ESAIM: Control, Optimisation and Calculus of Variations June 2001, Vol. 6, 467–488
URL: http://www.emath.fr/cocv/

CONTROL PROBLEMS FOR CONVECTION-DIFFUSION EQUATIONS WITH CONTROL LOCALIZED ON MANIFOLDS

Phuong Anh Nguyen[1] and Jean-Pierre Raymond[1]

Abstract. We consider optimal control problems for convection-diffusion equations with a pointwise control or a control localized on a smooth manifold. We prove optimality conditions for the control variable and for the position of the control. We do not suppose that the coefficient of the convection term is regular or bounded, we only suppose that it has the regularity of strong solutions of the Navier–Stokes equations. We consider functionals with an observation on the gradient of the state. To obtain optimality conditions we have to prove that the trace of the adjoint state on the control manifold belongs to the dual of the control space. To study the state equation, which is an equation with measures as data, and the adjoint equation, which involves the divergence of L^p-vector fields, we first study equations without convection term, and we next use a fixed point method to deal with the complete equations.

Mathematics Subject Classification. 49K20, 49J20, 35K57.

Received May 31, 2000. Revised February 15, 2001.

1. Introduction

We are interested in the following optimal control problem:

> Can we control the temperature distribution of a fluid in a three dimensional domain by heating sources localized on a network of wires?

More generally we are interested in heating sources (the control variables) concentrated on thin structures. For simplicity we consider the case of controls localized on a manifold γ included in a N-dimensional bounded domain $\overline{\Omega}$, but the case of a finite union of manifolds (a network of wires) can be considered as well. This optimal control problem clearly refers to a system of equations where the temperature and the fluid velocity are coupled. Such problems have been studied in the case of distributed or boundary controls (see for examples the references in [19]). The case of controls localized on thin structures, which is interesting for technological applications, has not yet been studied in the literature. As it is shown in [20], a fundamental step to tackle the complete Boussinesq system with controls localized on thin structures, first consists in studying a problem in which the fluid velocity is known. If y denotes the fluid temperature and $\vec{V}$ the fluid velocity, the temperature y

Keywords and phrases: Pointwise control, optimal control, convection-diffusion equation, control localized on manifolds.

[1] Université Paul Sabatier, UMR CNRS MIP, UFR MIG, 31062 Toulouse Cedex 4, France; e-mail: nguyen@mip.ups-tlse.fr and raymond@mip.ups-tlse.fr

is the solution to the following convection-diffusion equation:

$$\frac{\partial y}{\partial t} + Ay + \vec{V} \cdot \nabla y = u\delta_\gamma|_Q \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = u\delta_\gamma|_\Sigma \text{ on } \Sigma, \qquad y(0) = y_0 \text{ in } \Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a regular boundary Γ, $N \geq 2$, $Q = \Omega \times]0, T[$, $T > 0$ is given fixed, $\Sigma = \Gamma \times]0, T[$, A is a second order elliptic operator of the form $Ay = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_j y) + a_0(x)y$, $\gamma \subset \overline{\Omega}$ is a regular manifold of dimension $0 \leq D \leq N-2$, δ_γ denotes the Dirac distribution on γ, and u is a function from $\gamma \times]0, T[$ with values in $\mathbb{R}$.

Convection-diffusion equations are often refered to flow related models, and a computational approach in the case of Neumann boundary control is carried out in [5]. In [4,8], the case of pointwise controls (which is a particular type of thin structure) is considered for the one-dimensional Burgers' equation.

In order to next study the complete Boussinesq system we must suppose that $\vec{V}$ is not too regular. In control problems for convection-diffusion equation studied in the literature, it is often supposed that $\vec{V}$ is bounded [5,11,12]. However, it is not reasonable to suppose that the solution $(y, \vec{V})$ to the Boussinesq system is such that $\vec{V}$ belongs to $L^\infty(0, T; (L^\infty(\Omega))^N)$. Here we only suppose that $\vec{V}$ belongs to $L^{\tilde{m}}(0, T; (L^m(\Omega))^N)$, for some $\tilde{m} > 2$, $m > 2$ satisfying $\frac{1}{\tilde{m}} + \frac{N}{2m} \leq \frac{1}{2}$ (for $N = 2$ or $N = 3$, the limit cases $\frac{1}{\tilde{m}} + \frac{N}{2m} = \frac{1}{2}$, with $\tilde{m} = m = 4$ if $N = 2$, and $\tilde{m} = 8$ and $m = 4$ if $N = 3$, correspond to the regularity of strong solutions of the Navier–Stokes equations).

These assumptions are sufficient to next study problems where the heat equation is coupled with the Navier–Stokes equations in the two following cases [20]:

- the Boussinesq system linearized at $(z, \vec{U})$, where $(z, \vec{U})$ has the same regularity as strong solutions of the Boussinesq system;
- the two-dimensional nonlinear Boussinesq system.

As it is shown in [13], even in the case when $\vec{V} \equiv 0$, studying equation (1) is not completely obvious. (In [13], the domain is supposed to be a 3-dimensional cylindrical domain, and taking advantage of the particular form of the domain, the equation is split into a 2-dimensional elliptic equation with measures as data, and a heat equation with regular source terms.)

Here we shall use new regularity results for parabolic equations with measures as data obtained in [21], where we have studied optimal control problems with controls localized on thin structures for semilinear parabolic equations.

In the present paper, we first study the control problem

$$(P_1) \qquad \inf\{I(y, u) \mid (y, u) \in L^\kappa(0, T; W^{1,\kappa}(\Omega)) \times K_U,\ (y, u) \text{ satisfies (1)}\},$$

where

$$I(y, u) = C_Q \int_Q |\nabla y - V_d|^\kappa \mathrm{d}x\,\mathrm{d}t + C_\Omega \int_\Omega |y(T) - y_d|^\theta \mathrm{d}x + C_\gamma \int_0^T \left(\int_\gamma |u|^\sigma \mathrm{d}\zeta\right)^{\frac{q}{\sigma}} \mathrm{d}t.$$

(C_Q, C_Ω, C_γ are nonnegative constants.)

Next, we consider the case when γ is a point x_0 (that is $\delta_\gamma = \delta_{x_0}$. A finite union of points can be considered as well). In this case we are interested in characterizing the best location x_0 which minimizes the distance to an observed profile of temperature. The problem is formulated as follows:

$$(P_2) \qquad \inf\{J(y_{u,x_0}, u, x_0) \mid (y_{u,x_0}, u, x_0) \in L^1(0, T; W^{1,1}(\Omega)) \times K_U \times K_{\overline{\Omega}},\ (y, u, x_0) \text{ satisfies (1)}\},$$

where

$$J(y_{u,x_0}, u) = \int_\Omega |y_{u,x_0}(T) - y_d|^\theta \mathrm{d}x, \quad \text{and} \quad K_{\overline{\Omega}} \subset \overline{\Omega}\ .$$

This problem can be related to the identification of sources of pollution (see [16]).

Also mention that the techniques we have developed for parabolic equations with measures as data can be adapted to study the corresponding stationnary elliptic equations with measures as data. In both cases (elliptic and parabolic) these new results can be useful to tackle optimal control problems with point observations. In this case the Dirac measures are involved in the adjoint equations [9]. For a review on control problems with pointwise controls we refer to [17] (see also [3]).

Let us briefly present the difficulties encountered in studying (P_1) and (P_2). Equation (1) is an equation with measures as data which may be studied by the transposition method [6]. However the regularity results in the literature [6, 18] are not sufficient to deal with the control problems (P_1) and (P_2).

To obtain optimal regularity results for a convection-diffusion equation of the form (1), or for the adjoint equation associated with (P_1), throughout the paper the idea consists in studying equations firstly when $\vec{V} \equiv 0$, and next, by using a fixed point method, extending theses results to the general case. The fixed point method is developed in details in the proof of Proposition 2.7, and is next used for different propositions in the paper. We prove regularity results for the state equation in Section 2. The control problem (P_1) is studied in Section 3. The main difficulty to obtain optimality conditions for (P_1) is to prove that the trace of the adjoint state on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\sigma'}(\gamma))$. The adjoint equation for (P_1) is of the form

$$-\frac{\partial p}{\partial t} + Ap - \vec{V}\cdot\nabla p = -\operatorname{div}\vec{h} \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = \vec{h}\cdot\vec{n} \text{ on } \Sigma, \qquad p(T) = p_T \text{ in } \Omega.$$

Still using the fixed point method described above, we study the minimal regularity required on $\vec{h}$ and p_T to have $p|_{\gamma\times]0,T[} \in L^{q'}(0,T;L^{\sigma'}(\gamma))$ (Ths. 3.2, 3.3). Since in the adjoint equation $\vec{h}$ is equal to $\kappa C_Q|\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)$, where y_u is the solution to (1) corresponding to the optimal control u, and p_T is equal to $\theta C_\Omega|y_u(T) - y_d|^{\theta-2}(y_u(T) - y_d)$, the conditions on $\vec{h}$ and p_T to have $p|_{\gamma\times]0,T[} \in L^{q'}(0,T;L^{\sigma'}(\gamma))$, are satisfied under additional conditions on κ and θ (these conditions are stated in assumptions (A9, A10)). Optimality conditions for (P_1) are obtained in Theorem 3.5.

The control problem (P_2) is studied in Section 4. To obtain optimality conditions for (P_2), we prove that the adjoint state belongs to $L^1(0,T;C^{1,\nu}(\overline{\Omega}))$ (Th. 4.2). Next, we are able to characterize the optimal location of a pointwise control (Th. 4.3).

Numerical experiments for the computation of optimal solutions u for (P_1), and optimal pairs (u, x_0) for (P_2) are reported in [20] (Chap. 6).

2. State equation

2.1. Notation and assumptions

We make the following assumptions on the data.

(A1) The elliptic operator A is defined by $Ay = -\sum_{i,j=1}^{N} D_i(a_{ij}(x)D_jy) + a_0(x)y$. The coefficient a_0 is positive and belongs to $C(\overline{\Omega})$, the coefficients a_{ij} belong to $C^{1,\nu}(\overline{\Omega})$ with $0 < \nu \leq 1$, $a_{ij} = a_{ji}$, and they satisfy

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq m_0|\xi|^2 \quad \text{for every } \xi \in R^N \text{ and every } x \in \overline{\Omega}, \quad \text{with } m_0 > 0.$$

(A2) Γ is of class C^∞, and γ is a submanifold in $\overline{\Omega}$ of dimension $D \leq N-2$, of class $C^{\bar{k}}$ with $\bar{k} = \max\big(2, [\frac{N-D}{\sigma'}] + 1\big)$.

(A3) $\vec{V}$ belongs to $L^{\tilde{m}}(0,T;(L^m(\Omega))^N)$ and satisfies

$$\operatorname{div}\vec{V}=0 \text{ in } Q,\quad \vec{V}\cdot\vec{n}=0 \text{ on } \Sigma,\quad 2<\tilde{m}<\infty,\quad 2\le N<m<\infty,\quad \frac{1}{\tilde{m}}+\frac{N}{2m}\le\frac{1}{2}\,,$$

where "div" denotes the divergence operator with respect to $x\in\Omega$.

(A4) u belongs to $L^q(0,T;L^\sigma(\gamma))$ with $q\ge 2$, $\sigma\ge\frac{N}{N-1}$.

(A5) y_0 belongs to $L^\rho(\Omega)$ with $\rho=\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$.

(A6) K_U is a closed convex subset of $L^q(0,T;L^\sigma(\gamma))$. Either K_U is bounded in $L^q(0,T;L^\sigma(\gamma))$, or $C_\gamma>0$.

(A7) The function V_d belongs to $L^\kappa(0,T;(L^\kappa(\Omega))^N)$, the function y_d belongs to $L^\theta(\Omega)$, with $\kappa>1$, $\theta>1$.

Remark. For simplicity we have supposed that Γ is of class C^∞, but the results of the paper can be extended to less regular domains by using the techniques of [7] (Prop. 5).

Throughout the paper, we denote by T_γ (respectively $T_{\gamma\times]0,T[}$, T_Σ) the trace mapping on γ (respectively on $\gamma\times]0,T[$, on Σ). We denote by C, C_i, K, K_i for $i\in\mathbb{N}$, various constants depending on known quantities. The same letter may be used for different constants.

In [21] we have studied equation (1) in the case when $\vec{V}\equiv 0$. By the transposition method, we have proven the following regularity results.

Proposition 2.1. [21] *(Prop. 2.3) Suppose that $\vec{V}\equiv 0$ and $y_0\equiv 0$. Equation (1) admits a unique solution y_u in $L^1(0,T;W^{1,1}(\Omega))$. The mapping $u\mapsto y_u$ is continuous from $L^q(0,T;L^\sigma(\gamma))$ into $L^{\delta_1}(0,T;W^{1,d_1}(\Omega))$ for every (δ_1,d_1) satisfying:*

$$\begin{aligned}
&q\le\delta_1,\quad \sigma\le d_1<\frac{N}{N-\frac{D}{\sigma'}-1},\quad \frac{N-D}{2}+\frac{D}{2\sigma}+\frac{1}{q}<\frac{1}{\delta_1}+\frac{N}{2d_1}+\frac{1}{2},\qquad \text{if } \sigma<\frac{N-D}{N-D-1}\,,\\
&q\le\delta_1,\quad 1<d_1<\frac{N-D}{N-D-1},\quad \frac{N-D}{2}+\frac{1}{q}<\frac{1}{\delta_1}+\frac{N-D}{2d_1}+\frac{1}{2},\qquad \text{if } \sigma\ge\frac{N-D}{N-D-1}.
\end{aligned}\tag{2}$$

The mapping that associates y_u with u is continuous from $L^q(0,T;L^\sigma(\gamma))$ into $L^\infty(0,T;L^r(\Omega))$ for every $1\le r<\inf\left\{\frac{N-D}{N-D-\frac{2}{q'}},\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$. Moreover, y_u belongs to $C([0,T];L^r_w(\Omega))$ for every $1\le r<\inf\left\{\frac{N-D}{N-D-\frac{2}{q'}},\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$. ($C([0,T];L^r_w(\Omega))$ denotes the space of continuous functions from $[0,T]$ into $L^r(\Omega)$, endowed with its weak topology.)

Remark. The conditions expressed in (2) can be written in the following shorter form

$$q\le\delta_1,\quad \sigma\le d_1,\quad \frac{N-D}{2}+\frac{D}{2\sigma}+\frac{1}{q}<\frac{1}{\delta_1}+\frac{N}{2d_1}+\frac{1}{2}\,\cdot\tag{3}$$

Indeed if $d_1\ge\sigma$, taking $\delta_1=q$ in (3), we obtain:

$$\frac{N-D}{2}+\frac{D}{2d_1}<\frac{N-D}{2}+\frac{D}{2\sigma}<\frac{N}{2d_1}+\frac{1}{2}.$$

Therefore we have $\sigma\le d_1<\frac{N-D}{N-D-1}$. This means that (3) cannot be used if $\sigma\ge\frac{N-D}{N-D-1}$. Now, if $u\in L^q(0,T;L^\sigma(\gamma))$ with $\sigma\ge\frac{N-D}{N-D-1}$, then $u\in L^q(0,T;L^{\hat\sigma}(\gamma))$ for any $\hat\sigma<\frac{N-D}{N-D-1}$. Therefore y belongs to $L^{\delta_1}(0,T;W^{1,d_1}(\Omega))$ for every (δ_1,d_1) satisfying:

$$q\le\delta_1,\quad \hat\sigma=d_1<\frac{N-D}{N-D-1}\ \text{and}\ \frac{N-D}{2}+\frac{D}{2\hat\sigma}+\frac{1}{q}<\frac{1}{\delta_1}+\frac{N}{2d_1}+\frac{1}{2},$$

that is

$$\frac{N-D}{2}+\frac{1}{q}<\frac{1}{\delta_1}+\frac{N-D}{2d_1}+\frac{1}{2},$$

which is nothing else than the second condition in (2). Finally observe that the condition $d_1 < \frac{N}{N-\frac{D}{\sigma'}-1}$ follows from (3) by taking $\delta_1 = q$. Even if (2) and (3) are equivalent, using (2) avoids forgetting the condition $d_1 < \frac{N-D}{N-D-1}$.

Proposition 2.2. [21] *(Prop. 2.5) Suppose that $\vec{V} \equiv 0$ and $u \equiv 0$. Let y_0 belong to $L^\rho(\Omega)$ with $\rho = \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$, and let y be the solution to equation (1). The mapping $y_0 \mapsto y$ is continuous from $L^\rho(\Omega)$ into $L^\infty(0,T;L^\rho(\Omega)) \cap L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$, for every (δ_2, d_2) satisfying*

$$1<\delta_2<2, \quad \rho \le d_2 < \frac{N\rho}{N-\rho}, \quad 1+\frac{N}{2\rho}<\frac{1}{\delta_2}+\frac{N}{2d_2}+\frac{1}{2}\,\cdot \tag{4}$$

In this section we want to extend these results to the case when $\vec{V}$ satisfies (A3) (Props. 2.7 and 2.8). Due to the weak regularity of $\vec{V}$, we cannot use the transposition method. We obtain existence and regularity results for equation (1) by using a fixed point method. For this, we need some preliminary estimates that are stated below.

2.2. Preliminary estimates

First recall some results for analytic semigroups. We denote by $\tilde{A}$ the operator defined by

$$\mathcal{D}(\tilde{A}) = \left\{ y \in C^2(\overline{\Omega}) \mid \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma \right\}, \quad \tilde{A}y = Ay.$$

For $1 \le \ell < \infty$, we denote by A_ℓ the closure of $\tilde{A}$ in $L^\ell(\Omega)$. The operator $-A_\ell$ is the generator of a strongly continuous analytic semigroup $S_\ell(t)_{t\ge 0}$ in $L^\ell(\Omega)$ [2]. For $1 < \ell < \infty$ the domain of A_ℓ is $\mathcal{D}(A_\ell) = \left\{ y \in W^{2,\ell}(\Omega) \mid \frac{\partial y}{\partial n_A} = 0 \text{ on } \Gamma \right\}$. For any $1 \le \ell < \infty$, 0 belongs to the resolvent of $-A_\ell$ and there exists $\delta > 0$ such that $\operatorname{Re}\sigma(A_\ell) \ge \delta$ (it is a consequence of (A1) and of the fact that $\sigma(A_\ell)$ is independent of ℓ). Therefore, for $\alpha > 0$, there exists a constant $K = K(\ell, \alpha)$ such that

$$\|A_\ell^\alpha S_\ell(t)\varphi\|_{L^\ell(\Omega)} \le K t^{-\alpha} \|\varphi\|_{L^\ell(\Omega)},$$

for every $t > 0$ and every $\varphi \in L^\ell(\Omega)$ (see [14,22], A_ℓ^α is the α-power of A_ℓ). Thanks to this result the following lemma can be established.

Lemma 2.1. [2,24] *For every $1 \le \ell \le \lambda \le \infty$ with $\ell < \infty$, there exists a constant $K_1 = K_1(\lambda, \ell)$ such that*

$$\|S_\ell(t)\varphi\|_{L^\lambda(\Omega)} \le K_1 t^{-\frac{N}{2}(\frac{1}{\ell}-\frac{1}{\lambda})} \|\varphi\|_{L^\ell(\Omega)} \tag{5}$$

for every $\varphi \in L^\ell(\Omega)$ and every $t > 0$. For every $1 \le \ell \le \lambda \le \infty$ with $\ell < \infty$, and every $\alpha > 0$, there exists a constant $K_2 = K_2(\lambda, \ell, \alpha)$ such that

$$\|A_\ell^\alpha S_\ell(t)\varphi\|_{L^\lambda(\Omega)} \le K_2 t^{-\frac{N}{2}(\frac{1}{\ell}-\frac{1}{\lambda})-\alpha} \|\varphi\|_{L^\ell(\Omega)} \tag{6}$$

for every $\varphi \in L^\ell(\Omega)$ and every $t > 0$.

Proposition 2.3. [1] *(Th. 7.58) If assumption (A2) is satisfied, then T_γ is a continuous linear operator from $W^{r,p}(\Omega)$ into $L^q(\gamma)$ for all (r,p,q) such that $0 \le r \le \bar{k}$, $0 < N - rp < D$, $p \le q < \frac{Dp}{N-rp}$.*

Remark. Theorem 7.58 in [1] is stated with Ω and γ replaced by $\mathbb{R}^N$ and $\mathbb{R}^D$, but as it is noticed in [1], just before Theorem 7.58, the statement is also true for domains by using coverings, partitions of unity, and diffeomorphisms of class $C^{\bar{k}}$.

Proposition 2.4. [21] *(Prop. 2.1) Let ϕ be in $\mathcal{D}(\Omega)$, and w be the solution of the Cauchy problem:*

$$\frac{\partial w}{\partial t} + Aw = 0 \ \text{in}\ Q, \qquad \frac{\partial w}{\partial n_A} = 0 \ \text{on}\ \Sigma, \qquad w(0) = A^{\alpha}_{d'}\phi \ \text{in}\ \Omega, \tag{7}$$

where $0 \le \alpha \le 1$. The mapping that associates w with ϕ is continuous from $L^{d'}(\Omega)$ into $L^i(0,T;W^{r,j}(\Omega))$ for all (α, q, j, i, r, d) satisfying:

$$2 > r \ge 0, \quad i \ge 1, \quad j \ge d', \quad \alpha + \frac{r}{2} + \frac{N}{2d'} < \frac{1}{i} + \frac{N}{2j}\cdot \tag{8}$$

Proposition 2.5. *Let $\vec{f}$ be in $(\mathcal{D}(Q))^N$, and z be the solution of the equation:*

$$\frac{\partial z}{\partial t} + Az = \ \text{div}\ \vec{f} \ \text{in}\ Q, \qquad \frac{\partial z}{\partial n_A} = 0 \ \text{on}\ \Sigma, \qquad z(0) = 0 \ \text{in}\ \Omega. \tag{9}$$

The mapping that associates z with $\vec{f}$ is continuous from $L^{\tilde{\eta}}(0,T;(L^{\eta}(\Omega))^N)$ into $L^{\tilde{\eta}}(0,T;W^{1,\eta}(\Omega))$. It is also continuous from $L^{\tilde{\eta}}(0,T;(L^{\eta}(\Omega))^N)$ into $L^{\delta}(0,T;W^{r,d}(\Omega))$ for all $(r, \tilde{\eta}, \eta, \delta, d)$ satisfying:

$$0 \le r < 1,\ 1 \le \tilde{\eta} \le \delta,\ 1 < \eta \le d,\ \frac{r}{2} + \frac{1}{\tilde{\eta}} + \frac{N}{2\eta} < \frac{1}{\delta} + \frac{N}{2d} + \frac{1}{2}\cdot \tag{10}$$

The mapping that associates z with $\vec{f}$ is continuous from $L^{\tilde{\eta}}(0,T;(L^{\eta}(\Omega))^N)$ into $L^{\delta}(0,T;L^d(\Omega))$ for all $(\tilde{\eta}, \eta, \delta, d)$ satisfying:

$$1 < \tilde{\eta} < \delta,\ 1 < \eta \le d,\ \frac{1}{\tilde{\eta}} + \frac{N}{2\eta} \le \frac{1}{\delta} + \frac{N}{2d} + \frac{1}{2}\cdot \tag{11}$$

Remark. Since $\vec{f}$ belongs to $(\mathcal{D}(Q))^N$, equation (9) is defined in a classical sense.

Proof. The first continuity results are already proved in [21] and [27]. Here we only prove the second one. Let w be the solution of the Cauchy problem (7) when $\alpha = 0$, then:

$$\begin{aligned}
\int_\Omega z(x,t)\phi(x)\,\mathrm{d}x &= \int_0^t \left\{ \frac{d}{d\tau}\int_\Omega w(x,t-\tau)z(x,\tau)\,\mathrm{d}x \right\} \mathrm{d}\tau \\
&= \int_0^t \int_\Omega \left\{ -\frac{\partial w}{\partial t}(x,t-\tau)z(x,\tau) + w(x,t-\tau)\frac{\partial z}{\partial t}(x,\tau) \right\} \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int_\Omega \left\{ Aw(x,t-\tau)z(x,\tau) - w(x,t-\tau)Az(x,\tau) + \ \text{div}\ \vec{f}(x,\tau)w(x,t-\tau) \right\} \mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t \int_\Omega \nabla w(x,t-\tau)\cdot \vec{f}(x,\tau)\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Using (6) in Lemma 2.1 with $1 \leq d' \leq \eta' \leq \infty, d' < \infty$, we have:

$$\begin{aligned}\|z(t)\|_{L^d(\Omega)} &= \sup\left\{\left|\int_\Omega z(x,t)\phi(x)\,\mathrm{d}x\right|, \|\phi\|_{L^{d'}(\Omega)} = 1\right\}\\ &= \sup\left\{\left|\int_0^t\int_\Omega \nabla w(x,t-\tau)\cdot\vec{f}(x,\tau)\mathrm{d}x\mathrm{d}\tau\right|, \|\phi\|_{L^{d'}(\Omega)} = 1\right\}\\ &\leq K\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}\|\vec{f}(\tau)\|_{(L^\eta(\Omega))^N}\mathrm{d}\tau.\end{aligned}$$

The mapping $\tau \mapsto \|\vec{f}(\tau)\|_{(L^\eta(\Omega))^N}$ belongs to $L^{\tilde{\eta}}(0,T)$. We denote by $L^i_*(0,T)$ the weak-$L^i(0,T)$ space defined as follows ([23], p. 30):

$$L^i_*(0,T) := \left\{g : (0,T) \mapsto R \mid g \text{ is measurable, and } \sup_{\xi>0}(\xi^i\mathcal{L}^1\{t \,|\, |g(t)| > \xi\}) < \infty\right\},$$

where $\mathcal{L}^1$ denotes the Lebesgue measure on $(0,T)$. Then the mapping $t \mapsto t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}$ belongs to $L^i_*(0,T)$ with $i > 1$ defined by $\frac{1}{2}+\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d}) = \frac{1}{i}$. Due to (11), $1+\frac{1}{\delta} \geq \frac{1}{i}+\frac{1}{\tilde{\eta}}$. From the generalized Young inequality, it follows that the mapping $t \mapsto \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}\|\vec{f}(\tau)\|_{(L^\eta(\Omega))^N}\mathrm{d}\tau$ belongs to $L^\delta(0,T)$, and the proof is complete. □

Remark. Since $(\mathcal{D}(Q))^N$ is dense in $L^{\tilde{\eta}}(0,T;(L^\eta(\Omega))^N)$, the regularity result of Proposition 2.5 is also true for the solution z to the variational equation

$$-\int_Q z\frac{\partial\phi}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_Q\sum_{i,j=1}^N a_{ij}D_jzD_i\phi\,\mathrm{d}x\mathrm{d}t + \int_Q a_0z\phi\,\mathrm{d}x\mathrm{d}t\,\mathrm{d}x\mathrm{d}t = -\int_Q \vec{f}\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t\ ,$$

for all $\phi \in C^1(\overline{Q})$ such that $\phi(T) = 0$ on $\overline{\Omega}$, where $\vec{f} \in L^{\tilde{\eta}}(0,T;(L^\eta(\Omega))^N)$.

Proposition 2.6. *Let f be in $\mathcal{D}(Q)$, and z be the solution of the equation:*

$$\frac{\partial z}{\partial t} + Az = f \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega. \tag{12}$$

The mapping that associates z with f is continuous from $L^{\tilde{\eta}}(0,T;L^\eta(\Omega))$ into $L^\delta(0,T;W^{r,d}(\Omega))$ for all $(r,\tilde{\eta},\eta,\delta,d)$ satisfying:

$$0 \leq r < 2,\ 1 \leq \tilde{\eta} \leq \delta,\ 1 < \eta \leq d,\ \frac{r}{2}+\frac{1}{\tilde{\eta}}+\frac{N}{2\eta} < \frac{1}{\delta}+\frac{N}{2d}+1. \tag{13}$$

The mapping that associates z with f is continuous from $L^{\tilde{\eta}}(0,T;L^\eta(\Omega))$ into $L^\delta(0,T;W^{k,d}(\Omega))$ for all $(k,\tilde{\eta},\eta,\delta,d)$ satisfying:

$$k = 0 \text{ or } k = 1,\ 1 < \tilde{\eta} < \delta,\ 1 < \eta \leq d,\ \frac{k}{2}+\frac{1}{\tilde{\eta}}+\frac{N}{2\eta} \leq \frac{1}{\delta}+\frac{N}{2d}+1. \tag{14}$$

Proof. The first continuity result is already proved in [21]. Here we only prove the second one. Let us set $\alpha = \frac{k}{2}$. We have $D(A_d^0) = L^d(\Omega)$ and $D(A_d^{\frac{1}{2}}) = W^{1,d}(\Omega)$. Let w be the solution of the Cauchy problem (7), then:

$$\int_\Omega A_d^\alpha z(x,t)\phi(x)\,\mathrm{d}x = \int_0^t\int_\Omega A_{d'}^\alpha S_{d'}(t-\tau)\phi(x)f(x,\tau)\mathrm{d}x\mathrm{d}\tau.$$

Using (6) in Lemma 2.1 with $1 \le d' \le \eta' \le \infty, d' < \infty$, we have:

$$\begin{aligned}\|z(t)\|_{W^{k,d}(\Omega)} &\le C\|A_d^\alpha z(t)\|_{L^d(\Omega)} \le C \sup\left\{\left|\int_\Omega A_d^\alpha z(x,t)\phi(x)\,\mathrm{d}x\right|, \|\phi\|_{L^{d'}(\Omega)} = 1\right\}\\ &= C \sup\left\{\left|\int_0^t\int_\Omega A_{d'}^\alpha S_{d'}(t-\tau)\phi f(\tau)\,\mathrm{d}x\mathrm{d}\tau\right|, \|\phi\|_{L^{d'}(\Omega)} = 1\right\}\\ &\le C\int_0^t (t-\tau)^{-\alpha-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}\|f(\tau)\|_{L^\eta(\Omega)}\mathrm{d}\tau.\end{aligned}$$

The mapping $\tau \mapsto \|f(\tau)\|_{L^\eta(\Omega)}$ belongs to $L^{\tilde\eta}(0,T)$. The mapping $t \mapsto t^{-\alpha-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}$ belongs to $L_*^i(0,T)$ with $i > 1$ defined by $\alpha + \frac{N}{2}(\frac{1}{\eta} - \frac{1}{d}) = \frac{1}{i}$. Due to (14), $1 + \frac{1}{\delta} \ge \frac{1}{i} + \frac{1}{\tilde\eta}$. From the generalized Young inequality, it follows that the mapping $t \mapsto \int_0^t (t-\tau)^{-\alpha-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{d})}\|f(\tau)\|_{L^\eta(\Omega)}\mathrm{d}\tau$ belongs to $L^\delta(0,T)$. □

2.3. State equation

In this section, we prove regularity results for equation (1). We shall say that a function $y \in L^\delta(0,T;W^{1,d}(\Omega))$ is a weak solution to equation (1) if and only if $\delta \ge \tilde m'$, $d \ge m'$ and

$$-\int_Q y\frac{\partial\phi}{\partial t}\,\mathrm{d}x\mathrm{d}t + \int_Q\sum_{i,j=1}^N a_{ij}(x)D_jyD_i\phi\,\mathrm{d}x\mathrm{d}t + \int_Q a_0y\phi\,\mathrm{d}x\mathrm{d}t + \int_Q \vec V\cdot\nabla y\phi\,\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T\int_\gamma u\phi\,\mathrm{d}\zeta\,\mathrm{d}t + \int_\Omega \phi(0)y_0\,\mathrm{d}x \quad \text{for all } \phi \in C^1(\overline{Q}) \text{ such that } \phi(T) = 0 \text{ on } \overline{\Omega}. \tag{15}$$

To simplify the writing, throughout the sequel, we suppose that γ is included in Γ, but the results are true for $\gamma \subset \overline{\Omega}$.

Proposition 2.7. *We consider the equation*

$$\frac{\partial y}{\partial t} + Ay + \vec V\cdot\nabla y = 0 \text{ in } Q, \qquad \frac{\partial y}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma, \qquad y(0) = 0 \text{ in } \Omega. \tag{16}$$

Equation (16) admits a unique solution y in $L^{\hat\delta}(0,T;W^{1,\hat d}(\Omega))$ for all $(\hat\delta,\hat d)$ obeying $\tilde m' < \hat\delta \le q$, $m' < \hat d < \inf\left\{\frac{N}{N-\frac{D}{\sigma'}-1}, \frac{N-D}{N-D-1}\right\}$. Moreover, the mapping that associates y with u is continuous from $L^q(0,T;L^\sigma(\gamma))$ into $L^\delta(0,T;W^{1,d}(\Omega))$ for every (δ,d) satisfying:

$$\begin{aligned} &q \le \delta, \quad \sigma \le d < \frac{N}{N-\dfrac{D}{\sigma'}-1}, \quad \frac{N-D}{2} + \frac{D}{2\sigma} + \frac{1}{q} < \frac{1}{\delta} + \frac{N}{2d} + \frac{1}{2}, \qquad \text{if } \sigma < \frac{N-D}{N-D-1},\\ &q \le \delta, \quad m' < d < \frac{N-D}{N-D-1}, \quad \frac{N-D}{2} + \frac{1}{q} < \frac{1}{\delta} + \frac{N-D}{2d} + \frac{1}{2}, \qquad \text{if } \sigma \ge \frac{N-D}{N-D-1}.\end{aligned} \tag{17}$$

The mapping that associates y with u is continuous from $L^q(0,T;L^\sigma(\gamma))$ into $L^\infty(0,T;L^r(\Omega))$ for every $1 \le r < \inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$. Moreover, y belongs to $C([0,T];L_w^r(\Omega))$ for every $1 \le r < \inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$.

Proof. 1 - **Existence of a local solution.** Let us set $Q_{\bar t} := \Omega\times]0,\bar t[$, $\Sigma_{\bar t} := \Gamma\times]0,\bar t[$. Let (δ, d) be a pair obeying (17). By a fixed point method, we prove that the equation

$$\frac{\partial y}{\partial t} + Ay + \vec{V}\cdot\nabla y = 0 \text{ in } Q_{\bar t}, \qquad \frac{\partial y}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma_{\bar t}, \qquad y(0) = 0 \text{ in } \Omega, \tag{18}$$

admits a solution for $\bar t > 0$ small enough. Let ξ belong to $L^\delta(0,\bar t; W^{1,d}(\Omega))$, and y_ξ be the solution to the equation:

$$\frac{\partial y_\xi}{\partial t} + Ay_\xi = -\vec{V}\cdot\nabla\xi \text{ in } Q_{\bar t}, \quad \frac{\partial y_\xi}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma_{\bar t}, \quad y_\xi(0) = 0 \text{ in } \Omega. \tag{19}$$

Then $y_\xi = \hat y + \tilde y$, where $\hat y$ and $\tilde y$ are the solutions to the equations:

$$\frac{\partial \hat y}{\partial t} + A\hat y = 0 \text{ in } Q_{\bar t}, \quad \frac{\partial \hat y}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma_{\bar t}, \quad \hat y(0) = 0 \text{ in } \Omega,$$

$$\frac{\partial \tilde y}{\partial t} + A\tilde y = -\vec{V}\cdot\nabla\xi \text{ in } Q_{\bar t}, \quad \frac{\partial \tilde y}{\partial n_A} = 0 \text{ on } \Sigma_{\bar t}, \quad \tilde y(0) = 0 \text{ in } \Omega.$$

From Proposition 2.1, we know that $\hat y \in L^\delta(0,\bar t; W^{1,d}(\Omega))$. Since $\delta \ge q \ge 2 > \tilde m'$, and $d \ge \sigma \ge \frac{N}{N-1} > m'$, then $\frac{\tilde m\delta}{\tilde m+\delta} > 1$, $\frac{md}{m+d} > 1$, and $\vec V\cdot\nabla\xi \in L^{\frac{\tilde m\delta}{\tilde m+\delta}}(0,\bar t; L^{\frac{md}{m+d}}(\Omega))$. Due to Proposition 2.6, it follows that $\tilde y \in L^\delta(0,\bar t; W^{1,d}(\Omega))$. Thus y_ξ belongs to $L^\delta(0,\bar t; W^{1,d}(\Omega))$.

Let ξ_1 and ξ_2 belong to $L^\delta(0,\bar t; W^{1,d}(\Omega))$. Still with Proposition 2.6, we have

$$\|y_{\xi_1} - y_{\xi_2}\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))} \le C_1\|\vec V\|_{L^{\tilde m}(0,\bar t;L^m(\Omega)^N)}\|\xi_1-\xi_2\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))},$$

where C_1 can be chosen depending on T, but independent of $\bar t$. The mapping $t\mapsto C_1^{\tilde m}\int_0^t \|\vec V(x,\tau)\|^{\tilde m}_{L^m(\Omega)^N}\mathrm{d}\tau$ is absolutely continuous, then there exists $\bar t > 0$ such that $C_1(\int_t^{\min\{t+\bar t,T\}}\|\vec V(\cdot,\tau)\|^{\tilde m}_{L^m(\Omega)^N}\mathrm{d}\tau)^{\frac{1}{\tilde m}} \le C = \frac12$ for all $t\in[0,T]$. Thus the mapping $\xi\mapsto y_\xi$ is a contraction in the Banach space $L^\delta(0,\bar t; W^{1,d}(\Omega))$.

2 - **Estimate of the local solution.** Consider the sequence $(\xi_n)_n$ defined by $\xi_0 = 0$ and $\xi_n = y_{\xi_{n-1}}$. Then $(\xi_n)_n$ converges to the unique solution y of (18). From the definition of y_{ξ_0}, and due to Proposition 2.1, we deduce that

$$\|y_{\xi_0}\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))} \le K\|u\|_{L^q(0,\bar t;L^\sigma(\gamma))}.$$

Moreover for all n we have

$$\|y_{\xi_n}\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))} \le 2\|y_{\xi_0}\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))} \le K\|u\|_{L^q(0,T;L^\sigma(\gamma))}.$$

By letting n goes to ∞, we obtain:

$$\|y\|_{L^\delta(0,\bar t;W^{1,d}(\Omega))} \le K\|u\|_{L^q(0,T;L^\sigma(\gamma))}. \tag{20}$$

3 - **Existence of a global solution.** We prove that a solution exists in $L^\delta(0,T;W^{1,d}(\Omega))$, by repeating the above process. Let $\hat y$ be the solution constructed on $(0,\bar t)$ in Step 2. Let $(\hat\xi_1,\hat\xi_2)$ belong to $L^\delta(\bar t, 2\bar t; W^{1,d}(\Omega))$. Define (ξ_1,ξ_2) belonging to $L^\delta(0,2\bar t;W^{1,d}(\Omega))$ by $\xi_1=\xi_2=\hat y$ on $(0,\bar t)$, and $\xi_1 = \hat\xi_1$, $\xi_2 = \hat\xi_2$ on $(\bar t, 2\bar t)$. We still denote by y_{ξ_i} the solution to equation (19) on $(0,2\bar t)$ corresponding to ξ_i for $i=1,2$. As in Step 2, we have:

$$\|y_{\xi_1}-y_{\xi_2}\|_{L^\delta(0,2\bar t;W^{1,d}(\Omega))} = \|y_{\xi_1}-y_{\xi_2}\|_{L^\delta(\bar t,2\bar t;W^{1,d}(\Omega))} \le \frac12\|\xi_1-\xi_2\|_{L^\delta(0,2\bar t;W^{1,d}(\Omega))}.$$

Thus the mapping $\xi \mapsto y_\xi$ admits a unique fixed point in the metric space $\{\pi \in L^\delta(0,2\bar{t};W^{1,d}(\Omega)) \mid \pi = \hat{y}$ on $]0,\bar{t}[\ \}$. We want to estimate the solution in $L^\delta(0,2\bar{t};W^{1,d}(\Omega))$. Let $(\xi_n)_n$ be the sequence defined by:

$$\xi_0 = \hat{y} \text{ on } (0,\bar{t}),\ \xi_0 = 0 \text{ on } (\bar{t},2\bar{t}), \qquad \xi_n = y_{\xi_{n-1}} \text{ on } (0,2\bar{t})\ .$$

Then $(\xi_n)_n$ converges to y in $L^\delta(0,2\bar{t};W^{1,d}(\Omega))$. From the properties of the fixed point, we have

$$\|y_{\xi_n} - y_{\xi_0}\|_{L^\delta(0,2\bar{t};W^{1,d}(\Omega))} \leq \|y_{\xi_0} - \xi_0\|_{L^\delta(0,2\bar{t};W^{1,d}(\Omega))}.$$

Since y_{ξ_0} is the solution to the equation

$$\frac{\partial y}{\partial t} + Ay = -\vec{V}\cdot\nabla\xi_0 \text{ in } Q_{2\bar{t}}, \quad \frac{\partial y}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma_{2\bar{t}}, \quad y(0) = 0 \text{ in } \Omega,$$

then

$$\|y_{\xi_0}\|_{L^\delta(0,2\bar{t};W^{1,d}(\Omega))} \leq K\left(1 + \|\vec{V}\|_{L^{\tilde{m}}(0,T;(L^m(\Omega))^N)}\right)\|u\|_{L^q(0,T;L^\sigma(\gamma))}.$$

Therefore, we have:

$$\|y\|_{L^\delta(0,2\bar{t};W^{1,d}(\Omega))} \leq K\left(1 + \|\vec{V}\|_{L^{\tilde{m}}(0,T;(L^m(\Omega))^N)}\right)\|u\|_{L^q(0,T;L^\sigma(\gamma))}. \tag{21}$$

4 - **Estimate of the global solution in** $L^\delta(0,T;W^{1,d}(\Omega))$. By induction, it is easy to prove that

$$\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \leq K_n\left(1 + \|\vec{V}\|_{L^{\tilde{m}}(0,T;(L^m(\Omega))^N)} + ... + \|\vec{V}\|^{n-1}_{L^{\tilde{m}}(0,T;(L^m(\Omega))^N)}\right)\|u\|_{L^q(0,T;L^\sigma(\gamma))}, \tag{22}$$

where $n = [\frac{T}{\bar{t}}] + 1$, and where K_n depends on n (observe that n depends on $\bar{t}$, and $\bar{t}$ depends on $\vec{V}$). Therefore, there exists a constant $\tilde{C}$ depending on $\vec{V}$ and T, such that

$$\|y\|_{L^\delta(0,T;W^{1,d}(\Omega))} \leq \tilde{C}\|u\|_{L^q(0,T;L^\sigma(\gamma))}.$$

5 - **Estimate in** $L^\infty(0,T;L^r(\Omega))$. Observe that $y = y_1 + y_2$, where y_1 and y_2 are the solutions to the equations:

$$\frac{\partial y_1}{\partial t} + Ay_1 = 0 \text{ in } Q, \qquad \frac{\partial y_1}{\partial n_A} = u\delta_\gamma \text{ on } \Sigma, \qquad y_1(0) = 0 \text{ in } \Omega,$$

$$\frac{\partial y_2}{\partial t} + Ay_2 = -\vec{V}\cdot\nabla y \text{ in } Q, \qquad \frac{\partial y_2}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y_2(0) = 0 \text{ in } \Omega.$$

Due to Proposition 2.1, y_1 belongs to $L^\infty(0,T;L^r(\Omega))$ for every $1 \leq r < \inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\} \leq \inf\left\{\frac{N-D}{N-D-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$.

If $\sigma < \frac{N-D}{N-D-1}$, then $\inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\} = \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$, and $\sigma < \frac{N}{N-\frac{D}{\sigma'}-1}$. Let r satisfy $\sigma < \frac{N}{N-\frac{D}{\sigma'}-1} < r < \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$. Observe that $\frac{1}{r} < \frac{1}{m}+\frac{1}{\sigma}$, $\frac{1}{q}+\frac{N}{2r}-\frac{N}{2m} < \frac{N}{2r}+\frac{1}{2}$, and $\frac{1}{q}+\frac{1}{2}(N-\frac{D}{\sigma'}-1) = \frac{1}{2}(N-\frac{D}{\sigma'}-\frac{2}{q'})+\frac{1}{2} < \frac{N}{2r}+\frac{1}{2}$. Therefore, there exists d satisfying

$$\sup\left\{\frac{N}{2r} - \frac{N}{2m}\,,\, \frac{1}{2}\left(N - \frac{D}{\sigma'} - 1\right)\right\} < \frac{N}{2d} < \inf\left\{\frac{N}{2\sigma}\,,\, \frac{N}{2r} + \frac{1}{2} - \frac{1}{q}\right\}.$$

Since $\frac{1}{2}(N-\frac{D}{\sigma'}-1)<\frac{N}{2d}$, applying (17), we deduce that y belongs to $L^q(0,T;W^{1,d}(\Omega))$. Set $\frac{1}{\tilde\ell}=\frac{1}{\tilde m}+\frac{1}{q}$, and $\frac{1}{\ell}=\frac{1}{m}+\frac{1}{d}$. Then $\tilde\ell>1$, $\ell>1$, and $\vec V\cdot\nabla y$ belongs to $L^{\tilde\ell}(0,T;L^{\ell}(\Omega))$. Since $\frac{1}{r}-\frac{1}{m}<\frac{1}{d}$, we have $r\ge\ell$. From $\frac{N}{2d}<\frac{N}{2r}+\frac{1}{2}-\frac{1}{q}$ and $\frac{1}{\tilde m}+\frac{N}{2m}\le\frac{1}{2}$, it follows that $\frac{1}{\tilde\ell}+\frac{N}{2\ell}<\frac{N}{2r}+1$. Due to Proposition 2.6, we deduce that y_2 belongs to $L^\infty(0,T;L^r(\Omega))$ for all $r<\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$.

If $\sigma\ge\frac{N-D}{N-D-1}$, then $\inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}},\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}=\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}$. Let r satisfy $1<r<\frac{N}{N-\frac{D}{(N-D)}-\frac{2}{q'}}$. We have $q\ge 2$, $\frac{N-D-1}{N-D}<\frac{1}{m'}$, and $\frac{N(N-D-1)}{(N-D)}<\frac{N}{r}+1-\frac{2}{q}$. Thus there exists d satisfying

$$\sup\left\{\frac{N}{2r}-\frac{N}{2m},\frac{N(N-D-1)}{2(N-D)}\right\}<\frac{N}{2d}<\inf\left\{\frac{N}{2m'},\frac{N}{2r}+\frac{1}{2}-\frac{1}{q}\right\}.$$

Applying (17), we deduce that y belongs to $L^q(0,T;W^{1,d}(\Omega))$. Set $\frac{1}{\tilde\ell}=\frac{1}{\tilde m}+\frac{1}{q}$, and $\frac{1}{\ell}=\frac{1}{m}+\frac{1}{d}$. Then $\tilde\ell>1$, $\ell>1$, and $\vec V\cdot\nabla y$ belongs to $L^{\tilde\ell}(0,T;L^{\ell}(\Omega))$. Since $\frac{1}{r}-\frac{1}{m}<\frac{1}{d}$, we have $r\ge\ell$. From $\frac{N}{2d}<\frac{N}{2r}+\frac{1}{2}-\frac{1}{q}$ and $\frac{1}{\tilde m}+\frac{N}{2m}\le\frac{1}{2}$, it follows that $\frac{1}{\tilde\ell}+\frac{N}{2\ell}<\frac{N}{2r}+1$. Due to Proposition 2.6, we deduce that y_2 belongs to $L^\infty(0,T;L^r(\Omega))$ for all $r<\frac{N}{N-\frac{D}{(N-D)}-\frac{2}{q'}}$.

Using the same argument as in the proof of Proposition 2.1 in [10], we can prove that y belongs to $C([0,T];L^r_w(\Omega))$ for every $1\le r<\inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}},\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$.

6 - **Uniqueness.** If we consider the equation

$$\frac{\partial y}{\partial t}+Ay+\vec V\cdot\nabla y=0\text{ in }Q,\qquad\frac{\partial y}{\partial n_A}=0\text{ on }\Sigma,\qquad y(0)=0\text{ in }\Omega,$$

we can apply the above fixed point method to prove that $y\equiv 0$ is the unique solution to this equation in $L^{\hat\delta}(0,T;W^{1,\hat d}(\Omega))$, for all $(\hat\delta,\hat d)$ obeying $\tilde m'<\hat\delta\le q$, and all $m'<\hat d<\inf\{\frac{N}{N-\frac{D}{\sigma'}-1},\frac{N-D}{N-D-1}\}$. □

Proposition 2.8. *Consider the equation*

$$\frac{\partial y}{\partial t}+Ay+\vec V\cdot\nabla y=0\ \textit{in}\ Q,\qquad\frac{\partial y}{\partial n_A}=0\ \textit{on}\ \Sigma,\qquad y(0)=y_0\ \textit{in}\ \Omega,\tag{23}$$

where y_0 belongs to $L^\rho(\Omega)$ with $\rho=\frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}$. The mapping that associates the solution y with y_0 is continuous from $L^\rho(\Omega)$ into $L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$ for every (δ_2,d_2) satisfying

$$\tilde m'<\delta_2<2,\quad\rho\le d_2<\frac{N\rho}{N-\rho},\quad 1+\frac{N}{2\rho}<\frac{1}{\delta_2}+\frac{N}{2d_2}+\frac{1}{2}\,.\tag{24}$$

The mapping that associates the solution y with y_0 is continuous from $L^\rho(\Omega)$ into $L^\infty(0,T;L^r(\Omega))$ for every $1\le r<\rho$. Moreover, y belongs to $C([0,T];L^r(\Omega))$ for every $1\le r<\rho$.

Proof. 1 - We still use a fixed point method. Let (δ_2,d_2) be a pair satisfying (24). Let $\xi\in L^{\delta_2}(0,\bar t;W^{1,d_2}(\Omega))$, and y_ξ be the solution to the equation:

$$\frac{\partial y_\xi}{\partial t}+Ay_\xi=-\vec V\cdot\nabla\xi\text{ in }Q_{\bar t},\quad\frac{\partial y_\xi}{\partial n_A}=0\text{ on }\Sigma_{\bar t},\quad y_\xi(0)=y_0\text{ in }\Omega.\tag{25}$$

Then $y_\xi = \hat{y} + \tilde{y}$, where $\hat{y}$ and $\tilde{y}$ satisfy the following equations:

$$\frac{\partial \hat{y}}{\partial t} + A\hat{y} = 0 \text{ in } Q_{\bar{t}}, \quad \frac{\partial \hat{y}}{\partial n_A} = 0 \text{ on } \Sigma_{\bar{t}}, \quad \hat{y}(0) = y_0 \text{ in } \Omega,$$

$$\frac{\partial \tilde{y}}{\partial t} + A\tilde{y} = -\vec{V}\cdot\nabla\xi \text{ in } Q_{\bar{t}}, \quad \frac{\partial \tilde{y}}{\partial n_A} = 0 \text{ on } \Sigma_{\bar{t}}, \quad \tilde{y}(0) = 0 \text{ in } \Omega.$$

From Proposition 2.2, it follows that $\hat{y} \in L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$. Since $\delta_2 > \tilde{m}'$, and $d_2 > m'$, then $\frac{\tilde{m}\delta_2}{\tilde{m}+\delta_2} > 1$, $\frac{md_2}{m+d_2} > 1$, and $\vec{V}\cdot\nabla\xi$ belongs to $L^{\frac{\tilde{m}\delta_2}{\tilde{m}+\delta_2}}(0,\bar{t};L^{\frac{md_2}{m+d_2}}(\Omega))$. Using Proposition 2.6 for $k=1$, we deduce that $\tilde{y} \in L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$. Thus $y_\xi \in L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$.

We prove that the mapping $\xi \mapsto y_\xi$ admits a fixed point in $L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$ for some $\bar{t} > 0$. Let (ξ_1,ξ_2) belong to $L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$. Still with Proposition 2.6, we have:

$$\|y_{\xi_1} - y_{\xi_2}\|_{L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))} \le C_1\|\vec{V}\|_{L^{\tilde{m}}(0,\bar{t};L^m(\Omega)^N)}\|\xi_1-\xi_2\|_{L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))}.$$

The mapping $t \mapsto C_1^{\tilde{m}}\int_0^t \|\vec{V}(\tau)\|^{\tilde{m}}_{L^m(\Omega)^N}\mathrm{d}\tau$ is absolutely continuous, there exists $\bar{t}$ depending on $\vec{V}$, such that $C_1(\int_t^{\min\{t+\bar{t},T\}}\|\vec{V}(\tau)\|^{\tilde{m}}_{L^m(\Omega)^N}\mathrm{d}\tau)^{\frac{1}{\tilde{m}}} \le C = \frac{1}{2}$ for all $t \in [0,T]$. Therefore the mapping $\xi \mapsto y_\xi$ is a contraction in $L^{\delta_2}(0,\bar{t};W^{1,d_2}(\Omega))$, and it admits a unique fixed point in this space. Next, we can prove the existence of a unique global solution in $L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$ as in the proof of Proposition 2.7.

2 - Let $\frac{\rho m}{\rho+m} \le r < \rho$. Observe that $y = y_1 + y_2$, where y_1 and y_2 are the solutions to the equations:

$$\frac{\partial y_1}{\partial t} + Ay_1 = 0 \text{ in } Q, \qquad \frac{\partial y_1}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y_1(0) = y_0 \text{ in } \Omega,$$

$$\frac{\partial y_2}{\partial t} + Ay_2 = -\vec{V}\cdot\nabla y \text{ in } Q, \qquad \frac{\partial y_2}{\partial n_A} = 0 \text{ on } \Sigma, \qquad y_2(0) = 0 \text{ in } \Omega.$$

Due to Proposition 2.4, y_1 belongs to $L^\infty(0,T;L^\rho(\Omega))$. Since $\frac{1}{2}+\frac{N}{2\rho}-\frac{N}{2r}+\frac{N}{2m} < \frac{N}{2m}+\frac{1}{2}$, we can choose $\tilde{m}' < \delta_2 < 2$ such that $\frac{1}{2}+\frac{N}{2\rho}-\frac{N}{2r}+\frac{N}{2m} < \frac{1}{\delta_2} < \frac{N}{2m}+\frac{1}{2} \le \frac{1}{\tilde{m}'}$. We take $\frac{1}{d_2} = \frac{1}{r}-\frac{1}{m}$. Then (δ_2,d_2) satisfies (24), and y belongs to $L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$. Moreover $\frac{1}{\delta_2}+\frac{N}{2d_2} < \frac{N}{2r}+1-\frac{1}{\tilde{m}}-\frac{N}{2m}$. Set $\frac{1}{\tilde{\ell}} = \frac{1}{\tilde{m}}+\frac{1}{\delta_2}$, then $\tilde{\ell} > 1$, and $\vec{V}\cdot\nabla y$ belongs to $L^{\tilde{\ell}}(0,T;L^r(\Omega))$. Due to Proposition 2.6, we deduce that y_2 belongs to $C([0,T];L^r(\Omega))$. □

3. Control problem (P$_1$)

3.1. Existence of solutions to problem (P$_1$)

Theorem 3.1. *Assume that hypotheses (A1) to (A7) are satisfied. Suppose that there exist (δ,d) satisfying (17), and (δ_2,d_2) satisfying (24), such that $\kappa \le \delta$, $\kappa \le d$, $\kappa \le \delta_2$, and $\kappa \le d_2$. Suppose in addition that $\theta < \inf\left\{\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}, \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}}\right\}$. Then the control problem (P_1) admits solutions.*

Proof. Let $(u_n)_n$ be a minimizing sequence in K_U. Then $(u_n)_n$ is bounded in $L^q(0,T;L^\sigma(\gamma))$. We can suppose that $(u_n)_n$ converges to some $\bar{u}$ weakly-star in $L^q(0,T;L^\sigma(\gamma))$. Since K_U is convex and closed in $L^q(0,T;L^\sigma(\gamma))$, then $\bar{u} \in K_U$. Due to Propositions 2.7 and 2.8, the sequence $(y_n)_n$ is bounded in $L^\delta(0,T;W^{1,d}(\Omega)) + L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$ for all (δ,d) satisfying (17), and all (δ_2,d_2) satisfying (24). Therefore we can suppose that $(y_n)_n$ converges to some $\bar{y}$ for the weak topology of $L^\delta(0,T;W^{1,d}(\Omega)) + L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$ for all (δ,d) satisfying (17), and all (δ_2,d_2) satisfying (24). We can easily verify that $\bar{y}$ is the solution to equation (1) corresponding to $\bar{u}$. Since $\kappa \le \delta$, $\kappa \le d$, $\kappa \le \delta_2$, $\kappa \le d_2$ for some (δ,d) satisfying (17), and some (δ_2,d_2) satisfying (24), $(y_n)_n$ converges to $\bar{y}$ for the weak topology of $L^\kappa(0,T;W^{1,\kappa}(\Omega))$. Moreover, due to

Propositions 2.8 and 2.7, we can prove that $(y_n(T))_n$ is bounded in $L^\theta(\Omega)$, and that $(y_n(T))_n$ converges to $\bar{y}(T)$ for the weak topology of $L^\theta(\Omega)$. By classical arguments, we can prove that $(\bar{u},\bar{y})$ is a solution of (P_1). □

3.2. Regularity results for the adjoint equation

To study the control problem (P_1), we look for solutions to equation (1) belonging to $L^\kappa(0,T;W^{1,\kappa}(\Omega))$. Therefore we must have $\kappa \le \delta$, $\kappa \le d$, $\kappa \le \delta_2$, $\kappa \le d_2$ for some (δ, d) satisfying (17), and some (δ_2, d_2) satisfying (24). Observe that if $\sigma \ge \frac{N-D}{N-D-1}$, and if we take $\delta = q$, then (17) is satisfied for all $1 \le d < \frac{N-D}{N-D-1}$. If we take $d_2 = \rho$, then (24) is satisfied for all $1 \le \delta_2 < 2$. Due to these observations, to simplify the calculations, throughout the sequel we make the following additional assumptions.

(A8) $\sigma \ge \frac{N-D}{N-D-1}$, $q \ge \sigma$. The function V_d belongs to $L^q(0,T;(L^{\frac{N-D}{N-D-1}}(\Omega))^N) + (L^2(0,T;(L^\rho(\Omega))^N)$. The function y_d belongs to $L^{\hat{r}}(\Omega)$, where $\hat{r} = \frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})}$.

We consider the following terminal boundary value problem

$$-\frac{\partial p}{\partial t} + Ap - \vec{V}\cdot\nabla p = -\operatorname{div}\vec{h} \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = \vec{h}\cdot\vec{n} \text{ on } \Sigma, \qquad p(T) = p_T \text{ in } \Omega. \tag{26}$$

When $\vec{h}\cdot\vec{n}$ is not defined, equation (26) is a formal writing for the variational equation:

$$\int_Q \left(p\frac{\partial y}{\partial t} + \sum_{i,j=1}^N a_{ij}D_j p D_i y + a_0 p y - \vec{V}\cdot\nabla p\, y \right) \mathrm{d}x\mathrm{d}t = \int_Q \vec{h}\cdot\nabla y \,\mathrm{d}x\mathrm{d}t + \int_\Omega y(T)p_T \,\mathrm{d}x\ ,$$

for all $y \in C^1(\overline{Q})$ such that $y(0) = 0$. If u is a solution of (P_1), if y_u is the solution of (1) corresponding to u, if we set $\vec{h} = \kappa C_Q|\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)$, and $p_T = \theta C_\Omega |y_u(T) - y_d|^{\theta-2}(y_u(T) - y_d)$, then equation (26) corresponds to the adjoint equation for (P_1) associated with (u, y_u). Due to Propositions 2.7, 2.8, assumption (A8), and Lemma 4.1 in [21], the function $\vec{h} = \kappa C_Q|\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)$ belongs to $L^{\frac{q}{\kappa-1}}(0,T;(L^{\frac{d}{\kappa-1}}(\Omega))^N) + L^{\frac{\delta_2}{\kappa-1}}(0,T;(L^{\frac{\rho}{\kappa-1}}(\Omega))^N$ for all $1 \le d < \frac{N-D}{N-D-1}$ and all $1 \le \delta_2 < 2$. The function $p_T = \theta C_\Omega |y_u(T) - y_d|^{\theta-2}(y_u(T) - y_d)$ belongs to $L^\epsilon(\Omega)$ for all $1 \le \epsilon < \frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})(\theta-1)}$. This is the reason why we now study the regularity of the solution p to equation (26) when $\vec{h}$ and p_T satisfy such conditions. In particular, to prove the optimality conditions for (P_1), we establish that the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\sigma'}(\gamma))$. We study equation (26) for $p_T \equiv 0$ and $\vec{h} \not\equiv 0$ in Theorem 3.2 and Theorem 3.3, for $\vec{h} \equiv 0$ and $p_T \not\equiv 0$ in Theorem 3.3. We summarize these results in Theorem 3.4.

Theorem 3.2. *Suppose that $\vec{h}$ belongs to $L^{\tilde{\eta}}(0,T;(L^\eta(\Omega))^N)$ with $\tilde{\eta} > \tilde{m}'$ and $\eta > m'$. Consider the equation:*

$$-\frac{\partial p}{\partial t} + Ap - \vec{V}\cdot\nabla p = -\operatorname{div}\vec{h} \textit{ in } Q, \qquad \frac{\partial p}{\partial n_A} = \vec{h}\cdot\vec{n} \textit{ on } \Sigma, \qquad p(T) = 0 \textit{ in } \Omega. \tag{27}$$

Equation (27) admits a unique weak solution in $L^{\tilde{\eta}}(0,T;W^{1,\eta}(\Omega))$. If $\tilde{\eta} \ge q'$ and $\eta > N - D$, then the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\sigma'}(\gamma))$.

Proof. To study equation (27), we still use a fixed point method as in the proof of Proposition 2.7. Let ξ belong to $L^{\tilde{\eta}}(T-\bar{t},T;\,W^{1,\eta}(\Omega))$, and let p_ξ be the solution of the equation:

$$-\frac{\partial p}{\partial t} + Ap = -\operatorname{div}\vec{h} + \vec{V}\cdot\nabla\xi \text{ in } \Omega\times]T-\bar{t},T[, \qquad \frac{\partial p}{\partial n_A} = \vec{h}\cdot\vec{n} \text{ on } \Gamma\times]T-\bar{t},T[, \qquad p(T) = 0 \text{ in } \Omega.$$

Set $\frac{1}{\tilde{\ell}} = \frac{1}{\tilde{m}} + \frac{1}{\tilde{\eta}}$, $\frac{1}{\ell} = \frac{1}{m} + \frac{1}{\eta}$. Then $\tilde{\ell} > 1$, $\ell > 1$, and $\vec{V}\cdot\nabla\xi$ belongs to $L^{\tilde{\ell}}(T-\bar{t},T;L^\ell(\Omega))$. From Proposition 2.5, and Proposition 2.6 for $k = 1$, it follows that p_ξ belongs to $L^{\tilde{\eta}}(T-\bar{t},T;W^{1,\eta}(\Omega))$.

Let ξ_1 and ξ_2 belong to $L^{\tilde{\eta}}(T-\bar{t},T;W^{1,\eta}(\Omega))$. Still with Proposition 2.6, we have:

$$\|p_{\xi_1}-p_{\xi_2}\|_{L^{\tilde{\eta}}(T-\bar{t},T;W^{1,\eta}(\Omega))}\leq C\|\vec{V}\|_{L^{\tilde{m}}(T-\bar{t},T;(L^m(\Omega))^N)}\|\xi_1-\xi_2\|_{L^{\tilde{\eta}}(T-\bar{t},T;W^{1,\eta}(\Omega))},$$

where C is independent of $\bar{t}$. For $\bar{t}>0$ small enough, we have $C\|\vec{V}\|_{L^{\tilde{m}}(T-\bar{t},T;(L^m(\Omega))^N)}<1$.

Therefore, there exists $\bar{t}>0$ such that the mapping $\xi\mapsto p_\xi$ is a contraction in the Banach space $L^{\tilde{\eta}}(T-\bar{t},T;W^{1,\eta}(\Omega))$, and it admits a fixed point. As in the proof of Proposition 2.7, we can prove that equation (27) admits a unique global solution in $L^{\tilde{\eta}}(0,T;W^{1,\eta}(\Omega))$.

If $\tilde{\eta}\geq q'$ and $\eta>N-D$, using Proposition 2.3, the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\eta}(\gamma))$. Since $\eta>N-D\geq\sigma'$, the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\sigma'}(\gamma))$. □

Remark. Since $N-D\geq\frac{N}{N-1}>m'$, the condition $\eta>N-D$ is stronger than $\eta>m'$.

Theorem 3.3. *Consider the equation*

$$-\frac{\partial p}{\partial t}+Ap-\vec{V}\cdot\nabla p=0 \text{ in } Q,\qquad \frac{\partial p}{\partial n_A}=0 \text{ on } \Sigma,\qquad p(T)=p_T \text{ in } \Omega, \tag{28}$$

where p_T belongs to $L^{\epsilon}(\Omega)$, with ϵ satisfying $\frac{N}{2\epsilon}<\frac{1}{\tilde{m}'}+\frac{D}{2(N-D)}$ and $\frac{N}{2\epsilon}<\frac{1}{q'}+\frac{D}{2(N-D)}$. Equation (28) admits a unique solution in $L^{\tilde{k}}(0,T;W^{1,k}(\Omega))$ for some $(\tilde{k},k)$ satisfying

$$\tilde{k}>\tilde{m}',\quad k\geq\epsilon,\quad k>N-D>m',\quad \frac{N}{2\epsilon}+\frac{1}{2}<\frac{1}{\tilde{k}}+\frac{N}{2k}\cdot \tag{29}$$

Moreover, the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T;L^{\sigma'}(\gamma))$.

Proof. We distinguish the cases $\epsilon\leq N-D$ and $\epsilon>N-D$.

1 - First consider the case $\epsilon\leq N-D$. From the inequality $\frac{N}{2(N-D)}\leq\frac{N}{2\epsilon}<\frac{1}{q'}+\frac{D}{2(N-D)}$, it follows that $q'<2$. There exists $\max(q',\tilde{m}')<\tilde{k}<2$ such that

$$\frac{N}{2\epsilon}+1<\frac{1}{\tilde{k}}+\frac{D}{2(N-D)}+\frac{1}{2}+\frac{1}{2}=\frac{1}{\tilde{k}}+\frac{N}{2(N-D)}+\frac{1}{2}\cdot$$

Therefore there exists $k>N-D>m'$ such that

$$k\geq\epsilon,\quad \frac{N}{2\epsilon}+1<\frac{N}{2k}+\frac{1}{\tilde{k}}+\frac{1}{2}<\frac{N}{2(N-D)}+\frac{1}{\tilde{k}}+\frac{1}{2}\cdot$$

To study equation (28), we still use a fixed point argument. Let ξ belong to $L^{\tilde{k}}(T-\bar{t},T;W^{1,k}(\Omega))$, and p_ξ be the solution of the equation:

$$-\frac{\partial p}{\partial t}+Ap=\vec{V}\cdot\nabla\xi \text{ in } \Omega\times]T-\bar{t},T[,\qquad \frac{\partial p}{\partial n_A}=0 \text{ on } \Gamma\times]T-\bar{t},T[,\qquad p(T)=p_T \text{ in } \Omega. \tag{30}$$

Set $\frac{1}{\tilde{\ell}}=\frac{1}{\tilde{m}}+\frac{1}{\tilde{k}}$, $\frac{1}{\ell}=\frac{1}{m}+\frac{1}{k}$. Observe that $\vec{V}\cdot\nabla\xi$ belongs to $L^{\tilde{\ell}}(T-\bar{t},T;L^{\ell}(\Omega))$, with $\tilde{\ell}>1$, $\ell>1$. From Proposition 2.6, and Proposition 2.4 with $\alpha=0$, it follows that $p_\xi\in L^{\tilde{k}}(T-\bar{t},T;W^{1,k}(\Omega))$.

Let ξ_1 and ξ_2 belong to $L^{\tilde{k}}(T-\bar{t},T;W^{1,k}(\Omega))$. Still with Proposition 2.6, we have:

$$\|p_{\xi_1}-p_{\xi_2}\|_{L^{\tilde{k}}(T-\bar{t},T;W^{1,k}(\Omega))}\leq C\|\vec{V}\|_{L^{\tilde{m}}(T-\bar{t},T;L^m(\Omega))}\|\xi_1-\xi_2\|_{L^{\tilde{k}}(T-\bar{t},T;W^{1,k}(\Omega))},$$

where C is independent of $\bar{t}$. There exists $\bar{t} > 0$ such that the mapping $\xi \mapsto p_\xi$ is a contraction in the Banach space $L^{\tilde{k}}(T-\bar{t}, T; W^{1,k}(\Omega))$, and it admits a fixed point. As in the proof of Proposition 2.7, we can prove that equation (28) admits a unique solution in $L^{\tilde{k}}(0,T; W^{1,k}(\Omega))$.

Since $\tilde{k} \geq q'$, and $k > N-D \geq \sigma'$, the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T; L^{\sigma'}(\gamma))$.

2 - Now we study the case $\epsilon > N-D$. We choose $\frac{N-D}{\epsilon} < r < 1$, for example we can set $r = \frac{1}{2}\big(1 + \frac{N-D}{\epsilon}\big)$. We choose

$$\frac{1}{2} < \frac{1}{\tilde{k}} < \min\left(\frac{1}{\tilde{m}'}, 1 - \frac{r}{2}\right)\cdot \tag{31}$$

Still using a fixed point method, we prove that equation (28) admits a unique solution in $L^{\tilde{k}}(0,T; W^{1,\epsilon}(\Omega)) \cap L^2(0,T; W^{r,\epsilon}(\Omega))$. Let ξ belong to $L^{\tilde{k}}(T-\bar{t},T; W^{1,\epsilon}(\Omega)) \cap L^2(T-\bar{t},T; W^{r,\epsilon}(\Omega))$ be the solution of the equation:

$$-\frac{\partial p}{\partial t} + Ap = \vec{V}\cdot\nabla\xi \text{ in } \Omega\times]T-\bar{t},T[, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Gamma\times]T-\bar{t},T[, \qquad p(T) = p_T \text{ in } \Omega. \tag{32}$$

Set $\frac{1}{\tilde{\ell}} = \frac{1}{\tilde{m}} + \frac{1}{\tilde{k}}$, $\frac{1}{\ell} = \frac{1}{m} + \frac{1}{\epsilon}$. Then $\vec{V}\cdot\nabla\xi$ belongs to $L^{\tilde{\ell}}(T-\bar{t},T; L^{\ell}(\Omega))$, with $\tilde{\ell} > 1$, $\ell > 1$. Due to (31), we have

$$\frac{N}{2\epsilon} + 1 < \frac{N}{2\epsilon} + \frac{1}{\tilde{k}} + \frac{1}{2}, \quad \frac{N}{2\epsilon} + 1 < \frac{N}{2\epsilon} + \frac{1}{2} + 1 - \frac{r}{2},$$

and

$$\frac{N}{2\ell} + \frac{1}{\tilde{\ell}} \leq \frac{N}{2\epsilon} + \frac{1}{\tilde{k}} + \frac{1}{2}, \quad \frac{N}{2\ell} + \frac{1}{\tilde{\ell}} < \frac{N}{2\epsilon} + \frac{1}{2} + 1 - \frac{r}{2}\cdot$$

From Propositions 2.4 and 2.6, it follows that p_ξ belongs to $L^{\tilde{k}}(T-\bar{t},T; W^{1,\epsilon}(\Omega)) \cap L^2(T-\bar{t},T; W^{r,\epsilon}(\Omega))$. Let ξ_1 and ξ_2 belong to $L^{\tilde{k}}(T-\bar{t},T; W^{1,\epsilon}(\Omega)) \cap L^2(T-\bar{t},T; W^{r,\epsilon}(\Omega))$. Still with Proposition 2.6, we have:

$$\|p_{\xi_1} - p_{\xi_2}\|_{L^{\tilde{k}}(T-\bar{t},T;W^{1,\epsilon}(\Omega))} + \|p_{\xi_1} - p_{\xi_2}\|_{L^2(T-\bar{t},T;W^{r,\epsilon}(\Omega))} \leq C\|\vec{V}\|_{L^{\tilde{m}}(T-\bar{t},T;(L^m(\Omega))^N)}\|\xi_1 - \xi_2\|_{L^{\tilde{k}}(T-\bar{t},T;W^{1,\epsilon}(\Omega))},$$

where C is independent of $\bar{t}$. There exists $\bar{t} > 0$ such that the mapping $\xi \mapsto p_\xi$ is a contraction in the Banach space $L^{\tilde{k}}(T-\bar{t},T; W^{1,\epsilon}(\Omega)) \cap L^2(T-\bar{t},T; W^{r,\epsilon}(\Omega))$, and it admits a fixed point. As in the proof of Proposition 2.7, we can prove that equation (28) admits a unique solution in $L^{\tilde{k}}(0,T; W^{1,\epsilon}(\Omega)) \cap L^2(0,T; W^{r,\epsilon}(\Omega))$. Since $2 \geq q'$, and $r\epsilon > N-D \geq \sigma'$, the trace of p on $\gamma\times]0,T[$ belongs to $L^{q'}(0,T; L^{\sigma'}(\gamma))$. □

We have to study equation (26) in the case when $\vec{h}$ belongs to $L^{\frac{q}{\kappa-1}}(0,T; (L^{\frac{d}{\kappa-1}}(\Omega))^N) + L^{\frac{\delta_2}{\kappa-1}}(0,T; (L^{\frac{\rho}{\kappa-1}}(\Omega))^N$ for all $1 \leq d < \frac{N-D}{N-D-1}$ and all $1 \leq \delta_2 < 2$, and p_T belongs to $L^{\epsilon}(\Omega)$ for all $1 \leq \epsilon < \frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})(\theta-1)}$. In this case, due to Theorems 3.2 and 3.3, the trace of p on $\gamma\times]0,T[$ belong to $L^{q'}(0,T; L^{\sigma'}(\gamma))$ if the following conditions are satisfied:

$$\frac{N-D}{(N-D-1)(\kappa-1)} > N-D > m',$$

$$\frac{\rho}{\kappa-1} > N-D > m',$$

$$\left(N - \frac{D}{N-D} - \frac{2}{q'}\right)(\theta-1) < \frac{2}{q'} + \frac{D}{N-D}, \quad \left(N - \frac{D}{N-D} - \frac{2}{q'}\right)(\theta-1) < \frac{2}{\tilde{m}'} + \frac{D}{N-D},$$

$$\frac{q}{\kappa-1} > q', \qquad \frac{2}{\kappa-1} > q', \qquad \frac{2}{\kappa-1} > \tilde{m}', \qquad \frac{q}{\kappa-1} > \tilde{m}'.$$

The condition $\frac{N-D}{(N-D-1)(\kappa-1)} > N-D$ is equivalent to $\kappa < 1 + \frac{1}{N-D-1}$. If $\kappa < 1 + \frac{1}{N-D-1}$, then $\kappa < 2$, and the conditions $\frac{q}{\kappa-1} > q'$, $\frac{2}{\kappa-1} > q'$, $\frac{2}{\kappa-1} > \tilde{m}'$, and $\frac{q}{\kappa-1} > \tilde{m}'$ are automatically satisfied. Moreover, we have $N-D \geq \frac{N}{N-1} > m'$, and $\rho = \frac{N}{N-\frac{D}{\sigma'}-\frac{2}{q'}} \geq \frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}} > \frac{N-D}{N-D-1}$. Thus $\frac{\rho}{\kappa-1} > N-D$ if $\frac{N-D}{(N-D-1)(\kappa-1)} > N-D$.

We can summarize the above conditions in the assumptions stated below.

(A9) The exponent κ satisfies

$$1<\kappa<1+\frac{1}{N-D-1}\cdot \tag{33}$$

(A10) The exponent θ satisfies

$$(\theta-1)\left(N-\frac{D}{N-D}-\frac{2}{q'}\right)<\frac{2}{q'}+\frac{D}{N-D}\ , \tag{34}$$

$$(\theta-1)\left(N-\frac{D}{N-D}-\frac{2}{q'}\right)<\frac{2}{\tilde{m}'}+\frac{D}{N-D}\cdot \tag{35}$$

These assumptions are satisfied in the following cases. When $N=3$, $D=1$, $\tilde{m}=8$, $m=4$, $q=\sigma=2$, we can set $\rho=2$, and (A9, A10) are satisfied for all $1<\kappa<2$, and all $1<\theta<2$. In the case where $N=3$, $D=1$, $\tilde{m}=8$, $m=4$, $q=\sigma=\infty$, we can set $\rho=\infty$, and the previous assumptions are satisfied for all $1<\kappa<2$, and all $1<\theta<\frac{11}{2}$.

Theorem 3.4. *Set* $\vec{h}=\kappa C_Q|\nabla y_u-V_d|^{\kappa-2}(\nabla y_u-V_d)$, *and* $p_T=\theta C_\Omega|y_u(T)-y_d|^{\theta-2}(y_u(T)-y_d)$, *where* y_u *is the solution of (1) corresponding to* $u\in L^q(0,T;L^\sigma(\gamma))$. *Let* p *be the solution to the equation*

$$-\frac{\partial p}{\partial t}+Ap-\vec{V}\cdot\nabla p=-\mathrm{div}\ \vec{h}\ \text{in } Q,\qquad \frac{\partial p}{\partial n_A}=\vec{h}\cdot\vec{n}\ \text{on } \Sigma,\qquad p(T)=p_T\ \text{in } \Omega. \tag{36}$$

Then the trace of p *on* $\gamma\times]0,T[$ *belongs to* $L^{q'}(0,T;L^{\sigma'}(\gamma))$.

Proof. From assumption (33), it follows that $\vec{h}\in L^{\tilde{\eta}_1}(0,T;L^{\eta_1}(\Omega))+L^{\tilde{\eta}_2}(0,T;L^{\eta_2}(\Omega))$, for some $(\tilde{\eta}_1,\eta_1)$, $(\tilde{\eta}_2,\eta_2)$ satisfying $\tilde{\eta}_i>\tilde{m}'$, $\tilde{\eta}_i\geq q'$, $\eta_i>N-D>m'$ for $i=1,2$. From assumptions (34, 35), it follows that $p_T\in L^\epsilon(\Omega)$ for some ϵ satisfying $\frac{N}{2\epsilon}<\frac{1}{\tilde{m}'}+\frac{D}{2(N-D)}$ and $\frac{N}{2\epsilon}<\frac{1}{q'}+\frac{D}{2(N-D)}$. Thus, the theorem is a direct consequence of Theorems 3.2 and 3.3.

□

3.3. Optimality conditions for (P_1)

Proposition 3.1. *Let* $u\in L^q(0,T;L^\sigma(\gamma))$ *and* y_u *be the solution of (1) corresponding to* u. *Let* p *be the solution to the adjoint equation (36). Let* z *be the solution to the equation*

$$\frac{\partial z}{\partial t}+Az+\vec{V}\cdot\nabla z=0\ \text{in } Q,\quad \frac{\partial z}{\partial n_A}=u\delta_\gamma\ \text{on } \Sigma,\quad z(0)=0\ \text{in } \Omega. \tag{37}$$

Then

$$\int_0^T\int_\gamma pu\,\mathrm{d}\zeta\mathrm{d}t=\int_Q\nabla z\cdot\vec{h}\,\mathrm{d}x\mathrm{d}t+\int_\Omega p_T(x)z(x,T)\,\mathrm{d}x. \tag{38}$$

Proof. Observe that $z(T)$ belongs to $L^r(\Omega)$ for all $1\leq r<\frac{N}{N-\frac{D}{N-D}-\frac{2}{q'}}$. With condition (34), we can find ϵ satisfying $\epsilon<\frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})(\theta-1)}$, such that $z(T)$ belongs to $L^{\epsilon'}(\Omega)$. Thus $p_Tz(T)$ belongs to $L^1(\Omega)$. Due to

Proposition 2.7, z belongs to $L^q(0,T;W^{1,d}(\Omega))$ for all $d < \frac{N-D}{N-D-1}$. Set $\vec{h} = \vec{h^1} + \vec{h^2}$, where $\vec{h^1}$ (respectively $\vec{h^2}$) belongs to $L^{\frac{q}{\kappa-1}}(0,T;(L^{\frac{d}{\kappa-1}}(\Omega))^N)$ for all $d < \frac{N-D}{N-D-1}$ (respectively $L^{\frac{\delta_2}{\kappa-1}}(0,T;(L^{\frac{\rho}{\kappa-1}}(\Omega))^N)$ for all $\delta_2 < 2$). Let $(\vec{h^1_k})_k$ (respectively $(\vec{h^2_k})_k$) be a sequence of functions in $(\mathcal{D}(Q))^N$ converging to $\vec{h^1}$ (respectively $\vec{h^2}$) in $L^{\frac{q}{\kappa-1}}(0,T;(L^{\frac{d}{\kappa-1}}(\Omega))^N)$ for all $d < \frac{N-D}{N-D-1}$ (respectively $L^{\frac{\delta_2}{\kappa-1}}(0,T;(L^{\frac{\rho}{\kappa-1}}(\Omega))^N)$ for all $\delta_2 < 2$). Due to assumption (A9), there exists $\delta_2 < 2$ and $d < \frac{N-D}{N-D-1}$ obeying $\frac{\kappa-1}{\delta_2} + \frac{1}{q} \le 1$, $\frac{\kappa-1}{\rho} + \frac{1}{d} \le 1$. Then $\nabla z \cdot (\vec{h^1_k} + \vec{h^2_k})$ belongs to $L^1(Q)$, and converges, when k tends to infinity, to $\nabla z \cdot \vec{h}$ in $L^1(Q)$. Let $(p_T^k)_k$ be a sequence of regular functions converging to p_T in $L^\epsilon(\Omega)$ for all $1 \le \epsilon < \frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})(\theta-1)}$, and p_k be the solution to the equation:

$$-\frac{\partial p_k}{\partial t} + Ap_k - \vec{V} \cdot \nabla p = -\operatorname{div}\left(\vec{h^1_k} + \vec{h^2_k}\right) \text{ in } Q, \quad \frac{\partial p_k}{\partial n_A} = 0 \text{ on } \Sigma, \quad p_k(T) = p_T^k \text{ in } \Omega. \tag{39}$$

Then we have

$$\int_0^T \int_\gamma p_k(\zeta,t)u(\zeta,t)\,\mathrm{d}\zeta\mathrm{d}t = \int_Q \nabla z \cdot \left(\vec{h^1_k} + \vec{h^2_k}\right) \mathrm{d}x\mathrm{d}t + \int_\Omega p_T^k(x)z(x,T)\,\mathrm{d}x. \tag{40}$$

By passing to the limit when k tends to ∞, we obtain:

$$\int_0^T \int_\gamma p(\zeta,t)u(\zeta,t)\,\mathrm{d}\zeta\mathrm{d}t = \int_Q \nabla z \cdot \left(\vec{h^1} + \vec{h^2}\right) \mathrm{d}x\mathrm{d}t + \int_\Omega p_T(x)z(x,T)\,\mathrm{d}x. \tag{41}$$

□

Theorem 3.5. *If u is a solution of (P_1), then*

$$\int_0^T \int_\gamma p(v-u)\,\mathrm{d}\zeta\mathrm{d}t + qC_\gamma \int_0^T \left(\int_\gamma |u|^\sigma \mathrm{d}\zeta\right)^{\frac{q}{\sigma}-1} \left(\int_\gamma |u|^{\sigma-2}u(v-u)\,\mathrm{d}\zeta\right) \mathrm{d}t \ge 0 \tag{42}$$

for every $v \in K_U$, where p is the solution to

$$-\frac{\partial p}{\partial t} + Ap - \vec{V} \cdot \nabla p = -\kappa C_Q \operatorname{div}\left(|\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)\right) \text{ in } Q,$$

$$\frac{\partial p}{\partial n_A} = \kappa C_Q\left(|\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)\right) \cdot \vec{n} \text{ on } \Sigma, \quad p(T) = \theta C_\Omega |y_u(T) - y_d|^{\theta-2}(y_u(T) - y_d) \text{ in } \Omega,$$

where y_u is the solution to equation (1) corresponding to u.

Proof. Let v be in K_U, $\lambda > 0$, and denote by y_λ the solution of (1) corresponding to $u + \lambda(v-u)$. Due to (34), applying Proposition 2.7, y_λ and y_u belong to $C([0,T];L^\theta_w(\Omega))$.

1 - We set $w_\lambda = y_\lambda - y_u$, then w_λ is the solution to the equation:

$$\frac{\partial w}{\partial t} + Aw + \vec{V} \cdot \nabla w = 0 \text{ in } Q, \qquad \frac{\partial w}{\partial n_A} = \lambda(v-u)\delta_\gamma \text{ on } \Sigma, \qquad w(0) = 0 \text{ in } \Omega\,.$$

With Proposition 2.7 and condition (34), for all (δ, d) satisfying (17), we have the estimate:

$$\|y_\lambda - y_u\|_{L^\delta(0,T;W^{1,d}(\Omega))} + \|y_\lambda(T) - y_u(T)\|_{L^\theta(\Omega)} \le C\lambda\|v-u\|_{L^q(0,T;L^\sigma(\gamma))}\,. \tag{43}$$

2 - Let us set $z = (y_\lambda - y_u)/\lambda$. Observe that z is independent of λ. Due to Proposition 2.7, z belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ for all (δ,d) satisfying (17), and $z(T)$ belongs to $L^r(\Omega)$, for all $1 \le r < \frac{N}{(N-\frac{D}{N-D}-\frac{2}{q'})}\cdot$

3 - We want to calculate the gradient of the functional I. If we set $G(u) = \int_\Omega |y_u(T)-y_d|^\theta \mathrm{d}x + \int_Q |\nabla y_u - V_d|^\kappa \mathrm{d}x\mathrm{d}t$, from the convexity of the mapping $y \mapsto \int_\Omega |y(T)-y_d|^\theta \mathrm{d}x + \int_Q |\nabla y - V_d|^\kappa \mathrm{d}x\mathrm{d}t$, it follows that:

$$\theta C_\Omega \int_\Omega |y_u(T)-y_d|^{\theta-2}(y_u(T)-y_d)z(T)\mathrm{d}x + \kappa C_Q \int_Q |\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)\nabla z \mathrm{d}x$$

$$\le \frac{1}{\lambda}(G(u+\lambda(v-u)) - G(u))$$

$$\le \theta C_\Omega \int_\Omega |y_\lambda(T)-y_d|^{\theta-2}(y_\lambda(T)-y_d)z(T)\mathrm{d}x + \kappa C_Q \int_Q |\nabla y_\lambda - V_d|^{\kappa-2}(\nabla y_\lambda - V_d)\nabla z \mathrm{d}x.$$

We set $p_T = \theta C_\Omega |y_u(T)-y_d|^{\theta-2}(y_u(T)-y_d)$, and $p_T^\lambda = \theta C_\Omega |y_\lambda(T)-y_d|^{\theta-2}(y_\lambda(T)-y_d)$. From (43), it follows that p_T and p_T^λ belong to $L^{\theta'}(\Omega)$, and that $\int_\Omega p_T^\lambda z(T)\ \mathrm{d}x \longmapsto \int_\Omega p_T z(T)\mathrm{d}x$ as λ tends to zero.

We set $\vec{h} = \kappa C_Q |\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)$, and $\vec{h}_\lambda = \kappa C_Q |\nabla y_\lambda - V_d|^{\kappa-2}(\nabla y_\lambda - V_d)$. Since $q \ge 2$ we have $\kappa \le q$ and $\frac{\kappa-1}{2} + \frac{1}{q} < 1$. Due to assumptions (33), we can find d such that $\kappa \le d < \frac{N-D}{N-D-1}$, $\frac{\kappa-1}{\rho} + \frac{N-D-1}{N-D} < \frac{\kappa-1}{\rho} + \frac{1}{d} \le 1$. Thus we can choose $1 < \delta_2 < 2$ such that $\frac{\kappa-1}{2} + \frac{1}{q} < \frac{\kappa-1}{\delta_2} + \frac{1}{q} \le 1$. We can verify that $\vec{h}_\lambda \cdot \nabla z$ belongs to $L^1(Q)$, and $\int_Q \vec{h}_\lambda \cdot \nabla z\ \mathrm{d}x\mathrm{d}t \longmapsto \int_Q \vec{h} \cdot \nabla z\ \mathrm{d}x\mathrm{d}t$ as λ tends to zero. Therefore, if we set $F(u) = I(y_u, u)$, thanks to the above calculations, we obtain:

$$F'(u)(v-u) = \theta C_\Omega \int_\Omega |y_u(T)-y_d|^{\theta-2}(y_u(T)-y_d)z(T)\,\mathrm{d}x + \kappa C_Q \int_Q |\nabla y_u - V_d|^{\kappa-2}(\nabla y_u - V_d)\cdot\nabla z\,\mathrm{d}x$$

$$+ qC_\gamma \int_0^T \left(\int_\gamma |u|^\sigma \mathrm{d}\zeta\right)^{\frac{q}{\sigma}-1} \left(\int_\gamma |u|^{\sigma-2}u(v-u)\mathrm{d}\zeta\right)\mathrm{d}t.$$

Finally we can use the Green formula (38) to complete the proof. □

4. Control problem (P$_2$)

In this section we study the control problem (P_2). In this case δ_γ is replaced by δ_{x_0}, which corresponds to $D = 0$. We first prove the existence of an optimal pair (u, x_0), and next establish optimality conditions. For this, we make the following assumptions.
(A6') The function y_d belongs to $L^{\frac{N}{N-2}}(\Omega)$, K_U is a closed convex bounded subset of $L^\infty(0,T)$, and $K_{\overline{\Omega}}$ is a closed convex subset of $\overline{\Omega}$. (Observe that $q = \sigma = \infty$.)
(A7') The exponent θ satisfies the condition

$$\theta < \frac{N-1}{N-2}\cdot \tag{44}$$

4.1. **Existence of solutions to (P$_2$)**

Theorem 4.1. *Assume that conditions (A1–A5) and (A6') are fulfilled. Suppose that $\theta < \frac{N}{N-2}$. Then the optimal control problem (P_2) admits solutions.*

Proof. Let $(u_n, x_n)_n$ be a minimizing sequence in $K_U \times K_{\overline{\Omega}}$. Then $(u_n, x_n)_n$ is bounded in $L^\infty(0,T) \times R^N$. We can suppose that $(u_n)_n$ converges to some u weakly-star in $L^\infty(0,T)$, and $(x_n)_n$ converges in R^N to some $x_0 \in K_{\overline{\Omega}}$. Since K_U is convex and closed in $L^\infty(0,T)$, u belongs to K_U. Let y_n be the solution to (1) corresponding to (u_n, x_n), and y_{u,x_0} be the solution of (1) corresponding to (u, x_0). As in the proof of Theorem 3.1, we can prove that $(y_n)_n$ converges to y_{u,x_0} for the weak topology of $L^\delta(0,T;W^{1,d}(\Omega)) + L^{\delta_2}(0,T;W^{1,d_2}(\Omega))$ for all (δ, d) satisfying (17) with $D = 0$, and all (δ_2, d_2) satisfying (24) with $D = 0$. Moreover, we can prove that $(y_n(T))_n$ converges to $y_{u,x_0}(T)$ for the weak topology of $L^\theta(\Omega)$. By classical arguments, we can next prove that (u, x_0) is a solution of (P_2). □

4.2. Regularity results of the adjoint state

We consider the following terminal boundary value problem

$$-\frac{\partial p}{\partial t} + Ap - \vec{V}\cdot\nabla p = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = p_T \text{ in } \Omega, \tag{45}$$

where p_T belongs to $L^\epsilon(\Omega))$ for all $1 \le \epsilon < \frac{N}{(N-2)(\theta-1)}$. If $\bar{y}$ is the solution of (1) corresponding to u and x_0, if we set $p_T = \theta|\bar{y}(T) - y_d|^{\theta-2}(\bar{y}(T) - y_d)$, then equation (45) corresponds to the adjoint equation for J associated with (x_0, u).

Theorem 4.2. *The solution p to equation (45) belongs to $L^1(0,T;C^1(\overline{\Omega}))$. Moreover, for all $N < \epsilon < \frac{N}{(N-2)(\theta-1)}$, there exists $0 < \nu < 1$, such that the mapping that associates p with p_T is continuous from $L^\epsilon(\Omega)$ into $L^1(0,T;C^{1,\nu}(\overline{\Omega}))$.*

Proof. 1 - Due to condition (44) on θ, we can find ϵ such that $N < \epsilon < \frac{N}{(N-2)(\theta-1)}$. Then $\epsilon > m'$. We choose (α, ξ) satisfying

$$\alpha > m > N, \quad \frac{1}{2} + \frac{N}{2\alpha} < \frac{\xi}{2} < \inf\left\{1 + \frac{N}{2\alpha} - \frac{N}{2\epsilon}\,, \frac{3}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{N}{2\alpha} - \frac{N}{2\epsilon}\,, 1 - \frac{N}{2m} + \frac{N}{2\alpha}\right\}\cdot \tag{46}$$

Since $\frac{N}{2\alpha} - \frac{N}{2m} < 0$, then $\frac{\xi}{2} < 1 < 1 + \frac{1}{\tilde{m}'}$. Thus there exists $\tilde{\alpha}$ such that

$$\sup\left\{\frac{\xi}{2} - \frac{1}{\tilde{m}'}\,, \frac{1}{2}\right\} < \frac{1}{\tilde{\alpha}} < 1, \tag{47}$$

$$-\frac{1}{2} + \frac{\xi}{2} + \frac{1}{\tilde{m}} + \frac{N}{2m} - \frac{N}{2\alpha} + \frac{N}{2\epsilon} < \frac{1}{\tilde{\alpha}} < 1\cdot \tag{48}$$

With (48), we have $\frac{1}{2} - \frac{1}{\tilde{m}'} + \frac{N}{2\epsilon} < 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha}$, and $0 < \frac{1}{2} + \frac{N}{2\epsilon} < 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha}$. From (46), we deduce that $1 - \frac{\xi}{2} - \frac{N}{2m} + \frac{N}{2\alpha} > 0$ and $1 - \frac{\xi}{2} - \frac{N}{2m} + \frac{N}{2\alpha} > \frac{1}{2} - \frac{1}{\tilde{m}'} + \frac{N}{2\epsilon}$. Thus there exists k satisfying

$$\sup\left\{\frac{1}{2} - \frac{1}{\tilde{m}'} + \frac{N}{2\epsilon}\,, 0\right\} < \frac{N}{2k} < \inf\left\{\frac{N}{2\epsilon}\,, 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha}\,, 1 - \frac{\xi}{2} - \frac{N}{2m} + \frac{N}{2\alpha}\right\}\cdot \tag{49}$$

With (49), we obtain $\frac{1}{2} + \frac{N}{2\epsilon} - \frac{N}{2k} < \frac{1}{\tilde{m}'}$, and $-\frac{1}{\tilde{m}} + \frac{1}{\tilde{\alpha}} < 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha} - \frac{N}{2k}$. From (48), we deduce that $\frac{1}{2} + \frac{N}{2\epsilon} - \frac{N}{2k} < 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha} - \frac{N}{2k}$. Thus we can choose $\tilde{k}$ such that

$$\sup\left\{-\frac{1}{\tilde{m}} + \frac{1}{\tilde{\alpha}}\,, \frac{1}{2} + \frac{N}{2\epsilon} - \frac{N}{2k}\right\} < \frac{1}{\tilde{k}} < \inf\left\{\frac{1}{\tilde{m}'}\,, 1 - \frac{\xi}{2} - \frac{1}{\tilde{m}} - \frac{N}{2m} + \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha} - \frac{N}{2k}\right\}\cdot \tag{50}$$

With (49), we have $k > \epsilon > m'$. Due to (50), the pair $(\tilde{k}, k)$ obeys (29). From Theorem 3.3, we deduce that p belongs to $L^{\tilde{k}}(0,T;W^{1,k}(\Omega))$.

2 - We set $\frac{1}{\tilde{\ell}} = \frac{1}{\tilde{m}} + \frac{1}{\tilde{k}}$ and $\frac{1}{\ell} = \frac{1}{m} + \frac{1}{k}$, then $\tilde{\ell} > 1$. From $\frac{1}{m} + \frac{1}{k} < \frac{1}{m} + \frac{1}{\epsilon} < \frac{1}{m} + \frac{1}{m'} = 1$, it follows that $\ell > 1$ and $\vec{V} \cdot \nabla p \in L^{\tilde{\ell}}(0,T; L^{\ell}(\Omega))$. Let π be the solution to the equation:

$$-\frac{\partial \pi}{\partial t} + A\pi = \vec{V} \cdot \nabla p \text{ in } Q, \qquad \frac{\partial \pi}{\partial n_A} = 0 \text{ on } \Sigma, \qquad \pi(T) = 0 \text{ in } \Omega.$$

From the choice of $\tilde{k}$ and k, with $\frac{1}{\alpha} < \frac{1}{m} < \frac{1}{m} + \frac{1}{k}$, we obtain $\tilde{\alpha} \geq \tilde{\ell}$, $\alpha \geq \ell$, $\frac{1}{\tilde{\ell}} + \frac{N}{2\ell} < \frac{1}{\tilde{\alpha}} + \frac{N}{2\alpha} + 1 - \frac{\xi}{2}$. Due to Proposition 2.6, we deduce that π belongs to $L^{\tilde{\alpha}}(0,T; W^{\xi,\alpha}(\Omega))$.

3 - Let π_2 be the solution to the equation:

$$-\frac{\partial \pi_2}{\partial t} + A\pi_2 = 0 \text{ in } Q, \qquad \frac{\partial \pi_2}{\partial n_A} = 0 \text{ on } \Sigma, \qquad \pi_2(T) = p_T \text{ in } \Omega.$$

Let (α_2, ξ_2) be a pair satisfying

$$\alpha_2 \geq \epsilon, \quad 1 + \frac{N}{2\epsilon} < \frac{N}{2\alpha_2} + 2 - \frac{\xi_2}{2} < \frac{3}{2} \cdot \tag{51}$$

From Proposition 2.4, we deduce that π_2 belongs to $L^1(0,T; W^{\xi_2,\alpha_2}(\Omega))$, which is included in $L^1(0,T; C^1(\overline{\Omega}))$.

4 - Let (α, ξ) obey (46), and (α_2, ξ_2) obey (51). Then there exists ν such that

$$1 > \nu > 0, \quad \xi > \nu + 1 + \frac{N}{\alpha}, \quad \xi_2 > \nu + 1 + \frac{N}{\alpha_2} \cdot \tag{52}$$

We have $p = \pi + \pi_2 \in L^1(0,T; W^{\xi,\alpha}(\Omega)) + L^1(0,T; W^{\xi_2,\alpha_2}(\Omega))$, and $L^1(0,T; W^{\xi,\alpha}(\Omega)) + L^1(0,T; W^{\xi_2,\alpha_2}(\Omega))$ is included in $L^1(0,T; C^{1,\nu}(\overline{\Omega}))$. The estimate of p in $L^1(0,T; C^{1,\nu}(\overline{\Omega}))$ in function of $\|p_T\|_{L^\epsilon(\Omega)}$ may be deduced from the analysis of Step 1 and Step 2. □

4.3. **Optimality conditions**

Lemma 4.1. *Let p be the solution of (45), where p_T belongs to $L^\epsilon(\Omega))$ for all $1 \leq \epsilon < \frac{N}{(N-2)(\theta-1)}$. Let $(p_T^\lambda)_\lambda$ be a sequence of functions converging to p_T for the weak topology of the space $L^\epsilon(\Omega)$ for all $\epsilon < \frac{N}{(N-2)(\theta-1)}$. Let p_λ be the solution to the equation:*

$$-\frac{\partial p}{\partial t} + Ap - \vec{V} \cdot \nabla p = 0 \text{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \text{ on } \Sigma, \qquad p(T) = p_T^\lambda \text{ in } \Omega. \tag{53}$$

Then the sequence $(p_\lambda)_\lambda$ converges to p in $L^1(0,T; C^1(\overline{\Omega}))$.

Proof. From Theorem 4.2, we know that the sequence $(p_\lambda)_\lambda$ is bounded in $L^1(0,T; C^{1,\nu}(\overline{\Omega}))$ for some $0 < \nu < 1$. The identity mapping from $C^{1,\nu}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ is compact. The sequence $(\frac{dp_\lambda}{dt})_\lambda$ is bounded in $L^1(0,T; (W^{1,\beta}(\Omega))')$ for some β big enough. From a compactness result ([25], Cor. 5), we deduce that the identity mapping is compact from $L^1(0,T; C^{1,\nu}(\overline{\Omega})) \cap W^{1,1}(0,T; (W^{1,\beta}(\Omega))')$ into $L^1(0,T; C^1(\overline{\Omega}))$. Therefore the sequence $(p_\lambda)_\lambda$ converges to p in $L^1(0,T; C^1(\overline{\Omega}))$.

Lemma 4.2. *Let p and p_λ be defined as in Lemma 4.1. Then we have*

$$lim_{\lambda \to 0} \frac{1}{\lambda} \int_0^T (p_\lambda(x_0 + \lambda(x_1 - x_0)) - p_\lambda(x_0)) u \, \mathrm{d}t = \int_0^T \nabla p(x_0) \cdot (x_1 - x_0) u \, \mathrm{d}t.$$

Proof. We can write:

$$\frac{1}{\lambda}\int_0^T (p_\lambda(x_0+\lambda(x_1-x_0)) - p_\lambda(x_0))u\,\mathrm{d}t = \int_0^T\int_0^1 \nabla p_\lambda(x_0+\vartheta\lambda(x_1-x_0))\cdot(x_1-x_0)u\,\mathrm{d}\vartheta\mathrm{d}t$$

$$= \int_0^T\int_0^1 (\nabla p_\lambda(x_0+\vartheta\lambda(x_1-x_0)) - \nabla p_\lambda(x_0))\cdot(x_1-x_0)u\,\mathrm{d}\vartheta\mathrm{d}t + \int_0^T \nabla p_\lambda(x_0)\cdot(x_1-x_0)u\,\mathrm{d}t.$$

Due to Lemma 4.1, the sequence $(p_\lambda)_\lambda$ converges to p in $L^1(0,T;C^1(\overline{\Omega}))$. Thus we have:

$$\int_0^T \nabla p_\lambda(x_0)\cdot(x_1-x_0)u\,\mathrm{d}t \longrightarrow \int_0^T \nabla p(x_0)\cdot(x_1-x_0)u\,\mathrm{d}t \quad \text{as } \lambda\to 0\ .$$

Due to Theorem 4.2, $(p_\lambda)_\lambda$ is bounded in $L^1(0,T;C^{1,\nu}(\overline{\Omega}))$ for some $0<\nu<1$. Therefore we have

$$\left|\int_0^T\int_0^1 (\nabla p_\lambda(x_0+\vartheta\lambda(x_1-x_0)) - \nabla p_\lambda(x_0))\cdot(x_1-x_0)u\,\mathrm{d}\vartheta\mathrm{d}t\right|$$

$$\le \int_0^T\int_0^1 \frac{|(\nabla p_\lambda(x_0+\vartheta\lambda(x_1-x_0)) - \nabla p_\lambda(x_0))\cdot(x_1-x_0)|}{|x_0+\vartheta\lambda(x_1-x_0)-x_0|^\nu}\mathrm{d}\vartheta|\lambda(x_1-x_0)|^\nu|u|\,\mathrm{d}t$$
$$\le C\|u\|_{L^\infty(0,T)}\|p_\lambda\|_{L^1(0,T;C^{1,\nu}(\overline{\Omega}))}|\lambda(x_1-x_0)|^\nu \longrightarrow 0 \quad \text{as } \lambda\to 0\ .$$

The proof is complete. □

Theorem 4.3. *If (u,x_0) is a solution of (P_2), then*

$$\int_0^T p(x_0)(v-u)\,\mathrm{d}t \ge 0 \textit{ for all } v\in K_U,\ \textit{and}\ \int_0^T \nabla p(x_0)\cdot(x_1-x_0)u\,\mathrm{d}t\ge 0,\ \textit{for all } x_1\in K_{\overline{\Omega}},$$

where p is the solution to the equation

$$-\frac{\partial p}{\partial t} + Ap - \vec{V}\cdot\nabla p = 0 \textit{ in } Q, \qquad \frac{\partial p}{\partial n_A} = 0 \textit{ on } \Sigma, \qquad p(T) = \theta|\bar{y}(T)-y_d|^{\theta-2}(\bar{y}(T)-y_d) \textit{ in } \Omega,$$

and where $\bar{y}$ is the solution of (1) corresponding to u and x_0.

Proof. We only prove the optimality condition for x_0. For $\lambda>0$, we denote by y_λ the solution of (1) corresponding to u and $x_0+\lambda(x_1-x_0)$. Set $F(x) = J(y_{u,x},u,x)$. We set $z_\lambda = (y_\lambda - y)/\lambda$, then z_λ is the solution to the equation:

$$\frac{\partial z}{\partial t} + Az + \vec{V}\cdot\nabla z = 0 \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} = u(t)(\delta_{x_0+\lambda(x_1-x_0)} - \delta_{x_0})/\lambda \text{ on } \Sigma, \qquad z(0)=0 \text{ in } \Omega.$$

Under conditions (44) on θ, and applying Proposition 2.7, we deduce that y_λ and z_λ belong to $C([0,T];L^\theta_w(\Omega))$. Then $|y_\lambda(T)-y_d|^{\theta-2}(y_\lambda(T)-y_d)z_\lambda(T)$ belongs to $L^1(\Omega)$. From the convexity of the mapping $y\mapsto\int_\Omega|y(T)-y_d|^\theta\mathrm{d}x$, it follows that

$$0\le\frac{F(x_0+\lambda(x_1-x_0))-F(x_0)}{\lambda}\le\theta\int_\Omega|y_\lambda(T)-y_d|^{\theta-2}(y_\lambda(T)-y_d)z_\lambda(T)\mathrm{d}x.$$

Using the Green formula of Proposition 3.1 with $\vec{h}=0$, we obtain

$$0\le\frac{F(x_0+\lambda(x_1-x_0))-F(x_0)}{\lambda}\le\frac{1}{\lambda}\int_0^T(p_\lambda(x_0+\lambda(x_1-x_0))-p_\lambda(x_0))\,u\,\mathrm{d}t,$$

where p_λ is the solution to equation (53) with $p_T^\lambda = \theta|y_\lambda(T) - y_d|^{\theta-2}(y_\lambda(T) - y_d)$. We conclude with Lemma 4.2 by passing to the limit in the above inequality. □

References

[1] R.A. Adams, *Sobolev spaces*. Academic Press, New-York (1975).

[2] H. Amann, Dual semigroups and second order linear elliptic boundary value problems. *Israel J. Math.* **45** (1983) 225-254.

[3] S. Anita, Optimal control of parameter distributed systems with impulses. *Appl. Math. Optim.* **29** (1994) 93-107.

[4] M. Berggren, R. Glowinski and J.L. Lions, A Computational Approach to Controllability Issues for Flow-Related Models, Part 1. *Int. J. Comput. Fluid Dyn.* **7** (1996) 237-252.

[5] M. Berggren, R. Glowinski and J.L. Lions, A Computational Approach to Controllability Issues for Flow-Related Models, Part 2. *Int. J. Comput. Fluid Dyn.* **6** (1996) 253-247.

[6] E. Casas, J.-P. Raymond and H. Zidani, Pontryagin's principle for local solutions of control problems with mixed control-state constraints. *SIAM J. Control Optim.* **39** (2000) 1182-1203.

[7] E. Casas, M. Mateos and J.-P. Raymond, Pontryagin's principle for the control of parabolic equations with gradient state constraints. *Nonlinear Anal.* (to appear).

[8] E.J. Dean and P. Gubernatis, Pointwise Control of Burgers' Equation – A Numerical Approach. *Comput. Math. Appl.* **22** (1991) 93-100.

[9] Z. Ding, L. Ji and J. Zhou, Constrained LQR Problems in Elliptic distributed Control systems with Point observations. *SIAM* **34** (1996) 264-294.

[10] J. Droniou and J.-P. Raymond, Optimal pointwise control of semilinear parabolic equations. *Nonlinear Anal.* **39** (2000) 135-156.

[11] J.W. He and R. Glowinski, Neumann control of unstable parabolic systems: Numerical approach. *J. Optim. Theory Appl.* **96** (1998) 1-55.

[12] J.W. He, R. Glowinski, R. Metacalfe and J. Periaux, A numerical approach to the control and stabilization of advection-diffusion systems: Application to viscous drag reduction, Flow control and optimization. *Int. J. Comput. Fluid Dyn.* **11** (1998) 131-156.

[13] H. Henrot and J. Sokolowski, *Shape Optimization Problem for Heat Equation*. Rapport de recherche INRIA (1997).

[14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*. Springer-Verlag, Berlin/Heidelberg/New-York (1981).

[15] K.-H. Hoffmann and J. Sokolowski, Interface optimization problems for parabolic equations. *Control Cybernet.* **23** (1994) 445-451.

[16] J.-P. Kernevez, *The sentinel method and its application to environmental pollution problems*. CRC Press, Boca Raton (1997).

[17] J.-L. Lions, Pointwise control for distributed systems, in *Control and estimation in distributed parameters sytems*, edited by H.T. Banks. SIAM, Philadelphia (1992) 1-39.

[18] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Ural'ceva, Linear and quasilinear equations of parabolic type. AMS, Providence, RI, *Transl. Math. Monographs* **23** (1968).

[19] H.-C. Lee and O.Yu. Imanuvilov, Analysis of Neumann boundary optimal control problems for the stationary Boussinesq equations including solid media. *SIAM J. Control Optim.* **39** (2000) 457-477.

[20] P.A. Nguyen, *Optimal Control Localized on Thin Structure for Semilinear Parabolic Equations and the Boussinesq system*. Thesis, Toulouse (2000).

[21] P.A. Nguyen and J.-P. Raymond, Control Localized On Thin Structure For Semilinear Parabolic Equations. *Sém. Inst. H. Poincaré* (to appear).

[22] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, Berlin/Heidelberg/New-York (1983).

[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Tome 2, Fourier Analysis, Self-Adjointness. Academic Press, Inc. (1975).

[24] J.-P. Raymond and H. Zidani, Hamiltonian Pontryagin's principles for control problems governed by semilinear parabolic equations. *Appl. Math. Optim.* **39** (1999) 143-177.

[25] J. Simon, Compact Sets in the Space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* **196** (1987) 65-96.

[26] H. Triebel, *Interpolation Theory, Functions Spaces, Differential Operators*. North Holland Publishing Campany, Amsterdam/New-York/Oxford (1977).

[27] V. Vespri, *Analytic Semigroups Generated in $H^{-m,p}$ by Elliptic Variational Operators and Applications to Linear Cauchy Problems, Semigroup theory and applications*, edited by Clemens *et al.* Marcel Dekker, New-York (1989) 419-431.