In [7] Krabs has shown the exact controllability of a rotating Euler–Bernoulli beam. In this paper we consider the corresponding problem of boundary control at one end for a Timoshenko beam. The control is performed by the angular acceleration of the axis, to which the beam is clamped. In [8], Krabs and Sklyar have shown controllability from a position of rest to a position of rest for the Timoshenko beam for a special parameter value (namely $\gamma = 1$).

A similar problem with controls at both ends of the beam has been studied by Moreles in [9], where the Timoshenko equations and the Rayleigh equations are considered. In [5], boundary control on the free end of the beam is considered. A similar problem with nonhomogeneous parameters is treated in [11].

The equations of the Timoshenko beam can be transformed to a system where exactly one real parameter γ appears in the equations. This parameter is always positive. Apart from γ, the length of the beam also plays an important role. We show that the controllability of the beam can be guaranteed if a condition on the number-theoretic properties of the parameter γ is satisfied: if the parameter is rational with an even numerator and an odd denominator or *vice versa* and the rotation time is long enough and the beam is sufficiently short, the system is completely controllable.

Our proofs are based on the method of moments as described by Russel in [10]. A detailed exposition of the method of moments and its relation to problems of controllability is given in [1]. The controllability result depends on the asymptotic behaviour of the eigenvalues of the beam, and this behaviour depends of the properties of the parameter. If our condition on the parameter is satisfied, there is an asymptotic gap between the eigenvalues of the Timoshenko beam in the sense that the distance between different eigenvalues is uniformly bounded below by a positive constant.

Keywords and phrases: Rotating Timoshenko beam, exact controllability, eigenvalues, moment problem.

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2. The Timoshenko beam and the problem of controllability

We consider a Timoshenko beam as described by Timoshenko in [12], that is clamped to an axis that is perpendicular to the axis of the beam. The beam is controlled only at the clamped end, by the rotation of the axis. Its motion is governed by the following two equations for $\tilde{x} \in [0, \tilde{L}]$ and $\tilde{t} \geq 0$:

$$\rho \tilde{w}_{\tilde{t}\tilde{t}}(\tilde{x}, \tilde{t}) - K \tilde{w}_{\tilde{x}\tilde{x}}(\tilde{x}, \tilde{t}) + K \phi_{\tilde{x}}(\tilde{x}, \tilde{t}) = -\tilde{x} \rho \tilde{u}(\tilde{t}), \tag{1}$$

$$I_\rho \phi_{\tilde{t}\tilde{t}}(\tilde{x}, \tilde{t}) - K w_{\tilde{x}}(\tilde{x}, \tilde{t}) + K \phi(\tilde{x}, \tilde{t}) - EI \phi_{\tilde{x}\tilde{x}}(\tilde{x}, \tilde{t}) = -I_\rho \tilde{u}(\tilde{t}). \tag{2}$$

Here I is the moment of inertia of the cross section, ρ is the mass per unit length, E is Young's modulus, I_ρ is the mass moment inertia of the cross section and K is as in [5].

Moreover, $\tilde{w}$ denotes the displacement of the center line of the beam with respect to a reference configuration that rotates with the axis and $\phi(x, t)$ the rotation angle due to bending and shear with respect to the same reference configuration and $\tilde{u}(\tilde{t})$ denotes the angular acceleration of the axis at time $\tilde{t}$.

We use the transformation $x = \tilde{x}\sqrt{\rho}/\sqrt{I_\rho}$, $t = \tilde{t}\sqrt{\rho}\sqrt{EI}/I_\rho$ to obtain the equations in the form presented in [3] (p. 188), where only one real parameter γ appears. Define

$$\gamma = \rho EI/(K I_\rho),$$

$\psi(x, t) = (EI/I_\rho)\phi(\tilde{x}, \tilde{t})$, $w(x, t) = (EI/(I_\rho^{3/2})\tilde{w}(\tilde{x}, \tilde{t})$ and $u(t) = (I_\rho/\rho)\tilde{u}(\tilde{t})$. For the transformed length L we have $L = (\sqrt{\rho}/\sqrt{I_\rho})\tilde{L}$, and for $x \in [0, L]$ we have the equations

$$w_{tt} - \frac{1}{\gamma} w_{xx} + \frac{1}{\gamma} \psi_x = -x u(t), \tag{3}$$

$$\psi_{tt} - \frac{1}{\gamma} w_x + \frac{1}{\gamma} \psi - \psi_{xx} = -u(t). \tag{4}$$

The boundary conditions are

$$w(0, t) = \psi(0, t) = 0,\ w_x(L, t) = \psi(L, t),\ \psi_x(L, t) = 0. \tag{5}$$

In the sequel, assume that $\gamma > 0$ and $\gamma \neq 1$. In [8], the case $\gamma = 1$ and $L = 1$ is considered and controllability from rest to rest is proved.

The initial state of the beam is described by the conditions

$$w(x, 0) = w_0(x),\ w_t(x, 0) = w_1(x),\ \psi(x, 0) = \psi_0(x),\ \psi_t(x, 0) = \psi_1(x) \tag{6}$$

and

$$\theta(0) = \theta_0,\ \theta_t(0) = \theta_1, \tag{7}$$

where $\theta(t)$ denotes the angle of rotation at time t and $u = \theta_{tt}$ is the corresponding angular acceleration.

Assume that w_0, w_1, ψ_0, ψ_1 are continuous functions on the interval $(0, L)$.

We consider the following problem of exact controllability: given an arrival time $T > 0$, find a control function

$$\theta \in \{\xi \in L^2(0, T) : \xi_{tt} \in L^2(0, T),\ \xi(0) = \theta_0,\ \xi_t(0) = \theta_1\}$$

and such that the solution (w, ψ) of (3–7) with $u = \theta_{tt}$ satisfies the end conditions

$$w(x, T) = 0 = w_t(x, T),\ \psi(x, T) = 0 = \psi_t(x, T),\ \theta(T) = 0 = \theta_t(T) \tag{8}$$

for all $x \in [0, L]$. So we are looking for a control function such that if for all $t > T$, the control is $\theta(t) = 0$, the beam stays in a position of rest.

3. The eigenvalues of an ordinary differential operator

We define the Hilbert space $H = L^2(0, L) \times L^2(0, L)$ and the differential operator

$$A(y, \varphi) = \left(-\frac{1}{\gamma} y'' + \frac{1}{\gamma} \varphi',\ \frac{1}{\gamma} \varphi - \varphi'' - \frac{1}{\gamma} y' \right)$$

with the domain

$$D(A) = \{(y, \varphi) \in H :\ y'',\ \varphi'' \in L^2,\ y(0) = 0 = \varphi(0),\ y'(L) = \varphi(L),\ \varphi'(L) = 0\}\cdot$$

Using the operator A, we can write the system (3, 4) in the form (with $r_1(x, t) = -u(t)x$)

$$(w_{tt}(\cdot, t), \psi_{tt}(\cdot, t)) + A(w(\cdot, t), \psi(\cdot, t)) = (r_1(\cdot, t), -u(t)).$$

Theorem 1. *The operator A is self-adjoint in H and positive.*
If $\gamma > 1$ and $L^2 \leq 2\gamma - 1$, it satisfies the coerciveness inequality

$$\langle (y, \varphi), A(y, \varphi) \rangle_H \geq \frac{1}{L^2 \gamma} \|(y, \varphi)\|_H^2 + \frac{1}{2\gamma} \int_0^L (y' - 2\varphi)^2. \tag{9}$$

The resolvent is a Hilbert–Schmidt Operator. The eigenfunctions form a complete orthogonal system in H.

Proof. For $(y, \varphi) \in D(A)$, $(v, \alpha) \in \{(z, \xi) \in H : (z'', \xi'') \in H\}$ we have

$$\langle (v, \alpha),\ A(y, \varphi) \rangle_H = \frac{1}{\gamma} \overline{v(0)} y'(0) + \overline{\alpha(0)} \varphi'(0) + \frac{1}{\gamma} \overline{(v'(L) - \alpha(L))} y(L) + \overline{\alpha'(L)} \varphi(L) + \langle A(v, \alpha), (y, \varphi) \rangle_H. \tag{10}$$

For $(v, \alpha) \in D(A)$, this implies

$$\langle (v, \alpha),\ A(y, \varphi) \rangle_H = \langle A(v, \alpha), (y, \varphi) \rangle_H.$$

Thus the operator A is symmetric, hence $D(A) \subset D(A^*)$.

For $(v, \alpha) \in D(A^*)$, let $(w, \beta) = A^*(v, \alpha) \in H$. Then for all $(y, \varphi) \in D(A)$

$$\langle (w, \beta), (y, \varphi) \rangle_H = \langle (v, \alpha), A(y, \varphi) \rangle_H.$$

Hence for all functions $(y, \varphi) \in (C_c^\infty)^2$, we have

$$\begin{aligned} \int_0^L wy &= \int_0^L \left(-\frac{1}{\gamma} v y'' - \frac{1}{\gamma} \alpha y' \right) \\ \int_0^L \beta \varphi &= \int_0^L \left(\frac{1}{\gamma} v \varphi' + \frac{1}{\gamma} \alpha \varphi - \alpha \varphi'' \right). \end{aligned}$$

Thus in the sense of distributions we have the equations

$$w = -\frac{1}{\gamma} v'' + \frac{1}{\gamma} \alpha',\ \beta = -\frac{1}{\gamma} v' - \alpha'' + \frac{1}{\gamma} \alpha.$$

So $v'' - \alpha'$ is in L^2, hence $v' - \alpha$ is also in L^2 and $\alpha'' = -\frac{1}{\gamma}(v' - \alpha) - \beta$ is in L^2, too.

Thus v'' is also in L^2 and $(v'', \alpha'') \in H$. Now (10) implies that $(v, \alpha) \in D(A)$. So the operator A is self-adjoint.

For $(y, \varphi) \in D(A)$, we have

$$\langle A(y,\varphi),(y,\varphi)\rangle_H = \int_0^L |\varphi'|^2 + \frac{1}{\gamma}\int_0^L |y' - \varphi|^2.$$

If $\gamma > 1$, this implies that (9) holds, where the last inequality follows with the help of the Friedrichs inequality $\int_0^L |f'|^2 \geq (2/L^2)\int_0^L |f|^2$ if $f' \in L^2(0, L)$ and $f(0) = 0$.

The variation of constant formula shows that the resolvent of A is an integral operator with an L^2-kernel. □

3.1. The eigenvalue equation of the operator

Consider the eigenvalue equation

$$Az = \lambda z \tag{11}$$

that is equivalent to the equations

$$-\frac{1}{\gamma}y'' + \frac{1}{\gamma}\varphi' = \lambda y, \tag{12}$$

$$\frac{1}{\gamma}\varphi - \varphi'' - \frac{1}{\gamma}y' = \lambda\varphi. \tag{13}$$

If $\gamma > 1$ and $L^2 \leq 2\gamma - 1$, the coerciveness inequality (9) implies that all eigenvalues are greater than or equal to $1/(L^2\gamma)$.

Lemma 1. *Assume that $\gamma > 0$ and $L > 0$ are such that*

$$2 + (\gamma + 1/\gamma)\cos\left(L\sqrt{1+1/\gamma}\right) - L\sqrt{1+1/\gamma}\,\sin\left(L\sqrt{1+1/\gamma}\right) \neq 0. \tag{14}$$

Then the number $1/\gamma$ cannot be an eigenvalue of the operator A.

Proof. Suppose that $A(y, \varphi) = (y, \varphi)/\gamma$. Then (13) yields $\varphi'' = -y'/\gamma$ and (12) implies that $y + y'' = \varphi'$. This yields the equation

$$y''' = \varphi'' - y' = -(1 + 1/\gamma)y'.$$

So we have

$$y(x) = A\sin\left(\sqrt{1+1/\gamma}\,x\right) + B\cos\left(\sqrt{1+1/\gamma}\,x\right) + C$$

and

$$\varphi'(x) = y(x) + y''(x) = -\frac{1}{\gamma}A\sin\left(\sqrt{1+1/\gamma}\,x\right) - \frac{1}{\gamma}B\cos\left(\sqrt{1+1/\gamma}\,x\right) + C.$$

The boundary condition $y(0) = 0$ yields $C = -B$. The boundary condition $\varphi'(L) = 0$ yields the equation

$$A\sin\left(\sqrt{1+1/\gamma}\,L\right) + B\cos\left(\sqrt{1+1/\gamma}\,L\right) = \gamma C = -\gamma B.$$

We have

$$\varphi(x) = \frac{1}{\gamma}\frac{A}{\sqrt{1+1/\gamma}}\cos\left(\sqrt{1+1/\gamma}\,x\right) - \frac{1}{\gamma}\frac{B}{\sqrt{1+1/\gamma}}\sin\left(\sqrt{1+1/\gamma}\,x\right) + Cx + D.$$

The equation $\varphi(0) = 0$ implies $D\sqrt{1+1/\gamma} = -A/\gamma$. The boundary condition $y'(L) = \varphi(L)$ yields the condition

$$A\cos\left(\sqrt{1+1/\gamma}L\right) - B\sin\left(\sqrt{1+1/\gamma}\,L\right) = D\sqrt{1+1/\gamma} + C\sqrt{1+1/\gamma}L = -A/\gamma - B\sqrt{1+1/\gamma}L.$$

So A and B solve the following system of linear equations:

$$A\sin\left(\sqrt{1+1/\gamma}L\right) + B\left[\cos\left(\sqrt{1+1/\gamma}L\right) + \gamma\right] = 0.$$

$$A\left[\cos\left(\sqrt{1+1/\gamma}L\right) + 1/\gamma\right] + B\left[-\sin\left(\sqrt{1+1/\gamma}L\right) + \sqrt{1+1/\gamma}L\right] = 0.$$

If $A = 0 = B$, we have $C = 0$ and $D = 0$, so $y = 0 = \varphi$. So we are looking for a nontrivial solution that can only exist if the determinant of the above system vanishes, which is the case if

$$2 + (\gamma + 1/\gamma)\cos\left(\sqrt{1+1/\gamma}L\right) - \sqrt{1+1/\gamma}L\sin\left(\sqrt{1+1/\gamma}L\right) = 0.$$

This contradicts our assumption. □

Let a complex number ω be given and define

$$\begin{aligned} y(x) &= -\omega\sin(\omega x), \\ \varphi(x) &= (-\omega^2 + \gamma\lambda)\cos(\omega x). \end{aligned}$$

Then equation (12) holds and

$$\frac{1}{\gamma}\varphi - \varphi'' - \frac{1}{\gamma}y' - \lambda\varphi = -(\omega^4 + \omega^2(-\lambda - \gamma\lambda) + \gamma\lambda^2 - \lambda)\cos(\omega x).$$

Hence $z = (y, \varphi)$ satisfies the eigenvalue equation (11) if

$$\omega^4 + \omega^2(-\lambda)(1+\gamma) + \gamma\lambda^2 - \lambda = 0. \tag{15}$$

Now let

$$y(x) = \omega\cos(\omega x), \tag{16}$$
$$\varphi(x) = (-\omega^2 + \gamma\lambda)\sin(\omega x). \tag{17}$$

Then equation (12) holds and

$$\frac{1}{\gamma}\varphi - \varphi'' - \frac{1}{\gamma}y' - \lambda\varphi = -(\omega^4 + \omega^2(-\lambda - \gamma\lambda) + \gamma\lambda^2 - \lambda)\sin(\omega x).$$

Hence $z = (y, \varphi)$ as defined in (16, 17) satisfies the eigenvalue equation (11) if (15) holds.

3.2. The roots of a polynomial

For fixed $\lambda \in \mathbb{R}$, we consider the equation (15). Define

$$\begin{aligned} \alpha(\lambda) &= \lambda\frac{\gamma+1}{2} + \sqrt{\lambda + \frac{\lambda^2}{4}(\gamma-1)^2}, \\ \beta(\lambda) &= \lambda\frac{\gamma+1}{2} - \sqrt{\lambda + \frac{\lambda^2}{4}(\gamma-1)^2}. \end{aligned}$$

The solutions of (15) are $\sqrt{\alpha(\lambda)}$, $-\sqrt{\alpha(\lambda)}$, $\sqrt{\beta(\lambda)}$, $-\sqrt{\beta(\lambda)}$.

For $\lambda > 0$, we have $\alpha(\lambda) > 0$. For $\lambda > 1/\gamma$, we have

$$\beta(\lambda) = \frac{\lambda^2(\gamma+1)^2/4 - \lambda - \lambda^2(\gamma-1)^2/4}{\alpha(\lambda)} = (\lambda^2\gamma - \lambda)/\alpha(\lambda) > 0$$

hence in this case, the four solutions of (15) are all real numbers.

For $0 < \lambda < 1/\gamma$, we have $\alpha(\lambda) > 0$ and $\beta(\lambda) < 0$ hence in this case $\sqrt{\alpha(\lambda)}$ and $-\sqrt{\alpha(\lambda)}$ are real numbers and the solutions $i\sqrt{-\beta(\lambda)}$, $-i\sqrt{-\beta(\lambda)}$ are complex.

For $\lambda = 1/\gamma$, we have $\beta(\lambda) = 0$.

Define

$$\omega_1(\lambda) = \sqrt{\alpha(\lambda)}, \;\; \omega_2(\lambda) = \sqrt{\beta(\lambda)}. \tag{18}$$

If $\lambda > 1/\gamma$, we have $\omega_1(\lambda) > 0$ and $\omega_2(\lambda) > 0$.

For $0 < \lambda \le 1/\gamma$, we have $\omega_1(\lambda) > 0$ and $i\omega_2(\lambda) = \sqrt{-\beta(\lambda)}$ is a real number.

We have

$$\omega_1(\lambda)^2 + \omega_2(\lambda)^2 = \alpha(\lambda) + \beta(\lambda) = \lambda(\gamma+1) \tag{19}$$

and

$$\omega_1(\lambda)^2\omega_2(\lambda)^2 = \alpha(\lambda)\beta(\lambda) = \gamma\lambda^2 - \lambda. \tag{20}$$

Moreover, the following equation holds:

$$(-\omega_1(\lambda)^2 + \gamma\lambda)(-\omega_2(\lambda)^2 + \gamma\lambda) = \omega_1(\lambda)^2\omega_2(\lambda)^2 - \gamma\lambda(\omega_1(\lambda)^2 + \omega_2(\lambda)^2) + \gamma^2\lambda^2 = -\lambda. \tag{21}$$

Finally, we have

$$\begin{aligned} (-\omega_1(\lambda)^2 + \gamma\lambda)^2 + (-\omega_2(\lambda)^2 + \gamma\lambda)^2 &= (-\omega_1(\lambda)^2 + \gamma\lambda - \omega_2(\lambda)^2 + \gamma\lambda)^2 - 2(-\omega_1(\lambda)^2 + \gamma\lambda)(-\omega_2(\lambda)^2 + \gamma\lambda) \\ &= (2\gamma\lambda - (\omega_1(\lambda)^2 + \omega_2(\lambda)^2))^2 + 2\lambda = (2\gamma\lambda - \lambda(\gamma+1))^2 + 2\lambda \\ &= 2\lambda + (\gamma-1)^2\lambda^2. \end{aligned} \tag{22}$$

3.3. The boundary conditions

Now we return to the eigenvalue equation (11). It has the general solution

$$\begin{aligned} y(x) &= C_1(-\omega_1(\lambda))\sin(\omega_1(\lambda)x) + C_2\omega_1(\lambda)\cos(\omega_1(\lambda)x) \\ &\quad + C_3(-\omega_2(\lambda))\sin(\omega_2(\lambda)x) + C_4\omega_2(\lambda)\cos(\omega_2(\lambda)x) \\ \varphi(x) &= C_1(-\omega_1^2(\lambda) + \gamma\lambda)\cos(\omega_1(\lambda)x) + C_2(-\omega_1^2(\lambda) + \gamma\lambda)\sin(\omega_1(\lambda)x) \\ &\quad + C_3(-\omega_2^2(\lambda) + \gamma\lambda)\cos(\omega_2(\lambda)x) + C_4(-\omega_2^2(\lambda) + \gamma\lambda)\sin(\omega_2(\lambda)x). \end{aligned}$$

The boundary condition $y(0) = 0$ implies the equation

$$C_2\omega_1(\lambda) + C_4\omega_2(\lambda) = 0.$$

We have seen that for $\lambda \neq 1/\gamma$, $\omega_2(\lambda) \neq 0$. In the sequel, we assume that (14) holds. Then Lemma 1 implies that $\lambda \neq 1/\gamma$.

So we have the equation

$$C_4 = -(\omega_1(\lambda)/\omega_2(\lambda))C_2. \tag{23}$$

The boundary condition $\varphi(0) = 0$ implies the equation

$$C_1(-\omega_1(\lambda)^2 + \gamma\lambda) + C_3(-\omega_2(\lambda)^2 + \gamma\lambda) = 0,$$

hence $C_3 = -((-\omega_1(\lambda)^2 + \gamma\lambda)/(-\omega_2(\lambda)^2 + \gamma\lambda))C_1$. (Note that (21) implies that $-\omega_2(\lambda)^2 + \gamma\lambda \neq 0$.)

We have

$$\begin{aligned} \varphi'(x) &= C_1(-\omega_1(\lambda)^2 + \gamma\lambda)(-\omega_1(\lambda))\sin(\omega_1(\lambda)x) + C_2(-\omega_1(\lambda)^2 + \gamma\lambda)(\omega_1(\lambda))\cos(\omega_1(\lambda)x) \\ &\quad + C_3(-\omega_2(\lambda)^2 + \gamma\lambda)(-\omega_2(\lambda))\sin(\omega_2(\lambda)x) + C_4(-\omega_2(\lambda)^2 + \gamma\lambda)(\omega_2(\lambda))\cos(\omega_2(\lambda)x). \end{aligned} \tag{24}$$

Hence the following equation holds:

$$\begin{aligned} 0 = \varphi'(L) &= C_1[(-\omega_1(\lambda)^2 + \gamma\lambda)(-\omega_1(\lambda))\sin(\omega_1(\lambda)L) + (-\omega_1(\lambda)^2 + \gamma\lambda)\omega_2(\lambda)\sin(\omega_2(\lambda)L)] \\ &\quad + C_2[(-\omega_1(\lambda)^2 + \gamma\lambda)(\omega_1(\lambda))\cos(\omega_1(\lambda)L) + (-\omega_2(\lambda)^2 + \gamma\lambda)(-\omega_1(\lambda))\cos(\omega_2(\lambda)L)] \\ &= C_1(-\omega_1(\lambda)^2 + \gamma\lambda)[-\omega_1(\lambda)\sin(\omega_1(\lambda)L) + \omega_2(\lambda)\sin(\omega_2(\lambda)L)] \\ &\quad + C_2\omega_1(\lambda)[(-\omega_1(\lambda)^2 + \gamma\lambda)\cos(\omega_1(\lambda)L) - (-\omega_2(\lambda)^2 + \gamma\lambda)\cos(\omega_2(\lambda)L)]. \end{aligned} \tag{25}$$

The derivative of the function y is given by the equation

$$\begin{aligned} y'(x) \quad = \quad & C_1(-\omega_1^2(\lambda))\cos(\omega_1(\lambda)x) + C_2(-\omega_1^2(\lambda))\sin(\omega_1(\lambda)x) + C_3(-\omega_2^2(\lambda))\cos(\omega_2(\lambda)x) \\ & + C_4(-\omega_2^2(\lambda))\sin(\omega_2(\lambda)x). \end{aligned}$$

Moreover, the boundary conditions imply that

$$0 = \varphi(L) - y'(L) = C_1\gamma\lambda\cos(\omega_1(\lambda)L) + C_2\gamma\lambda\sin(\omega_1(\lambda)L) + C_3\gamma\lambda\cos(\omega_2(\lambda)L) + C_4\gamma\lambda\sin(\omega_2(\lambda)L).$$

Hence the following equation holds.

$$\begin{aligned} 0 &= C_1\cos(\omega_1(\lambda)L) + C_2\sin(\omega_1(\lambda)L) + C_3\cos(\omega_2(\lambda)L) + C_4\sin(\omega_2(\lambda)L) \\ &= C_1\left[\cos(\omega_1(\lambda)L) - \frac{-\omega_1(\lambda)^2 + \gamma\lambda}{-\omega_2(\lambda)^2 + \gamma\lambda}\cos(\omega_2(\lambda)L)\right] + C_2\left[\sin(\omega_1(\lambda)L) - \frac{\omega_1(\lambda)}{\omega_2(\lambda)}\sin(\omega_2(\lambda)L)\right]. \end{aligned}$$

Thus we have

$$\begin{aligned} 0 &= C_1[\omega_2(\lambda)(-\omega_2(\lambda)^2 + \gamma\lambda)\cos(\omega_1(\lambda)L) - (-\omega_1(\lambda)^2 + \gamma\lambda)\omega_2(\lambda)\cos(\omega_2(\lambda)L)] \\ &\quad + C_2[\omega_2(\lambda)(-\omega_2(\lambda)^2 + \gamma\lambda)\sin(\omega_1(\lambda)L) - (-\omega_2(\lambda)^2 + \gamma\lambda)\omega_1(\lambda)\sin(\omega_2(\lambda)L)]. \end{aligned} \tag{26}$$

We are looking for a nonzero solution (C_1, C_2, C_3, C_4) and (25) and (26) imply that it can only exist if the vectors

$$((-\omega_1(\lambda)^2+\gamma\lambda)[-\omega_1(\lambda)\sin(\omega_1(\lambda)L)+\omega_2(\lambda)\sin(\omega_2(\lambda)L)],$$

$$\omega_1(\lambda)[(-\omega_1(\lambda)^2+\gamma\lambda)\cos(\omega_1(\lambda)L)-(-\omega_2(\lambda)^2+\gamma\lambda)\cos(\omega_2(\lambda)L)])$$

and

$$(\omega_2(\lambda)[(-\omega_2(\lambda)^2+\gamma\lambda)\cos(\omega_1(\lambda)L)-(-\omega_1(\lambda)^2+\gamma\lambda)\cos(\omega_2(\lambda)L)],$$

$$(-\omega_2(\lambda)^2+\gamma\lambda)[\omega_2(\lambda)\sin(\omega_1(\lambda)L)-\omega_1(\lambda)\sin(\omega_2(\lambda)L)])$$

are linearly dependent.

Lemma 2. *All the eigenvalues of the operator A are simple, that is the space generated by the corresponding eigenfunctions has dimension 1.*

Proof. For an eigenvalue λ, the above vectors are linearly dependent. To show the assertion, we show that these vectors cannot both be the zero vector. Suppose that both vectors equal zero, then

$$\begin{aligned}\omega_1(\lambda)\sin(\omega_1(\lambda)L) &= \omega_2(\lambda)\sin(\omega_2(\lambda)L),\\ \omega_2(\lambda)\sin(\omega_1(\lambda)L) &= \omega_1(\lambda)\sin(\omega_2(\lambda)L)\end{aligned}$$

hence $0=(\omega_1(\lambda)^2-\omega_2(\lambda)^2)\sin(\omega_1(\lambda)L)$ and since $(\omega_1(\lambda)^2-\omega_2(\lambda)^2)\neq 0$ this implies that $\sin(\omega_1(\lambda)L)=0$. Analogously, we can show that $\cos(\omega_1(\lambda)L)=0$, which is a contradiction. □

3.4. **The spectral equation**

So for the eigenvalues of the operator A that are unequal to $1/\gamma$, we obtain the spectral equation

$$\begin{aligned}0 &= (-\omega_1(\lambda)^2+\gamma\lambda)[-\omega_1(\lambda)\sin(\omega_1(\lambda)L)+\omega_2(\lambda)\sin(\omega_2(\lambda)L)]\\ &\quad\times(-\omega_2(\lambda)^2+\gamma\lambda)[\omega_2(\lambda)\sin(\omega_1(\lambda)L)-\omega_1(\lambda)\sin(\omega_2(\lambda)L)])\\ &\quad-\omega_1(\lambda)[(-\omega_1(\lambda)^2+\gamma\lambda)\cos(\omega_1(\lambda)L)-(-\omega_2(\lambda)^2+\gamma\lambda)\cos(\omega_2(\lambda)L)]\\ &\quad\times\omega_2(\lambda)[(-\omega_2(\lambda)^2+\gamma\lambda)\cos(\omega_1(\lambda)L)-(-\omega_1(\lambda)^2+\gamma\lambda)\cos(\omega_2(\lambda)L)].\end{aligned}$$

The equation can be written in the simpler form

$$\begin{aligned}0 &= (-\omega_1(\lambda)^2+\gamma\lambda)(-\omega_2(\lambda)^2+\gamma\lambda)[-\omega_1(\lambda)\omega_2(\lambda)(\sin^2(\omega_1(\lambda)L)+\sin^2(\omega_2(\lambda)L))\\ &\quad+(\omega_1(\lambda)^2+\omega_2(\lambda)^2)\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L)]\\ &\quad-\omega_1(\lambda)\omega_2(\lambda)[(-\omega_1(\lambda)^2+\gamma\lambda)(-\omega_2(\lambda)^2+\gamma\lambda)(\cos^2(\omega_1(\lambda)L)+\cos^2(\omega_2(\lambda)L))\\ &\quad-((-\omega_1(\lambda)^2+\gamma\lambda)^2+(-\omega_2(\lambda)^2+\gamma\lambda)^2)\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L)].\end{aligned}$$

Further simplification yields the form

$$\begin{aligned}0 &= (-\omega_1(\lambda)^2+\gamma\lambda)(-\omega_2(\lambda)^2+\gamma\lambda)(-\omega_1(\lambda)\omega_2(\lambda))\\ &\quad\times(\sin^2(\omega_1(\lambda)L)+\cos^2(\omega_1(\lambda)L)+\sin^2(\omega_2(\lambda)L)+\cos^2(\omega_2(\lambda)L))\\ &\quad+(-\omega_1(\lambda)^2+\gamma\lambda)(-\omega_2(\lambda)^2+\gamma\lambda)(\omega_1(\lambda)^2+\omega_2(\lambda)^2)\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L)\\ &\quad+\omega_1(\lambda)\omega_2(\lambda)((-\omega_1(\lambda)^2+\gamma\lambda)^2+(-\omega_2(\lambda)^2+\gamma\lambda)^2)\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L).\end{aligned}$$

Using (19–22), this can be written in the form

$$0 = -\lambda\left(-\sqrt{\gamma\lambda^2-\lambda}\right)2 + (-\lambda)\lambda(\gamma+1)\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L)$$
$$+\sqrt{\gamma\lambda^2-\lambda}(2\lambda+(\gamma-1)^2\lambda^2)\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L).$$

Dividing by $\lambda^2\sqrt{\gamma\lambda^2-\lambda}$ yields the spectral equation

$$0 = (\gamma-1)^2\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L) + \frac{2}{\lambda}(1+\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L)) - \frac{\gamma+1}{\sqrt{\gamma\lambda^2-\lambda}}\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L). \tag{27}$$

3.5. **The asymptotic behaviour of the eigenvalues**

Define

$$\delta(\lambda) = \frac{2}{(\gamma-1)^2\lambda}\left[1+\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L)\right] - \frac{\gamma+1}{(\gamma-1)^2}\frac{1}{\sqrt{\gamma\lambda^2-\lambda}}\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L). \tag{28}$$

Since we have assumed that $\gamma \neq 1$, the function δ is well-defined for $\lambda > 0$, $\lambda \neq 1/\gamma$.

We have seen that the eigenvalues of the operator A unequal to $1/\gamma$ are the solutions of the equation

$$0 = \cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L) + \delta(\lambda), \tag{29}$$

where $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are defined in (18).

The definition of $\delta(\lambda)$ implies that for $\lambda > 1/\gamma$,

$$|\delta(\lambda)| \leq \frac{1}{(\gamma-1)^2}\left(\frac{4}{\lambda} + \frac{\gamma+1}{\sqrt{\gamma\lambda^2-\lambda}}\right). \tag{30}$$

Hence for λ sufficiently large, we have $\delta(\lambda) \in [-1,1]$ and it is easy to show that the following statement holds: if the equation

$$0 = \cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L) + \delta(\lambda)$$

is valid, then $|\cos(\omega_1(\lambda))L| \leq \sqrt{|\delta(\lambda)|}$ or $|\cos(\omega_2(\lambda))L| \leq \sqrt{|\delta(\lambda)|}$, which implies that

$$L\omega_1(\lambda) \in \left[j\pi + \arccos\left(\sqrt{|\delta(\lambda)|}\right), j\pi + \arccos\left(-\sqrt{|\delta(\lambda)|}\right)\right]$$

for some $j \in \mathbb{N}$ or

$$L\omega_2(\lambda) \in \left[k\pi + \arccos\left(\sqrt{|\delta(\lambda)|}\right), k\pi + \arccos\left(-\sqrt{|\delta(\lambda)|}\right)\right]$$

for some $k \in \mathbb{N}$.

Define

$$\varepsilon(\lambda) = \pi/2 - \arccos\left(\sqrt{|\delta(\lambda)|}\right).$$

Note that $\varepsilon(\lambda) = \arcsin\left(\sqrt{|\delta(\lambda)|}\right) = \arccos\left(-\sqrt{|\delta(\lambda)|}\right) - \pi/2$, that $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$ and that we have

$$L\omega_1(\lambda) \in [j\pi + \pi/2 - \varepsilon(\lambda), j\pi + \pi/2 + \varepsilon(\lambda)]$$

for some $j \in \mathbb{N}$ or

$$L\omega_2(\lambda) \in [k\pi + \pi/2 - \varepsilon(\lambda), k\pi + \pi/2 + \varepsilon(\lambda)]$$

for some $k \in \mathbb{N}$.

There exists a constant $C > 0$ such that $\varepsilon(\lambda) \leq C/\sqrt{\lambda}$ for λ sufficiently large.

In fact, for $\lambda \geq 2/\gamma$ we have $\gamma\lambda^2 - \lambda \geq (\gamma/2)\lambda^2$ and thus

$$\varepsilon(\lambda) \leq \frac{\pi}{2}\sqrt{|\delta(\lambda)|} \leq \frac{\pi}{2}\frac{1}{|\gamma - 1|}\sqrt{\frac{4}{\lambda} + \frac{\gamma + 1}{\sqrt{(\gamma/2)\lambda^2}}}$$

hence we can choose the constant as

$$C = \frac{\pi}{2}\sqrt{\frac{4}{(\gamma - 1)^2} + \frac{\gamma + 1}{(\gamma - 1)^2\sqrt{(\gamma/2)}}}.$$

This implies that we have

$$L\omega_1(\lambda) \in [j\pi + \pi/2 - C/\sqrt{\lambda}, j\pi + \pi/2 + C/\sqrt{\lambda}]$$

for some $j \in \mathbb{N}$ or

$$L\omega_2(\lambda) \in \left[k\pi + \pi/2 - C/\sqrt{\lambda}, k\pi + \pi/2 + C/\sqrt{\lambda}\right]$$

for some $k \in \mathbb{N}$.

Lemma 3. *Let $D = 1/|\gamma - 1|$.*
If $\gamma > 1$, for all $\lambda > 1/\gamma$ we have

$$\omega_1(\lambda) \in \left(\sqrt{\gamma\lambda},\ \sqrt{\gamma\lambda} + D/\sqrt{\gamma\lambda}\right), \tag{31}$$

$$\omega_2(\lambda) \in \left(\sqrt{\lambda} - D/\sqrt{\lambda},\ \sqrt{\lambda}\right). \tag{32}$$

If $\gamma \in (0, 1)$, for all $\lambda > 1/\gamma$ we have

$$\omega_1(\lambda) \in \left(\sqrt{\lambda},\ \sqrt{\lambda} + D/\sqrt{\lambda}\right), \tag{33}$$

$$\omega_2(\lambda) \in \left(\sqrt{\gamma\lambda} - D/\sqrt{\gamma\lambda},\ \sqrt{\gamma\lambda}\right). \tag{34}$$

Proof. If $\gamma > 1$, we have the equalities

$$\begin{aligned}\omega_1(\lambda) - \sqrt{\gamma\lambda} &= \frac{\left(\omega_1(\lambda) - \sqrt{\gamma\lambda}\right)\left(\omega_1(\lambda) + \sqrt{\gamma\lambda}\right)}{\omega_1(\lambda) + \sqrt{\gamma\lambda}} = \frac{\alpha(\lambda) - \gamma\lambda}{\omega_1(\lambda) + \sqrt{\gamma\lambda}} \\ &= \frac{\sqrt{\lambda + \lambda^2(\gamma-1)^2/4} - \lambda(\gamma-1)/2}{\omega_1(\lambda) + \sqrt{\gamma\lambda}} \\ &= \frac{\lambda^2(\gamma-1)^2/4 + \lambda - \lambda^2(\gamma-1)^2/4}{\left(\omega_1(\lambda) + \sqrt{\gamma\lambda}\right)\left(\sqrt{\lambda + \lambda^2(\gamma-1)^2/4} + \lambda(\gamma-1)/2\right)} \\ &= \frac{1}{\left(\omega_1(\lambda) + \sqrt{\gamma\lambda}\right)\left((\gamma-1)/2 + \sqrt{1/\lambda + (\gamma-1)^2/4}\right)}.\end{aligned}$$

Hence $\omega_1(\lambda) > \sqrt{\gamma\lambda}$ and the assertion for ω_1 follows.

Analogously we can show that

$$\omega_2(\lambda) - \sqrt{\lambda} = \frac{\beta(\lambda) - \lambda}{\omega_2(\lambda) + \sqrt{\lambda}} = \frac{-1}{(\omega_2(\lambda) + \sqrt{\lambda})((\gamma-1)/2 + \sqrt{1/\lambda + (\gamma-1)^2/4})}.$$

Thus $\omega_2(\lambda) < \sqrt{\lambda}$ and the assertion for ω_2 follows.

The proof for the case $\gamma \in (0,1)$ works analogously. □

Lemma 3 implies that for every eigenvalue λ of the operator A that is sufficiently large we have

$$L\sqrt{\gamma\lambda} \in \left[j\pi + \pi/2 - (C + LD)/\sqrt{\gamma\lambda}, j\pi + \pi/2 + (C + LD)/\sqrt{\gamma\lambda}\right]$$

for some $j \in \mathbb{N}$ or

$$L\sqrt{\lambda} \in \left[k\pi + \pi/2 - (C + LD)/\sqrt{\lambda}, k\pi + \pi/2 + (C + LD)/\sqrt{\lambda}\right]$$

for some $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, $t > 0$ define $a_k = k\pi + \pi/2$, $b_k(t) = \sqrt{a_k^2/(L^2 t) - 4(C + LD)/(Lt)}$, $c_k(t) = \sqrt{a_k^2/(L^2 t) + 4(C + LD)/(Lt)}$.

Lemma 4. *There exist $k_0 \in \mathbb{N}$ and $\lambda_0 > 0$ such that all the square roots of the eigenvalues of the operator A that are greater than λ_0 are contained in the intervals*

$$I_k = \left[\frac{a_k/(L\sqrt{\gamma}) + b_k(\gamma)}{2}, \ \frac{a_k/(L\sqrt{\gamma}) + c_k(\gamma)}{2}\right],$$
$$J_k = \left[\frac{a_k/L + b_k(1)}{2}, \ \frac{a_k/L + c_k(1)}{2}\right]$$

for $k \geq k_0$.

Proof. If λ is sufficiently large, the inequality $L\sqrt{\gamma\lambda} \geq a_k - (C + LD)/\sqrt{\gamma\lambda}$ implies that

$$\left(\sqrt{\lambda}\right)^2 - a_k\sqrt{\lambda}/\left(L\sqrt{\gamma}\right) + (C + LD)/(L\gamma) \geq 0,$$

hence

$$\sqrt{\lambda} \geq \left(a_k/\left(L\sqrt{\gamma}\right) + \sqrt{a_k^2/(L^2\gamma) - 4(C + LD)/(L\gamma)}\right)/2.$$

Moreover, the inequality $L\sqrt{\gamma\lambda} \leq a_k + (C+LD)/\sqrt{\gamma\lambda}$ implies that

$$\sqrt{\lambda} \leq \left(a_k/\left(L\sqrt{\gamma}\right) + \sqrt{a_k^2/(L^2\gamma) + 4(C+LD)/(L\gamma)}\right)/2.$$

Hence if λ is large enough, the statement $L\sqrt{\gamma\lambda} \in \left[a_k - (C+LD)/\sqrt{\gamma\lambda}, a_k + (C+LD)/\sqrt{\gamma\lambda}\right]$ implies that $\sqrt{\lambda} \in \left[(a_k/(L\sqrt{\gamma}) + b_k(\gamma))/2,\ (a_k/(L\sqrt{\gamma}) + c_k(\gamma))/2\right]$.

The assertion for the other family of intervals follows analogously. □

Lemma 5. *Let $\gamma > 1$. There exists a number $k_0 \in \mathbb{N}$ such that for all natural numbers k greater than k_0 the function $t \mapsto \cos(\omega_1(t^2)L)$ changes its sign in the interval I_k and the function $t \mapsto \cos(\omega_2(t^2)L)$ changes its sign in the interval J_k.*

For $\gamma \in (0,1)$, there exists a number $k_0 \in \mathbb{N}$ such that for all natural numbers k greater than k_0 the function $t \mapsto \cos(\omega_1(t^2)L)$ changes its sign in the interval J_k and the function $t \mapsto \cos(\omega_2(t^2)L)$ changes its sign in the interval I_k.

Proof. We only proof the statement for the case $\gamma > 1$. For a fixed $k \in \mathbb{N}$, define

$$\begin{aligned} \alpha &= \left(a_k/\left(L\sqrt{\gamma}\right) + b_k(\gamma)\right)/2, \\ \beta &= \left(a_k/\left(L\sqrt{\gamma}\right) + c_k(\gamma)\right)/2. \end{aligned}$$

Then $I_k = [\alpha,\ \beta]$, $a_k - L\sqrt{\gamma}\alpha = (C+LD)/\left(\sqrt{\gamma}\alpha\right)$ and $L\sqrt{\gamma}\beta - a_k = (C+LD)/\left(\sqrt{\gamma}\beta\right)$.

Now let $k \in \mathbb{N}$ be such that $\alpha^2 > 1/\gamma$. Then Lemma 3 implies that $\omega_1(\alpha^2) - \sqrt{\gamma}\alpha \leq D/\left(\sqrt{\gamma}\alpha\right)$, hence

$$\begin{aligned} d_1 &:= \omega_1(\alpha^2) - a_k/L = \omega_1(\alpha^2) - \sqrt{\gamma}\alpha + \sqrt{\gamma}\alpha - a_k/L \leq D/\left(\sqrt{\gamma}\alpha\right) - (C+LD)/\left(L\sqrt{\gamma}\alpha\right) \\ &= -C/\left(L\sqrt{\gamma}\alpha\right) < 0. \end{aligned}$$

Moreover, $\omega_1(\beta^2) - \sqrt{\gamma}\beta \geq -D/(\sqrt{\gamma}\beta)$, hence

$$\begin{aligned} d_2 &:= \omega_1(\beta^2) - a_k/L = \omega_1(\beta^2) - \sqrt{\gamma}\beta + \sqrt{\gamma}\beta - a_k/L \geq -D/(\sqrt{\gamma}\beta) + (C+LD)/(L\sqrt{\gamma}\beta) \\ &= C/(L\sqrt{\gamma}\beta) > 0. \end{aligned}$$

Thus we have

$$\begin{aligned} \cos(\omega_1(\alpha^2)L)\cos(\omega_1(\beta^2)L) &= \cos(a_k + Ld_1)\cos(a_k + Ld_2) \\ &= (\cos(Ld_2 - Ld_1) + \cos(2a_k + Ld_2 + Ld_1))/2 \\ &= (\cos(Ld_2 - Ld_1) + \cos(\pi + Ld_2 + Ld_1))/2 \\ &= (\cos(Ld_2 + L|d_1|) - \cos(Ld_2 - L|d_1|))/2 \\ &= -\sin(Ld_2)\sin(L|d_1|). \end{aligned}$$

If k is sufficiently large we have $Ld_1 \in (-\pi, 0)$ and $Ld_2 \in (0, \pi)$, hence $-\sin(Ld_2)\sin(L|d_1|) < 0$, and the assertion for the interval I_k follows.

The assertion for the interval J_k follows analogously. □

Lemma 6. *Assume that $\sqrt{\gamma} = q/p$ with $p,\ q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*

Then for all $k,\ j \in \mathbb{N}$ we have

$$|a_k - a_j/\sqrt{\gamma}| \geq \pi/(2q). \tag{35}$$

Moreover there exists $k_0 \in \mathbb{N}$ such that the intervals $(I_k, J_k)_{k\geq k_0}$ are all disjoint and the distance between the intervals is uniformly bounded below by a positive number, hence we have

$$\inf_{j,k\geq k_0} \inf_{s\in I_k, t\in J_j} |s-t| =: \rho > 0. \tag{36}$$

Proof. From the definition of a_k we have

$$a_k - a_j/\sqrt{\gamma} = k\pi + \pi/2 - (j\pi + \pi/2)p/q = \pi((2k+1)q - (2j+1)p)/(2q).$$

Since one of the numbers $(2j+1)p$, $(2k+1)q$ is even and the other is odd, the absolute value of the difference is always greater than or equal to 1, *i.e.*

$$|(2k+1)q - (2j+1)p| \geq 1$$

and hence (35) follows.

There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the lengths $l(I_k)$, $l(J_k)$ satisfy the inequalities $l(I_k) < \pi/(8qL)$ and $l(J_k) < \pi/(8qL)$. This implies that for all k, $j \geq k_0$ the intersections $I_k \cap I_j$, $J_k \cap J_j$, $I_k \cap J_j$ are all empty, and that (36) is valid. □

Lemma 7. *Assume that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even. There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the function*

$$t \mapsto \cos(\omega_1(t^2)L)\cos(\omega_2(t^2)L) + \delta(t^2)$$

is strictly monotone on the intervals I_k, J_k and thus the intervals I_k, J_k contain at most one square root of an eigenvalue of the operator A.

Proof. Define the function $F(t) = \cos(\omega_1(t^2)L)\cos(\omega_2(t^2)L) + \delta(t^2)$. Then F is differentiable and the derivative is

$$F'(t) = -\sin(\omega_1(t^2)L)\cos(\omega_2(t^2)L)\omega_1'(t^2)2tL - \cos(\omega_1(t^2)L)\sin(\omega_2(t^2)L)\omega_2'(t^2)2tL + \delta'(t^2)2t.$$

We have

$$\omega_1'(\lambda) = \frac{(\gamma+1)/2 + \left[1 + \lambda(\gamma-1)^2/2\right] / \left[2\sqrt{\lambda + \lambda^2(\gamma-1)^2/4}\right]}{2\sqrt{\lambda(\gamma+1)/2 + \sqrt{\lambda + \lambda^2(\gamma-1)^2/4}}},$$

$$\omega_2'(\lambda) = \frac{(\gamma+1)/2 - \left[1 + \lambda(\gamma-1)^2/2\right] / \left[2\sqrt{\lambda + \lambda^2(\gamma-1)^2/4}\right]}{2\sqrt{\lambda(\gamma+1)/2 - \sqrt{\lambda + \lambda^2(\gamma-1)^2/4}}}.$$

Hence if $\gamma > 1$, $\lim_{t\to\infty} \omega_1'(t^2)2t = \sqrt{\gamma}$ and $\lim_{t\to\infty} \omega_2'(t^2)2t = 1$, and if $\gamma \in (0,1)$, the roles of ω_1 and ω_2 are exchanged.

Moreover we have

$$\begin{aligned}\delta'(\lambda) &= \frac{2}{(\gamma-1)^2}\left(-\frac{1}{\lambda^2}\right)(1+\cos(\omega_1(\lambda)L)\cos(\omega_2(\lambda))L)\\ &+ \frac{2}{(\gamma-1)^2}\frac{1}{\lambda}(-\sin(\omega_1(\lambda)L)\omega_1'(\lambda)L\cos(\omega_2(\lambda)L) - \cos(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L)\omega_2'(\lambda)L)\\ &- \frac{\gamma+1}{(\gamma-1)^2}\left(-\frac{1}{2}\frac{1}{(\gamma\lambda^2-\lambda)^{3/2}}\right)(2\gamma\lambda-1)\sin(\omega_1(\lambda)L)\sin(\omega_2(\lambda)L)\\ &- \frac{\gamma+1}{(\gamma-1)^2}\frac{1}{\sqrt{\gamma\lambda^2-\lambda}}(\cos(\omega_1(\lambda)L)\omega_1'(\lambda)L\sin(\omega_2(\lambda)L) + \sin(\omega_1(\lambda)L)\cos(\omega_2(\lambda)L)\omega_2'(\lambda)L),\end{aligned}$$

hence $\lim_{t\to\infty}\delta'(t^2)2t = 0$.

Now assume that $\gamma > 1$, and let $t \in I_k$ be given, then for k sufficiently large we have

$$|\omega_1(t^2) - a_k/L| \le |\omega_1(t^2) - \sqrt{\gamma}t| + |\sqrt{\gamma}t - a_k/L| \le O(1/t),$$

hence

$$|\cos(\omega_1(t^2)L)| = |\cos(a_k + \omega_1(t^2)L - a_k)| = |\sin(\omega_1(t^2)L - a_k)| \le O(1/t),$$

and

$$|\sin(\omega_1(t^2)L)|^2 \ge 1 - O(1/t^2). \tag{37}$$

Since all the sufficiently large roots of the function $t \mapsto \cos(\omega_2(t^2)L)$ are contained in the intervals $(J_k)_{k\in\mathbb{N}}$, equation (36) implies that there exists k_0 such that

$$\inf_{k\ge k_0}\inf_{t\in I_k}|\cos(\omega_2(t^2)L)| =: M_2 > 0. \tag{38}$$

Hence we can conclude that there exists k_0 such that for $k \ge k_0$ and $t \in I_k$ the sign of $F'(t)$ is equal to the sign of

$$-\sin(\omega_1(t^2)L)\cos(\omega_2(t^2)L)$$

and since the function $\sin(\omega_1(t^2)L)\cos(\omega_2(t^2)L)$ does not have a root in the interval I_k, this implies that the derivative F' has constant sign in I_k. Hence the function F is strictly monotone on the interval I_k.

The assertion for the intervals J_k and for the case $\gamma \in (0,1)$ can be shown analogously.

Due to (29), the assertion for the roots of the eigenvalues of the operator A follows. □

Lemma 8. *Assume that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*

There exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ the intervals I_k and J_k contain one square root of an eigenvalue of the operator A.

Proof. We only prove the statement for the case $\gamma > 1$.

For $t \in I_k$, define the function

$$H(t) = \cos(\omega_1(t^2)L) + \delta(t^2)/\cos(\omega_2(t^2)L).$$

Then the function H is well-defined since due to Lemma 6 the function $t \mapsto \cos(\omega_2(t^2)L)$ does not have a root in I_k.

Inequality (30) implies that there exists a constant $M_3 > 0$ such that for $t \geq 1$ we have

$$|\delta(t^2)| \leq M_3/t^2.$$

As in the proof of Lemma 5, let $\alpha = (a_k/(L\sqrt{\gamma}) + b_k(\gamma))/2$ and $d_1 = \omega_1(\alpha^2) - a_k/L \in (-(C+2LD)/(L\sqrt{\gamma}\alpha), -C/(L\sqrt{\gamma}\alpha))$. Then

$$\cos(\omega_1(\alpha^2)L) = \cos(a_k + Ld_1) = \cos(a_k)\cos(Ld_1) - \sin(a_k)\sin(Ld_1) = (-1)^{k+1}\sin(Ld_1).$$

Now (38) implies that

$$\begin{aligned} H(\alpha)\cos(\omega_1(\alpha^2)L) &= \cos^2(\omega_1(\alpha^2)L) + \cos(\omega_1(\alpha^2)L)\delta(\alpha^2)/\cos(\omega_2(\alpha^2)L) \\ &\geq \sin^2\left(C/\left(\sqrt{\gamma}\alpha\right)\right) - \sin\left((C+2LD)/\left(\sqrt{\gamma}\alpha\right)\right)\ \delta(\alpha^2)/M_2 \\ &\geq C^2/(2\gamma\alpha^2) - M_3(C+2LD)/\left(M_2\sqrt{\gamma}\alpha^3\right) > 0 \end{aligned}$$

if α is sufficiently large. Therefore the sign of $H(\alpha)$ is equal to the sign of $\cos(\omega_1(\alpha^2)L)$ if α is sufficiently large.

In a similar way, we can show that for $\beta = (a_k/(L\sqrt{\gamma}) + c_k(\gamma))/2$ the sign of $H(\beta)$ is equal to the sign of $\cos(\omega_1(\beta^2)L)$. Lemma 5 states that the function $t \mapsto \cos(\omega_1(t^2)L)$ changes its sign in the interval I_k, hence the function H changes its sign in the interval $I_k = [\alpha, \beta]$ and due to the continuity of H this implies that the interval I_k contains a root of H and thus by equation (29) the square root of an eigenvalue of the operator A.

The assertion for the interval J_k follows analogously. □

The following theorem summarizes the preceeding lemmas.

Theorem 2. *Assume that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*

Then there exists $\lambda_0 > 0$, $k_0 \in \mathbb{N}$ such that all the square roots of the eigenvalues that are greater than λ_0 are contained in the intervals I_k, J_k ($k \geq k_0$) and each of these intervals contains exactly one square root of an eigenvalue.

Moreover, there is an asymptotic gap between the eigenvalues in the sense that for all $\varepsilon > 0$ there exists $\bar{\lambda} > 0$ such that the distance between all square roots of eigenvalues that are greater than $\bar{\lambda}$ is greater than $\pi/(2qL) - \varepsilon$.

4. Controllability and moment problems

Now we return to the question of controllability that was introduced in Section 2. From the given intial state, we want to steer the system to rest with the given arrival time $T > 0$.

The control function is the angular acceleration of the axis $u = \theta_{tt}$. For the angle of rotation, we have the initial conditions (7) and the end conditions

$$\theta(T) = 0 = \theta_t(T). \tag{39}$$

These conditions can be replaced by the two moment equations

$$\int_0^T u(s)\,\mathrm{d}s = -\theta_1, \tag{40}$$

$$\int_0^T su(s)\,\mathrm{d}s = \theta_0. \tag{41}$$

If u satisfies (40) and (41), then

$$\theta(t) = t\left(\int_0^t u(s)\,\mathrm{d}s\right) + t\theta_1 - \int_0^t su(s)\,\mathrm{d}s + \theta_0$$

satisfies (7) and (39).

For a given control function $u \in L^2(0,T)$, the solution of the intitial boundary value problem (3–6) can be written in the form

$$y(x,t) = \sum_{j=1}^{\infty} y_j(t)\phi_j(x),$$

where the functions ϕ_j are eigenfunctions of the operator A which are normalized to form a complete orthonormal system (see Th. 1). The corresponding eigenvalues are denoted as λ_j, and we assume that they are ordered in such a way that the sequence $(\lambda_j)_{j\in\mathbb{N}}$ is increasing. By Lemma 2 we can assume that the sequence is strictly increasing.

With the expansions of the initial values

$$(w_0,\psi_0) = \sum_{j=1}^{\infty} y_j^0\phi_j,\ (w_1,\psi_1) = \sum_{j=1}^{\infty} y_j^1\phi_j$$

and the expansion

$$(-x,-1) = \sum_{j=1}^{\infty} r_j\phi_j(x),\ x\in(0,L)$$

we obtain for the solution the expression

$$y_j(t) = y_j^0\cos\left(\sqrt{\lambda_j}t\right) + \frac{y_j^1}{\sqrt{\lambda_j}}\sin\left(\sqrt{\lambda_j}t\right) + \int_0^t \frac{u(s)r_j}{\sqrt{\lambda_j}}\sin\left(\sqrt{\lambda_j}(t-s)\right)\,\mathrm{d}s$$

(see for example [1]). The end conditions (8) are equivalent to the sequence of equations

$$y_j(T) = 0 = y_j'(T),\ j\in\mathbb{N}.$$

This means that for all $j\in\mathbb{N}$

$$\begin{aligned} y_j^0\cos\left(\sqrt{\lambda_j}T\right) + \frac{y_j^1}{\sqrt{\lambda_j}}\sin\left(\sqrt{\lambda_j}T\right) + \int_0^T \frac{u(s)r_j}{\sqrt{\lambda_j}}\sin\left(\sqrt{\lambda_j}(T-s)\right)\,\mathrm{d}s &= 0,\\ -y_j^0\sin\left(\sqrt{\lambda_j}T\right) + \frac{y_j^1}{\sqrt{\lambda_j}}\cos\left(\sqrt{\lambda_j}T\right) + \int_0^T \frac{u(s)r_j}{\sqrt{\lambda_j}}\cos\left(\sqrt{\lambda_j}(T-s)\right)\,\mathrm{d}s &= 0. \end{aligned}$$

By trigonometric identities, this yields the sequence of moment equations

$$\int_0^T u(s)r_j\sin\left(\sqrt{\lambda_j}s\right)\,\mathrm{d}s = y_j^0\sqrt{\lambda_j} \tag{42}$$

$$\int_0^T u(s)r_j\cos\left(\sqrt{\lambda_j}s\right)\,\mathrm{d}s = -y_j^1,\ \ j\in\mathbb{N}. \tag{43}$$

In the next section, we will compute the numbers r_j. We show that for $\gamma > 1$ for all j with $\lambda_j \geq 1/\gamma$, these numbers are unequal to zero. If the beam is sufficently short, we can guarantee that the numbers r_j are all unequal to zero.

Hence the set of successful controls that steer the system from the given initial state to the desired target state with the given arrival time is equal to the solution set of the moment problem (40–43).

4.1. The Fourier coefficients of the right-hand side

In this section we compute the values of the numbers r_j. Let the eigenfunction $\phi_j = (y_j, \varphi_j)$ be given. We have

$$r_j = \langle(-x, -1), \phi_j\rangle_H = \frac{1}{\lambda_j}\langle(-x, -1), A\phi_j\rangle_H.$$

Now (10) with $v(x) = -x$, $\alpha(x) = -1$ implies

$$r_j = \frac{1}{\lambda_j}\left(-\varphi_j'(0) + \langle A(-x, -1), \phi_j\rangle_H\right) = -\varphi_j'(0)/\lambda_j,$$

since $A(-x, -1) = 0$.

Lemma 9. *If $\gamma > 1$ and condition (14) holds, for all $j \in \mathbb{N}$ with $\lambda_j \geq 1/\gamma$, we have $\varphi_j'(0) \neq 0$ and thus $r_j \neq 0$.*

Proof. Suppose that $\varphi_j'(0) = 0$. Then equation (24) implies that

$$0 = C_2(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)\omega_1(\lambda_j) + C_4(-\omega_2(\lambda_j)^2 + \gamma\lambda_j)\omega_2(\lambda_j) = C_2\omega_1(\lambda_j)(\omega_2(\lambda_j)^2 - \omega_1(\lambda_j)^2),$$

where the last line follows from equation (23). Since $\omega_1(\lambda_j) \neq 0 \neq \omega_2(\lambda_j)^2 - \omega_1(\lambda_j)^2$, this implies that $C_2 = 0$. Thus $C_1 \neq 0$ and equation (25) implies that

$$\omega_1(\lambda_j)\sin(\omega_1(\lambda_j)L) = \omega_2(\lambda_j)\sin(\omega_2(\lambda_j)L). \tag{44}$$

Moreover, equation (26) implies that

$$(-\omega_2(\lambda_j)^2 + \gamma\lambda_j)\cos(\omega_1(\lambda_j)L) = (-\omega_1(\lambda_j)^2 + \gamma\lambda_j)\cos(\omega_2(\lambda_j)L). \tag{45}$$

We introduce the notation

$$h_1(\lambda_j) = (-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2, \quad h_2(\lambda_j) = (-\omega_2(\lambda_j)^2 + \gamma\lambda_j)^2.$$

Due to (21), we know that $-\omega_2(\lambda_j)^2 + \gamma\lambda_j \neq 0$, so we obtain the equation

$$\begin{aligned} 1 - \sin^2(\omega_1(\lambda_j)L) &= \cos^2(\omega_1(\lambda_j)L) = \frac{(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2}{(-\omega_2(\lambda_j)^2 + \gamma\lambda_j)^2}\cos^2(\omega_2(\lambda_j)L) \\ &= \frac{h_1(\lambda_j)}{h_2(\lambda_j)}(1 - \sin^2(\omega_2(\lambda_j)L) = \frac{h_1(\lambda_j)}{h_2(\lambda_j)}\left(1 - \frac{\omega_1(\lambda_j)^2}{\omega_2(\lambda_j)^2}\sin^2(\omega_1(\lambda_j)L)\right). \end{aligned}$$

Thus we have the following equation

$$\sin^2(\omega_1(\lambda_j)L)\left[1 - \frac{\omega_1(\lambda_j)^2 h_1(\lambda_j)}{\omega_2(\lambda_j)^2 h_2(\lambda_j)}\right] = 1 - \frac{h_1(\lambda_j)}{h_2(\lambda_j)} = 1 - \frac{\omega_1(\lambda_j)^2}{\omega_2(\lambda_j)^2}\frac{h_1(\lambda_j)}{h_2(\lambda_j)} + \left(\frac{\omega_1(\lambda_j)^2}{\omega_2(\lambda_j)^2} - 1\right)\frac{h_1(\lambda_j)}{h_2(\lambda_j)}\cdot \tag{46}$$

This implies the equation

$$\sin^2(\omega_1(\lambda_j)L) = 1 + \left(\frac{\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2}{\omega_2(\lambda_j)^2}\right) / \left(\frac{h_2(\lambda_j)}{h_1(\lambda_j)} - \frac{\omega_1(\lambda_j)^2}{\omega_2(\lambda_j)^2}\right).$$

We have

$$\frac{h_2(\lambda_j)}{h_1(\lambda_j)} - \frac{\omega_1(\lambda_j)^2}{\omega_2(\lambda_j)^2} = \frac{\omega_2(\lambda_j)^2(-\omega_2(\lambda_j)^2 + \gamma\lambda_j)^2 - \omega_1(\lambda_j)^2(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2}{\omega_2(\lambda_j)^2(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2}.$$

Due to (19), we have

$$\begin{aligned}
&\omega_2(\lambda_j)^2(-\omega_2(\lambda_j)^2 + \gamma\lambda_j)^2 - \omega_1(\lambda_j)^2(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2 \\
&\quad = \omega_2(\lambda_j)^2(\omega_1(\lambda_j)^2 - \lambda_j)^2 - \omega_1(\lambda_j)^2(\omega_2(\lambda_j)^2 - \lambda_j)^2 \\
&\quad = \omega_2(\lambda_j)^2(\omega_1(\lambda_j)^4 - 2\lambda_j\omega_1(\lambda_j)^2 + \lambda_j^2) - \omega_1(\lambda_j)^2((\omega_2(\lambda_j)^4 - 2\lambda_j\omega_2(\lambda_j)^2 + \lambda_j^2) \\
&\quad = \omega_1(\lambda_j)^2\omega_2(\lambda_j)^2(\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2) + \lambda_j^2(\omega_2(\lambda_j)^2 - \omega_1(\lambda_j)^2) \\
&\quad = (\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)(\omega_1(\lambda_j)^2\omega_2(\lambda_j)^2 - \lambda_j^2) \\
&\quad = (\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)((\gamma - 1)\lambda_j^2 - \lambda_j) \qquad (47)
\end{aligned}$$

where the last line follows from equation (20). So we have

$$\sin^2(\omega_1(\lambda_j)L) = 1 + \left(\frac{1}{\omega_2(\lambda_j)^2}\right) / \left(\frac{(\gamma-1)\lambda_j^2 - \lambda_j}{\omega_2(\lambda_j)^2(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2}\right) = 1 + \frac{(-\omega_1(\lambda_j)^2 + \gamma\lambda_j)^2}{\lambda_j((\gamma-1)\lambda_j - 1)}.$$

If $\lambda_j > 1/(\gamma - 1)$, we have $(\gamma - 1)\lambda_j - 1 > 0$.

This implies that $\sin^2(\omega_1(\lambda_j)L) > 1$, which is a contradiction.

To consider the other case, namely $\lambda_j \in (1/\gamma, 1/(\gamma - 1)]$, we use equation (46)

$$\sin^2(\omega_1(\lambda_j)L)\left[1 - \frac{\omega_1(\lambda_j)^2 h_1(\lambda_j)}{\omega_2(\lambda_j)^2 h_2(\lambda_j)}\right] = 1 - \frac{h_1(\lambda_j)}{h_2(\lambda_j)}$$

to obtain the equation

$$\sin^2(\omega_1(\lambda_j)L)[\omega_2(\lambda_j)^2 h_2(\lambda_j) - \omega_1(\lambda_j)^2 h_1(\lambda_j)] = \omega_2(\lambda_j)^2 h_2(\lambda_j) - \omega_2(\lambda_j)^2 h_1(\lambda_j) = \omega_2(\lambda_j)^2[h_2(\lambda_j) - h_1(\lambda_j)].$$

Due to (47), this yields the equation

$$\sin^2(\omega_1(\lambda_j)L)(\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)\lambda_j((\gamma-1)\lambda_j - 1) = \omega_2(\lambda_j)^2[h_2(\lambda_j) - h_1(\lambda_j)].$$

Now using (19), we get the equation

$$\begin{aligned}
h_1(\lambda_j) - h_2(\lambda_j) &= (\omega_2(\lambda_j)^2 - \lambda_j)^2 - (\omega_1(\lambda_j)^2 - \lambda_j)^2 \\
&= \omega_2(\lambda_j)^4 - 2\omega_2(\lambda_j)^2\lambda_j + \lambda_j^2 - \omega_1(\lambda_j)^4 + 2\omega_1(\lambda_j)^2\lambda_j - \lambda_j^2 \\
&= \omega_2(\lambda_j)^4 - \omega_1(\lambda_j)^4 + 2\lambda_j(\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2) \\
&= (\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)[2\lambda_j - (\omega_1(\lambda_j)^2 + \omega_2(\lambda_j)^2)] \\
&= (\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)[2\lambda_j - \lambda_j(\gamma + 1)] \\
&= (\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)(1 - \gamma)\lambda_j.
\end{aligned}$$

Thus the following equation holds:

$$\sin^2(\omega_1(\lambda_j)L)((\gamma-1)\lambda_j-1)=(\gamma-1)\omega_2(\lambda_j)^2. \tag{48}$$

For $\gamma>1$ and $\lambda_j\in(1/\gamma,1/(\gamma-1)]$, we have $(\gamma-1)\lambda_j-1\le 0$, which gives a contradiction, since the left hand side of equation (48) is negative and the right-hand side is strictly positive.

We have assumed (14), so Lemma 1 implies that $1/\gamma$ cannot be an eigenvalue of A. □

Lemma 10. *If $\gamma>1$ and condition (14) holds and $L<\pi/\sqrt{1+1/\gamma}$, then for all $j\in\mathbb{N}$, we have $\varphi_j'(0)\neq 0$ and thus $r_j\neq 0$.*

Proof. Lemma 9 gives the assertion for $\lambda_j\ge 1/\gamma$. So only the case $\lambda_j<1/\gamma$ remains.

For $\lambda<1/\gamma$, we have $\beta(\lambda)<0$ and $\omega_2(\lambda)=\sqrt{\beta(\lambda)}=i\sqrt{-\beta(\lambda)}$.

Hence $\sin(\omega_2(\lambda)L)=\sin\left(i\sqrt{-\beta(\lambda)}L\right)=i\sinh\left(\sqrt{-\beta(\lambda)}L\right)$.

Suppose that $\varphi_j'(0)=0$. Then equation (44) yields

$$\omega_1(\lambda)\sin(\omega_1(\lambda)L)=\omega_2(\lambda)\sin(\omega_2(\lambda)L)=-\sqrt{-\beta(\lambda)}\sinh\left(\sqrt{-\beta(\lambda)}L\right).$$

We have $\alpha(1/\gamma)=1+1/\gamma$, so the definition of $\omega_1(\lambda)$ implies that for $\lambda\in(0,1/\gamma)$ we have

$$0<\omega_1(\lambda)L\le\omega_1(1/\gamma)L=L\sqrt{1+1/\gamma}<\pi,$$

hence $\sin(\omega_1(\lambda)L)>0$. This implies the following inequality:

$$0<\omega_1(\lambda)\sin(\omega_1(\lambda)L)=-\sqrt{-\beta(\lambda)}\sinh(\sqrt{-\beta(\lambda)}L)<0,$$

which is a contradiction. □

Lemma 9 allows us to prove a lemma about the coefficients of the eigenfunctions. For every j, the lemma states that at least one of two formulas that give the coefficient C_1 in terms of the coefficient C_2 is valid.

Lemma 11. *If $\gamma>1$ and condition (14) holds, for all $j\in\mathbb{N}$ with $\lambda_j\ge 1/\gamma$ we have*

$$\sin(\omega_2(\lambda_j)L)-\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\sin(\omega_1(\lambda_j)L)\neq 0$$

or

$$\cos(\omega_1(\lambda_j)L)-\frac{\omega_2(\lambda_j)^2-\lambda_j}{\omega_1(\lambda_j)^2-\lambda_j}\cos(\omega_2(\lambda_j)L)\neq 0.$$

Thus for the coefficients of the eigenfunctions $\phi_j=(y_j,\varphi_j)$ we have

$$C_1=C_2\,\frac{\omega_1[\cos(\omega_1(\lambda_j)L)-(\omega_1(\lambda_j)^2-\lambda_j)/(\omega_2(\lambda_j)^2-\lambda_j)\cos(\omega_2(\lambda_j)L)]}{\omega_2(\lambda_j)[\sin(\omega_2(\lambda_j)L)-(\omega_1(\lambda_j)/\omega_2(\lambda_j))\sin(\omega_1(\lambda_j)L)]} \tag{49}$$

or

$$C_1=C_2\frac{\sin(\omega_1(\lambda_j)L)-(\omega_1(\lambda_j)/\omega_2(\lambda_j))\sin(\omega_2(\lambda_j)L)}{\cos(\omega_1(\lambda_j)L)-[(\omega_2(\lambda_j)^2-\lambda_j)/\omega_1(\lambda_j)^2-\lambda_j)]\cos(\omega_2(\lambda_j)L)}.$$

Proof. Let $j \in \mathbb{N}$ be given. Suppose that

$$\sin(\omega_2(\lambda_j)L) - \frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)} \sin(\omega_1(\lambda_j)L) = 0 \tag{50}$$

and

$$\cos(\omega_1(\lambda_j)L) - \frac{\omega_2(\lambda_j)^2 - \lambda_j}{\omega_1(\lambda_j)^2 - \lambda_j} \cos(\omega_2(\lambda_j)L) = 0. \tag{51}$$

By the proof of Lemma 9, we have $C_2 \neq 0$, hence (25) and (50) yield

$$0 = (-\omega_1(\lambda_j)^2 + \gamma\lambda_j)\cos(\omega_1(\lambda_j)L) - (-\omega_2(\lambda_j)^2 + \gamma\lambda_j)\cos(\omega_2(\lambda_j)L).$$

By (51) and (19), this yields

$$\frac{\omega_2(\lambda_j)^2 - \lambda_j}{\omega_1(\lambda_j)^2 - \lambda_j} \cos(\omega_2(\lambda_j)L) = \cos(\omega_1(\lambda_j)L) = \frac{\omega_1(\lambda_j)^2 - \lambda_j}{\omega_2(\lambda_j)^2 - \lambda_j} \cos(\omega_2(\lambda_j)L).$$

Moreover, $C_2 \neq 0$ and (26) and (51) yield the equation

$$0 = \sin(\omega_1(\lambda_j)L) - \frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)} \sin(\omega_2(\lambda_j)L) = 0.$$

By (50), this yields

$$\frac{\omega_2(\lambda_j)}{\omega_1(\lambda_j)} \sin(\omega_2(\lambda_j)L) = \sin(\omega_1(\lambda_j)L) = \frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)} \sin(\omega_2(\lambda_j)L).$$

Now we consider two cases.
Case 1: If $\sin(\omega_2(\lambda_j)L) \neq 0$, we have

$$\frac{\omega_2(\lambda_j)}{\omega_1(\lambda_j)} = \frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)},$$

hence $\omega_1(\lambda_j)^2 = \omega_2(\lambda_j)^2$, which is a contradiction, since $\lambda_j > 0$.
Case 2: If $\sin(\omega_2(\lambda_j)L) = 0$, we have $\cos(\omega_2(\lambda_j)L) \neq 0$, hence

$$(\omega_1(\lambda_j)^2 - \lambda_j)^2 = (\omega_2(\lambda_j)^2 - \lambda_j)^2.$$

Thus

$$\omega_2(\lambda_j)^4 - \omega_1(\lambda_j)^4 + 2\lambda_j(\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2) = 0.$$

By (19), this yields the equation $(\omega_1(\lambda_j)^2 - \omega_2(\lambda_j)^2)\lambda_j(1-\gamma) = 0$, which is a contradiction. □

In the next two lemmas we obtain lower bounds for $|\varphi_j'(0)|$. The first lemma treats the case $\sqrt{\lambda_j} \in I_k$ and the second the case $\sqrt{\lambda_j} \in J_k$.

Lemma 12. *If* $\gamma > 1$ *and (14) holds, we have*

$$\inf_{j\in\mathbb{N}:\ \sqrt{\lambda_j}\in I_k \text{ for some } k\in\mathbb{N} \text{ and } \lambda_j \geq 1/\gamma} \left|\varphi_j'(0)\right| > 0. \tag{52}$$

Proof. Let $j \in \mathbb{N}$ be given. We use the notation $\lambda = \lambda_j$, $\omega_1 = \omega_1(\lambda_j)$, $\omega_2 = \omega_2(\lambda_j)$, $y(x) = y_j(x)$, $\varphi(x) = \varphi_j(x)$. Then the results from 3.3 imply that

$$y(x) = C_1 \left(-\omega_1 \sin(\omega_1 x) + \omega_2 \frac{-\omega_1^2 + \gamma\lambda}{-\omega_2^2 + \gamma\lambda} \sin(\omega_2 x) \right) + C_2 \omega_1 (\cos(\omega_1 x) - \cos(\omega_2 x)),$$

$$\varphi(x) = C_1(\omega_2^2 - \lambda)(\cos(\omega_1 x) - \cos(\omega_2 x)) + C_2 \left((\omega_2^2 - \lambda)\sin(\omega_1 x) - \frac{\omega_1}{\omega_2}(\omega_1^2 - \lambda)\sin(\omega_2 x) \right),$$

where we have used the equation $-\omega_1^2 + \gamma\lambda = \omega_2^2 - \lambda$ that follows from (19).

Hence for all $x \in (0, L)$, we have

$$|y(x)| \le |C_1| \left(\omega_1 + \omega_2 \left| \frac{\omega_2^2 - \lambda}{\omega_1^2 - \lambda} \right| \right) + 2|C_2|\omega_1.$$

This implies the inequality

$$\int_0^L y(x)^2 \, \mathrm{d}x \le \left[|C_1| \left(\omega_1 + \omega_2 \left| \frac{\omega_2^2 - \lambda}{\omega_1^2 - \lambda} \right| \right) + 2|C_2|\omega_1 \right]^2 L. \tag{53}$$

Moreover, for all $x \in (0, L)$ we have

$$|\varphi(x)| \le 2|C_1||\omega_2^2 - \lambda| + |C_2| \left(|\omega_2^2 - \lambda| + \frac{\omega_1}{\omega_2}|\omega_1^2 - \lambda| \right).$$

This yields the inequality

$$\int_0^L \varphi(x)^2 \, \mathrm{d}x \le \left[2|C_1||\omega_2^2 - \lambda| + |C_2| \left(|\omega_2^2 - \lambda| + \frac{\omega_1}{\omega_2}|\omega_1^2 - \lambda| \right) \right]^2 L. \tag{54}$$

For $\gamma > 1$ and $\sqrt{\lambda} \in I_k$ and k sufficiently large, (37) implies

$$|\sin(\omega_1(\lambda)L)|^2 \ge 1 - O(1/\lambda) \tag{55}$$

and $|\cos(\omega_1(\lambda)L)|^2 \le O(1/\lambda)$.

Moreover, Lemma 3 implies the ineqality

$$\frac{\omega_1(\lambda)}{\omega_2(\lambda)} \ge \frac{\sqrt{\gamma\lambda}}{\sqrt{\lambda}} = \sqrt{\gamma}.$$

This implies that

$$\left| \sin(\omega_2 L) - \frac{\omega_1}{\omega_2} \sin(\omega_1 L) \right| \ge \frac{\omega_1}{\omega_2} |\sin(\omega_1 L)| - |\sin(\omega_2 L)| \ge \sqrt{\gamma} |\sin(\omega_1 L)| - 1.$$

Hence (55) implies that for λ sufficiently large

$$\left| \sin(\omega_2 L) - \frac{\omega_1}{\omega_2} \sin(\omega_1 L) \right| > 0.$$

Hence (25) yields the equation

$$C_1 \ = C_2\ \frac{\omega_1[(\omega_2^2-\lambda)\cos(\omega_1 L)-(\omega_1^2-\lambda)\cos(\omega_2 L)]}{\omega_2(\omega_2^2-\lambda)[\sin(\omega_2 L)-(\omega_1/\omega_2)\sin(\omega_1 L)]}\,. \tag{56}$$

Hence the following inequality holds:

$$|C_1| \le |C_2|\frac{\omega_1}{\omega_2}\ \frac{|\cos(\omega_1 L)|+(|\omega_1^2-\lambda|/|\omega_2^2-\lambda|)}{\sqrt{\gamma}|\sin(\omega_1 L)|-1}.$$

We have

$$\lim_{k\to\infty,\sqrt{\lambda}\in I_k}\left(\frac{\omega_1(\lambda)}{\omega_2(\lambda)}\frac{|\cos(\omega_1(\lambda L))|+(|\omega_1(\lambda)^2-\lambda|/|\omega_2(\lambda)^2-\lambda|)}{\sqrt{\gamma}|\sin(\omega_1(\lambda)L)|-1}\right)/\lambda$$
$$= \frac{\sqrt{\gamma}}{\sqrt{\gamma}-1}\lim_{\lambda\to\infty}(|\omega_1^2(\lambda)-\lambda|/(\lambda|\omega_2^2(\lambda)-\lambda|) = \frac{\sqrt{\gamma}}{\sqrt{\gamma}-1}\lim_{\lambda\to\infty}\frac{|\omega_1^2(\lambda)-\lambda|^2}{\lambda^2} = \frac{\sqrt{\gamma}}{\sqrt{\gamma}-1}(\gamma-1)^2.$$

Thus for all $\varepsilon > 0$, the inequality

$$|C_1| \le |C_2|(\frac{\sqrt{\gamma}}{\sqrt{\gamma}-1}(\gamma-1)^2+\varepsilon)\lambda$$

holds for λ sufficiently large. In particular $|C_1| \le |C_2|\ O(\lambda)$.

So (53) implies the inequality

$$\int_0^L y^2/L \le |C_2|^2\left[\frac{\omega_1}{\omega_2}\ \frac{|\cos(\omega_1 L)|+(|\omega_1^2-\lambda|/|\omega_2^2-\lambda|)}{\sqrt{\gamma}|\sin(\omega_1 L)|-1}\left(\omega_1+\omega_2\frac{|\omega_2^2-\lambda|}{|\omega_1^2-\lambda|}\right)+2\omega_1\right]^2.$$

So we have

$$\int_0^L y^2 \le |C_2|^2[O(\lambda^{3/2})]^2.$$

Moreover, equation (54) yields the inequality

$$\int_0^L \varphi^2/L \le |C_2|^2\left[\frac{\omega_1}{\omega_2}\ \frac{|\cos(\omega_1 L)|+(|\omega_1^2-\lambda|/|\omega_2^2-\lambda|)}{\sqrt{\gamma}|\sin(\omega_1 L)|-1}2|\omega_2^2-\lambda|+|\omega_2^2-\lambda|+\frac{\omega_1}{\omega_2}|\omega_1^2-\lambda|\right]^2.$$

So we have

$$\int_0^L \varphi^2 \le |C_2|^2[O(\lambda)]^2.$$

Since the eigenfunctions are normalized, this yields

$$1 = \left(\int_0^L y^2+\varphi^2\right)^{1/2} \le |C_2|O(\lambda^{3/2}).$$

Thus there exists a constant $\hat{M} > 0$ that only depends on γ such that $|C_2| \geq \hat{M}/(\lambda^{3/2})$. Since by Lemma 3 we have $\omega_1^2(\lambda) - \omega_2^2(\lambda) \geq (\gamma - 1)\lambda$, this implies

$$|\varphi_j'(0)| = |C_2|(\omega_1^2 - \omega_2^2)\omega_1 \geq \hat{M}(\gamma - 1)\lambda\sqrt{\gamma\lambda}/\lambda^{3/2}$$

if $\sqrt{\lambda_j} \in I_k$ is sufficiently large, and the assertion follows with Lemma 9. □

Lemma 13. *If $\gamma > 1$, (14) holds and $\sqrt{\gamma} = q/p$ with p even and q odd or vice versa we have*

$$\inf_{j\in\mathbb{N}:\sqrt{\lambda_j}\in J_k \text{ for some } k\in\mathbb{N} \text{ and } \lambda_j \geq 1/\gamma} \left|\varphi_j'(0)\right| / \sqrt{\lambda_j} > 0. \tag{57}$$

Proof. For k sufficiently large and $\sqrt{\lambda} \in J_k$ we have

$$\cos(\omega_2(\lambda)L) = O\left(1/\sqrt{\lambda}\right).$$

Moreover, equation (36) implies that there exists $k_0 \in \mathbb{N}$ such that

$$\inf_{k\geq k_0} \inf_{t\in J_k} |\cos(\omega_1(t^2)L)| =: M_1 > 0.$$

Hence for λ sufficently large and $\sqrt{\lambda} \in J_k$ we have

$$\left|\cos(\omega_1 L) - \frac{\omega_2^2 - \lambda}{\omega_1^2 - \lambda}\cos(\omega_2 L)\right| \geq M_1 - \left|\frac{\omega_2^2 - \lambda}{\omega_1^2 - \lambda}\right| |\cos(\omega_2 L)| \geq M_1 - \frac{\lambda}{(\gamma - 1)^2\lambda^2}|\cos(\omega_2 L)| > 0.$$

Hence (26) implies the equation

$$C_1 \;=\; C_2 \frac{\sin(\omega_1 L) - (\omega_1/\omega_2)\sin(\omega_2 L)}{\cos(\omega_1 L) - [(\omega_2^2 - \lambda)/(\omega_1^2 - \lambda)]\cos(\omega_2 L)}.$$

We have

$$\limsup_{k\to\infty, \sqrt{\lambda}\in J_k} \left|\frac{\sin(\omega_1(\lambda)L) - (\omega_1(\lambda)/\omega_2(\lambda))\sin(\omega_2(\lambda)L)}{\cos(\omega_1(\lambda)L) - ((\omega_2^2(\lambda) - \lambda)/\omega_1^2(\lambda) - \lambda))\cos(\omega_2(\lambda)L)}\right| \leq \frac{1 + \sqrt{\gamma}}{M_1}.$$

Hence $|C_1| \leq [(1 + \sqrt{\gamma})/M_1]\ |C_2|$.

So (53) implies the inequality

$$\int_0^L y^2/L \leq |C_2|^2 \left[\frac{1 + \sqrt{\gamma}}{M_1}\left(\omega_1 + \omega_2 \frac{|\omega_2^2 - \lambda|}{|\omega_1^2 - \lambda|}\right) + 2\omega_1\right]^2.$$

So we have

$$\int_0^L y^2 \leq |C_2|^2[O(\sqrt{\lambda})]^2.$$

Moreover, equation (54) yields the inequality

$$\int_0^L \varphi^2/L \leq |C_2|^2 \left[2\frac{1 + \sqrt{\gamma}}{M_1}|\omega_2^2 - \lambda| + |\omega_2^2 - \lambda| + \frac{\omega_1}{\omega_2}|\omega_1^2 - \lambda|\right]^2.$$

So we have

$$\int_0^L \varphi^2 \leq |C_2|^2 [O(\lambda)]^2.$$

This yields

$$1 = \left(\int_0^L y^2 + \varphi^2 \right)^{1/2} \leq |C_2| \; O(\lambda).$$

Thus there exists a constant $\hat{M} > 0$ that only depends on γ such that $|C_2| \geq \hat{M}/\lambda$. This implies

$$|\varphi_j'(0)| = |C_2|(\omega_1^2 - \omega_2^2)\omega_1 \geq \hat{M}(\gamma - 1)\lambda\sqrt{\gamma\lambda}/\lambda = \hat{M}(\gamma - 1)\sqrt{\gamma}\sqrt{\lambda_j}$$

if $\sqrt{\lambda_j} \in J_k$ is sufficiently large, and the assertion follows with Lemma 9. □

4.2. Controllability

We have seen at the beginning of Section 4 that the question of controllability is equivalent to the question: For which right-hand sides $(c_j^1, c_j^2)_{j \geq 0}$ is the moment problem

$$\int_0^T u(s)\,\mathrm{d}s = c_0^1, \tag{58}$$

$$\int_0^T u(s)s\,\mathrm{d}s = c_0^2, \tag{59}$$

$$\int_0^T u(s)\sin\left(\sqrt{\lambda}_j s\right)\,\mathrm{d}s = c_j^1, \tag{60}$$

$$\int_0^T u(s)\cos\left(\sqrt{\lambda}_j s\right)\,\mathrm{d}s = c_j^2 \tag{61}$$

solvable? This approach to controllability is well-established, see the bibliographical remarks in [6] (p. 78).

For the convenience of the reader we state Theorem 1.2.22 from [6] (p. 74) in our notation.

Theorem 3. *Assume that*

$$\liminf_{j\to\infty} \sqrt{\lambda}_j - \sqrt{\lambda}_{j-1} > 2\pi/T. \tag{62}$$

For $x > 0$, let $d(x)$ = number of $\sqrt{\lambda}_j < x$. Assume that

$$\limsup_{y\to\infty} \limsup_{x\to\infty} (\mathrm{d}(x+y) - \mathrm{d}(x))/y < T/(2\pi). \tag{63}$$

Then for every sequence $(c_j^1, c_j^2)_{j\geq 0}$ such that

$$\sum_{j=0}^{\infty} (c_j^1)^2 + (c_j^2)^2 < \infty,$$

there is exactly one minimum norm solution in $L^2(0,T)$ of the moment problem (58–61).

For the controllability of the rotating Timoshenko beam, this yields the following result.

Theorem 4. *Assume that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*
Assume that for all $j \in \mathbb{N}$, we have $\varphi_j'(0) \neq 0$. Assume that

$$T > \max\{4qL,\ 2\,(1+\sqrt{\gamma})\,L\}\cdot$$

Then for all sequences $(y_j^0, y_j^1)_{j\in\mathbb{N}}$ such that

$$\sum_{j=1}^{\infty}(\lambda_j (y_j^0)^2 + (y_j^1)^2)\lambda_j^2/(\varphi_j'(0))^2 < \infty, \tag{64}$$

there is exactly one minimum norm solution in $L^2(0,T)$ of the moment problem (40–43) and the problem of null-controllability with arrival time T is solvable.

Proof. Let $\varepsilon = \pi/(2qL) - (2\pi)/T$. Since $T > 4qL$, we have $\varepsilon = \pi(T-4qL)/(2qTL) > 0$. Theorem 2 implies that if j is large enough we have

$$\sqrt{\lambda}_j - \sqrt{\lambda}_{j-1} > \frac{\pi}{2qL} - \frac{\varepsilon}{2} > \frac{\pi}{2qL} - \varepsilon = \frac{2\pi}{T},$$

hence (62) holds. Moreover, if x is sufficiently large, Lemma 4 implies that

$$\mathrm{d}(x+y) - \mathrm{d}(x) \leq Ly\left(\frac{1}{\pi} + \frac{\sqrt{\gamma}}{\pi}\right) + 2 = Ly\frac{1}{\pi}\,(1+\sqrt{\gamma}) + 2.$$

Hence we obtain

$$\limsup_{y\to\infty}\limsup_{x\to\infty}\frac{\mathrm{d}(x+y)-\mathrm{d}(x)}{y} \leq \frac{L}{\pi}(1+\sqrt{\gamma}).$$

Since $L\left(1+\sqrt{\gamma}\right) < T/2$, this implies that (63) holds. Thus Theorem 3 implies the assertion. □

If $\gamma > 1$, the assumptions can be weakened: in this case for eigenvalues λ_j that are greater than $1/\gamma$, Lemma 9 implies that we have $\varphi_j'(0) \neq 0$. Moreover, if in addition $L < \pi/\sqrt{1+1/\gamma}$, we have $\varphi_j'(0) \neq 0$ for all $j \in \mathbb{N}$ by Lemma 10.

Theorem 5. *Let $\gamma > 1$ be given such that (14) holds. Assume that $L < \pi/\sqrt{1+1/\gamma}$ and that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*
Moreover, assume that

$$T > \max\{4qL,\ 2\,(1+\sqrt{\gamma})\,L\}\cdot$$

Then for all sequences $(y_j^0, y_j^1)_{j\in\mathbb{N}}$ such that (64) holds, there is exactly one minimum norm solution in $L^2(0,T)$ of the moment problem (40–43) and the problem of null-controllability with arrival time T is solvable.

Now we can use the lower bounds for $(\varphi_j'(0))^2$ from Lemma 12 and Lemma 13 to obtain the following theorem:

Theorem 6. *Let $\gamma > 1$ be given such that (14) holds. Assume that $L < \pi/\sqrt{1+1/\gamma}$ and that $\sqrt{\gamma} = q/p$ with p, $q \in \mathbb{N}$ with p even and q odd or with q odd and p even.*
Moreover, assume that

$$T > \max\{4qL,\ 2\,(1+\sqrt{\gamma})\,L\}\cdot$$

Then for all $(w_0,\psi_0)\in D(A)$ with $A(w_0,\psi_0)\in D(A)$ and for all $(w_1,\psi_1)\in D(A)$ there is exactly one minimum norm solution in $L^2(0,T)$ of the moment problem(40–43) and the problem of null-controllability with arrival time T is solvable.

Remark 1. The assumption $L<\pi/\sqrt{1+1/\gamma}$ can be replaced by the condition $\varphi_j'(0)\neq 0$ or $y_j^0=0=y_j^1$ for all $\lambda_j<1/\gamma$.

Proof. We have

$$y_j^1=\langle (w_1,\psi_1),\phi_j\rangle=\frac{1}{\lambda_j}\langle (w_1,\psi_1),A\phi_j\rangle=\frac{1}{\lambda_j}\langle A(w_1,\psi_1),\phi_j\rangle\cdot$$

Since $A(w_1,\psi_1)$ is in H, Theorem 1 implies that the coefficients $\langle A(w_1,\psi_1),\phi_j\rangle$ are in l^2. Hence the sequence $(y_j^1\lambda_j)$ is in l^2, that is

$$\sum_{j=1}^{\infty}(y_j^1)^2\lambda_j^2<\infty.$$

Moreover, we have

$$y_j^0=\langle (w_0,\psi_0),\phi_j\rangle=\frac{1}{\lambda_j^2}\langle (w_0,\psi_0),A^2\phi_j\rangle=\frac{1}{\lambda_j^2}\langle A^2(w_0,\psi_0),\phi_j\rangle\cdot$$

Since $A^2(w_0,\psi_0)$ is in H, Theorem 1 implies that the coefficients $\langle A^2(w_0,\psi_0),\phi_j\rangle$ are in l^2. Hence the sequence $(y_j^0\lambda_j^2)$ is in l^2, that is

$$\sum_{j=1}^{\infty}(y_j^0)^2\lambda_j^4<\infty.$$

By Lemma 12 and Lemma 13, we have $\inf_{j\in\mathbb{N}}|\varphi_j'(0)|>0$. Hence condition (64) holds, and the assertion follows. □

It is interesting that the above controllability results for the Timoshenko beam are completely different from the result that holds for the Euler–Bernoulli beam where the fast growth of the eigenvalues implies that the system is controllable for arbitrarily small arrival times without additional restrictions to the parameters (see for example [4]).

If the assumptions of the controllability result are valid, the numerical algorithm for the computation of time-optimal controls introduced in [4] can be used for the rotating Timoshenko beam.

Remark 2. The condition $(w_1,\psi_1)\in D(A)$ in Theorem 6 can be interpreted as a compatibility condition. This can be seen as follows.

The boundary condition $w(0,t)=0$ implies that $w_t(0,t)=0$ for all $t\geq 0$, thus for $(w_1,\psi_1)\in D(A)$ we have $w_t(0,0)=w_1(0)=0$, and in the same way, the boundary condition $\psi(0,t)=0$ implies that $\psi_t(0,0)=\psi_1(0)=0$.

If the functions (w,ψ) are sufficiently regular, for example $C^{(2)}$-functions, the condition $w_x(L,t)-\psi(L,t)=0$ implies by the theorem of Schwarz the equations

$$0=w_{xt}(L,t)-\psi_t(L,t)=w_{tx}(L,t)-\psi_t(L,t).$$

Hence $0=w_1'(L)-\psi_1(L)$.

Similarly, we obtain the condition $0=\psi_{xt}(L,t)=\psi_{tx}(L,t)$, hence $\psi_1'(L)=0$.

In this situation, three of the equations of the condition $A(w_0,\psi_0)\in D(A)$ can be explained in a similar way.

Let (w_0, ψ_0) with $A(w_0, \psi_0) \in D(A)$ be given. Define $(w_2, \psi_2) := A(w_0, \psi_0)$. Then (3) for $t = 0$ implies the equation

$$w_2(x) = -u(0)x - w_{tt}(x, 0).$$

Hence we have $w_2(0) = -\frac{\mathrm{d}}{\mathrm{d}t} w_t(0, t)|_{t=0} = w_1(0) = 0$.

Equation (4) implies that

$$\psi_2(x) = -u(0) - \psi_{tt}(x, 0),$$

hence if ψ is a $C^{(3)}$-function, we have $\psi_2'(L) = -\frac{\mathrm{d}}{\mathrm{d}t}\psi_{xt}(L, t)|_{t=0} = 0$.

If w is also a $C^{(3)}$-function, we have

$$w_2'(L) = -u(0) - \frac{\mathrm{d}}{\mathrm{d}t} w_{tx}(L, t)|_{t=0} = -u(0) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(L, t)|_{t=0} = -u(0) - \psi_{tt}(L, 0) = \psi_2(L).$$

So only the condition $0 = \psi_2(0) = -\psi_0''(0) - \frac{1}{\gamma} w_0'(0)$ is not explained as a compatibility condition.

Hence if w and ψ are $C^{(3)}$-functions, the condition $A(w_0, \psi_0) \in D(A)$ requires only that $\psi_0''(0) + (1/\gamma) w_0'(0) = 0$.

Remark 3. Note that the physical meaning of the parameter γ is given by the equation $\gamma = E\rho I/(KI_\rho)$. With the parameters given in [5] for a solid aluminium bar, we have $\gamma > 1$.

Remark 4. The assumption $\sqrt{\gamma} = q/p$ should be interpreted as a relation between bending and shear.

5. Conclusion

We have shown that the rotating Timoshenko beam is exactly controllable if the parameter γ is a rational number greater than one with even numerator and odd denominator or *vice versa*, the time-interval is long enough and the beam is short enough.

The last assumption was only necessary since we had to secure that $\varphi_j'(0) \neq 0$ for all $j \in \mathbb{N}$.

We have shown this for the eigenvalues larger than $1/\gamma$. For the eigenvalues smaller than $1/\gamma$, we have only shown $\varphi_j'(0) \neq 0$ if the beam is sufficently short. The author does not expect that this assumption is really necessary, but the question is open.

In our analysis, we have given intervals that contain the eigenvalues of the beam. These intervals are very useful for numerical computations of the eigenvalues, since each of them contains exactly one eigenvalue.

It is interesting to compare the controllability result with the result for the Euler–Bernoulli beam. For this model, which is also known as the beam of Euler–Bernoulli–Navier, the rapid growth of the eigenvalues implies controllability for arbitrarily short time-intervals. This situation is related to the fact that in the beam of Euler–Bernoulli–Navier, propagation of waves with infinite speed occurs.

This is in contrast to the situation for the Timoshenko beam, where the waves propagate in the beam with finite speed only, and the time interval has to be long enough to assure controllability.

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