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REMARKS ON EXACT CONTROLLABILITY FOR THE NAVIER-STOKES EQUATIONS*

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Abstract. We study the local exact controllability problem for the Navier-Stokes equations that describe an incompressible fluid flow in a bounded domain Ω with control distributed in a subdomain $\omega \subset \Omega \subset \mathbb{R}^n, n \in \{2,3\}$. The result that we obtained in this paper is as follows. Suppose that $\hat{v}(t,x)$ is a given solution of the Navier-Stokes equations. Let $v_0(x)$ be a given initial condition and $\|\hat{v}(0,\cdot)-v_0\| < \varepsilon$ where ε is small enough. Then there exists a locally distributed control $u, \operatorname{supp} u \subset (0,T) \times \omega$ such that the solution $v(t,x)$ of the Navier-Stokes equations:

$$\partial_t v - \Delta v + (v,\nabla)v = \nabla p + u + f,\ \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0,\ v|_{t=0} = v_0$$

coincides with $\hat{v}(t,x)$ at the instant $T: v(T,x) \equiv \hat{v}(T,x)$.

Résumé. On étudie le problème de contrôlabilité locale exacte pour les équations de Navier-Stokes incompressibles dans un domaine Ω borné avec un contrôle réparti dans un sous-domaine $\omega \subset \Omega \subset \mathbb{R}^n, n \in \{2,3\}$. On obtient le résultat suivant. Supposons que $\hat{v}(t,x)$ soit une solution des équations de Navier-Stokes et $v_0(x)$ une condition initiale telle que $\|\hat{v}(0,\cdot) - v_0\| < \varepsilon$ pour ε assez petit. On montre alors qu'il existe un contrôle localement réparti u, $\operatorname{supp} u \subset [0,T] \times \omega$, tel que la solution $v(t,x)$ des équations de Navier-Stokes:

$$\partial_t v - \Delta v + (v,\nabla)v = \nabla p + u + f, \quad \operatorname{div} v = 0, \quad v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0$$

coïncide avec $\hat{v}(t,x)$ au temps T: $v(T,x) \equiv \hat{v}(T,x)$.

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The paper is concerned on the local exact controllability of the Navier-Stokes equations, defined on the bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2,3)$ with boundary $\partial\Omega \in C^2$. More precisely, the investigated problem is as follows. Let us consider the nonstationary Navier-Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)\, v + \nabla p = f + \chi_\omega u \ \text{ in } \Omega, \quad \operatorname{div} v = 0, \tag{1}$$

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with boundary and initial conditions

$$v|_{\Sigma} = 0, \quad v\,|_{t=0} = v_0(x), \tag{2}$$

where $v(t,x) = (v_1(t,x),...,v_n(t,x))$ is a velocity of fluid, p is a pressure, $f(t,x) = (f_1(t,x),...,f_n(t,x))$ is a density of external forces, $u(t,x)$ is a control distributed in some an arbitrary fixed subdomain ω of the domain Ω, and χ_ω- is a characteristic function of the set ω:

$$\chi_\omega(x) = \begin{cases} 1, & \text{for } x \in \omega \\ 0, & \text{for } x \in \Omega \setminus \omega. \end{cases}$$

Let $(\hat v(t,x), \hat p(t,x))$ be a solution of the Navier-Stokes equations with the right hand side f exactly same as in (1):

$$\partial_t \hat v - \Delta \hat v + (\hat v, \nabla)\hat v + \nabla \hat p = f \text{ in } \Omega, \quad \operatorname{div} \hat v = 0, \quad \hat v|_{\Sigma} = 0 \tag{3}$$

close enough to the initial condition v_0 at the moment $t = 0$

$$\|v_0 - \hat v(0,\cdot)\|_{V^1(\Omega)} \le \varepsilon, \qquad \text{(the parameter } \varepsilon \text{ is sufficiently small)} \tag{4}$$

where $V^1(\Omega) = \{v(x) = (v_1, \dots, v_n) \in (W_2^1(\Omega))^n : \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0\}$.

One needs to find a control u such that for the given $T > 0$ equality holds

$$v(T,x) = \hat v(T,x). \tag{5}$$

We assume:

Condition 1. *The boundary $\partial\Omega = \Gamma = \cup_{i=1}^N \Gamma_i \in C^2$, ($\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \ne j$) where Γ_i is a n-1 dimensional compact connected manifold of class C^2.*

In order to formulate our results we introduce the following functional spaces

$$V^0(\Omega) = \{v(x) = (v_1, \dots, v_n) \in (L^2(\Omega))^n : \operatorname{div} v = 0,\ (v,\nu)|_{\partial\Omega} = 0\},$$

$$V^{1,2}(Q) = \{v(t,x) \in (W_2^{1,2}(Q))^n : \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0\},$$

where $\nu = \nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the outward unit normal to $\partial\Omega$.

The main result of this paper is the following theorem:

Theorem 1. *Let $v_0 \in V^1(\Omega), f \in L^2(0,T;V^0(\Omega))$ and suppose that the pair $(\hat v, \hat p) \in W^1_\infty(0,T;(V^1(\Omega) \bigcap (W^1_\infty(\Omega))^n)) \times L^2(0,T;W_2^1(\Omega))$ is a given solution of the Navier-Stokes equations* (3) *with the right hand side f. Then for sufficiently small $\varepsilon > 0$ there exists a solution $(v,p,u) \in V^{1,2}(Q) \times L^2(0,T;W_2^1(\Omega)) \times (L^2(Q_\omega))^n$ to problem (1, 2, 4, 5).*

The aim of this paper is to remove some technical conditions which appeared in the same result previously proved by the author in [19]. The result of the previous paper is improved now in several directions. First we omit the condition that the function $\hat v$ vanished in some neighborhood of the boundary $\partial\Omega$. Second, we do not assume that the domain Ω diffeomorphic to the unit sphere. And finally, now the function $\hat v$ is not supposed to be a steady-state solution to the Navier-Stokes equations.

This paper is organized as follows. To prove Theorem 1 we used a version of the implicit function theorem. The only one nontrivial condition to be checked is to show that the derivative of the corresponding mapping at some point is an epimorphism. In our case this problem is equivalent to the zero controllability of the linearization of the Navier-Stokes equations at the point $\hat v$ (see problem (3.1–3.3)). Sections 1–3 are devoted to this problem. One of the usual ways to solve the controllability problem is to reduce it to a observability one for the adjoint equation. So in Section 2 we introduced a linear operator (see Eq. (2.1)) which after change $t \to -t$ is formally the adjoint of the derivative of the Navier-Stokes equations at the point $\hat{v}(t,x)$. The observability problem for this operator is solved in three steps. First in Theorem 1.2 we got an appropriate estimate for the pressure p. Then in Theorem 2.1 we obtained a Carleman estimate for the velocity v of the fluid *via* a weighted L^2-norm of density of external forces f and L^2-norm of the pressure p on the subdomain $(0,T) \times \omega$. And finally in Theorem 2.2 we proved an estimate (non Carleman type) for the velocity where in the right hand side pressure and an initial condition are absent. In the Section 3 this observability estimate was converted into the controllability result of Theorem 3.1. In Section 4 all conditions required by the implicit function theorem are checked.

We close this introductory section by mentioning some of the previous results regarding our problems. The case of the local exact controllability for the Navier-Stokes equations with boundary and local distributed control has been studied in papers [9,11,12,21] and for the Boussinesq system in [10,14]. On the other hand, in pioneering works [3–5] Coron proved the global approximate controllability for the 2-D Euler equations and the 2-D Navier-Stokes equations with slip boundary conditions. Later this result was extended for the case of 3-dimensional Euler equation in [15,16]. In [6] combining results on global approximate and local exact controllability results, Coron and Fursikov obtained the global exact controllability for the Navier-Stokes system on a 2-D manifold without boundary. The similar result for the Boussinesq system on torus was obtained recently in [14]. In [7] Fabre obtained an approximate controllability of “cut off” Navier-Stokes equations.

1. Estimate of the pressure

It is well-known (see [30]) that for the Stokes system the pressure p is a harmonic function in x for each fixed t. If we consider the linearization of the incompressible Navier-Stokes system at the point $\hat{v}$ the pressure is the solution of the Laplace equation which right hand side some function of $\hat{v}, v$ and derivatives of these functions. Unfortunately in the case when the velocity v has zero boundary conditions, there are no explicit boundary condition on the pressure p. To solve observability problem in this section we will prove a new Carleman estimate for the general second order elliptic equation. The main purpose of this Carleman estimate is to “minimize” (in terms of power of s) the weight which corresponds to the term with the L^2-norm of the trace of the pressure on the boundary (see estimate (1.7)). To achieve this goal we will sacrifice the weight corresponding to the L^2-norm of the right hand side of the elliptic operator. The whole chapter is devoted to establish this estimate.

Let ω be an arbitrary subdomain of Ω. Denote $Q = (0,T) \times \Omega, Q_\omega = (0,T) \times \omega, \Sigma = (0,T) \times \partial\Omega$. In this paper we use the following functional spaces. Recall, that $W_p^k(\Omega),\ k \geq 0,\ 1 \leq p < \infty$ is the Sobolev space of functions with finite norm

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega \left| \partial^{|\alpha|} u(x) \,/ \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \right|^p dx \right)^{1/p},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)\,,\ |\alpha| = \alpha_1 + \dots + \alpha_n$,

$$W^k_{p,Per}(K) = \left\{ u \in W_p^k(K) | u(..., x_i + 2\ell, ..) = u(x)\ i \in \{1, \dots, n\} \right\},$$

where $K = \prod_{i=1}^n [-\ell, \ell]$,

$$W_2^{1,2}(Q) = \left\{ w(t,x) | w \in L^2(0,T; W_2^2(\Omega)),\ \frac{\partial w}{\partial t} \in L^2(0,T; L^2(\Omega)) \right\},$$

$$V^{-1}(\Omega) = (V^1(\Omega))^*,$$

$$L^2(Q,\rho)=\left\{v(t,x):\int_Q \rho v^2\,dx\,dt<\infty\right\}.$$

In the domain Ω we consider the elliptic equation

$$Ay=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial y}{\partial x_i}+c(x)y=f\quad\text{in }\Omega,\tag{1.1}$$

$$y|_{\partial\Omega}=g.\tag{1.2}$$

In the above problem we assume

$$a_{ij}\in C^2(\overline{\Omega}),\quad a_{ij}(x)=a_{ji}(x),\quad b_i\in C^1(\overline{\Omega}),\quad c\in L^\infty(\Omega),\tag{1.3}$$

where $i,j=\{1,\dots,n\}$ and the uniform ellipticity: there exists $\beta>0$ such that

$$a(x,\zeta,\zeta)=\sum_{i,j=1}^{n}a_{ij}(x)\zeta_i\zeta_j\geq\beta|\zeta|^2\quad\forall\zeta\in\mathbb{R}^n,\quad x\in\Omega.\tag{1.4}$$

To formulate our Carleman estimate we need a special weight function.

We have:

Lemma 1.1 [2, 20]. *Let $\omega_1\Subset\omega$ be an arbitrary fixed subdomain of Ω. Then there exists a function $\psi\in C^2(\overline{\Omega})$ such that*

$$\psi(x)>0\;\forall x\in\Omega,\;\psi|_{\partial\Omega}=0,\quad|\nabla\psi(x)|>0\quad\forall x\in\Omega\setminus\omega_1.\tag{1.5}$$

Using the function $\psi(x)$ constructed in Lemma 1.1 we introduce the function φ by formula

$$\varphi(x)=e^{\lambda\psi(x)},\tag{1.6}$$

where $\lambda>1$ is a parameter to be fixed below.

Also we need the following result:

Theorem 1.1 [2, 20]. *Let (1.3, 1.4) be fulfilled, $\psi\in C^2(\overline{\Omega}),|\nabla\psi(x)|\neq0$, in $\overline{\Omega\setminus\omega_1}$ and let φ be the function defined by (1.6). Then there exists a number $\hat\lambda>1$ such that for an arbitrary $\lambda\geq\hat\lambda$ there exists $s_0(\lambda)$ such that for each $s\geq s_0(\lambda)$*

$$\int_\Omega(s|\nabla y|^2+s^3y^2)e^{2s\varphi}dx\leq C_1\left(\int_\Omega|Ay|^2e^{2s\varphi}dx+\int_{\Gamma_1}\left|\frac{\partial y}{\partial\nu}\right|^2e^{2s\varphi}d\,\sigma+\int_{\omega_1}s^3y^2e^{2s\varphi}\,dx\right)$$
$$\forall\,y\in W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega),$$

where the constant C_1 depends continuously on λ and $\Gamma_1=Int(\partial\Omega\setminus\{x\in\partial\Omega|a(x,\nu,\nabla\psi)\leq0\})$.

As it was mentioned above the main purpose of this chapter is to prove the following theorem:

Theorem 1.2. *Let (1.3, 1.4) be fulfilled and functions ψ, φ, be defined as in (1.5, 1.6). Then there exists a number $\hat\lambda>0$ such that for an arbitrary $\lambda\geq\hat\lambda$ there exists $s_0(\lambda)$ such that for each $s\geq s_0(\lambda)$ the solutions* *to problem (1.1, 1.2) satisfy the following inequality:*

$$\int_\Omega (s^{\frac{5}{4}}y^2 + s^{-\frac{3}{4}}|\nabla y|^2)e^{2s\varphi}dx \le C_2\left(\|g\|^2_{W_2^{\frac{1}{2}}(\partial\Omega)}e^{2s} + \int_\Omega s^{-\frac{1}{4}}f^2e^{2s\varphi}dx + \int_{\omega_1} s^2y^2e^{2s\varphi}dx\right), \tag{1.7}$$

where constant C_2 is independent of s.

To keep the proof of this theorem transparent we will separate it on several steps. To prove Theorem 1.2 let us first consider the following auxiliary problem

$$Lz = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial z}{\partial x_i} + c(x)z = f \quad \text{in } G, \tag{1.8}$$

$$z(x_1,\dots,x_i+2\ell,\dots) = z(x_1,\dots,x_i\dots) \quad i\in\{2,\dots,n\}, \tag{1.9}$$

$$z(0,x') = g(x'), \quad \frac{\partial z(-1,x')}{\partial x_1} = z(-1,x') = 0, \tag{1.10}$$

where $x' = (x_2,\dots,x_n)$, $K = \prod_{i=1}^{n-1}[-\ell,\ell], G = [-1,0]\times K$, and $\ell > 0$ be an arbitrary fixed number.
Assume the following condition:

Condition 1.1. *We assume that*

$$\begin{aligned} a_{11}(x)\equiv 1, \quad a_{ij}(x_1,\dots,x_k+2\ell,\dots) &= a_{ij}(x), \quad b_i(x_1,\dots,x_k+2\ell,\dots) = b_i(x),\\ c(x_1,\dots,x_k+2\ell,\dots) &= c(x) \quad \forall\, i,j\in\{1,\dots,n\}, \quad k\in\{2,\dots,n\},\\ a_{ij}\in C^2(\overline{G}), \quad a_{ij}(x) &= a_{ji}(x), \quad b_i\in C^1(\overline{G}), \quad c\in L^\infty(G), \end{aligned}$$

where $i,j\in\{1,\dots,n\}$ and the uniform ellipticity: there exists $\beta>0$ such that

$$a(x,\zeta,\zeta) = \sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j \ge \beta|\zeta|^2 \quad \forall\zeta\in\mathbb{R}^n, \quad x\in G.$$

Denote

$$\phi(x_1) = e^{\lambda\tilde\psi(x_1)}, \quad \tilde\psi(x_1) = -x_1, \tag{1.11}$$

where $\lambda > 1$ is some parameter.

Lemma 1.2. *Let Condition 1.1 be fulfilled and the function ϕ be defined as in (1.11). Then there exists a number $\hat\lambda > 1$ such that for an arbitrary $\lambda \ge \hat\lambda$ there exists $s_0(\lambda)$ that for $s \ge s_0(\lambda)$ the solutions to problem (1.8–1.10) satisfy the following inequality:*

$$\int_G \left(s^{\frac{1}{4}}|\nabla z|^2 + s^{\frac{9}{4}}z^2\right)e^{2s\phi}\,dx \le C_3\left(\int_K (|\nabla_{x'}g|^2+g^2)e^{2s\phi(0)}dx' + \int_G s^{-\frac{1}{4}}f^2e^{2s\phi}\,dx\right), \tag{1.12}$$

where constant C_3 is independent of s.

Proof. Denote $w = e^{s\phi}z, \tilde f = e^{s\phi}f$. Then by (1.8)

$$\mathbf{L}w = e^{s\phi}Le^{-s\phi}w = \tilde f.$$

The operator $\mathbf{L}$ can be written explicitly as follows

$$\mathbf{L}w = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) - 2s\lambda\phi a(x,\vec{e}_1,\nabla w) + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)w$$
$$- s^2\lambda^2\phi^2 a(x,\vec{e}_1,\vec{e}_1)w + \sum_{i=1}^{n} b_i\frac{\partial w}{\partial x_i} + cw - s\lambda\phi\sum_{i=1}^{n}\frac{\partial a_{i1}}{\partial x_i}w + s\lambda\phi b_1 w. \tag{1.13}$$

We recall that the quadratic form $a(x,\xi,\eta)$ was defined in (1.4). We introduce the operators L_1, L_2 by formulas

$$L_1 w = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) - s^2\lambda^2\phi^2 a(x,\vec{e}_1,\vec{e}_1)w, \tag{1.14}$$
$$L_2 w = -2s\lambda\phi a(x,\vec{e}_1,\nabla w) + 2s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)w. \tag{1.15}$$

By (1.13–1.15), using the new notations we have

$$L_1 w + L_2 w = f_s \quad \text{in} \quad G, \tag{1.16}$$

where

$$f_s(x) = fe^{s\phi} + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)w - \sum_{i=1}^{n} b_i\frac{\partial w}{\partial x_i} - cw + s\lambda\phi\sum_{i=1}^{n}\frac{\partial a_{i1}}{\partial x_i}w - s\lambda\phi b_1 w. \tag{1.17}$$

We set $G_\tau = [-1,\tau]\times K$, $\tau\in(-1,0)$.

Taking the $L^2(G_\tau)$ – norm of both sides of (1.16) we obtain

$$||f_s||^2_{L^2(G_\tau)} = ||L_1 w||^2_{L^2(G_\tau)} + ||L_2 w||^2_{L^2(G_\tau)} + 2(L_1 w, L_2 w)_{L^2(G_\tau)}. \tag{1.18}$$

By (1.14) and (1.15) we have the following equality:

$$(L_1 w, L_2 w)_{L^2(G_\tau)} = \left(-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) - \lambda^2 s^2\phi^2 a(x,\vec{e}_1,\vec{e}_1)w, 2s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)w\right)_{L^2(G_\tau)}$$
$$+ \int_{G_\tau} 2\lambda^3 s^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)wa(x,\vec{e}_1,\nabla w)dx$$
$$+ \int_{G_\tau}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right)2s\lambda\phi a(x,\vec{e}_1,\nabla w)dx = A_0 + A_1 + A_2. \tag{1.19}$$

Integrating by parts in the first term of the right hand side of (1.19) we obtain

$$
\begin{aligned}
A_0 &= \left(- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial w}{\partial x_j} \right) - \lambda^2 s^2 \phi^2 a(x, \vec{e}_1, \vec{e}_1) w, 2s\lambda^2 \phi a(x, \vec{e}_1, \vec{e}_1) w \right)_{L^2(G_\tau)} \\
&= \int_{G_\tau} \Big(- 2s^3 \phi^3 \lambda^4 a(x, \vec{e}_1, \vec{e}_1)^2 w^2 + 2s\lambda^2 \phi a(x, \vec{e}_1, \vec{e}_1) a(x, \nabla w, \nabla w) \\
&\quad + 2s\lambda^2 w \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial}{\partial x_i} (\phi a(x, \vec{e}_1, \vec{e}_1)) \Big) dx \\
&\quad - \int_{\{\tau\} \times K} 2s\lambda^2 \phi a(x, \vec{e}_1, \vec{e}_1) w a(x, \vec{e}_1, \nabla w) dx'.
\end{aligned} \tag{1.20}
$$

Integrating by parts in the second term of the right-hand-side of (1.19), we have

$$
\begin{aligned}
A_1 &= \int_{G_\tau} 2\lambda^3 s^3 \phi^3 w a(x, \vec{e}_1, \vec{e}_1) a(x, \vec{e}_1, \nabla w) dx \\
&= \int_{G_\tau} \lambda^3 s^3 \phi^3 a(x, \vec{e}_1, \vec{e}_1) a(x, \vec{e}_1, \nabla w^2) dx \\
&= \int_{G_\tau} \left(3\lambda^4 s^3 \phi^3 a(x, \vec{e}_1, \vec{e}_1)^2 w^2 - w^2 \phi^3 \lambda^3 s^3 \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a_{i1} a(x, \vec{e}_1, \vec{e}_1)) \right) dx \\
&\quad + \int_{\{\tau\} \times K} \lambda^3 s^3 \phi^3 a(x, \vec{e}_1, \vec{e}_1)^2 w^2 \, dx'.
\end{aligned} \tag{1.21}
$$

Finally, integrating by parts for the third term of right-hand-side of (1.19) we have

$$
\begin{aligned}
A_2 &= \int_{G_\tau} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial w}{\partial x_j} \right) 2s\lambda \phi a(x, \vec{e}_1, \nabla w) dx \\
&= \int_{G_\tau} \Bigg(2s\lambda^2 \phi a(x, \vec{e}_1, \nabla w)^2 - 2s\lambda\phi \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{\ell=1}^{n} \frac{\partial a_{1\ell}}{\partial x_j} \frac{\partial w}{\partial x_\ell} \right) \\
&\quad - 2s\lambda\phi \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_i} \sum_{\ell=1}^{n} a_{1\ell} \frac{\partial^2 w}{\partial x_j \partial x_\ell} \Bigg) dx + \int_{\{\tau\} \times K} 2s\lambda\phi a(x, \vec{e}_1, \nabla w)^2 \, dx' \\
&= \int_{G_\tau} \Bigg(2s\lambda^2 \phi a(x, \vec{e}_1, \nabla w)^2 - 2s\lambda\phi \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{\ell=1}^{n} \frac{\partial a_{1\ell}}{\partial x_j} \frac{\partial w}{\partial x_\ell} \right) \\
&\quad + s\lambda\phi \sum_{\ell=1}^{n} a_{1\ell} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_\ell} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} - s\lambda\phi \sum_{\ell=1}^{n} a_{1\ell} \frac{\partial}{\partial x_\ell} \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \Bigg) dx \\
&\quad + \int_{\{\tau\} \times K} 2s\lambda\phi a(x, \vec{e}_1, \nabla w)^2 \, dx'.
\end{aligned} \tag{1.22}
$$

Integrating in (1.22) by parts once again, we obtain

$$A_2 = \int_G \Bigg(2s\lambda^2\phi a(x,\vec{e}_1,\nabla w)^2 - 2s\lambda\phi \sum_{i,j=1}^n \left(a_{ij}\frac{\partial w}{\partial x_i}\sum_{\ell=1}^n \frac{\partial a_{1\ell}}{\partial x_j}\frac{\partial w}{\partial x_\ell}\right) + s\lambda\phi\sum_{\ell=1}^n a_{1\ell}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}$$
$$- s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w) + a(x,\nabla w,\nabla w)s\lambda\phi\sum_{\ell=1}^n \frac{\partial a_{1\ell}}{\partial x_\ell}\Bigg)dx$$
$$+ \int_{\{\tau\}\times K} \left(-s\lambda\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w) + 2s\lambda\phi a(x,\vec{e}_1,\nabla w)^2\right)\, dx'. \tag{1.23}$$

Using (1.19–1.21) and (1.23) one can rewrite (1.18) as follows

$$||f_s||^2_{L^2(G_\tau)} = ||L_1 w||^2_{L^2(G_\tau)} + ||L_2 w||^2_{L^2(G_\tau)}$$
$$+ 2\int_{G_\tau} (\lambda^4 s^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2 + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w)$$
$$+ 2s\lambda^2\phi a(x,\vec{e}_1,\nabla w)^2)dx + \mathcal{S}(\tau,w) + X_1(\tau) + X_2(\tau), \tag{1.24}$$

where

$$\mathcal{S}(\tau,w) = 2\int_{\{\tau\}\times K} (2s\lambda\phi a(x,\vec{e}_1,\nabla w)^2 - s\lambda\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w)$$
$$- 2s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)wa(x,\vec{e}_1,\nabla w) + s^3\lambda^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2)\, dx'$$

and

$$X_1(\tau) = 2\int_{G_\tau} \Bigg(- 2s\lambda\phi\sum_{i,j=1}^n \left(a_{ij}\frac{\partial w}{\partial x_i}\sum_{\ell=1}^n \frac{\partial a_{1\ell}}{\partial x_j}\frac{\partial w}{\partial x_\ell}\right) - w^2\phi^3\lambda^3 s^3\sum_{i=1}^n \frac{\partial}{\partial x_i}(a_{i1}a(x,\vec{e}_1,\vec{e}_1))$$
$$+ s\lambda\phi\sum_{\ell=1}^n a_{1\ell}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + a(x,\nabla w,\nabla w)s\lambda\phi\sum_{\ell=1}^n\frac{\partial a_{1\ell}}{\partial x_\ell}\Bigg)dx;$$

$$X_2 = 2\int_{G_\tau} 2s\lambda^2 w\sum_{i,j=1}^n a_{ij}\frac{\partial w}{\partial x_j}\frac{\partial}{\partial x_i}(\phi a(x,\vec{e}_1,\vec{e}_1))dx.$$

Taking the parameter $\hat{\lambda}$ sufficiently large we have

$$||f_s||^2_{L^2(G_\tau)} \geq ||L_2 w||^2_{L^2(G_\tau)} + ||L_1 w||^2_{L^2(G_\tau)}$$
$$+ \frac{11}{8}\int_{G_\tau} (\lambda^4 s^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2 + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w)$$
$$+ 2s\lambda^2\phi a(x,\vec{e}_1,\nabla w)^2)dx + \mathcal{S}(\tau,w) + X_2(\tau)$$

for all $\lambda > \hat{\lambda}$ and $s > 1$.

Now let $\lambda > \hat{\lambda}$ be fixed. Taking the parameter $s_0(\lambda)$ sufficiently large we obtain from the previous inequality

$$||f_s||^2_{L^2(G_\tau)} \geq ||L_2 w||^2_{L^2(G_\tau)} + ||L_1 w||^2_{L^2(G_\tau)}$$
$$+ \int_{G_\tau} (\lambda^4 s^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2 + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w)$$
$$+ 2s\lambda^2\phi a(x,\vec{e}_1,\nabla w)^2)dx + \mathcal{S}(\tau,w) \quad \forall\ s \geq s_0(\lambda). \tag{1.25}$$

By (1.15, 1.10) and Cauchy-Bunyakovskii inequality

$$\frac{1}{4}\|L_2 w\|^2_{L^2(G_\tau)} + \frac{1}{4}\int_{G_\tau} s^3\lambda^4\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2\,dx \ge \kappa s^{\frac{5}{2}}\int_K w(\tau,x')^2 dx' \quad \forall\, \tau\in[-1,0], \tag{1.26}$$

where constant $\kappa > 0$ is independent of s, τ.

Denote by

$$\tau^*(w) = \max_{\tau\in[-1,0]}\left\{\tau|\mathcal{S}(\tau,w) \ge -\kappa s^{\frac{5}{2}}\int_K w(\tau,x')^2\,dx'\right\}\cdot \tag{1.27}$$

By (1.25–1.27)

$$\begin{aligned}||f_s||^2_{L^2(G_{\tau^*})} \ge \frac{3}{4}\int_{G_{\tau^*}} &(\lambda^4 s^3\phi^3 a(x,\vec{e}_1,\vec{e}_1)^2 w^2 + s\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w)\\ &+ 2s\lambda^2\phi a(x,\vec{e}_1,\nabla w)^2)dx \quad \forall\, s\ge s_0(\lambda).\end{aligned} \tag{1.28}$$

If $\tau^*(w) = 0$ the lemma is proved. Now let us consider the case $\tau^*(w) < 0$. Multiplying (1.16) by $sa_{11}w$ scalary in $L^2(G)$ and integrating by parts we have

$$\begin{aligned}\int_G \tilde{f} s a_{11} w\,dx = &\int_G (sa(x,\vec{e}_1,\vec{e}_1)a(x,\nabla w,\nabla w) - s^3\lambda^2\phi^2 a(x,\vec{e}_1,\vec{e}_1)^2 w^2)dx\\ &+ \int_G sa(x,\nabla w,\nabla a_{11})w dx - \int_{\{0\}\times K} sa_{11}a(x,\vec{e}_1,\nabla w)w\,dx'\\ &- \int_{\{0\}\times K} s^2\lambda\phi a_{11}a(x,\vec{e}_1,\vec{e}_1)w^2 d\,x' + \frac{1}{2}\int_{\{0\}\times K} sa_{11}b_1 w^2 dx' + X_3,\end{aligned} \tag{1.29}$$

where

$$\begin{aligned}X_3 = \int_G \Bigg(&s^2\lambda^2\phi a(x,\vec{e}_1,\vec{e}_1)^2 w^2 + \sum_{j=1}^n s^2\lambda\frac{\partial}{\partial x_j}(\phi a_{11}a_{1j})w^2\\ &+ sa_{11}\left(-\sum_{i=1}^n \frac{\partial a_{i1}}{\partial x_i}s\lambda\phi + c + s\lambda\phi b_1\right)w^2 - \frac{s}{2}\sum_{i=1}^n\frac{\partial(b_i a_{11})}{\partial x_i}w^2\Bigg)dx.\end{aligned}$$

Obviously

$$|X_3| \le C_4 s^2\int_G w^2\,dx. \tag{1.30}$$

Note that

$$\begin{aligned}&-\int_{\{0\}\times K} sa_{11}a(x,\vec{e}_1,\nabla w)w\,dx' - \int_{\{0\}\times K} s^2\lambda\phi a_{11}a(x,\vec{e}_1,\vec{e}_1)w^2\,dx'\\ &= -\int_{\{0\}\times K} sa_{11}a(x,\vec{e}_1,\nabla z e^{s\phi} - s\lambda\phi\vec{e}_1 w)w\,dx'\\ &\quad -\int_{\{0\}\times K} s^2\lambda\phi a_{11}a(x,\vec{e}_1,\vec{e}_1)w^2\,dx' = -\int_{\{0\}\times K} sa_{11}a(x,\vec{e}_1,\nabla z)we^{s\phi}\,dx'\\ &= -\int_{\{0\}\times K} sa_{11}^2\frac{\partial z}{\partial x_1}e^{s\phi}w\,dx' + s\int_{\{0\}\times K}\sum_{j=2}^n \frac{1}{2}\frac{\partial(a_{11}a_{1j})}{\partial x_j}w^2\,dx'.\end{aligned} \tag{1.31}$$

Denote

$$X_4 = \frac{1}{2}\int_{\{0\}\times K} sa_{11}b_1 w^2 dx' + s\int_{\{0\}\times K}\sum_{j=2}^{n}\frac{1}{2}\frac{\partial(a_{1j}a_{11})}{\partial x_j}w^2\,dx'. \tag{1.32}$$

Using (1.31) one could rewrite (1.29) as

$$\int_G sa_{11}\tilde f w\,dx = \int_G (sa(x,\vec e_1,\vec e_1)a(x,\nabla w,\nabla w) - s^3\lambda^2\phi^2 a(x,\vec e_1,\vec e_1)^2 w^2)dx \\ + \int_G sa(x,\nabla w,\nabla a_{11})wdx - \int_{\{0\}\times K} sa_{11}^2\frac{\partial z}{\partial x_1}we^{s\phi}\,dx' + X_3 + X_4. \tag{1.33}$$

One could rewrite (1.33) as

$$-\int_{-1}^{0}\frac{1}{2\lambda\phi}S(x_1,w)dx_1 + \int_G (2sa(x,\vec e_1,\nabla w)^2 - 2s\lambda a(x,\vec e_1,\vec e_1)wa(x,\vec e,\nabla w))dx \\ = \int_G sa_{11}\tilde f wdx + \int_{\{0\}\times K} sa_{11}^2\frac{\partial z}{\partial x_1}we^{s\phi}dx' - X_3 - X_4 - \int_G sa(x,\nabla w,\nabla a_{11})w\,dx.$$

From this equality, by (1.27) we have

$$\hat\kappa_1\int_{(\tau^*,0)\times K} s^{\frac{5}{2}}w^2\,dx - \int_{-1}^{\tau^*}\frac{1}{2\lambda\phi}S(x_1,w)dx_1 + \int_G 2sa(x,\vec e_1,\nabla w)^2dx \\ \le \int_G 2s\lambda a(x,\vec e_1,\vec e_1)wa(x,\vec e,\nabla w)dx + \int_G sa_{11}\tilde f wdx \\ + \int_{\{0\}\times K} sa_{11}^2\frac{\partial z}{\partial x_1}we^{s\phi}dx' - \int_G sa(x,\nabla w,\nabla a_{11})wdx - X_3 - X_4,$$

where $\hat\kappa_1 = \kappa/(2\lambda e^{\lambda})$. Hence by (1.28) we have

$$\hat\kappa_2\int_G (s^{\frac{5}{2}}w^2 + sa(x,\nabla w,\vec e_1)^2)dx \le C_5\Big(\int_{\{0\}\times K} s\left|a_{11}^2\frac{\partial z}{\partial x_1}w\right|e^{s\phi}\,dx' \\ + |X_3| + |X_4| + \int_G (|\nabla w|^2 + s^2w^2)dx + s|\int_G a_{11}\tilde f wdx| + \|\tilde f\|^2_{L^2(G)}\Big) \\ \le \varepsilon\int_{\{0\}\times K} s^{\frac{7}{4}}w^2\,dx' + C_6\Big(\frac{1}{\varepsilon}\int_{\{0\}\times K} s^{\frac{1}{4}}|\frac{\partial z}{\partial x_1}|^2e^{2s\phi}\,dx' + \|\tilde f\|^2_{L^2(G)} \\ + \int_G (|\nabla w|^2 + s^2w^2)dx + |X_3| + |X_4|\Big) \quad \forall\, s \ge s_0(\lambda), \tag{1.34}$$

where $\hat\kappa_2 = \frac{1}{2}\min\{\hat\kappa_1, 2\}$ is some constant independent of s. Note that by (1.10)

$$\int_G (s^{\frac{5}{2}}w^2 + sa(x,\vec e_1,\nabla w)^2)d\,x \ge \hat\kappa_3 s^{\frac{7}{4}}\int_K w^2(0,x')dx', \tag{1.35}$$

where $\hat\kappa_3 > 0$ is independent of w, s.

Taking in (1.34) parameter ε sufficiently small and increasing $s_0(\lambda)$ if necessary, by (1.30, 1.35) we obtain

$$\int_G (s^{\frac{5}{2}} w^2 + sa(x, \vec{e}_1, \nabla w)^2) dx \le C_7 \Big(|X_4| + \|\tilde{f}\|^2_{L^2(G)} + \int_G |\nabla w|^2 \, dx + s^{\frac{1}{4}} \int_{\{0\}\times K} \left| \frac{\partial z}{\partial x_1} \right|^2 e^{2s\phi} \, dx' \Big) \quad \forall \, s \ge s_0(\lambda). \tag{1.36}$$

Increasing parameter $s_0(\lambda)$ if it is necessary we obtain from (1.35, 1.36)

$$\int_G (s^{\frac{5}{2}} w^2 + sa(x, \vec{e}_1, \nabla w)^2) dx \le C_8 \Big(\|\tilde{f}\|^2_{L^2(G)} + \int_G |\nabla w|^2 dx + s^{\frac{1}{4}} \int_{\{0\}\times K} \left| \frac{\partial z}{\partial x_1} \right|^2 e^{2s\phi} \, dx' \Big) \quad \forall \, s \ge s_0(\lambda). \tag{1.37}$$

Since

$$\mathcal{S}(0, w) \le 0,$$

for all values of the parameter s large enough we have

$$\int_K \left| \frac{\partial z(0, x')}{\partial x_1} \right|^2 e^{2s\phi(0)} dx' \le C_9 \int_K (|\nabla_{x'} g|^2 + g^2) e^{2s\phi(0)} dx'. \tag{1.38}$$

By (1.37, 1.38)

$$\int_G (s^{\frac{5}{2}} w^2 + sa(x, \vec{e}_1, \nabla w)^2) dx \le C_{10} \Big(\|\tilde{f}\|^2_{L^2(G)} + \int_G |\nabla w|^2 \, dx + s^{\frac{1}{4}} \int_K (|\nabla g_{x'}|^2 + g^2) e^{2s\phi(0)} dx' \Big) \quad \forall \, s \ge s_0(\lambda). \tag{1.39}$$

By (1.33)

$$\int_G s^{\frac{1}{2}} |\nabla w|^2 dx \le C_{11} \Big(\int_G s^{\frac{5}{2}} w^2 dx + \int_{\{0\}\times K} s^{\frac{1}{2}} \left| \frac{\partial z}{\partial x_1} e^{s\phi} w \right| dx' + \|\tilde{f}\|^2_{L^2(G)} + \int_G |\nabla w|^2 dx + |X_3| + |X_4| \Big) \quad \forall \, s \ge s_0(\lambda).$$

From this inequality, by (1.30, 1.32, 1.38, 1.39) we obtain (1.12). □

In the right hand side of estimate (1.12) we have the W^1_2-norm of the function g. This norm is too strong for our purposes. So now we would like to obtain from this estimate the similar one but with the L^2-norm in the right hand side First we formulate the following result:

Proposition 1.1. *Let $g = 0$, Condition 1.1 be fulfilled and function ϕ be defined as in (1.11). Then there exists a number $\hat{\lambda} > 0$ such that for an arbitrary $\lambda \ge \hat{\lambda}$ there exists $s_0(\lambda)$ such that for each $s \ge s_0(\lambda)$ the solutions of problem (1.8–1.10) satisfy the following inequality:*

$$\int_G (s^{1-2\ell} |\nabla z|^2 + s^{3-2\ell} |z|^2) e^{2s\phi} \, dx \le C_{12} \int_{-1}^0 \|f(x_1, \cdot)\|^2_{W_2^{-\ell}(K)} e^{2s\phi} dx_1, \quad \forall \, \ell \in [0, 1]$$

where constant C_{12} is independent of s.

The proof of this proposition is exactly the same as the proof of Theorem 2.1 as we can see from [22].

Now we are ready to prove the following lemma:

Lemma 1.3. *Let Condition 1.1 be fulfilled and function ϕ be defined as in (1.11). Then there exists a number $\hat\lambda > 0$ such that for an arbitrary $\lambda \ge \hat\lambda$ there exists $s_0(\lambda)$ such that for each $s \ge s_0(\lambda)$ the solutions of problem (1.8–1.10) satisfy the following inequality:*

$$\int_G s^{\frac14}|z|^2 e^{2s\phi}\,dx + s^{\frac14}\int_{-1}^{0}\left(\left\|\frac{\partial z}{\partial x_1}(x_1,\cdot)\right\|^2_{W_2^{-1}(K)} + s^{\frac94}\,\|Z(x_1,\cdot)\|^2_{W_2^{-1}(K)}\right)e^{2s\phi(x_1)}\,dx_1$$
$$\le C_{12}\left(\int_K g^2 e^{2s\phi(0)}dx' + \int_{-1}^{0} s^{-\frac14}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)} e^{2s\phi}\,dx_1\right), \tag{1.40}$$

where constant C_{12} is independent of s.

Proof. Denote $P = -\sum_{i=2}^{n}\frac{\partial^2}{\partial x_i^2} + 1$. There exists $P^{-1} : W^s_{2,Per}(K) \to W^{s+2}_{2,Per}(K)$ for all $s \in \{0,-1\}$. Let $R_1 = P^{-1}, R_i = \frac{\partial}{\partial x_i}P^{-1}$ $i \in \{2,\dots,n\}$ and $u_i = R_i\, z, f_i = R_i f$. Obviously

$$\sum_{i=2}^{n} -\frac{\partial u_i}{\partial x_i} + u_1 = z \quad \text{in } K \tag{1.41}$$

and

$$\|z(x_1,\cdot)\|_{L^2(K)} \le C_{13}\sum_{i=1}^{n}\|u_i(x_1,\cdot)\|_{W_2^1(K)};$$
$$\left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}\right\|_{W_2^{-1}(K)} \le C_{13}\sum_{i=1}^{n}\left\|\frac{\partial u_i(x_1,\cdot)}{\partial x_1}\right\|_{L^2(K)};$$
$$\|z(x_1,\cdot)\|_{W_2^{-1}(K)} \le C_{14}\sum_{i=1}^{n}\|u_i(x_1,\cdot)\|_{L^2(K)}. \tag{1.42}$$

By (1.8–1.10)

$$Lu_i + [L,R_i]z = f_i \quad \text{in } G, \tag{1.43}$$
$$u_i(0,x') = R_i g, \quad u_i(x_1,..,x_j + 2\ell,..) = u_i(x_1,...,x_j,..) \quad j \in \{2,\dots,n\}, \tag{1.44}$$
$$u_i(-1,x') = \frac{\partial u_i(-1,x')}{\partial x_1} = 0 \quad \text{in } K, \tag{1.45}$$

where $[A,B] = BA - AB$ is the commutator of operators A and B, $x' = (x_2,\dots,x_n)$. Note that

$$\|e^{s\phi}[L,R_i]z(x_1,\cdot)\|_{L^2(K)} \le C_{15}\left(\|z(x_1,\cdot)e^{s\phi}\|_{L^2(K)} + \left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}e^{s\phi}\right\|_{W_2^{-1}(K)}\right), \tag{1.46}$$

where the constant C_{15} is independent of s and x_1. Then by (1.46) and Lemma 1.2 there exists $\hat{\lambda} > 1$ such that for all $\lambda > \hat{\lambda}$ there exists $s_0(\lambda)$ such that

$$\int_G (s^{\frac{1}{4}}|\nabla u_i|^2 + s^{\frac{9}{4}}u_i^2)e^{2s\phi}\,dx \le C_{16}\Big(\int_K |g|^2 e^{2s\phi(0)}\,dx' + \int_{-1}^0 s^{-\frac{1}{4}}\left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}\right\|^2_{W_2^{-1}(K)} e^{2s\phi(x_1)}\,dx_1 + \int_G s^{-\frac{1}{4}}|z|^2 e^{2s\phi}\,dx + \int_{-1}^0 s^{-\frac{1}{4}}e^{2s\phi(x_1)}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)}\,dx_1\Big) \quad \forall\, s \ge s_0(\lambda). \tag{1.47}$$

By (1.42, 1.47) for all $s \ge s_0$

$$\begin{aligned}\int_G s^{\frac{1}{4}} z^2 e^{2s\phi}\,dx &+ \int_{-1}^0 \left(s^{\frac{1}{4}}\left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}\right\|^2_{W_2^{-1}(K)} + s^{\frac{9}{4}}\|z(x_1,\cdot)\|^2_{W_2^{-1}(K)}\right)e^{2s\phi(x_1)}\,dx_1\\
&\le C_{17}\sum_{i=1}^n \int_G (s^{\frac{1}{4}}|\nabla u_i|^2 + s^{\frac{9}{4}}u_i^2)e^{2s\phi}\,dx\\
&\le C_{18}\Big(\int_K g^2 e^{2s\phi(0)}\,dx' + \int_G s^{-\frac{1}{4}}z^2 e^{2s\phi}dx\\
&\quad + \int_{-1}^0 s^{-\frac{1}{4}}\left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}\right\|^2_{W_2^{-1}(K)} e^{2s\phi(x_1)}\,dx_1\\
&\quad + \int_{-1}^0 s^{-\frac{1}{4}}e^{2s\phi(x_1)}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)}d\,x_1\Big).\end{aligned}$$

Then by increasing the parameter $s_0(\lambda)$ if necessary we obtain

$$\begin{aligned}\int_G s^{\frac{1}{4}} z^2 e^{2s\phi}\,dx \quad &+ \int_{-1}^0 \left(s^{\frac{1}{4}}\left\|\frac{\partial z(x_1,\cdot)}{\partial x_1}\right\|^2_{W_2^{-1}(K)} + s^{\frac{9}{4}}\|z(x_1,\cdot)\|^2_{W_2^{-1}(K)}\right)e^{2s\phi(x_1)}dx_1\\
&\le \quad C_{18}\sum_{i=1} \int_G (s^{\frac{1}{4}}|\nabla u_i|^2 + s^{\frac{9}{4}}u_i^2)e^{2s\phi}dx \le C_{19}\Big(\int_K g^2 e^{2s\phi(0)}\,dx'\\
&\quad + \int_{-1}^0 s^{-\frac{1}{4}}e^{2s\phi(x_1)}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)}d\,x_1\Big)\end{aligned} \tag{1.48}$$

for all $s \ge s_0$. The proof of this lemma is complete. □

We have:

Lemma 1.4. *Let Condition 1.1 be fulfilled and function ϕ be defined as in (1.11) . Then there exists a number $\hat{\lambda} > 0$ such that for an arbitrary $\lambda \ge \hat{\lambda}$ there exists $s_0(\lambda)$ such that for each $s \ge s_0(\lambda)$ the solutions of problem (1.8–1.10) satisfy the following inequality:*

$$\begin{aligned}\int_G ((s^{\frac{3}{4}}x_1^{20} + s^{-\frac{3}{4}})|\nabla z|^2 + s^{\frac{5}{4}}|z|^2)e^{2s\phi}\,dx + s^{\frac{1}{4}}\int_{-1}^0 \left\|\frac{\partial z}{\partial x_1}(x_1,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi(x_1)}\,dx_1\\
\le C_{25}\Big(\|g\|^2_{W_2^{\frac{1}{2}}(K)} e^{2s} + \int_G s^{-\frac{1}{4}}x_1^{10}|f|^2 e^{2s\phi}\,dx + \int_{-1}^0 s^{-\frac{1}{4}}\|f(x_1,\cdot)\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi}dx_1\Big),\end{aligned} \tag{1.49}$$

where constant C_{25} is independent of s.

Proof. Let $\tilde z = \Lambda z = \int_{\mathbb{R}^{n-1}} \frac{\mu(x')e^{i<x',\xi'>}}{(1+|\xi'|^2)^{\frac14}}\widehat{\mu_1 z}d\xi'$, where $\xi'=(\xi_2,\dots,\xi_n)$, $\mu(x')\in C_0^\infty(K_1)$, $K_1=\Pi_{i=2}^n[-2\ell,2\ell]$, $\mu_1(x')\in C_0^\infty(K_2)$, $K_2=\Pi_{i=2}^n\left[-\frac32\ell,\frac32\ell\right]$, $\mu_1,\ \mu|_K=1$. Since z is the periodic function in variable x' we have

$$\|z\|_{W_2^{\frac12}(K)}\le C_{26}(\|\tilde z\|_{W_2^1(K_1)}+\|z\|_{W_2^{-1}(K)}),$$

$$\left\|\frac{\partial z}{\partial x_1}\right\|_{W_2^{-\frac12}(K)}\le C_{27}\left(\left\|\frac{\partial \tilde z}{\partial x_1}\right\|_{W_2^1(K_1)}+\left\|\frac{\partial z}{\partial x_1}\right\|_{W_2^{-1}(K)}\right).$$

By (1.40) we obtain from these estimates

$$\begin{aligned}\|ze^{s\phi}\|^2_{L^2(-1,0;W_2^{\frac12}(K))}&\le C_{28}(\|\tilde z e^{s\phi}\|^2_{L^2(-1,0;W_2^1(K_1))}\\&+s^{-\frac14}\left(\int_K g^2e^{2s\phi(0)}dx'+\int_{-1}^0 s^{-\frac14}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)}e^{2s\phi}\,dx_1\right).\end{aligned}\tag{1.50}$$

$$\begin{aligned}\left\|\frac{\partial z}{\partial x_1}e^{s\phi}\right\|^2_{L^2(-1,0;W_2^{-\frac12}(K))}&\le C_{29}\Bigg(\left\|\frac{\partial \tilde z}{\partial x_1}e^{s\phi}\right\|^2_{L^2(-1,0;W_2^1(K_1))}\\&+s^{-\frac14}\left(\int_K g^2e^{2s\phi(0)}\,dx'+\int_{-1}^0 s^{-\frac14}\|f(x_1,\cdot)\|^2_{W_2^{-1}(K)}e^{2s\phi}\,dx_1\right)\Bigg).\end{aligned}\tag{1.51}$$

By (1.8–1.10)

$$L\tilde z+[L,\Lambda]z=\Lambda f\quad \text{in } G_1=[-1,0]\times K_1,\tag{1.52}$$

$$\tilde z(0,x')=\Lambda g,\quad \tilde z(x_1,..,x_j+4\ell,..)=\tilde z(x_1,...,x_j,..)\quad j\in\{2,\dots,n\},\tag{1.53}$$

$$\tilde z(-1,x')=\frac{\partial \tilde z(-1,x')}{\partial x_1}=0\quad \text{in } K_1.\tag{1.54}$$

Note that (see [30])

$$\|[L,\Lambda]z\|_{L^2(K_1)}\le C_{30}\left(\|z\|_{W_2^{\frac12}(K)}+\left\|\frac{\partial z}{\partial x_1}\right\|_{W_2^{-\frac12}(K)}\right).\tag{1.55}$$

Applying to problem (1.52–1.54) the Carleman estimate (1.12) and keeping in mind (1.55) we have

$$\begin{aligned}\int_{G_1}\left(s^{\frac14}|\nabla\tilde z|^2+s^{\frac94}\tilde z^2\right)e^{2s\phi}\,dx&\le C_{31}\Bigg(\int_{K_1}(|\nabla_{x'}\Lambda g|^2+|\Lambda g|^2)e^{2s\phi(0)}\,dx'\\&+\int_{G_1}(s^{-\frac14}|\Lambda f|^2+|[L,\Lambda]z|^2)e^{2s\phi}dx\Bigg)\\&\le C_{32}\Bigg(\|g\|^2_{W_2^{\frac12}(K)}e^{2s}+\int_{G_1}s^{-\frac14}|\Lambda f|^2e^{2s\phi}\,dx\\&+s^{-\frac14}\int_{-1}^0\left(\|z\|^2_{W_2^{\frac12}(K)}+\left\|\frac{\partial z}{\partial x_1}\right\|^2_{W_2^{-\frac12}(K)}\right)e^{2s\phi}\,dx_1\Bigg).\end{aligned}$$

From this inequality, thanks to (1.40, 1.50, 1.51) we obtain

$$\int_G s^{\frac{5}{4}}|z|^2 e^{2s\phi}\,dx + s^{\frac{1}{4}}\int_{-1}^{0}\left\|\frac{\partial z}{\partial x_1}(x_1,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi(x_1)}\,dx_1 \leq C_{33}\left(\|g\|^2_{W_2^{\frac{1}{2}}(K)} e^{2s} + \int_{-1}^{0} s^{-\frac{1}{4}}\|f(x_1,\cdot)\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi}\,dx\right), \tag{1.56}$$

where constant C_{33} is independent of s.

Multiplying (1.8) by $s^{-\frac{3}{4}} z e^{2s\phi}$ scalary in $L^2(G)$ we have

$$\int_G s^{-\frac{3}{4}} a(x,\nabla z,\nabla z) e^{2s\phi}\,dx = -2\int_G s^{\frac{1}{4}}\lambda\phi z a(x,\nabla z,\nabla\psi) e^{2s\phi}\,dx - \int_K s^{-\frac{3}{4}} a(x,\nu,\nabla z) z e^{2s\phi(0)}\,dx' + \int_G s^{-\frac{3}{4}}\left(f - cz - \sum_{i=1}^{n} b_i\frac{\partial z}{\partial x_i}\right) z e^{2s\phi}\,dx. \tag{1.57}$$

Using known *a priori* estimates for elliptic equations (see [28]) we have

$$e^s\left\|\frac{\partial z(0,x')}{\partial x_1}\right\|_{W_2^{-\frac{1}{2}}(K)} \leq C_{37} e^s\left(\|f\|_{L^2(-1,0;W_2^{-\frac{1}{2}}(K))} + \|g\|_{W_2^{\frac{1}{2}}(K)} + \|z\|_{L^2(G)}\right) \leq C_{38}\left(\|f e^{s\phi}\|_{L^2(-1,0;W_2^{-\frac{1}{2}}(K))} + \|g e^s\|_{W_2^{\frac{1}{2}}(K)} + \|z e^{s\phi}\|_{L^2(G)}\right). \tag{1.58}$$

From (1.57) using (1.56, 1.58) and Cauchy-Bunyakovskii inequality we obtain

$$\int_G (s^{-\frac{3}{4}}|\nabla z|^2 + s^{\frac{5}{4}}|z|^2) e^{2s\phi}\,dx \leq C_{39}\left(\|g\|^2_{W_2^{\frac{1}{2}}(K)} e^{2s} + \int_{-1}^{0} s^{-\frac{1}{4}}\|f(x_1,\cdot)\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi}\,dx\right). \tag{1.59}$$

Let us introduce the function $\tilde{z}_1(x) = x_1^{10} z(x)$. By (1.9, 1.10) this function satisfies the equations

$$L\,\tilde{z}_1 + [L, x_1^{10}]z = x_1^{10} f,\ \ \tilde{z}_1(0,x') = \tilde{z}_1(-1,x') = 0, \tag{1.60}$$

$$\tilde{z}_1(x_1,...,x_i+2\ell,..) = \tilde{z}_1(x)\ \forall\ i\in\{2,\dots,n\},\ \ \frac{\partial\tilde{z}_1(-1,x')}{\partial x_1} = 0. \tag{1.61}$$

Obviously

$$|[L, x_1^{10}]z(x)| \leq C_{40}(|x_1|^9|\nabla z(x)| + |z(x)|) \quad \forall x\in G.$$

Therefore, by (1.59)

$$\|[L, x_1^{10}]z e^{s\phi}\|^2_{L^2(G)} \leq C_{41}(s^{-\frac{3}{8}}\||\nabla z|e^{s\phi}\|_{L^2(G)} s^{\frac{3}{8}}\||\nabla\tilde{z}_1|e^{s\phi}\|_{L^2(G)} + \|z\|^2_{L^2(G)}) \leq C_{42}(s^{\frac{3}{4}}\||\nabla\tilde{z}_1|e^{s\phi}\|_{L^2(G)} + \|g\|^2_{W_2^{\frac{1}{2}}(K)} e^{2s} + \int_{-1}^{0} s^{-\frac{1}{4}}\|f(x_1,\cdot)\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi}\,dx). \tag{1.62}$$

Applying to (1.60, 1.61) the Carleman estimate from Proposition 1.1 with $\ell = 0$ and using (1.62) we obtain

$$\int_G (s|\nabla\tilde{z}_1|^2 + s^3|\tilde{z}_1|^2) e^{2s\phi}\,dx \leq C_{43}\left(\int_{-1}^{0}\|f(x_1,\cdot)\|^2_{W_2^{-\frac{1}{2}}(K)} e^{2s\phi}\,dx_1 + \|x_1^{10} f e^{s\phi}\|^2_{L^2(G)} + \|g\|^2_{W_2^{\frac{1}{2}}(K)} e^{2s}\right). \tag{1.63}$$

Hence estimate (1.49) follows from (1.59, 1.63). The proof of Lemma 1.4 is complete. □

Let Γ be an $(n-1)$ dimensional compact connected C^2 manifold without boundary in $\mathbb{R}^n$. We consider the following elliptic equation on the manifold $[-1,0]\times\Gamma$.

$$\tilde{A}z = -\sum_{i,j=0}^{n}\frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}(\tilde{x})\frac{\partial z}{\partial x_j}\right)+\sum_{i=0}^{n}\tilde{b}_i(\tilde{x})\frac{\partial z}{\partial x_i}+\tilde{c}(\tilde{x})z = f(\tilde{x}) \quad \text{in } [-1,0]\times\Gamma, \tag{1.64}$$

$$z(0,\cdot)|_\Gamma = g(x), \quad \frac{\partial z(-1,\cdot)}{\partial x_0}|_\Gamma = z(-1,\cdot)|_\Gamma = 0, \tag{1.65}$$

where $\tilde{x}=(x_0,x)=(x_0,x_1,\dots,x_n)$ and we assume that

$$\tilde{a}_{ij}\in C^2([-1,0]\times\Gamma),\ \ \tilde{b}_i\in C^1([-1,0]\times\Gamma),\ \ \tilde{c}\in L^\infty([-1,0]\times\Gamma). \tag{1.66}$$

We have:

Lemma 1.5. *Let the operator $\tilde{A}$ be elliptic on the manifold $[-1,0]\times\Gamma$, condition (1.66) be fulfilled and the function ϕ be defined as in* (1.11). *Then there exists a number $\hat{\lambda}>0$ such that for an arbitrary $\lambda\ge\hat{\lambda}$ there exists $s_0(\lambda)$ such that for each $s\ge s_0(\lambda)$ the solutions of problem (1.64–1.66) satisfy the following inequality:*

$$\int_{[-1,0]\times\Gamma}((x_0^{20}s^{\frac{3}{4}}+s^{-\frac{3}{4}})|\nabla z|^2+s^{\frac{5}{4}}|z|^2)e^{2s\phi(x_0)}dx+s^{\frac{1}{4}}\int_{-1}^{0}\left\|\frac{\partial}{\partial x_0}z(x_0,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(\Gamma)}e^{2s\phi(x_0)}d\,x_0$$
$$\le C_{25}\left(\int_\Gamma g^2e^{2s\phi(0)}d\sigma+\int_{-1}^{0}s^{-\frac{1}{4}}\|f(x_0,\cdot)\|^2_{W_2^{-\frac{1}{2}}(\Gamma)}e^{2s\phi(x_0)}dx_0+s^{-\frac{1}{4}}\|x_0^{10}fe^{s\phi}\|^2_{L^2(G)}\right), \tag{1.67}$$

where constant C_{25} is independent of s.

Proof. Let $\{U_k\}_{k=1}^K$ be a covering of the manifold Γ, h_k – are the local coordinates which corresponds to this system, and $\{e_k\}_{k=1}^K$ is the partition of unity such that $e_k\in C_0^2(U_k)$, $\sum_{i=1}^K e_k(x)=1$ for all $x\in\Gamma$. Using the atlas $\{U_k,h_k\}$ we can introduce the new atlas $\{\tilde{U}_k,\tilde{h}_k\}$ on the manifold $[-1,0]\times\Gamma$ as follows $\tilde{U}_k=[-1,0]\times U_k$, $\tilde{h}_k=(\tilde{h}_1^{(k)},\dots\tilde{h}_{n+1}^{(k)})$ where $\tilde{h}_1^{(k)}=x_0,\tilde{h}_2^{(k)}=h_1^{(k)}(x),\dots,\tilde{h}_{n+1}^{(k)}=h_n^{(k)}(x)$. Obviously the partition of unity $\{e_k\}_{k=1}^K$ has the same properties on manifold $[-1,0]\times\Gamma$ as on manifold Γ: $e_k\in C^2(\tilde{U}_k), supp\, e_k\subset[-1,0]\times U_k, \sum_{i=1}^k e_k(x)=1$ for all $x\in[-1,0]\times U_k$. Also

$$\|[\tilde{A},e_k]z(x_0,\cdot)\|_{W_2^{-\frac{1}{2}}(\Gamma)}\le C_{26}\left(\|z(x_0,\cdot)\|_{W_2^{\frac{1}{2}}(\Gamma)}+\left\|\frac{\partial z}{\partial x_0}(x_0,\cdot)\right\|_{W_2^{-\frac{1}{2}}(\Gamma)}\right), \tag{1.68}$$

where the constant C_{26} is independent of x_0. Denote $z_k=ze_k$. Then by (1.64, 1.65) function z_k satisfies to equations

$$\tilde{A}z_k+[\tilde{A},e_k]z=fe_k \quad \text{in } [-1,0]\times\Gamma,$$
$$z_k(0,\cdot)|_\Gamma=e_kg, \quad \frac{\partial z_k(-1,\cdot)}{\partial x_0}|_\Gamma=z_k(-1,\cdot)|_\Gamma=0.$$

In the local coordinates $\tilde{h}_k$ the function $\tilde{z}_k = z_k(\tilde{h}_k^{-1}(x))$ satisfies to equations

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}(x)\frac{\partial \tilde{z}_k}{\partial x_j}\right) + \sum_{i=1}^{n} \tilde{b}_i(x)\frac{\partial \tilde{z}_k}{\partial x_i} + \tilde{c}(x)\tilde{z}_k = (fe_k - [\tilde{A}, e_k]z) \circ \tilde{h}_k^{-1} \quad \text{in } [-1,0] \times h_k(U_k), \tag{1.69}$$

$$\tilde{z}_k|_{[-1,0]\times\partial h_k(U_k)} = \nabla \tilde{z}_k|_{[-1,0]\times\partial h_k(U_k)} = 0, \quad \tilde{z}_k(0,\cdot) = ge_k, \quad \frac{\partial \tilde{z}_k(-1,\cdot)}{\partial x_0} = \tilde{z}_k(-1,\cdot) = 0. \tag{1.70}$$

Taking sufficiently large cube $K = \prod_{i=1}^{n-1}[-\ell, \ell]$ we may assume that $h_k(U_k) \subset K$. Then by extending the functions $\tilde{z}_k, ge_k \circ h_k^{-1}, (fe_k - [\tilde{A}, e_k]z) \circ h_k^{-1}$ by zero on the whole cube K, and the coefficients $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}$ up to periodic functions in K by taking into account properties of regularity (1.4) and ellipticity (1.5) we obtain that the function $\tilde{z}_k$ satisfies to problem (1.1–1.3) where $f = (fe_k - [\tilde{A}, e_k]z)\circ \tilde{h}_k^{-1}, g = ge_k \circ \tilde{h}_k^{-1}$ for all $k \in \{1, \dots, K\}$. Thus by Lemma 1.4 and estimate (1.68) we have

$$\begin{aligned}
&\int_{[-1,0]\times\Gamma} ((x_0^{20} s^{\frac{3}{4}} + s^{-\frac{3}{4}})|\nabla z_k|^2 + s^{\frac{5}{4}}|z_k|^2) e^{2s\phi(x_0)}\, d\tilde{x} + s^{\frac{1}{4}} \int_{-1}^{0} \left\| \frac{\partial z_k}{\partial x_0}(x_0,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} e^{2s\phi(x_0)}\, dx_0 \\
&\le C_{27}\Bigg(\|g\|^2_{W_2^{\frac{1}{2}}(\Gamma)} e^{2s} + \int_{-1}^{0} s^{-\frac{1}{4}} \left(\|f(x_0,\cdot)\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} + \left\|\frac{\partial z}{\partial x_0}(x_0,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} + \|z(x_0,\cdot)\|^2_{W_2^{\frac{1}{2}}(\Gamma)} \right) e^{2s\phi(x_0)}\, dx_0 \\
&\quad + s^{-\frac{1}{4}} \int_G x_1^{20}(|\nabla z|^2 + |f|^2) e^{2s\phi(x_0)} dx \Bigg), \ \forall s \ge s_0(k,\lambda).
\end{aligned} \tag{1.71}$$

Summing up inequalities (1.71) respect to the index k we have

$$\begin{aligned}
&\int_{[-1,0]\times\Gamma} ((x_0^{20} s^{\frac{3}{4}} + s^{-\frac{3}{4}})|\nabla z|^2 + s^{\frac{5}{4}}|z|^2) e^{2s\phi(x_0)}\, d\tilde{x} \\
&\quad + s^{\frac{1}{4}} \int_{-1}^{0} \left\| \frac{\partial z}{\partial x_0}(x_0,\cdot)\right\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} e^{2s\phi(x_0)}\, dx_0 \\
&\le C_{28}\Bigg(\|g\|^2_{W_2^{\frac{1}{2}}(\Gamma)} e^{2s\phi(0)} + \int_{-1}^{0} s^{-\frac{1}{4}} (\|f(x_0,\cdot)\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} \\
&\quad + \|\frac{\partial z}{\partial x_0}(x_0,\cdot)\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} + \|z(x_0,\cdot)\|^2_{W_2^{\frac{1}{2}}(\Gamma)}) e^{2s\phi(x_0)}\, dx_0 \\
&\quad + s^{-\frac{1}{4}} \int_G x_1^{20}(|\nabla z|^2 + |f|^2) e^{2s\phi(x_0)} dx \Bigg), \ \forall s \ge s_0 = \max_k s_0(k,\lambda).
\end{aligned}$$

Thus, taking parameter s_0 sufficiently large for all $s \ge s_0$ we obtain

$$\begin{aligned}
&\int_{[-1,0]\times\Gamma} ((x_0^{20} s^{\frac{3}{4}} + s^{-\frac{3}{4}})|\nabla z|^2 + s^{\frac{5}{4}}|z|^2) e^{2s\phi(x_0)}\, dx \\
&\quad + s^{\frac{1}{4}} \int_{-1}^{0} \left\| \frac{\partial}{\partial x_0} z(x_0,\cdot)\right\|^2_{W_2^{-1}(\Gamma)} e^{2s\phi(x_0)}\, dx_0 \le C_{29}\Bigg(\|g^2 e^{s\phi(0)}\|^2_{W_2^{\frac{1}{2}}(\Gamma)} \\
&\quad + \int_{-1}^{0} s^{-\frac{1}{4}} \|f(x_0,\cdot)\|^2_{W_2^{-\frac{1}{2}}(\Gamma)} e^{2s\phi(x_0)}\, dx_0 + s^{-\frac{1}{4}} \int_G x_1^{20} |f|^2 e^{2s\phi(x_0)}\, dx \Bigg).
\end{aligned} \tag{1.72}$$

The proof of the lemma is finished. □

To obtain the Carleman estimate (1.7) by means of cutoff functions we divide the problems in two cases. In the first case the support of a function y is concentrated near the boundary, and in the second case when y has a compact support in Ω. In order to deal with the first case we are going to use the Carleman estimate (1.67). Problem (1.1, 1.2) will become (1.64, 1.65) after a special change of variables which existence established in the following lemma:

Lemma 1.6. *Under Condition 1 there exists $\varepsilon^* > 0$ such that the set $\Omega^*_\varepsilon = \{x | 0 \le \psi(x) \le \varepsilon^*\}$ is C^2-diffeomorphic to the manifold $[-1, 0] \times \Gamma$.*

Proof. First we note that by Condition 1 there exists ε_0 sufficiently small such, that for all $\varepsilon \in (0, \varepsilon_0)$ the set $\Omega_\varepsilon = \{x | 0 \le \psi(x) \le \varepsilon\}$ consists of N connected components Ω^i_ε such that $\Omega^i_\varepsilon \cap \Omega^j_\varepsilon = \{\emptyset\}$ for $i \ne j$. We assume that the set Ω^i_ε contains Γ_i. To prove this lemma it suffices to establish the existence of diffeomorphism of Ω^i_ε on the manifold $[-\varepsilon, 0] \times \Gamma_i$ for some $\varepsilon > 0$. We construct a diffeomorphism $\eta(x) = (-\psi(x), \rho(x))$ from the set Ω^i_ε into $[-\varepsilon, 0] \times \Gamma_i$ where $\rho(x) = u$ is the solution to the extremal problem

$$J(y) = \frac{1}{2}\|x - y\|^2 \to \inf, \quad \psi(y) = 0. \tag{1.73}$$

Let us prove that for all $x \in \Omega^i_\varepsilon$ with ε sufficiently small this problem has only one solution. First, the existence of at least one solution $u \in \Gamma_i$ could be proved using standard arguments (see [25]). Applying the Lagrange principle to problem (1.73) we have

$$x - u = \lambda \nabla \psi(u), \tag{1.74}$$

where the Lagrange multiplier $\lambda \in \mathbb{R}^1$ is some constant. Thus, by (1.5)

$$x - u = \|x - u\| \nabla \psi(u) / |\nabla \psi(u)|. \tag{1.75}$$

Suppose that for some $x \in \Omega^i_\varepsilon$ problem (1.73) has two solutions u_i. Obviously $\|x - u_1\| = \|x - u_2\|$, thus by (1.75)

$$u_2 - u_1 = \|x - u_1\| (\nabla \psi(u_1)/|\nabla \psi(u_1)| - \nabla \psi(u_2)/|\nabla \psi(u_2)|). \tag{1.76}$$

From (1.5, 1.75) we have

$$\|u_2 - u_1\| \le \|x - u_1\| \; \hat{C} \; \|u_2 - u_1\|, \tag{1.77}$$

where constant $\hat{C}$ only depends on the norm $\|\psi\|_{C^2(\overline{\Omega})}$. Thus, taking ε_0 small enough such that $\max_{x \in \Omega^i_\varepsilon}$ $\mathrm{dist}(x, \Gamma_i) \le \frac{1}{\hat{C}+1}$ we obtain from (1.77) that $u_1 = u_2$.

One can easily check, that for all sufficiently small ε $\eta(\Omega^i_\varepsilon) \subset [-\varepsilon, 0] \times \Gamma_i$. Let us show that the mapping η is the one-to-one mapping. Our proof is by contradiction. Suppose that there exists the sequence $\{x^{(1)}_k, x^{(2)}_k\}^\infty_{k=1} \subset \mathbb{R}^n$ such that $\eta(x^{(1)}_k) = \eta(x^{(2)}_k)$ with $x^{(j)}_k \in \Omega^i_{\frac{1}{k}}$ and $x^{(1)}_k \ne x^{(2)}_k$. Without loss of generality one can assume that $x^{(j)}_k \to \tilde{x} \in \Gamma_i$. The equality $\eta(x^{(1)}_k) = \eta(x^{(2)}_k)$ implies $\psi(x^{(1)}_k) = \psi(x^{(2)}_k)$. Thus by considering the restriction of the function $\psi(x)$ on the line orthogonal to Γ^i at the point $\rho(x^{(1)}_k)$ there exists at least one point $x^{(3)}_k \in [x^{(1)}_k, x^{(2)}_k]$ such that $\frac{\partial \psi(x^{(3)}_k)}{\partial \nu(\rho(x^{(1)}_k))} = 0$. Hence $\frac{\partial \psi(\tilde{x})}{\partial \nu(\tilde{x})} = 0$ and $\nabla \psi(\tilde{x}) = 0$. But this contradicts to (1.5).

Obviously the mapping $\eta(x)$ is continuous. Our aim is to prove that $\eta \in C^2(\Omega^i_\varepsilon; [-\varepsilon, 0] \times \Gamma_i)$. This statement is trivial for the first component of the vector-function η. Hence now we should prove $\rho \in C^2(\Omega^i_\varepsilon; [-\varepsilon, 0] \times \Gamma_i)$. Let $\hat{x}$ be an arbitrary fixed point in Ω^i_ε, and $\mathcal{U}$ be a sufficiently small neighborhood of this point in $\mathbb{R}^n$ and

$\theta(u) : B_r(0) \subset \mathbb{R}^{n-1} \to \Gamma_i$ the local coordinate system on the manifold Γ_i in the neighborhood of $\rho(\hat{x})$: $\rho(\hat{x}) \in \theta(B_r(0))$. Then if $\rho(\hat{x}) = \theta(\hat{v})$, the point $\hat{v}$ is the solution to the extremal problem

$$J_1(v) = \|\hat{x} - \theta(v)\| \to \inf, \quad v \in B_r(0)$$

and by the Fermat theorem the pair $(\hat{x}, \hat{v})$ satisfies to the system of linear equations

$$\sum_{i=1}^{n} (\hat{x}_i - \theta_i(\hat{v})) \frac{\partial \theta_i(\hat{v})}{\partial v_\ell} = 0 \quad \ell \in \{1, \ldots, n-1\} \cdot \tag{1.78}$$

Moreover there exists an open set $\tilde{U} \subset U$ such that for all $x \in \tilde{U}$ exists $v \in B_r(0)$ that $\rho(x) = \theta(v)$ and equations (1.78) hold true with (x, v) exchanged for $(\hat{x}, \hat{v})$.

We introduce the new mapping

$$F(x, v) : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1},$$

where $F(x, v) = (F_1(x, v), \ldots, F_{n-1}(x, v))$ and

$$F_\ell(x, v) = \sum_{i=1}^{n} (x_i - \theta_i(v)) \frac{\partial \theta_i(v)}{\partial v_\ell} \cdot$$

Note that $\frac{\partial F(x,v)}{\partial v}$ is an epimorphism for all $v \in B_r(0)$ and $x \in \mathcal{U}$. In fact

$$\frac{\partial F_\ell}{\partial v_j}(x, v) = -\sum_{i=1}^{n} \frac{\partial \theta_j(v)}{\partial v_j} \frac{\partial \theta_i(v)}{\partial v_\ell} - \sum_{i=1}^{n} (x_i - \theta_i(v)) \frac{\partial^2 \theta_i(v)}{\partial v_j \partial v_\ell} \cdot$$

So the matrix $\{\frac{\partial F_\ell}{\partial v_j}\}$ is the sum of two matrices $A(v) = \{a_{\ell j}\} = \{-\sum_{i=1}^{n} \frac{\partial \theta_j(v)}{\partial v_j} \frac{\partial \theta_i(v)}{\partial v_\ell}\}$ and $B(v) = \{b_{\ell j}\} = \{-\sum_{i=1}^{n} (x_i - \theta_i(v)) \frac{\partial^2 \theta_i(v)}{\partial v_j \partial v_\ell}\}$. For each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\|B(v)\| \leq \delta.$$

On the other hand rank $A = n - 1$. Thus taking ε_0 sufficiently small we have

$$\text{rank} \left\{ \frac{\partial F_\ell}{\partial v_j} \right\} = n - 1.$$

Hence by implicit function theorem there exists a unique mapping $v = \mu(x)$ of class C^2such that $F(x, \mu(x)) = 0$.

On the other hand, since by Condition 1 the manifold Γ_i is compact one can choose $\varepsilon_0 > 0$ such that $\eta \in C^2(\Omega^i_\varepsilon, [-\varepsilon, 0] \times \Gamma_i)$ for all $x \in \Omega^i_\varepsilon$ with $\varepsilon \in (0, \varepsilon_0)$.

Finally let us show that $\eta(\Omega^i_\varepsilon) = [-\varepsilon, 0] \times \Gamma_i$ for all ε sufficiently small. Our proof is by contradiction. Suppose that there exists a sequence $\{y_k\}_1^\infty$ such that $y_k \in [-\frac{1}{k}, 0] \times \Gamma_i$ and $y_k = (y_{0,k}, y'_k) \notin \eta(\Omega^i_{\frac{1}{k}})$. Without loss of generality we can assume that $y'_k \to \tilde{y}'$. On the other hand on the line orthogonal to Γ_i at the point y'_k one can find $\hat{x}_k \in \Omega^i_{\frac{1}{k}}$ such that $\psi(\hat{x}_k) = y_{0,k}$. But in this case equality (1.74) holds true with x, u changed for $\hat{x}_k, y'_k$. Moreover for k sufficiently large one can choose the local coordinate system $\theta(u)$ on the manifold Γ_i in the neighborhood of y'_k such that $\rho(x_k) \in \theta(B_r(0))$, $\tilde{y}' = \theta(0)$. Let $y'_k = \theta(v'_k)$ and $\rho(\hat{x}_k) = \theta(\hat{v}_k)$. Then $v'_k \to 0$ and $\hat{v}_k \to 0$. Thus $F(\hat{x}_k, \hat{v}_k) = F(\hat{x}_k, v'_k) = 0$. But in the small neighborhood of $(\tilde{y}', 0)$ this mapping has only one solution. We arrived to contradiction. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Condition 1 implies that for ε sufficiently small the set $\Omega_\varepsilon = \{x | 0 \leq \psi(x) \leq \varepsilon\}$ consists of N connected components. By Lemma 1.6 one can choose ε^* small enough so that each component is diffeomorphic to manifold $[-1,0]\times\Gamma_i$. Hence there exists diffeomorphism $\eta(x):\Omega_{\varepsilon^*}\to[-1,0]\times\Gamma$ of class C^2. Let $e_i(x)$ $i\in\{1,2\}$ be a set of functions such that $e_1(x)\in C^2(\overline{\Omega})$, $e_2(x)\in C_0^2(\overline{\Omega\setminus\Omega_{\varepsilon^*/4}})$, $e_1(x)=1$ for all $x\in\Omega_{3\varepsilon^*/4}$, and $e_1(x)=0$ in $\Omega\setminus\Omega_{7\varepsilon^*/8}$, $e_2(x)=1$ for all $x\in\Omega\setminus\Omega_{\varepsilon^*/2}$ and $e_i(x)\geq 0$ in Ω. Obviously $|y(x)|\leq|y_1(x)|+|y_2(x)|$, where $y_i(x)=y(x)e_i(x)$. By (1.1–1.3) we have

$$Ay_i+[A,e_i]y=fe_i \quad \text{in } \Omega, \quad y_i|_{\partial\Omega}=ge_i,$$

where $[A,e_i]$ is a first order differential operator with coefficients in $L^\infty(\Omega)$. It is well known (see [29], p. 102) that the diffeomorphism η transfers the elliptic operator A into the elliptic operator on manifold $[-1,0]\times\partial\Omega$ of the form

$$\tilde{A}=\frac{\partial^2}{\partial x_0^2}+\sum_{j=0}^{1}A_j(x_0,x)\frac{\partial^j}{\partial x_0^j},$$

where $A_j(x_0,x)$ is a differential operator of order $2-j$ on $\partial\Omega$.

Hence the function $\tilde{y}=y_1\circ\eta^{-1}$ is the solution to the boundary value problem

$$\tilde{A}\tilde{y}=(fe_i-[A,e_i]y)\circ\eta^{-1} \quad \text{in } [-1,0]\times\partial\Omega,$$

$$\tilde{y}|_{x_0=0}=(ge_1)\circ\eta^{-1}, \quad \tilde{y}(-1,\cdot)=\frac{\partial\tilde{y}}{\partial x_0}(-1,\cdot)=0.$$

Thus, by Lemma 1.5 the estimate (1.67) for the function $\tilde{y}$ holds true. Moreover, it follows from the definition of the functions e_i, that all coefficients of the commutator $[A,e_i]$ have a compact support in Ω. Thus by (1.1–1.3)

$$\sum_{|\alpha|\leq 1}\int_\Omega\left(s^{\frac{5-8|\alpha|}{4}}+|\psi|^{20|\alpha|}\right)|D^\alpha y|^2\,dx\leq\sum_{i=1}^{2}\sum_{|\alpha|\leq 1}\int_\Omega\left(s^{\frac{5-8|\alpha|}{4}}+|\psi|^{20|\alpha|}\right)|D^\alpha y_i|^2\,dx$$

$$\leq C_{31}\left(\sum_{i=1}^{2}\int_{\partial\Omega}|ge_i|^2e^{2s\varphi}d\sigma+s^{-\frac14}\int_\Omega(|fe_i|^2+|[A,e_i]y|^2)e^{2s\varphi}\,dx\right.$$

$$\left.+\int_{\omega_1}s^2y^2e^{2s\varphi}\,dx\right)\leq C_{32}\left(\|g\|^2_{W_2^{\frac12}(\partial\Omega)}e^{2s}+s^{-\frac14}\int_\Omega(|f|^2+\psi^{10}|\nabla y|^2+y^2)e^{2s\varphi}\,dx+\int_{\omega_1}s^2y^2e^{2s\varphi}\,dx\right)\quad\forall\, s\geq s_0,$$

where we used Theorem 1.1 in order to estimate y_2 . This inequality imply immediately Carleman estimate (1.7). □

2. Carleman estimate for the Stokes system

In this section we will solve the observability problem for the linearized Navier-Stokes system. First let us consider the system of partial differential equations obtained from differential operator adjoint to the linearized Navier-Stokes system at point $\hat{v}$ by the change of variables $t\to -t$.

$$\frac{\partial y}{\partial t}-\Delta y+B^*(y,\hat{v})+B^*(\hat{v},y)=\nabla p+f \text{ in } Q, \tag{2.1}$$

$$\operatorname{div} y=0, \quad y|_\Sigma=0, \quad y(0,\cdot)=y_0, \tag{2.2}$$

where the operators $B^*(\cdot,\hat{v}),B^*(\hat{v},\cdot)$ are defined by formulas

$$B^*(y,\hat{v})=\left(\left(y,\frac{\partial\hat{v}}{\partial x_1}\right),\ldots,\left(y,\frac{\partial\hat{v}}{\partial x_n}\right)\right), \quad B^*(\hat{v},y)=-(\hat{v},\nabla)y. \tag{2.3}$$

We have:

Lemma 2.1. *Let $\hat v \in W^1_\infty(0,T;(V^1(\Omega)\bigcap(W^1_\infty(\Omega))^n))$. Then for each $y_0 \in V^1(\Omega), f \in L^2(0,T;V^0(\Omega))$ there exists a unique solution $y \in V^{1,2}(Q)$ to problem (2.1, 2.2). Moreover this solution satisfies to the estimate*

$$\|y\|_{V^{1,2}(Q)} \le C_1(\|y_0\|_{V^1(\Omega)} + \|f\|_{L^2(0,T;V^0(\Omega))}).$$

This lemma can be proved by standard arguments (see for example [31]). In order to formulate our Carleman estimate let us first introduce the weight functions

$$\alpha(t,x) = \frac{e^{\lambda\psi(x)} - e^{\lambda^2\|\psi\|_{C(\overline{\Omega})}}}{(t(T-t))^8}, \;\; \hat\alpha(t) = \alpha(t,x)|_{\partial\Omega} = \frac{1 - e^{\lambda^2\|\psi\|_{C(\overline{\Omega})}}}{(t(T-t))^8}, \;\; \hat\varphi(t) = \frac{1}{(t(T-t))^8} \tag{2.4}$$

where $\lambda > 1$ is some parameter to be fixed below. By (1.5, 2.4)

$$\hat\alpha(t) \le \alpha(t,x) \quad \forall (t,x) \in Q. \tag{2.5}$$

First let us remind some standard Carleman estimates for the heat equation

$$\frac{\partial z}{\partial t} - \Delta z = f \text{ in } Q, \quad z|_\Sigma = 0. \tag{2.6}$$

We have:

Lemma 2.2 [20]. *Let $z \in L^2(Q)$ be solution to (2.6) and $f \in L^2(Q)$. Then there exists a $\hat\lambda > 1$ such that for any $\lambda > \hat\lambda$ one can find $s_0(\lambda)$ such that the following inequality holds*

$$\int_Q \left(\frac{1}{s\hat\varphi}\left(\left|\frac{\partial z}{\partial t}\right|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 z}{\partial x_i \partial x_j}\right|^2\right) + s\hat\varphi|\nabla z|^2 + s^3\hat\varphi^3 z^2\right) e^{2s\alpha}\,dx\,dt$$
$$\le C_2\left(\int_Q f^2 e^{2s\alpha} dxdt + \int_{Q_{\omega_1}} s^3\hat\varphi^3 z^2 e^{2s\alpha}\,dx\,dt\right) \quad \forall s \ge s_0, \tag{2.7}$$

where the constant C_2 is independent of s and ω_1 is the subdomain from the Lemma 1.1.

Now we can prove the following theorem:

Theorem 2.1. *Let $y \in L^2(0,T;V^0(\Omega))$ be solution to (2.1, 2.2) and $f \in L^2(0,T;V^0(\Omega))$. Then there exists a $\hat\lambda > 1$ such that for any $\lambda > \hat\lambda$ one can find $s_0(\lambda)$ such that the following inequality holds*

$$I(s) = \int_Q \left(\frac{1}{s\hat\varphi}\left(\left|\frac{\partial y}{\partial t}\right|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 y}{\partial x_i \partial x_j}\right|^2\right) + s\hat\varphi|\nabla y|^2 + s^3\hat\varphi^3 y^2\right) e^{2s\alpha}\,dx\,dt$$
$$\le C_3\left(\int_Q f^2(e^{2s\alpha} + (s\hat\varphi)^{\frac{3}{4}} e^{2s\hat\alpha})\,dx\,dt + \int_{Q_{\omega_1}} (s^3\hat\varphi^3 y^2 + (s\hat\varphi)^{\frac{11}{4}} p^2) e^{2s\alpha} dxdt\right) \quad \forall s \ge s_0, \tag{2.8}$$

where the constant C_3 is independent of s and ω_1 is the subdomain from the Lemma 1.1.

Proof. Applying the Carleman estimate (2.7) to the linearized Navier-Stokes system and keeping in mind the inequality

$$\|B^*(y(t,\cdot),\hat v(t,\cdot))\|^2_{(L^2(\Omega))^n} + \|B^*(\hat v(t,\cdot),y(t,\cdot))\|^2_{(L^2(\Omega))^n} \le C_4 \int_\Omega (|\nabla y(t,x)|^2 + |y(t,x)|^2)\,dx$$

we have

$$I(s) \le C_5 \left(\int_{Q_{\omega_1}} s^3 \hat{\varphi}^3 y^2 e^{2s\alpha}\, dx\, dt + \int_Q |\nabla p|^2 e^{2s\alpha}\, dx\, dt + \int_Q f^2 e^{2s\alpha}\, dx\, dt \right) \; \forall s \ge s_0(\lambda). \tag{2.9}$$

Applying the divergence operator div to both parts of equation (2.1) we obtain

$$\Delta p(t,\cdot) = \operatorname{div}(B^*(\hat{v}(t,\cdot), y(t,\cdot)) + B^*(y(t,\cdot), \hat{v}(t,\cdot))) \text{ in } \Omega \tag{2.10}$$

for a.e. $t \in [0,T]$.

Note that

$$|\operatorname{div}(B^*(\hat{v}(t,x), y(t,x)) + B^*(y(t,x), \hat{v}(t,x)))| \le C_6(|\nabla y(t,x)| + |y(t,x)|) \quad \forall (t,x) \in Q. \tag{2.11}$$

From (2.10), using the estimates (1.7, 2.11) we can estimate the gradient of the pressure as follows

$$\int_\Omega |\nabla p(t,\cdot)|^2 e^{2s\alpha}\, dx \le C_7 \Bigg(\|p(t,\cdot)\|^2_{W_2^{\frac12}(\partial\Omega)} (s\hat{\varphi}(t))^{\frac34} e^{2s\hat{\alpha}(t)} + \int_\Omega (s\hat{\varphi})^{\frac12} (|\nabla y|^2 + |y|^2) e^{2s\alpha}\, dx + \int_{\omega_1} (s\hat{\varphi})^{\frac{11}{4}} p^2 e^{2s\alpha}\, dx \Bigg), \tag{2.12}$$

where before applying (1.7) we made the change $s \to s/(t(T-t))^8$ and we multiplied the inequality (1.7) by $\exp\{-\exp\{\lambda^2 \|\psi\|_{C(\overline{\Omega})}\}/(t(T-t))^8\}$. Inequalities (2.9, 2.12) imply

$$I(s) \le C_8 \Bigg(\int_{Q_{\omega_1}} (s^3 \hat{\varphi}^3 y^2 + (s\hat{\varphi})^{\frac{11}{4}} p^2) e^{2s\alpha}\, dx\, dt + \int_0^T (s\hat{\varphi}(t))^{\frac34} \|p(t,\cdot)\|^2_{W_2^{\frac12}(\partial\Omega)} e^{2s\hat{\alpha}} dt + \int_Q f^2 e^{2s\alpha}\, dx\, dt \Bigg) \; \forall\, s \ge s_0(\lambda). \tag{2.13}$$

To estimate the norm of the trace of the pressure on the boundary we introduce the new function $w(t,x) = y(t,x)\hat{\varphi}^{\frac38}(t) e^{s\hat{\alpha}(t)}$. By (2.1, 2.2) this function satisfies to the system of equations

$$Lw - \left(s\hat{\alpha}_t + \frac{\partial \hat{\varphi}^{\frac38}}{\partial t} \hat{\varphi}^{-\frac38} \right) w = \nabla p \hat{\varphi}^{\frac38} e^{s\hat{\alpha}} + f \hat{\varphi}^{\frac38} e^{s\hat{\alpha}} \quad \text{in } \Omega, \tag{2.14}$$

$$w|_\Sigma = 0, \; \operatorname{div} w = 0, \; w(0,\cdot) = 0. \tag{2.15}$$

By the Sobolev embedding Theorem and the standard energy *a priori* estimates for solutions of (2.14, 2.15) we have

$$\int_0^T \|p(t,\cdot)\|^2_{W_2^{\frac12}(\partial\Omega)} (s\hat{\varphi})^{\frac34} e^{2s\hat{\alpha}}\, dt \le C_9 \int_Q (|\nabla p|^2 + p^2)(s\hat{\varphi})^{\frac34} e^{2s\hat{\alpha}}\, dx\, dt$$
$$\le C_{10} \left(\int_Q \left((s\hat{\varphi})^{\frac34} |s\hat{\alpha}_t + \frac{\partial \hat{\varphi}^{\frac38}}{\partial t} \hat{\varphi}^{-\frac38}|^2 y^2 e^{2s\hat{\alpha}} + |f|^2 (s\varphi)^{\frac34} e^{2s\hat{\alpha}} \right) dx\, dt + \int_{Q_\omega} p^2 (s\hat{\varphi})^{\frac34} e^{2s\hat{\alpha}}\, dx\, dt \right).$$

Taking into account that

$$\hat{\varphi}^{\frac34}(t) \left(|\hat{\alpha}_t(t)|^2 + \left| \frac{\partial \hat{\varphi}^{\frac38}(t)}{\partial t} \hat{\varphi}^{-\frac38}(t) \right|^2 \right) \le C_{11} \hat{\varphi}^3(t) \quad \forall t \in [0,T]$$

we obtain from the previous inequality

$$\int_0^T (s\hat{\varphi})^{\frac{3}{4}} \|p(t,\cdot)\|^2_{W_2^{\frac{1}{2}}(\partial\Omega)} e^{2s\hat{\alpha}}\, dt \leq C_{12}\Bigg(\int_0^T (s\hat{\varphi})^{\frac{3}{4}} \|p(t,\cdot)\|^2_{L^2(\omega_1)} e^{2s\hat{\alpha}}\, dt + \int_Q \left(s^{\frac{11}{4}} \hat{\varphi}^3 y^2 e^{2s\alpha} + |f|^2 (s\hat{\varphi})^{\frac{3}{4}} e^{2s\hat{\alpha}} \right) dx\, dt \Bigg). \tag{2.16}$$

From (2.5, 2.13, 2.16) the Carleman estimate (2.8) follows immediately. □

Let us consider the system of partial differential equations which is obtained from (2.1, 2.2) by change of variables $t \to -t$:

$$L^* z = -\frac{\partial z}{\partial t} - \Delta z + B^*(z, \hat{v}) + B^*(\hat{v}, z) = \nabla q + f \text{ in } Q, \tag{2.17}$$

$$\operatorname{div} z = 0, \quad z|_\Sigma = 0, \quad z(T,\cdot) = z_0, \tag{2.18}$$

where the operators $B^*(\hat{v},\cdot), B^*(\cdot,\hat{v})$ are defined in (2.3). A short calculation shows, that L^* is the adjoint of the differential operator which corresponds to the linearization of the Navier-Stokes equations at the point $\hat{v}$.

We need the following technical lemma:

Lemma 2.3. *Let $z_0 \in V^0(\Omega)$ and $\hat{v} \in W^1_\infty(0,T;(V^1(\Omega) \cap (W^1_\infty(\Omega))^n)$. Then the solutions of problem (2.17, 2.18) satisfy the estimate*

$$\left\| \frac{\partial z}{\partial t} \right\|_{L^2(0,T/2;V^0(\Omega))} + \|z\|_{L^2(0,T/2;V^2(\Omega))} \leq C_{13}(\|z\|_{L^2(0,3T/4;V^1(\Omega))} + \|f\|_{L^2(0,3T/4;V^0(\Omega))}). \tag{2.19}$$

The proof of this lemma is given in Appendix.

Let us consider the boundary value problem for the stationary Stokes system

$$\Delta v = \nabla q + g \text{ in } \Omega, \ \operatorname{div} v = 0, \ v|_{\partial\Omega} = 0. \tag{2.20}$$

The following lemma is proved in [31]:

Lemma 2.4. *For any $g \in V^{-1}(\Omega)$ there exists a unique solution $v \in V^1(\Omega)$ of problem (2.20) and this solution satisfies the estimate*

$$\|v\|_{V^1(\Omega)} \leq C_{14} \|g\|_{V^{-1}(\Omega)}. \tag{2.21}$$

Let us introduce the function $\kappa(t,x)$ by the formula

$$\kappa(t,x) = (e^{\tilde{\lambda}\psi} - e^{\tilde{\lambda}^2 ||\psi||_{C(\overline{\Omega})}})/(\ell(t)(T-t))^8, \hat{\kappa}(t) = \min_{x\in\overline{\Omega}} \kappa(t,x), \qquad \tilde{\kappa}(t) = \max_{x\in\overline{\Omega}} \kappa(t,x), \tag{2.22}$$

$$\ell \in C^\infty[0,T], \quad \ell(t) = t \ \ \forall t \in [3T/2, T], \quad \ell(t) > 0 \quad \forall t \in [0,T]. \tag{2.23}$$

Let us take the parameter $\tilde{\lambda}$ such that

$$\tilde{\lambda} > \hat{\lambda},$$

where $\hat{\lambda}$ defined in Theorem 2.1 and

$$\max_{x\in\overline{\Omega}} \kappa(t,x) < \frac{9}{10} \min_{x\in\overline{\Omega}} \kappa(t,x) \quad \forall t \in [0,T]. \tag{2.24}$$

Note that

$$\kappa(t,x)=\alpha_{\tilde\lambda}(t,x)\quad\forall\,(t,x)\in\left(\frac{3}{2}T,T\right)\times\Omega.$$

Now we are ready to prove an observability inequality for system (2.17, 2.18). We have

Theorem 2.2. *Let pair $(z,q)\in V^{1,2}(Q)\times L^2(0,T;W_2^1(\Omega))$ satisfies (2.17, 2.18), $f\in L^2(0,T;V^0(\Omega))$. Then there exists $\hat s>0$ such that inequality holds*

$$\|z(0,\cdot)\|^2_{V^0(\Omega)}+\int_Q(T-t)^8|z|^2e^{\hat s\kappa}\,dx\,dt\le C_{15}\left(\int_{Q_\omega}|z|^2e^{\frac{9}{10}\hat s\hat\kappa}\,dx\,dt+\int_Q|f|^2e^{\frac{9}{10}\hat s\hat\kappa}\,dx\,dt\right),\tag{2.25}$$

where the constant C_{15} is independent of f,z.

Proof. Let us introduce the functions $r,g,\tilde f$ by formulas

$$r(t,x)=\int_{\frac{T}{2}}^t z(\tau,x)\,d\tau,\quad g(t,x)=\int_{\frac{T}{2}}^t q(\tau,x)\,d\tau,\quad \tilde f(t,x)=\int_{\frac{T}{2}}^t f(\tau,x)\,d\tau.$$

Short calculations shows that the pair (r,g) satisfies the equations

$$L^*r=\nabla g+\int_{\frac{T}{2}}^t B^*\left(r(s,\cdot),\frac{\partial\hat v(s,\cdot)}{\partial s}\right)ds+\int_{\frac{T}{2}}^t B^*\left(\frac{\partial\hat v(s,\cdot)}{\partial s},r(s,\cdot)\right)ds-z\left(\frac{T}{2},\cdot\right)+\tilde f\text{ in }Q,\tag{2.26}$$

$$\operatorname{div}r=0,\quad r|_\Sigma=0.\tag{2.27}$$

Let us show that the function g satisfies the estimate

$$\|g(t,\cdot)\|_{L^2(\omega_1)}\le C_{16}\Bigg(\|z\left(\frac{T}{2},\cdot\right)\|_{(L^2(\omega))^n}+\|z\|_{(L^2((\frac{T}{2},t)\times\omega))^n}+\|z(t,\cdot)\|_{(L^2(\omega))^n}$$
$$+\|r(t,\cdot)\|_{(L^2(\omega))^n}+\|\tilde f(t,\cdot)\|_{(L^2(\omega))^n}\Bigg),\tag{2.28}$$

where C_{16} is independent of $t\in[0,T]$. Using the definition of the function r we can rewrite equation (2.26) as follows

$$-\Delta r=\nabla g-z\left(\frac{T}{2},\cdot\right)+z(t,\cdot)-B^*(r,\hat v)-B^*(\hat v,r)$$
$$+\int_{\frac{T}{2}}^t B^*\left(r(s,\cdot),\frac{\partial\hat v(s,\cdot)}{\partial s}\right)ds+\int_{\frac{T}{2}}^t B^*\left(\frac{\partial\hat v(s,\cdot)}{\partial s},r(s,\cdot)\right)ds+\tilde f\quad\text{in }\Omega.\tag{2.29}$$

Note that the function g in (2.29) is defined up to a constant. To fix it we set

$$\int_{\omega_1}q(t,x)dx=0\quad\forall t\in[0,T].$$

This equality implies

$$\int_{\omega_1}g(t,x)dx=0\quad\forall t\in[0,T].\tag{2.30}$$

We are looking for the functions r, g in the form $r = r_1 + r_2, g = g_1 + g_2$, where

$$-\Delta r_1 = \nabla g_1 - z\left(\frac{T}{2},\cdot\right) + z(t,\cdot) - B^*(r,\hat{v}) - B^*(\hat{v},r) + \int_{\frac{T}{2}}^t B^*\left(r(s,\cdot),\frac{\partial\hat{v}(s,\cdot)}{\partial s}\right)ds + \int_{\frac{T}{2}}^t B^*\left(\frac{\partial\hat{v}(s,\cdot)}{\partial s},r(s,\cdot)\right)ds + \tilde{f} \text{ in } \omega,$$

$$\operatorname{div} r_1 = 0,\ r_1|_{\partial\omega} = 0,\ \int_{\omega_1} g_1(t,x)dx = 0\ t \in (0,T). \quad (2.31)$$

By Lemma 2.4 the unique solution of problem (2.31) exists and satisfies the estimate

$$\|r_1\|_{V^1(\omega)} + \|g_1\|_{L^2(\omega)} \le C_{17}\Bigg(\left\|z\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n} + \|r\|_{(L^2((\frac{T}{2},t)\times\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n}\Bigg). \quad (2.32)$$

By virtue of (2.29, 2.27, 2.31) the functions r_2, q_2 should satisfy the equations

$$\Delta r_2 = \nabla g_2 \text{ in } \omega,\ \operatorname{div} r_2 = 0. \quad (2.33)$$

Thus

$$\Delta g_2 = 0 \quad \text{in } \omega.$$

Applying the Laplace operator Δ to this last equation we obtain $\Delta^2 r_2 = 0$. Thus thanks to (2.33) and from well known estimates about interior regularity of solutions of elliptic equations (see [24]) we have

$$\|r_2(t)\|_{(C^2(\overline{\omega}_1))^n} \le C_{18}\|r(t) - r_1(t)\|_{(L^2(\omega))^n} \le C_{19}\Bigg(\left\|z\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\omega))^n} + \|r\|_{(L^2((\frac{T}{2},t)\times\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n} + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n}\Bigg). \quad (2.34)$$

By (2.34) equality (2.33) implies the estimate

$$\|\nabla g_2(t,\cdot)\|_{(C(\overline{\omega}_1))^n} \le C_{20}\Bigg(\left\|z\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|r\|_{(L^2((\frac{T}{2},t)\times\omega))^n} + \|\tilde{f}(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n}\Bigg). \quad (2.35)$$

By (2.30, 2.31) we obtain

$$\int_{\omega_1} g_2(t,x)dx = 0\ \forall\, t \in [0,T].$$

Thus inequality (2.35) yields

$$\|g_2(t,\cdot)\|_{L^2(\omega_1)} \le C_{21}\Bigg(\left\| z\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\omega))^n} + \|r(t,\cdot)\|_{(L^2(\omega))^n} + \|r\|_{(L^2((\frac{T}{2},t)\times\omega))^n} + \|\tilde f(t,\cdot)\|_{(L^2(\omega))^n} + \|z(t,\cdot)\|_{(L^2(\omega))^n}\Bigg). \tag{2.36}$$

This inequality and (2.32) imply (2.28).

Note that, since for each fixed x the function $\alpha(t,x)$ reaches it maximum at $T/2$ we have

$$\int_Q \left| \int_{\frac{T}{2}}^t B^*\left(r(s,\cdot),\frac{\partial \hat v(s,\cdot)}{\partial s}\right) ds + \int_{\frac{T}{2}}^t B^*\left(\frac{\partial \hat v(s,\cdot)}{\partial s}, r(s,\cdot)\right) ds\right|^2 e^{2s\alpha}\,dx\,dt \le C_{22}\int_Q \left(|\nabla r|^2 + r^2\right) e^{2s\alpha}\,dx\,dt, \tag{2.37}$$

where the constant C_{22} is independent of s.

Applying Carleman inequality (2.8) to equations (2.26, 2.27) and taking into account (2.37) we have

$$\int_Q \left(\frac{1}{s\hat\varphi}\left(|z|^2 + \sum_{i,j=1}^n \left|\frac{\partial^2 r}{\partial x_i \partial x_j}\right|^2\right) + s\hat\varphi|\nabla r|^2 + s^3\hat\varphi^3|r|^2\right) e^{2s\alpha}\,dx\,dt$$
$$\le C_{23}(\tilde\lambda)\Bigg(\int_{Q_{\omega_1}} (s\varphi)^{\frac{11}{4}} g^2 e^{2s\alpha}\,dx\,dt + \int_{Q_{\omega_1}} s^3\hat\varphi^3|r|^2 e^{2s\alpha}\,dx\,dt + \int_Q (s\hat\varphi)^{\frac{3}{4}}\left(|\tilde f|^2 + \left|z\left(\frac{T}{2},x\right)\right|^2\right) e^{2s\alpha}\,dx\,dt\Bigg), \tag{2.38}$$

where $s \ge s_0(\tilde\lambda)$.

Parameter $s_0(\tilde\lambda)$ is defined in Theorem 2.1. Set $\hat s = 2s_0(\tilde\lambda)$. Using *a priori* estimate (2.19) for system (2.17, 2.18) in the right hand side of inequality (2.38) we can replace the function α by κ and the function $\tilde\varphi$ by $(T-t)^{-8}$ and the constant C_{23} by $C_{24}(s)$:

$$\int_Q (T-t)^8|z|^2 e^{s\kappa}\,dx\,dt + \left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^1(\Omega)} \le C_{24}(s)\Bigg(\int_{Q_{\omega_1}} \frac{g^2}{(T-t)^{22}} e^{s\kappa}\,dx\,dt + \int_{Q_{\omega_1}} \frac{|r|^2}{(T-t)^{24}} e^{s\kappa}\,dx\,dt + \int_Q \frac{1}{(T-t)^6}|\tilde f|^2 e^{s\kappa}\,dx\,dt + \left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^0(\Omega)}\Bigg), \tag{2.39}$$

where $s \ge \hat s$.

Using estimate (2.28) one can rewrite (2.39) as follows

$$\int_Q (T-t)^8|z|^2 e^{\hat s\kappa}\,dx\,dt + \left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^1(\Omega)} \le C_{25}\Bigg(\int_{Q_{\omega_1}} (T-t)^{-24}|r|^2 e^{\hat s\tilde\kappa}\,dx\,dt + \int_{Q_\omega} \frac{(|\tilde f|^2 + |z|^2 + |r|^2)}{(T-t)^{24}} e^{\hat s\tilde\kappa}\,dxdt + \int_0^T \frac{\|z\|^2_{(L^2((\frac{T}{2},t)\times\omega))^n}}{(T-t)^{24}} e^{\hat s\tilde\kappa}\,dt + \int_Q \frac{1}{(T-t)^6}|\tilde f|^2 e^{\hat s\kappa}\,dx\,dt + \left\|z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^0(\Omega)}\Bigg). \tag{2.40}$$

Note that by (2.22–2.24)

$$\int_{Q_\omega} \frac{(|\tilde f|^2 + |z|^2 + |r|^2)}{(T-t)^{24}} e^{\hat s\tilde\kappa}\, dx\, dt + \int_Q \frac{|\tilde f|^2}{(T-t)^6} e^{\hat s\kappa}\, dx\, dt + \int_0^T \frac{\|z\|^2_{(L^2((\frac{T}{2},t)\times\omega))^n}}{(T-t)^{24}} e^{\hat s\tilde\kappa}\, dt \le C_{26} \left(\int_Q |f|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dx\, dt + \int_{Q_\omega} |z|^2 e^{\frac{9}{10}\hat s\hat\kappa} dxdt \right),$$

where C_{26} is independent constant. Thanks to this last inequality we deduce from (2.40)

$$\int_Q |z|^2 (T-t)^8 e^{\hat s\kappa}\, dx\, dt + \left\| z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^1(\Omega)} \le C_{27} \Bigg(\int_{Q_\omega} |z|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dx\, dt + \int_Q |f|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dx\, dt + \left\| z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^0(\Omega)} \Bigg). \tag{2.41}$$

Let us finish the proof by contradiction. By Lemma 2.1 instead of showing the estimate (2.25) it suffices to prove

$$\int_Q |z|^2 (T-t)^8 e^{\hat s\kappa}\, dx\, dt + \left\| z\left(\frac{T}{2},\cdot\right)\right\|^2_{V^1(\Omega)} \le C_{28} \left(\int_{Q_\omega} |z|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dx\, dt + \int_Q |f|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dx\, dt \right). \tag{2.42}$$

If inequality (2.42) is not true, then by (2.41) there exists a sequence (z_k, q_k, f_k) such that

$$L^* z_k = \nabla q_k + f_k \text{ in } Q,\ \operatorname{div} z_k = 0,\ z_k|_\Sigma = 0,\ \lim_{k\to\infty} \left\| z_k\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\Omega))^n} > 0, \tag{2.43}$$

$$f_k \to 0 \text{ in } (L^2(Q, e^{\frac{9}{10}\hat s\hat\kappa}))^n,\ \int_{Q_\omega} |z_k|^2 e^{\frac{9}{10}\hat s\hat\kappa}\, dxdt \to 0 \text{ as } k\to\infty,$$
$$z_k\left(\frac{T}{2},\cdot\right) \to z\left(\frac{T}{2},\cdot\right) \text{ in } (L^2(\Omega))^n,\ z_k \to z \text{ weakly in } V^{1,2}((0, T-\epsilon)\times\Omega) \tag{2.44}$$

for all $\epsilon \in (0,T)$. Passing to the limit in (2.43), and taking into account (2.44) we obtain

$$L^* z = \nabla q \text{ in } Q, \quad \operatorname{div} z = 0, \quad z|_\Sigma = 0, \quad z|_{Q_\omega} \equiv 0, \tag{2.45}$$

$$\left\| z\left(\frac{T}{2},\cdot\right)\right\|_{(L^2(\Omega))^n} \ne 0. \tag{2.46}$$

From (2.8, 2.45) we obtain

$$z \equiv 0,$$

but this is impossible by virtue of (2.46). Therefore the proof of lemma is complete. □

3. Solvability of linear controllability problem

Let us consider the problem of exact controllability of the linearized Navier-Stokes equations:

$$Ly = \frac{\partial y}{\partial t} - \Delta y + B(y, \hat{v}) + B(\hat{v}, y) = \nabla p + f + \chi_\omega u \text{ in } Q, \tag{3.1}$$

$$\operatorname{div} y = 0, \quad y|_\Sigma = 0, \quad y(0, x) = y_0(x), \tag{3.2}$$

$$y(T, x) = 0, \tag{3.3}$$

where the functions y_0, f are given and u is a control taken in the space $(L^2(Q_\omega))^n$. Before studying solvability of problem (3.1–3.3) let us recall some existence theorems for boundary value problem (3.1, 3.2), assuming that u is a fixed function.

Let us set

$$\eta(t, x) = -\hat{s}\kappa(t, x), \tag{3.4}$$

where the function κ defined in (2.22) and the parameter $\hat{s}$ have been taken from Theorem 2.2. Since the function $\kappa(t, x)$ is negative, $\eta(t, x)$ is positive. Moreover $\lim_{t \to T-0} \eta(t, x) = +\infty$.

Later on we use the following weight function:

$$\theta(t, x) = (1 - \chi_\omega)\frac{e^\eta}{(T - t)^8} + \chi_\omega. \tag{3.5}$$

To formulate our results we need to introduce some non-standard functional spaces

$$F(Q, \theta) = \{f \in (L_2(Q))^n : \exists \ f_1 \in (L_2(Q, \theta))^n, \\ \exists \ f_2 \in L_2(0, T; W_2^1(\Omega)) \text{ such that } f = f_1 + \nabla f_2\} \cdot$$

The norm of the space $F(Q, \theta)$ is defined by the relation

$$\|f\|_{F(Q,\theta)} = \inf_{\substack{f_1, \nabla f_2 \\ f = f_1 + \nabla f_2}} \left(\|f_1\|^2_{(L_2(Q,\theta))^n} + \|\nabla f_2\|^2_{(L_2(Q))^n}\right)^{1/2}. \tag{3.6}$$

We are looking for solution of the controllability problem in the following space

$$Y(Q) = \left\{y \in V^{1,2}(Q) | Ly \in F(Q, \theta), e^{-\frac{2}{5}\hat{s}\hat{\kappa}} y \in V^{1,2}(Q)\right\}$$

equipped with the norm

$$||y||^2_{Y(Q)} = ||Ly||^2_{F(Q,\theta)} + \|e^{-\frac{2}{5}\hat{s}\hat{\kappa}} y\|^2_{V^{1,2}(Q)}.$$

where the function κ introduced in (2.22) and the operator L determined in (3.1). Now we can convert observability inequality (2.25) into the controllability result for the linearized Navier-Stokes system (3.1–3.3). We have

Theorem 3.1. *Let $f \in F(Q, \theta)$, $y_0 \in V^1(\Omega)$. Then there exists a solution of problem (3.1–3.3) $(y, p, u) \in Y(Q) \times L^2(0, T; W_2^1(\Omega)) \times (L^2(Q_\omega))^n$ which satisfies the estimate*

$$||(y, p, u)||_{Y(Q) \times L^2(0,T;W_2^1(\Omega)) \times (L^2(Q_\omega))^n} \leq C_1(||y_0||_{V^1(\Omega)} + ||f||_{F(Q,\theta)}). \tag{3.7}$$

Proof. We first assume that $f \in L^2(Q,\theta)$ and $f|_{Q^\omega} \equiv 0$. Let us consider the extremal problem

$$\mathcal{J}_k(y,u) = \frac{1}{2}\int_Q \rho_k |y|^2\,dx\,dt + \frac{1}{2}\int_Q m_k|u|^2 dxdt \to \inf, \tag{3.8}$$

$$Ly = u + \nabla p + f \quad \text{in} \quad Q,\ \operatorname{div} y = 0,\ y\big|_\Sigma = 0, \quad y(0,x) = y_0(x), \quad y(T,x) = 0, \tag{3.9}$$

where

$$\rho_k(t) = e^{\frac{-9\hat{s}\hat{\kappa}(t)(T-t)^8}{10(T-t+1/k)^8}}, \qquad m_k(t,x) = \begin{cases} e^{\frac{-9}{10}\frac{\hat{s}\hat{\kappa}(T-t)^8}{(T-t+1/k)^8}}, x \in \overline{\omega}, \\ k, x \in \Omega \setminus \overline{\omega}. \end{cases}$$

Obviously the functions ρ_k, m_k are bounded in Q for every $k > 0$.

By Lemma 2.1 the exists an admissible element to problem (3.8, 3.9). So it is easy to prove (see [25,26]) that problem (3.8, 3.9) has a unique solution, which we denote as $(\hat{y}_k, \hat{p}_k, \hat{u}_k) \in V^{1,2}(Q) \times L^2(0,T; W_2^1(\Omega)) \times (L^2(Q))^n$.

Thus, applying the Lagrange principle to extremal problem (3.8, 3.9) (see [1,9]) we obtain

$$L\hat{y}_k = f + \nabla p_k + \hat{u}_k \text{ in } Q, \operatorname{div} \hat{y}_k = 0,\ \hat{y}_k\big|_\Sigma = 0,\ \hat{y}_k(T,\cdot) \equiv 0,\ \hat{y}_k(0,\cdot) = y_0, \tag{3.10}$$

$$L^* z_k = \nabla q_k + \rho_k \hat{y}_k \text{ in } Q, \quad z_k\big|_\Sigma = 0,\ \operatorname{div} z_k = 0,\ z_k = -m_k \hat{u}_k \text{ in } Q, \tag{3.11}$$

where the operator L^* defined in (2.17) is an operator conjugate to the operator L.

Since the function ρ_k only depends on the variable t only, by Lemma 2.1 $\rho_k \hat{y}_k \in L^2(0,T; V^1(\Omega))$. So we can apply the estimate (2.25) to equation (3.11) in order to obtain:

$$\int_Q (T-t)^8 e^{\hat{s}\kappa} |z_k|^2\,dx\,dt + \|z_k(0,\cdot)\|^2_{(L^2(\Omega))^n} \le C_2 \left(\int_Q \rho_k^2 e^{\frac{9}{10}\hat{s}\hat{\kappa}} |\hat{y}_k|^2 dxdt + \int_Q e^{\frac{9}{10}\hat{s}\hat{\kappa}} |z_k|^2\,dx\,dt \right). \tag{3.12}$$

We observe that $|\rho_k(t) e^{\frac{9}{10}\hat{s}\hat{\kappa}(t)}| \le 1$ for all $(t,x) \in Q$ and $|m_k(t,x) e^{\frac{9}{10}\hat{s}\hat{\kappa}(t)}| \le 1$ for all $(t,x) \in Q_\omega$. In fact

$$-\frac{9\hat{s}\hat{\kappa}(t)(T-t)^8}{10(T-t+1/k)^8} + \frac{9}{10}\hat{s}\hat{\kappa}(t) = -\frac{9}{10}\frac{\hat{s}}{\ell(t)} \max_{x\in\overline{\Omega}} \left(e^{\tilde{\lambda}^2 \|\psi\|_{C(\overline{\Omega})}} - e^{\tilde{\lambda}\psi(x)} \right) \left(\frac{1}{(T-t)^8} - \frac{1}{(T-t+1/k)^8} \right) < 0.$$

Keeping in mind these inequalities and substituting the function z_k by using the forth equation in (3.11) in the last integral of equality (3.12) we have

$$\int_\Omega |z_k(0,x)|^2 dx + \int_Q |z_k|^2 (T-t)^8 e^{\hat{s}\kappa}\,dx\,dt \le C_3 \left(\int_Q \rho_k |\hat{y}_k|^2\,dx\,dt + \int_{Q_\omega} m_k |\hat{u}_k|^2\,dx\,dt \right). \tag{3.13}$$

Multiplying (3.11_1) by $\hat{y}_k$ scalarly in $(L^2(Q))^n$, and integrating by parts with respect to t and x, bearing in mind (3.10) after simplifications we have

$$0 = (L^* z_k - \nabla q_k - \rho_k \hat{y}_k, \hat{y}_k)_{(L^2(Q))^n} = -\int_Q \rho_k |\hat{y}_k|^2\,dx\,dt + (z_k, L\hat{y}_k)_{(L^2(Q))^n} + (z_k(0,\cdot), \hat{y}_k(0,\cdot))_{(L^2(\Omega))^n}$$
$$= -\int_Q \rho_k |\hat{y}_k|^2\,dx\,dt - \int_Q m_k |\hat{u}_k|^2\,dx\,dt + \int_Q (f, z_k)\,dx\,dt + (z_k(0,\cdot), y_0)_{(L^2(\Omega))^n}.$$

Hence

$$\mathcal{J}_k(\hat{y}_k, \hat{u}_k) = \frac{1}{2}\int_Q \left(\rho_k |\hat{y}_k|^2 + m_k |\hat{u}_k|^2 \right)\,dx\,dt = \frac{1}{2}\left(\int_Q (f, z_k)\,dx\,dt + (z_k(0,\cdot), y_0)_{(L^2(\Omega))^n} \right). \tag{3.14}$$

Note that

$$\left|\int_Q (f, z_k)dx\,dt\right| \leq C_4\|f\|_{(L^2(Q,\theta))^n}\|(T-t)^4 e^{\hat{s}\kappa/2} z_k\|_{(L^2(Q))^n}. \tag{3.15}$$

By (3.13–3.15) we obtain

$$\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \leq C_5(||f||_{F(Q,\theta)} + ||y_0||_{(L^2(\Omega))^n})\sqrt{\mathcal{J}_k(\hat{y}_k, \hat{u}_k)}.$$

Hence

$$\mathcal{J}_k(\hat{y}_k, \hat{u}_k) \leq C_5^2(||f||^2_{F(Q,\theta)} + ||y_0||^2_{(L^2(\Omega))^n}). \tag{3.16}$$

By virtue of (3.16, 3.4) we have a subsequence $\{(\hat{y}_k, \hat{u}_k)\}_{k=1}^{\infty}$ such that

$$\begin{aligned}
&(\hat{y}_k, \hat{u}_k) \to (y, u) \quad \text{weakly in} \quad V^{1,2}(Q) \times (L^2(Q))^n, \\
&\hat{u}_k \to 0 \quad \text{in} \quad (L^2((0,T) \times (\Omega \setminus \omega)))^n, \\
&\int_{Q_\omega} m_k|\hat{u}_k|^2\,dx\,dt + \int_Q \rho_k|\hat{y}_k|^2\,dx\,dt \leq C_6.
\end{aligned} \tag{3.17}$$

Also, by (3.13, 3.17) it follows from (3.11)

$$\|m_k\hat{u}_k\|_{V^{1,2}((0,T-\varepsilon)\times\Omega)} \leq C_7(\varepsilon)$$

for all $\varepsilon \in (0, T)$. Hence without loss of generality one can assume

$$\hat{u}_k(t, x) \to u(t, x) \quad \text{almost everywhere in } Q_\omega. \tag{3.18}$$

Using (3.17), we pass to the limit in (3.10) to obtain that pair (y, u) is a solution of problem (3.1–3.3). The relations (3.16, 3.17) and Lemma 2.1 imply the estimate

$$||(y, p, u)||_{V^{1,2}(Q)\times L^2(0,T;W_2^1(\Omega))\times(L^2(Q_\omega))^n} \leq C_8(||y_0||_{V^1(\Omega)} + ||f||_{F(Q,\theta)}). \tag{3.19}$$

Also by (3.17, 3.18) and Fatou's theorem (see [23], p. 307) we have

$$\|(y, u)\|_{(L^2(Q,e^{-\frac{9}{10}\hat{s}\hat{\kappa}}))^n\times(L^2(Q_\omega,e^{-\frac{9}{10}\hat{s}\hat{\kappa}}))^n} \leq C_6. \tag{3.20}$$

Now to prove (3.7) we need only to estimate the norm of the function $e^{-\frac{2}{5}\hat{s}\hat{\kappa}}y$ in the space $V^{1,2}(Q)$. Let us make the following change of variables in (3.1). Set $\tilde{y} = ye^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{p} = pe^{-\frac{2}{5}\hat{s}\hat{\kappa}}, g = (f - \frac{2}{5}\hat{s}\frac{\partial\hat{\kappa}}{\partial t}y)e^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{u} = ue^{-\frac{2}{5}\hat{s}\hat{\kappa}}, \tilde{y}_0 = y_0e^{-\frac{2}{5}\hat{s}\hat{\kappa}(0)}$. Note that by (3.19, 3.20)

$$\|g\|_{(L^2(Q))^n} + \|\tilde{u}\|_{(L^2(Q))^n} \leq C_9(\|y_0\|_{V^0(\Omega)} + \|f\|_{F(Q,\theta)}). \tag{3.21}$$

We observe, that the pair $(\tilde{y}, \tilde{p})$ satisfies the equation

$$L\tilde{y} = \nabla\tilde{p} + g + \chi_\omega\tilde{u} \text{ in } Q, \tag{3.22}$$

$$\operatorname{div}\tilde{y} = 0, \quad \tilde{y}|_\Sigma = 0, \; \tilde{y}(0, x) = \tilde{y}_0(x). \tag{3.23}$$

Thus (3.19–3.21) and *a priori* estimate taken from Lemma 2.1 implies (3.7).

Now let $f \in F(Q,\theta)$ be an arbitrary function. Then there exist functions $f_1 \in (L^2(Q,\theta))^n$ and $f_2 \in L^2(0,T;W_2^1(\Omega))$ such that $f = f_1 + \nabla f_2$. Above we proved that there exists a solution (y,p,u) of the exact controllability problem (3.1–3.3) with initial datum $(y_0,(1-\chi_\omega)f_1)$ which satisfy the estimate (3.7). Obviously $(y, p+f_2, u+\chi_\omega f_1)$ is a solution of this problem for the initial datum (y_0, f). □

4. Proof of the main theorem

Now our nearest aim is to reduce the proof of Theorem 1 to the case of a linear controllability problem. We look for a solution of problem (1, 2, 4, 5) in the form

$$v(t,x) = \hat{v}(t,x) + y(t,x). \tag{4.1}$$

Substitution of (4.1) into equations (1, 2, 5) and then subtraction from them of the same equation for $\hat{v}$ yields

$$\mathcal{N}(y,q,u) = \partial_t y(t,x) - \Delta y + B(\hat{v},y) + B(y,\hat{v}) + B(y,y) - \nabla q - \chi_\omega u = 0 \text{ in } \Omega, \tag{4.2}$$

$$\operatorname{div} y = 0, \tag{4.3}$$

$$y(0,x) = v_0(x) - \hat{v}(0,x), \tag{4.4}$$

$$y(T,x) = 0. \tag{4.5}$$

We will solve the problem (4.2–4.5) with help of the following variant of the implicit function Theorem (see [1]).

Theorem 4.1 (on a right inverse operator). *Suppose that X, Z are Banach spaces and*

$$\mathcal{A} : X \to Z$$

is a continuously differentiable map. We assume that for $x_0 \in X$, $z_0 \in Z$ the equality

$$\mathcal{A}(x_0) = z_0 \tag{4.6}$$

holds and the derivative $\mathcal{A}'(x_0) : X \to Z$ of the map $\mathcal{A}$ at x_0 is epimorphism. Then there exists $\epsilon > 0$ such that for any $z \in Z$ which satisfies the condition

$$\|z - z_0\|_Z < \epsilon$$

there exists a solution $x \in X$ of the equation

$$\mathcal{A}(x) = z.$$

In our case the space

$$X = Y(Q) \times L^2(0,T;W_2^1(\Omega)) \times (L^2(Q_\omega))^n \tag{4.7}$$

and

$$Z = F(Q,\theta) \times V^1(\Omega). \tag{4.8}$$

The operator $\mathcal{A}$ is defined by the formula

$$\mathcal{A}(y,q,u) = (\mathcal{N}(y,q,u), y(0,\cdot)). \tag{4.9}$$

We have:

Lemma 4.1. *Let $\hat{v} \in W^1_\infty(0,T;V^1(\Omega) \bigcap (W^1_\infty(\Omega))^n)$, then $\mathcal{A} \in C^1(X,Y)$.*

Proof. It follows directly from the definitions of the spaces X, Z that the operator

$$(y,q,u) \to (\partial_t y(t,x) - \Delta y + B(\hat{v},y) + B(y,\hat{v}) - \nabla q - \chi_\omega u, y(0,\cdot)) : X \to Z$$

is continuous and moreover continuously differentiable in virtue of linearity. The operator B is a bilinear one. So to prove this lemma it is sufficient to establish continuity of the bilinear operator

$$B : Y(Q) \times Y(Q) \to (L^2(Q,\theta))^n,$$

where the function θ was defined in (3.6). Note that from (2.4, 2.22, 3.5)

$$\eta(t,x) \le -\hat{s}\hat{\kappa}(t,x) \quad \forall\,(t,x) \in Q. \tag{4.10}$$

Then (4.10) and simple transformations give the estimate

$$\begin{aligned}\|B(y_1,y_2)\|^2_{(L^2(Q,\theta))^n} &\le C_1 \sum_{i,j=1}^{2} \left(\int_{Q\setminus Q_\omega} \frac{|y_i|^2|\nabla y_j|^2 e^\eta}{(T-t)^8}\,dx\,dt \right.\\ &\left. + \int_{Q_\omega} |y_i|^2|\nabla y_j|^2\,dx\,dt \right) \le C_2 \sum_{i,j=1}^{2} \int_Q \frac{|y_i|^2|\nabla y_j|^2}{(T-t)^8} e^{-\hat{s}\hat{\kappa}}\,dx\,dt\\ &\le C_3 \|e^{-\frac{4}{5}\hat{s}\hat{\kappa}} y_1\|^2_{V^{1,2}(Q)} \|e^{-\frac{4}{5}\hat{s}\hat{\kappa}} y_2\|^2_{V^{1,2}(Q)}.\end{aligned}$$

This inequality proves the theorem. □

Proof of the Theorem 1. Firstly we apply the inverse operator theorem to the problem (4.2–4.5). Let $\mathcal{A}$ be the operator defined by formulas (4.9, 4.2), and let the spaces X, Z be defined as in (4.7, 4.8). Set $x_0 = (0,0,0)$, $z_0 = (0,0)$. Then equation (4.6) obviously holds. By Lemma 4.1 $\mathcal{A} \in C^1(X,Z)$ and by Theorem 3.1 $Im\mathcal{A}'(0) = Z$. So all necessary conditions to apply the theorem about the inverse operator are fulfilled. Therefore there exists such $\varepsilon > 0$ such that for any initial data (4.4) satisfying the inequality

$$\|y_0\|_{V^1(\Omega)} \le \varepsilon$$

the problem (4.2–4.5) has a solution $(y,p,u) \in X$. Then the triple $(y+\hat{v}, p+\hat{p}, u)$ is a solution of the problem (1–3, 5). □

Appendix

Proof of Lemma 2.3. Making the change of the variable $t \to (T-t)$ in (2.17, 2.18) we obtain the system

$$\frac{\partial z}{\partial t} - \Delta z + B^*(z,\mathbf{v}) + B^*(\mathbf{v},z) = \nabla q + \mathbf{f} \text{ in } Q, \tag{1}$$

$$\operatorname{div} z = 0, \quad z|_\Sigma = 0, \quad z(0,\cdot) = z_0, \tag{2}$$

where $\mathbf{v}(t,x) = \hat{v}(T-t,x), \mathbf{f}(t,x) = f(T-t,x)$. Now instead of (2.19) it suffices to prove

$$\left\|\frac{\partial z}{\partial t}\right\|_{L^2(T/2,T;V^0(\Omega))} + \|z\|_{L^2(T/2,T;V^2(\Omega))} \le C(\|z\|_{L^2(T/4,T;V^1(\Omega))} + \|\mathbf{f}\|_{L^2(T/4,T;V^0(\Omega))}). \tag{3}$$

The Sobolev embedding theorem implies the following estimates

$$\|B^*(z,\mathbf{v})+B^*(\mathbf{v},z)\|_{L^2(0,T;V^{-1}(\Omega))} \le C\|z\|_{L^2(0,T;V^0(\Omega))}\|\mathbf{v}\|_{L^\infty(0,T;(W^1_\infty(\Omega))^n)}$$
$$\|B^*(z,\mathbf{v})+B^*(\mathbf{v},z)\|_{L^2(0,T;V^0(\Omega))} \le C\|z\|_{L^2(0,T;V^1(\Omega))}\|\mathbf{v}\|_{L^\infty(0,T;(W^1_\infty(\Omega))^n)}. \tag{4}$$

Thanks to inequalities (4) using the Faedo-Galerkin method and arguing exactly same way as in [31] (p. 255) we can construct a solution to problem (1, 2) such that

$$s\left\|\frac{\partial z}{\partial t}\right\|_{L^2(0,T;V^0(\Omega))} + \|z\|_{L^2(0,T;V^1(\Omega)\cap(W_2^{1+s}(\Omega))^n)} \le C(\|z_0\|_{V^s(\Omega)} + \|\mathbf{f}\|_{L^2(0,T;V^{-1+s}(\Omega))}) \quad s\in\{0,1\}\,. \tag{5}$$

Thus for $z \in V^0(\Omega)$, $\mathbf{f}\in L^2(0,T;V^0(\Omega))$ there exists a solution to problem (1, 2) a function $z\in L^2(0,T;V^1(\Omega))$. Denote $\mathbf{z}=\rho z$, where $\rho\in C^\infty[0,T]$, $\rho(t)=1$ for all $t\in[T/2,T]$ and $\rho(t)=0$ for all $t\in[0,T/4]$. The function $\mathbf{z}$ satisfies to the system of equations

$$\frac{\partial \mathbf{z}}{\partial t} - \Delta\mathbf{z} + B^*(\mathbf{z},\mathbf{v}) + B^*(\mathbf{v},\mathbf{z}) = \nabla(\rho q) + z\frac{\partial\rho}{\partial t} + \rho\mathbf{f} \text{ in } Q, \tag{6}$$

$$\operatorname{div}\mathbf{z}=0, \quad \mathbf{z}|_\Sigma = 0, \quad \mathbf{z}(0,\cdot)=0, \tag{7}$$

Thus applying to problem (6, 7) the apriori estimate (5) with $s=0$ we obtain (2.19). The proof of the lemma is complete. □

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