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REMARKS ON WEAK STABILIZATION OF SEMILINEAR WAVE EQUATIONS

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Abstract. If a second order semilinear conservative equation with esssentially oscillatory solutions such as the wave equation is perturbed by a possibly non monotone damping term which is effective in a non negligible sub-region for at least one sign of the velocity, all solutions of the perturbed system converge weakly to 0 as time tends to infinity. We present here a simple and natural method of proof of this kind of property, implying as a consequence some recent very general results of Judith Vancostenoble.

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1. Introduction

Following a recent work of Vancostenoble [20], we investigate the weak stabilization to 0 of solutions to the equations

$$u_{tt} + Au + Q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{1.1}$$

$$u_{tt} + Au + g(u) + Q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{1.2}$$

where A is a linear positive selfadjoint operator of elliptic type on $H = L^2(\Omega)$, Ω is a bounded open domain of $\mathbb{R}^N$, the term $-Q(u_t)$ represents a possibly non monotone feedback dissipation acting on a "non negligible" part Y of $\overline{\Omega}$ and $g(u)$ stands for the Nemytsckii operator associated to some numerical function $g \in C^1(\mathbb{R})$. Concerning (1.1), the original proof from [20] was inspired both by the work of Slemrod [19] and the techniques of Conrad and Pierre [10]; here we present a new simplified proof relying on almost periodicity of generalized solutions to

$$u_{tt} + Au = 0 \quad \text{on} \quad \mathbb{R}$$

which implies some essential oscillatory behavior of those solutions on $\mathbb{R} \times Y$. Weak convergence is proved, following the philosophy introduced in [11] (*cf.* also [12,13]) under the hypothesis that the damping is effective at least for one sign of the velocity (one-sided dissipation). This method is applicable to more complicated problems of the form (1.2) when solutions of

$$u_{tt} + Au + g(u) = 0 \quad \text{on} \quad \mathbb{R}$$

are known to be oscillatory on $\mathbb{R} \times Y$. A typical example is the nonlinear string equation

$$u_{tt} - u_{xx} + g(u) + a(x)q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \Omega$$

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with homogeneous Dirichlet boundary conditions on $\partial\Omega$ with $\Omega = (0, L)$ when g is odd nonincreasing, $a \geq 0$, $a > 0$ on an open subdomain ω and q satisfies:

$$q \in C^1, \quad q(v)v \geq 0 \quad \text{on} \quad \mathbb{R}, \quad q(v) > 0 \quad \text{for all} \quad v > 0.$$

Here even if q is monotone, compactness of trajectories in the energy space is not known. The consideration of more general similar examples sheds a new light on the interest of oscillatory behavior of semilinear conservative systems.

2. Internal damping

In this section, we consider the case of equation (1.1) with internal damping, which means that $Y = \omega$, an open subset of Ω. In other terms we consider the equation

$$u_{tt} + Au + a(x)q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{2.1}$$

where $a \in L^\infty(\Omega), a \geq 0$ a.e. in Ω and $a \geq \eta > 0$ a.e. in ω. The function $q \in C(\mathbb{R})$ satisfies

$$q(v)v \geq 0 \quad \text{on} \quad \mathbb{R}, \quad q(v) > 0 \quad \text{for all} \quad v > 0. \tag{2.2}$$

We consider a Hilbert space $V \subset H = L^2(\Omega)$ with compact and dense imbedding. The linear operator $A : V \to V'$ satisfies the following conditions

$$A \in \mathcal{L}(V, V'); \quad \forall v \in V, < Av, v > \geq \alpha \|v\|^2$$

where $\alpha > 0$ and $\|v\|$ denotes the norm of v in V. Assume that

$$W = L^\infty(\Omega) \cap V \quad \text{is dense in} \quad V.$$

We say that a function $u : \mathbb{R}^+ \to V$ is a solution of (2.1) if u satisfies the following conditions:

$$u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^+, V'), \qquad a(x)q(u') \in L^1_{\text{loc}}(\mathbb{R}^+, L^1(\Omega))$$

$$\forall \varphi \in W, \quad < u"(t) + Au(t) + a(x)q(u'(t)), \varphi >= 0 \quad a.e. \quad \text{on} \quad \mathbb{R}^+.$$

In addition, we say that u satisfies the energy inequality if

$$\forall T > 0, \quad E(T) + \int_0^T \int_\Omega a(x)q(u'(t, x))u'(t, x)\mathrm{d}x\mathrm{d}t \leq E(0)$$

with

$$\forall t \geq 0, \quad E(t) := \frac{1}{2}\{|u'|^2(t) + < Au(t), u(t) >\}\cdot$$

Finally we say that unique determination of eigenfunctions of A holds in ω if

$$\forall \lambda > 0, \forall \varphi \in V, \quad A\varphi = \lambda\varphi \quad \text{and} \quad \varphi \equiv 0 \quad \text{in } \omega \quad \Longrightarrow \quad \varphi \equiv 0 \quad \text{in } \Omega$$

The main result of this section is:

Theorem 2.1. *Under the above hypotheses, let u be a solution of (2.1) satisfying the energy inequality and assume that unique determination of eigenfunctions of A holds in ω. Then as $t \to \infty$:*

$$(u(t), u_t(t)) \rightharpoonup (0, 0) \quad in\, V \times H.$$

Proof. Let t_n be a sequence of positive real numbers tending to $+\infty$ with n and $u_n(t,x) = u(t+t_n,x)$ for all $(t,x) \in [-t_n,+\infty) \times \Omega$. Given any $\tau > 0$, the function $u_n(t,x)$ is well defined a.e. on Ω as an element of V for all $t \in [-\tau,\tau] =: J_\tau$ as soon as $t_n \geq \tau$. In addition it follows easily from the energy inequality that u_n is bounded uniformly in $C(J_\tau,V) \cap C^1(J_\tau,H)$ for $n \geq \tau$. In particular, by Ascoli–Arzela's theorem, we can assume that a certain subsequence $u_{n_k} =: z_k$ converges in $C(J_\tau,H)$ for all $\tau > 0$, to a certain limiting function $z \in C(\mathbb{R},H)$. Moreover z is bounded in H and weakly differentiable $\mathbb{R} \to H$ with bounded derivative. From the energy inequality it also follows, by using continuity of q at 0, that

$$\forall \tau > 0, \quad a(x)q(u_n'(t,x)) \to 0 \quad \text{in} \quad L^1(J_\tau \times \Omega) \quad \text{as} \quad n \to \infty$$

for all $\tau > 0$. By using as test functions the eigenfunctions of A, it follows easily that z is in fact a solution of

$$z \in C(\mathbb{R},V) \cap C^1(\mathbb{R},H) \cap C^2(\mathbb{R},V'), \qquad z'' + Az = 0.$$

In particular, z is a C^1 almost periodic vector function: $\mathbb{R} \to H$, *cf.* *e.g.* [1,3,16]. From (2.2) we infer that in fact

$$z' = z_t \leq 0 \quad \text{a.e. on} \quad \mathbb{R} \times \omega. \tag{2.3}$$

Assuming (2.3) the conclusion follows easily. Indeed then the trace of z on ω is a non-increasing function: $\mathbb{R} \to L^2(\omega)$. Classically, such a function has to remain constant with respect to t for almost all $x \in \omega$ (this can be checked easily on multiplying by any smooth nonnegative function supported in ω and applying a classical recurrence property of real-valued almost periodic vector function, *cf.* *e.g.* [1,3] or Cor. 4.2.6, p. 50 of [16], or even Cor. I.3.1.6 of [15]), therefore if we consider (*cf.* *e.g.* [1,18]) the Fourier–Bohr expansion of z given by the formula

$$z(t,x) = \sum_{n=1}^{\infty} \left[\varphi_n(x)\cos\left(t\sqrt{\lambda_n}\right) + \psi_n(x)\sin\left(t\sqrt{\lambda_n}\right)\right]$$

where $\{\lambda_n\}_{n\geq 1}$ is the increasing sequence of eigenvalues of A and

$$\varphi_n(x) = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \cos(t\sqrt{\lambda_n})z(t,x)\mathrm{d}x$$

$$\psi_n(x) = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \sin(t\sqrt{\lambda_n})z(t,x)\mathrm{d}x$$

the functions $\varphi_n(x)$ and $\psi_n(x)$ are eigenfunctions of A which vanish in ω. By the unique determination of eigenfunctions of A in ω the result follows at once. Now (2.3) will follow as an easy consequence of the following

Lemma 2.2. *Let (U,μ) be any finitely measured space and $w_n \in L^p(U,\mathrm{d}\mu)$ with $p > 1$. Assume*

$$w_n \rightharpoonup w \quad \text{in } L^p(U,\mathrm{d}\mu) \quad \text{as} \quad n \to \infty \tag{2.4}$$

$$\mu\{w_n \geq 0\} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Then we have:

$$\mu\{w > 0\} = 0. \tag{2.6}$$

Proof of Lemma 2.2. Let $y_n = \inf\{w_n, 0\} = -w_n^- \leq 0$. We have

$$\|y_n - w_n\|_{L^1(U)} \leq \|w_n\|_{L^p(U)} \left[\mu\{w_n \geq 0\}\right]^{1-\frac{1}{p}} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.7}$$

In particular we have

$$y_n \rightharpoonup w \quad \text{in } L^1(U,\mathrm{d}\mu) \quad \text{as} \quad n \to \infty. \tag{2.8}$$

Since by construction, $y_n \leq 0, \quad \mu.a.e. \quad \text{on} \quad U, \quad$ (2.6) follows immediately.

End of proof of Theorem 2.1. From (2.2) we finally deduce (2.3) as follows. Let $\tau > 0$ be fixed and set $U = J_\tau \times \omega$ and denote by μ the Lebesgue measure on U in $\mathbb{R}^{N+1}$. We establish that $w = z' \leq 0$, a.e. on U. In order to do that it is sufficient to establish, for any given $\varepsilon > 0$, the inequality $w = z' \leq \varepsilon$, μ – a.e. on U. First, given any $\delta > 0$ we select $M = M(\delta)$ such that

$$\forall n \geq \tau, \quad \mu\{(t,x) \in U, z_n'(t,x) \geq M\} \leq \delta.$$

This is made possible by boundedness of u' in $L^2(\Omega)$. In particular we have

$$\forall n \geq \tau, \quad \mu\{(t,x) \in U, z_n'(t,x) \geq \varepsilon\} \leq \delta + \mu\{(t,x) \in U, \varepsilon \leq z_n'(t,x) \leq M\}\cdot$$

As a consequence of (2.2) and by compactness of $[\varepsilon, M]$ it now follows easily from the properties $a \geq \eta > 0$ a.e. in ω and $a(x)q(u_n'(t,x)) \to 0 \quad \text{in}\, L^1(U) \quad \text{as} \quad n \to \infty$, that

$$\lim_{n\to\infty} \mu\{(t,x) \in U, \varepsilon \leq z_n'(t,x) \leq M\} = 0.$$

Therefore

$$\limsup_{n\to\infty} \mu\{(t,x) \in U, z_n'(t,x) \geq \varepsilon\} \leq \delta.$$

Since $\delta > 0$ is arbitrary, this means

$$\lim_{n\to\infty} \mu\{(t,x) \in U, z_n'(t,x) \geq \varepsilon\} = 0.$$

By Lemma 2.2 applied with $w_n = z_n' - \varepsilon$ we deduce $z' \leq \varepsilon, \quad \mu$ – a.e. on U. The proof is now complete. □

3. The general case

In this section, we consider the case of equation (1.1) with a damping possibly distributed on a lower dimensional subset. For instance Y can be a relatively open subset of $\partial\Omega$, in which case (1.1) can take the form of a wave equation with boundary dissipation

$$u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^+ \times \Omega; \quad \frac{\partial u(t,x)}{\partial \nu} + a(x)q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega \tag{3.1}$$

considered in [21] by Vancostenoble.

In the general case we consider a function $q \in C(\mathbb{R})$ satisfying (2.2) and the stronger condition

$$\forall \varepsilon > 0, \quad \inf_{s \geq \varepsilon} q(s) > 0. \tag{3.2}$$

We consider a Hilbert space $V \subset H = L^2(\Omega)$ with compact and dense imbedding. The linear operator $A : V \to V'$ satisfies the following conditions

$$A \in \mathcal{L}(V, V'); \quad \forall v \in V, < Av, v > \geq \alpha \|v\|^2$$

where $\alpha > 0$ and $\|v\|$ denotes the norm of v in V. Assume that

$$W = C(\overline{\Omega}) \cap V \quad \text{is dense in} \quad V.$$

In addition we consider a compact subset Y of $\overline{\Omega}$ and a nonegative bounded measure $\mu \in M_B(Y)$. We say that a function $u : \mathbb{R}^+ \to V$ is a solution of (1.1) if u satisfies the following conditions:

$$u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+, V')$$

$$a(y)q(u'(t,y)) \in L^1_{\mathrm{loc}}(\mathbb{R}^+, L^1(Y, \mathrm{d}\mu))$$

$$\forall \varphi \in W, \quad \langle u"(t) + Au(t), \varphi\rangle + \int_Y a(y)q(u'(t,y))\varphi(y)\mathrm{d}\mu(y) = 0 \quad a.e. \quad \text{on} \quad \mathbb{R}^+.$$

In addition, we say that u satisfies the energy inequality if

$$\forall T > 0, \quad E(T) + \int_0^T \int_Y a(y)q(u'(t,y))u'(t,y)\mathrm{d}\mu(y)\mathrm{d}t \leq E(0)$$

with

$$\forall t \geq 0, \quad E(t) := \frac{1}{2}\{|u'|^2(t) + \langle Au(t), u(t)\rangle\}\cdot$$

Finally we say that unique determination of eigenfunctions of A holds in $\omega \subset Y$ if

$$\forall \lambda > 0, \forall \varphi \in V, \quad A\varphi = \lambda\varphi \quad \text{and} \quad \varphi \equiv 0 \quad \mu-\text{a.e.} \quad \text{in } \omega \quad \Longrightarrow \quad \varphi \equiv 0 \quad \text{in } \Omega.$$

The main result of this section is:

Theorem 3.1. *Under the above hypotheses, assume that unique determination of eigenfunctions of A holds in ω with $\inf_{y\in\omega} a(y) > 0$. In addition assume that the trace $z \longrightarrow z|Y$ is well defined and continuous: $V \longrightarrow L^1(Y, \mathrm{d}\mu)$. Let u be a solution of (1.1) satisfying the energy inequality. Then as $t \to \infty$:*

$$(u(t), u_t(t)) \rightharpoonup (0,0) \quad in V \times H.$$

Proof of Theorem 3.1. Let t_n be a sequence of positive real numbers tending to $+\infty$ with n and $u_n(t,x) = u(t + t_n, x)$ for all $(t,x) \in [-t_n, +\infty) \times \Omega$. Keeping the notation of Section 2, by Ascoli–Arzela's theorem, we can assume that a certain subsequence $u_{n_k} =: z_k$ converges in $C(J_\tau, H)$ for all $\tau > 0$, to a limiting function $z \in C(\mathbb{R}, H)$. Moreover z is bounded in H and weakly differentiable $\mathbb{R} \to H$ with bounded derivative. From the energy inequality it also follows, by using continuity of q at 0, that

$$\forall \tau > 0, \quad a(y)q(u_n'(t,y)) \to 0 \quad \text{in} \quad L^1(J_\tau \times Y) \quad \text{as} \quad n \to \infty.$$

By using as test functions the eigenfunctions of A, it follows easily that z is in fact a solution of

$$z \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H) \cap C^2(\mathbb{R}, V'), \qquad z" + Az = 0.$$

In particular, z is a C^1 almost periodic vector function: $\mathbb{R} \to H$. However in the general case the analog of (2.3) is more delicate to establish and in fact, in order to use the trace operator: $V \longrightarrow L^1(Y, \mathrm{d}\mu)$ we shall rely on a smoothing procedure replacing $u_{n_k} =: z_k$ by some auxiliary functions which have bounded time-derivatives in V. For any $\delta > 0$, we consider

$$u_\delta(t) := \int_t^{t+\delta} u(s)\mathrm{d}s$$

and we define accordingly $u_{\delta,n}(t)$ and $z_\delta(t)$. From (2.2) and (3.2) we infer that in fact

$$z_\delta'(t,y) \leq 0 \quad \mu-\text{a.e. on} \quad \mathbb{R} \times \omega. \tag{3.3}$$

In order to establish (3.3), first of all from the energy inequality we deduce

$$\int_{-\tau}^{\tau}\int_{\omega}(u_n' - \varepsilon)^+(t,y)\mathrm{d}\mu(y)\mathrm{d}t \to 0$$

valid for all $\varepsilon > 0$. On the other hand we have for each $\delta \in (0,\tau)$

$$u_{\delta,n}'(t,y) - \delta\varepsilon \le \int_t^{t+\delta}(u_n' - \varepsilon)^+(t,y)\mathrm{d}s$$

almost-everywhere on Ω and in particular for any nonnegative function $\zeta \in L^\infty(\omega, \mathrm{d}\mu)$ we find, since ε is arbitrarily small

$$\forall t \in \mathbb{R}, \quad \limsup_{n\to\infty}\int_{\omega}(u_{\delta,n}'(t,y)\zeta(y)\mathrm{d}\mu(y)\mathrm{d}t \le 0.$$

Now since $u_{\delta,n}'(t,x) = u_n(t+\delta,x) - u_n(t,x)$, the convergence of to $z(t,.)$ in V weak implies the convergence pointwise in t of $u_{\delta,n}'(t,.)$ to $z_\delta'(t,.)$ in V weak. Since V is a Hilbert space, there is, for each given t, a convex combination of the functions $u_{\delta,n}'(t,.)$ which converges in fact to $z_\delta'(t,.)$ in V strong. By continuity of the trace: $V \to L^1(Y,\mathrm{d}\mu)$ we obtain (3.3), more precisely we find

$$\forall t \in \mathbb{R}, \quad \forall \zeta \in L_+^\infty(\omega,\mathrm{d}\mu), \quad \int_{\omega} z_\delta'(t,y)\zeta(y)\mathrm{d}\mu(y)\mathrm{d}t \le 0.$$

Now the conclusion follows easily. Indeed then the trace of z_δ on ω is a non-increasing almost periodic function: $\mathbb{R} \to L^1(\omega,\mathrm{d}\mu)$ which is also the trace of a solution of the linear equation. Classically, such a function has to remain constant, and by the unique determination of eigenfunctions of A in ω, reasoning as in the proof of Theorem 2.1, we find that $z_\delta = 0$ for all $\delta > 0$. By letting $\delta \to 0$ we obtain $z = 0$. Since the result is valid for any convergent subsequence of (u_n, u_n') we conclude easily.

4. Additional results and remarks

The method of proof of Theorems 2.1 and 3.1 is applicable to more complicated problems of the form (1.2) when solutions of

$$u_{tt} + Au + g(u) = 0 \quad \text{on} \quad \mathbb{R}$$

are known to be oscillatory on $\mathbb{R} \times Y$. As a typical example we consider the nonlinear string equation

$$u_{tt} - u_{xx} + g(u) + a(x)q(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times (0,L); \quad u(t,0) = u(t,L) \quad \text{on} \quad \mathbb{R}^+ \tag{4.1}$$

when g is odd nonincreasing, $a \in L^\infty(0,L)$, $a \ge 0$, $a(x) \ge \alpha > 0$ on some open subdomain ω and q satisfies:

$$q \in C^1, \quad q(v)v \ge 0 \quad \text{on} \quad \mathbb{R}, \quad q(v) > 0 \quad \text{for all} \quad v > 0.$$

Here we obtain:

Theorem 4.1. *Under the above hypotheses, let u be a solution of (4.1) satisfying the energy inequality*

$$\forall T > 0, \quad E(T) + \int_0^T\int_0^L a(x)q(u_t(t,x))u_t(t,x)\mathrm{d}x\mathrm{d}t \le E(0)$$

with

$$\forall t \ge 0, \quad E(t) := \frac{1}{2}\int_0^L \{u_t^2(t,x) + u_x^2(t,x)\}\mathrm{d}x + \int_0^L G(u(t,x)\mathrm{d}x$$

where

$$G(r) := \int_0^r g(s)\mathrm{d}s$$

Then as $t \to \infty$*:*

$$(u(t), u_t(t)) \rightharpoonup (0,0) \quad in\, V \times H$$

with

$$V = H_0^1(0,L) \quad and \quad H = L^2(0,L).$$

Proof of Theorem 4.1. Let $\Omega = (0,L)$, let t_n be a sequence of positive real numbers tending to $+\infty$ with n and $u_n(t,x) = u(t+t_n, x)$ for all $(t,x) \in [-t_n, +\infty) \times \Omega$. Keeping the notation of Section 2, we obtain that a certain subsequence $u_{n_k} =: z_k$ converges in $C(J_\tau, H)$ for all $\tau > 0$, to a certain limiting function $z \in C(\mathbb{R}, H)$. From the energy inequality it also follows, by using continuity of q at 0, that

$$\forall \tau > 0, \quad a(x)q(u_n'(t,x)) \to 0 \quad \text{in} \quad L^1(J_\tau \times \Omega) \quad \text{as} \quad n \to \infty.$$

It follows easily that z is in fact a solution of

$$z \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H) \cap C^2(\mathbb{R}, V'),$$

$$z_{tt} - z_{xx} + g(z) = 0 \quad \text{on} \quad \mathbb{R} \times (0,L)$$

with in addition

$$z' = z_t \leq 0 \quad \text{a.e. on} \quad \mathbb{R} \times \omega.$$

As a consequence of [2, 3], it follows that $z = 0$. The conclusion then follows easily from the fact that any sequence $(u(t_n), u'(t_n))$ has a subsequence converging weakly to (0, 0).

Remark 4.2. Here even if q is monotone, compactness of trajectories in the energy space is not known.

Remark 4.3. When $q(s) = cs$ for some $c > 0$, compactness of positive trajectories in the energy space is satisfied as a special case of the classical theorem of Webb [22]. Indeed then the equation

$$u_{tt} - u_{xx} + ca(x)(u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times (0,L); \quad u(t,0) = u(t,L) \quad \text{on} \quad \mathbb{R}^+$$

generates an exponentially damped linear semi-group in $V \times H$ and the Nemytskii operator $u \to g(u)$ is compact $V \to H$.

Remark 4.4. The method of proof of Theorems 2.1 and 2.2 applies also to the more qeneral case of the equation

$$u_{tt} + Au + Q(t, u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \tag{4.2}$$

where $Q(t, u_t)$ is realized in the form

$$a(t,y)q(t,y,u_t)$$

with

$$\inf_{y \in \omega, t \geq 0} a(t,y) > 0$$

when q satisfies the uniform conditions

$$\forall \varepsilon > 0, \quad \inf_{s \geq \varepsilon, y \in \omega, t \geq 0} q(t,y,s) > 0$$

and

$$\lim_{\varepsilon \to 0} \sup_{|s| \leq \varepsilon, y \in \omega, t \geq 0} |q(t,y,s)| = 0.$$

This is in particular applicable to the problems

$$u_{tt} - \Delta u + a(x)\tilde{q}(x, \nabla u, u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \Omega; \quad u(t,x) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega$$

and

$$u_{tt} - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^+ \times \Omega; \quad \frac{\partial u(t,x)}{\partial \nu} + a(x)\tilde{q}(x, \nabla u, u_t) = 0 \quad \text{on} \quad \mathbb{R}^+ \times \partial\Omega$$

with

$$\tilde{q}(x, \nabla u, u_t) = \tilde{q}(y, \nabla u(t,y), u_t(t,y)) =: q(t, y, u_t).$$

In this case we recover some recent results of Vancostenoble [21] which generalize Slemrod [19].

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