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RELAXATION OF QUASILINEAR ELLIPTIC SYSTEMS *VIA* A-QUASICONVEX ENVELOPES

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Abstract. We consider the weak closure WZ of the set Z of all feasible pairs (solution, flow) of the family of potential elliptic systems

$$\operatorname{div}\left(\sum_{s=1}^{s_0}\sigma_s(x)F_s'(\nabla u(x)+g(x))-f(x)\right)=0\text{ in }\Omega,$$
$$u=(u_1,\dots,u_m)\in H_0^1(\Omega;\mathbf{R}^m),\ \sigma=(\sigma_1,\dots,\sigma_{s_0})\in S,$$

where $\Omega\subset\mathbf{R}^n$ is a bounded Lipschitz domain, F_s are strictly convex smooth functions with quadratic growth and $S=\{\sigma\,\textit{measurable}\mid\sigma_s(x)=0\text{ or }1,\ s=1,\dots,s_0,\ \sigma_1(x)+\dots+\sigma_{s_0}(x)=1\}$. We show that WZ is the zero level set for an integral functional with the integrand $Q\mathcal{F}$ being the $\mathbf{A}$-quasiconvex envelope for a certain function $\mathcal{F}$ and the operator $\mathbf{A}=(\text{curl},\text{div})^m$. If the functions F_s are isotropic, then on the characteristic cone Λ (defined by the operator $\mathbf{A}$) $Q\mathcal{F}$ coincides with the $\mathbf{A}$-polyconvex envelope of $\mathcal{F}$ and can be computed by means of rank-one laminates.

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1. Introduction

We consider the problem of weak closure for the set of solutions of a family of quasilinear elliptic systems. The origin of this investigation is optimal material layout (or optimal design) problems. Mathematically such problems often can be formulated, see *e.g.* Kohn and Strang [4] or Tartar [9], as

$$\begin{cases} I(u)\to\min,\\ \operatorname{div}\left(\sum_{s=1}^{s_0}\sigma_s(x)F_s'(\nabla u(x)+g(x))-f(x)\right)=0\text{ in }\Omega,\\ u=(u_1,\dots,u_m)\in H_0^1(\Omega;\mathbf{R}^m),\ \sigma\in S,\end{cases}\tag{1.1}$$

where $\Omega\subset\mathbf{R}^n$ is a bounded Lipschitz domain, F_s' are gradients of given functions F_s, I is weakly continuous (with respect to H_0^1-topology) and the control set $\mathcal{S}$ is defined as

$$\mathcal{S}=\Big\{\sigma\in L_\infty(\mathbf{R}^n;\mathbf{R}^{s_0})\mid\sigma=(\sigma_1,\dots,\sigma_{s_0}),\,\sigma_s(x)=0\text{ or }1,\ s=1,\dots,s_0,\ \sigma_1(x)+\dots+\sigma_{s_0}(x)=1\Big\}\cdot$$

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In this setting, σ_s corresponds to the characteristic function of a domain occupied by s th material and F'_s corresponds to the constitutive law for the s-th material. The problem is to lay out these materials throughout a given domain Ω with the aim to minimize the functional I associated with the state of the assembled medium. Such problems, as a rule, have no optimal solutions and the minimizing sequences lead to highly oscillating functions which, in the limit, can be associated with homogenized media. A well known procedure for the relaxation of such problems is the passage to the G-closure of the set of initial operators, which, for the case of linear constitutive laws, leads to the G-closure of a given set of $nm \times nm$-matrices. We recall that the notion of G-closure was introduced by Lurie *et al.* [5] and by G-closure is understood the closure of a given set of admissible matrix-valued functions (or Nemitskii operators) with respect to the topology induced by G-convergence, see *e.g.* Zhikov *et al.* [10]. From the point of view of optimal design, the G-closure is the set of all possible effective tensors obtainable by mixing a given set of materials. The knowledge of the corresponding G-closure is not necessary, in general, for the relaxation of the optimal control problem at hand, see *e.g.* Tartar [9]. This observation leads to the another problem: find a direct description of the weak closure of the set of all feasible states (solutions of Eq. (1.1)), preferably in the form of the level set for some integral functional. The first question here is the existence of such integral functionals with more or less analytically defined integrands. We do not know the existence of such integrands if only the states are involved, but we have a positive answer for the weak closure of the set of all feasible pairs (state, flow).

More precisely, the state equation (1.1) can be rewritten as the equation

$$\sum_{s=1}^{s_0} \sigma_s(x) F'_s\left(v(x) + g(x)\right) - f(x) - \eta(x) = 0 \text{ in } \Omega \tag{1.2}$$

with trespect to $(v, \eta) \in \mathcal{V} \times \mathcal{N}$, where

$$\begin{aligned} \mathcal{V} &= \Big\{ v \in L_2(\Omega; \mathbf{R}^{nm}) \mid v = (v_1, \ldots, v_m),\ v_j = \nabla u_j,\ u_j \in H_0^1(\Omega),\ j = 1, \ldots, m \Big\}, \\ \mathcal{N} &= L_2(\Omega; \mathbf{R}^{nm}) \ominus \mathcal{V}. \end{aligned}$$

Then, as we shall show in Section 4, the week closure of the set of all solutions of (1.2) with $\sigma \in S$ can be represented as

$$\left\{ (v, \eta) \in \mathcal{V} \times \mathcal{N} \;\middle|\; \int_\Omega Q\mathcal{F}\big(v(x) + g(x), \eta(x) + f(x)\big) \mathrm{d}x \le 0 \right\},$$

where the function $Q\mathcal{F}$ does not depend on the choice of g and f and $Q\mathcal{F}$ is the $\mathbf{A}$-quasiconvex envelope (for the operator $\mathbf{A} = (\text{curl, div})^m$) of the function $\mathcal{F}$,

$$\mathcal{F}(\xi', \xi'') = \min_s \Big\{ F_s(\xi') + F_s^*(\xi'') - \langle \xi', \xi'' \rangle \Big\}$$

with F_s^* being the conjugate function to F_s, *i.e.*

$$F_s^*(\xi'') = \sup_{z \in \mathbf{R}^{nm}} \Big\{ \langle z, \xi'' \rangle - F_s(z) \Big\},\ s = 1, \ldots, s_0.$$

For the definition and properties of $\mathbf{A}$-quasiconvex functions see Fonseca and Müller [3]. The necessary corresponding results for the case of $\mathbf{A} = (\text{curl}, \text{div})^m$ are given in Section 2. What concerns assumptions imposed on the functions F_s, then we assume that F_s are smooth convex functions with quadratic growth and that the corresponding gradients F'_s are strongly monotone mappings. These assumptions are formulated in Section 2.

The next, and more serious, problem is to obtain appropriate approximations or estimates for the function $Q\mathcal{F}$. For comparison of complexity of this problem, we point out here that for the linear case, *i.e.* $F'_s(\xi) = A_s\xi$, $s = 1, \ldots, s_0$, where A_s are symmetric $nm \times nm$-matrices, the G-closure of the set $\{A_s\}$ can be described by means of analogous to $Q\mathcal{F}$ functions, see *e.g.* Raitums [8], the difference is only in the dimension of the problem.

In Sections 5, 6 we shall show that for the case of isotropic functions F_s, *i.e.* $F_s(z) = \varphi_s(|z|),\ s = 1, \dots, s_0$, the function $Q\mathcal{F}$ on the characteristic cone Λ (for precise definition see Murat [7] or Fonseca and Müller [3]) coincides with the **A**-polyconvex envelope $\mathcal{PF}$ of $\mathcal{F}$ and can be computed analytically or by means of rank-one laminates.

Finally, in Section 6, we show that analogous results are valid for infinite sets of admissible functions F and that, in addition, the function $Q\mathcal{F}$ belongs to C^1 provided some additional smoothness properties of the functions F.

2. Preliminaries

Let $n \geq 2,\ m \geq 1$ be integers, let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain homeomorphic to the unit ball and let

$$F_s\ :\ \mathbf{R}^{nm} \to \mathbf{R},\ s = 1, \dots, s_0,$$

be given functions.

Denote, for a given function $F\ :\ \mathbf{R}^{nm} \to \mathbf{R}$, by F^* its conjugate function, *i.e.*

$$F^*\ :\ \mathbf{R}^{nm} \to \mathbf{R},\ F^*(z) = \sup_{\xi \in \mathbf{R}^{nm}} [\langle z, \xi \rangle - F(\xi)],\ z \in \mathbf{R}^{nm}.$$

Here and in sequal by $\langle \cdot, \cdot \rangle$ we denote the scalar product in Euclidean spaces. The standard Euclidean norm will be denoted by $|\cdot|$. The elements $z \in \mathbf{R}^{nm}$ we shall often represent as $z = (z^1, \dots, z^m)$ with $z^j \in \mathbf{R}^n$, $j = 1, \dots, m$.

Throughout the paper we always suppose that the functions F_s, F_s^*, $s = 1, \dots, s_0$, satisfy the following hypotheses:

H1. F_s, $s = 1, \dots, s_0$, are convex and continuously differentiable on $\mathbf{R}^{nm}$.

H2. There exist constants $\nu_1 > 0$ and $\nu_2 > 0$ such that for all $s = 1, \dots, s_0$, and all $z \in \mathbf{R}^{nm}$

$$\begin{aligned}
&\nu_1 |z|^2 \leq F_s(z) \leq \nu_2(1 + |z|^2),\\
&\nu_1 |z|^2 \leq F_s^*(z) \leq \nu_2(1 + |z|^2),\\
&F_s(0) = F_s^*(0) = 0.
\end{aligned}$$

H3. There exists a constant ν_3 such that for all $s = 1, \dots, s_0$ and all $z \in \mathbf{R}^{nm}$

$$|F_s'(z)| \leq \nu_3(1 + |z|),\ |F_s^{*\prime}(z)| \leq \nu_3(1 + |z|).$$

H4. There exists a constant $\nu_4 > 0$ such that for all $s = 1, \dots, s_0$ and all $z,\ \xi \in \mathbf{R}^{nm}$

$$\begin{aligned}
&\langle F_s'(z + \xi) - F_s'(z), \xi \rangle \geq \nu_4 |\xi|^2,\\
&\langle F_s^{*\prime}(z + \xi) - F_s^{*\prime}(z), \xi \rangle \geq \nu_4 |\xi|^2.
\end{aligned}$$

Here and in what follows by F_s' and $F_s^{*\prime}$ we denote the gradient of F_s and F_s^* respectively.

In the last section we shall involve the additional hypothesis.

H5. There exist a constant ν_5 and a continuous increasing function $\gamma_0\ :\ \mathbf{R} \to \mathbf{R}$ with $\gamma_0(0) = 0$ such that for all $s = 1, \dots, s_0$ and all $z,\ \xi \in \mathbf{R}^{nm}$

$$|F_s'(z + \xi) - F_s'(z)| + |F_s^{*\prime}(z + \xi) - F_s^{*\prime}(z)| \leq \nu_5(1 + |z|)\gamma_0(|\xi|).$$

Remark 2.1. It is easy to see that H3 and H5 are straight consequences from H1 and H4. The reason why H3 and H5 are involved is to indicate more exactly which properties and where are exploited. All hypotheses H1–H5 can be formulated in terms of the functions F_s as: $F_s : \mathbf{R}^{nm} \to \mathbf{R},\ s = 1, \dots, s_0$, are convex and continuously differentiable and there exists constants $\nu > 0$ and $L > 0$ such that $0 = F_s(0) \le F_s(z)$, $|F_s'(z) - F_s'(\xi)| \le L|z-\xi|$, $\langle F_s'(z+\xi) - F_s'(z), \xi\rangle \ge \nu|\xi|^2$ for all $z, \xi \in \mathbf{R}^{nm}$, $s = 1, \dots, s_0$.

Define

$$S_0 = \Big\{\theta \in \mathbf{R}^{s_0} \ \mid\ \theta = (\theta_1, \dots, \theta_{s_0}),\ \theta_j = 0 \,\text{or}\, 1,\ j = 1, \dots, s_0,\ \theta_1 + \dots + \theta_{s_0} = 1\Big\},$$
$$S = \Big\{\sigma \in L_\infty(\mathbf{R}^n; \mathbf{R}^{s_0}) \ \mid\ \sigma = (\sigma_1, \dots, \sigma_{s_0}),\ \sigma(x) \in S_0 \text{ a.e. } x \in \mathbf{R}^n\Big\}$$

and let the function $\mathcal{F}$,

$$\mathcal{F}\ :\ \overline{co}\, S_0 \times \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R},\ \mathcal{F} = \mathcal{F}(\theta, \xi', \xi''),$$

be defined as

$$\mathcal{F}(\theta, \xi', \xi'') = \sum_{s=1}^{s_0} \theta_s F_s(\xi') + \sum_{s=1}^{s_0} \theta_s F_s^*(\xi'') - \langle \xi', \xi''\rangle. \tag{2.1}$$

By hypothesis H2 and Young's inequality for all $\theta \in \overline{co}\, S_0$ and all ξ', $\xi'' \in \mathbf{R}^{nm}$

$$0 = \mathcal{F}(\theta, 0, 0) \le \mathcal{F}(\theta, \xi', \xi''),\ \nu_1(|\xi'|^2 + |\xi''|^2) - \langle \xi', \xi''\rangle \le \mathcal{F}(\theta, \xi', \xi'') \le (\nu_2 + 1)(|\xi'|^2 + |\xi''|^2 + 2). \tag{2.2}$$

Denote

$$\mathcal{F}_0(\xi', \xi'') = \min_{\theta \in S_0} \mathcal{F}(\theta, \xi', \xi''),\ \xi',\ \xi'' \in \mathbf{R}^{nm}. \tag{2.3}$$

Obviously, $\mathcal{F}_0$ is continuous and satisfies inequalities (2.2).

Let the spaces $\mathcal{V}$ and $\mathcal{N}$ be defined as

$$\mathcal{V} = \Big\{v \in L_2(\Omega; \mathbf{R}^{nm}) \ \mid\ v = (v^1, \dots, v^m),\ v^j = \nabla u_j,\ u_j \in H_0^1(\Omega),\ j = 1, \dots, m\Big\},$$
$$\mathcal{N} = L_2(\Omega; \mathbf{R}^{nm}) \ominus \mathcal{V}.$$

Let the elements g, $f \in L_2(\Omega; \mathbf{R}^{nm})$ be fixed. Denote, for a chosen $\sigma \in S$, by $(v(\sigma), \eta(\sigma))$ a pair $(v(\sigma), \eta(\sigma)) \in \mathcal{V} \times \mathcal{N}$ such that

$$\sum_{s=1}^{s_0} \sigma_s(x) F_s'(v(\sigma)(x) + g(x)) = \eta(\sigma)(x) + f(x) \text{ a.e. } x \in \Omega. \tag{2.4}$$

Obviously, if such a pair exists, then

$$\operatorname{div}\left(\sum_{s=1}^{s_0} \sigma_s(x) F_s'(v(\sigma)(x) + g(x)) - f(x)\right) = 0 \text{ in } \Omega$$

in the sense of distributions and $v(\sigma)$ is the minimizer of the functional

$$v \to \int_\Omega \left\{\sum_{s=1}^{s_0} \sigma_s(x) F_s(v(x) + g(x)) - \langle v(x), f(x)\rangle\right\} \mathrm{d}x$$

over $v \in \mathcal{V}$.

By construction and by virtue of H1–H4, such minimizer $v(\sigma)$ always exists and is unique. Since $\mathcal{N}$ is the orthogonal complement of $\mathcal{V}$, then for every $\sigma \in S$ there exists an unique pair $(v(\sigma), \eta(\sigma)) \in \mathcal{V} \times \mathcal{N}$ that satisfies (2.4). Denote the set of all such pairs with $\sigma \in S$ by $Z(g, f)$, *i.e.*

$$Z(g, f) = \Big\{(v(\sigma), \eta(\sigma)) \in \mathcal{V} \times \mathcal{N} \ \mid\ \sigma \in S\Big\}.$$

We are interested to find a description for the closure $wcl\, Z(g, f)$ of the set $Z(g, f)$ in the weak topology.

Let $K \subset \mathbf{R}^n$ be the unit cube, *i.e.* $K = (0,1)^n$.

Definition 2.1. A function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is said to be K-*periodic* if for all integers $k_1, \ldots, k_n$ and all $x \in \mathbf{R}^n$

$$\varphi(x_1 + k_1, \ldots, x_n + k_n) = \varphi(x).$$

Introduce the spaces

$$\mathcal{V}^{\#} = \Big\{ v \in L_2(K; \mathbf{R}^{nm}) \mid v = (v^1, \ldots, v^m),\ v^j = \nabla u_j,\ u_j \in W^1_{2\,\mathrm{loc}}(\mathbf{R}^n),\ u_j \,\mathrm{is}\, K-\mathrm{periodic},\ j = 1, \ldots, m \Big\},$$

$$\mathcal{N}^{\#} = \Big\{ \eta \in L_2(K; \mathbf{R}^{nm}) \mid \eta = (\eta^1, \ldots, \eta^m),\ \eta^j = \sum_{l=1}^{l_0} T_l \nabla u_{jl},\ u_{jl} \in W^1_{2\,\mathrm{loc}}(\mathbf{R}^n),\ u_{jl} \,\mathrm{is}\, K-\mathrm{periodic}, \quad l = 1, \ldots, l_0 = n(n-1)/2,\ j = 1, \ldots, m \Big\} \cdot \tag{2.6}$$

Here by T_l, $l = 1, \ldots, l_0$, we denote arranged in a given order all skew-symmetric $n \times n$-matrices with only two nonzero entries equal to $+1$ and -1 respectively.

It is well known, see, for instance, Zhikov *et al.* [10], that

$$L_2(K; \mathbf{R}^{nm}) = \mathcal{V}^{\#} \oplus \mathcal{N}^{\#} \oplus \mathbf{R}^{nm}$$

and that there exists a constant c_0 such that the elements u_{jl} in (2.6) can be chosen so that

$$\|\, |\nabla u_{jl}|\, \|_{L_2(K)} \le c_0 \|\eta\|,\ l = 1, \ldots, l_0;\ j = 1, \ldots, m.$$

By construction, $\mathcal{V}^{\#} \times \mathcal{N}^{\#}$ is the closure in $L_2(K; \mathbf{R}^{nm}) \times L_2(K; \mathbf{R}^{nm})$ of the kernel of the operator $\mathbf{A} = (\mathrm{curl}, \mathrm{div})^m$ in the space of K-periodic functions from $C^\infty(\mathbf{R}^n; \mathbf{R}^{nm}) \times C^\infty(\mathbf{R}^n; \mathbf{R}^{nm})$ with zero mean value.

Denote, for $(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$,

$$Q\mathcal{F}_0(\xi', \xi'') = \inf_{v \in \mathcal{V}^{\#}} \inf_{\eta \in \mathcal{N}^{\#}} \int_K \mathcal{F}_0(\xi' + v(x), \xi'' + \eta(x))\mathrm{d}x. \tag{2.7}$$

By virtue of H1–H4, the function $Q\mathcal{F}_0$ is the $\mathbf{A}$-quasiconvex envelope of $\mathcal{F}_0$ for the operator $\mathbf{A} = (\mathrm{curl}, \mathrm{div})^m$. We emphasize that $\mathbf{A} = (\mathrm{curl}, \mathrm{div})^m$ has a constant rank, see Murat [7], what is essential for Proposition 2.2 below.

Let us recall, for convenience of the reader, the results on $\mathbf{A}$-quasiconvexity from Fonseca and Müller [3], reformulated for the case $\mathbf{A} = (\mathrm{curl}, \mathrm{div})^m$.

Definition 2.2. A continuous function $F : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ is said to be $\mathbf{A}$-*quasiconvex* if

$$F(\xi', \xi'') \le \int_K F(\xi' + v(x), \xi'' + \eta(x))\mathrm{d}x$$

for all $(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ and all $(v, \eta) \in C^\infty(\mathbf{R}^n; \mathbf{R}^{nm} \times \mathbf{R}^{nm})$ such that

$$\mathbf{A}(v, \eta) = 0,\ (v, \eta) \,\mathrm{is}\, K-\mathrm{periodic},\ \int_K (v(x), \eta(x))\mathrm{d}x = 0.$$

Definition 2.3. Given a continuous function $F : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ we define the **A**-*quasiconvex envelope* of F at $(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ as

$$QF(\xi',\xi'') = \inf\Bigg\{\int_K F(\xi' + v(x), \xi'' + \eta(x))\mathrm{d}x \;\mid\; (v,\eta) \in C^\infty(\mathbf{R}^n; \mathbf{R}^{nm} \times \mathbf{R}^{nm}),\ \mathbf{A}(v,\eta) = 0,$$
$$(v,\eta)\,\text{is}\,K-\text{periodic},\ \int_K (v(x),\eta(x))dx = 0\Bigg\}\cdot$$

Proposition 2.1 (Fonseca and Müller [3]). *If $F : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ is upper semicontinuous then QF is* **A**-*quasiconvex and upper semicontinuous. Moreover, QF is Λ-convex, i.e. for all $0 < t < 1$*

$$QF(t\xi' + (1-t)z', t\xi'' + (1-t)z'') \le tQF(\xi',\xi'') + (1-t)QF(z',z'') \quad \text{whenever} \quad (\xi' - z', \xi'' - z'') \in \Lambda$$

where

$$\Lambda = \bigcup_{e\in\mathbf{R}^n,\,|e|=1} \Big\{(\xi',\xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm} \mid \xi' = (\alpha_1 e, \dots, \alpha_m e),$$
$$\alpha_j \in \mathbf{R},\ j = 1,\dots,m,\ \xi'' = (\xi''^1, \dots, \xi''^m),\ \langle \xi''^j, e\rangle = 0,\ j = 1,\dots,m\Big\}\cdot \tag{2.8}$$

Proposition 2.2 (Fonseca and Müller [3]). *Let $\Omega' \subset \mathbf{R}^n$ be a bounded open domain, let $1 \le q < \infty$ and suppose that $F : \Omega' \times \mathbf{R}^l \times \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ is measurable in $x \in \Omega'$ and continuous in $(z, \xi', \xi'') \in \mathbf{R}^l \times \mathbf{R}^{nm} \times \mathbf{R}^{nm}$, and that for a.e. $x \in \Omega'$ and all $z \in \mathbf{R}^l$ the mapping $(\xi', \xi'') \to F(x, z, \xi', \xi'')$ is* **A**-*quasiconvex. Assume further that there exists a locally bounded nonnegative function $\varphi : \Omega' \times \mathbf{R}^l \to \mathbf{R}$ such that*

$$0 \le F(x,z,\xi',\xi'') \le \varphi(x,z)(1 + |\xi'|^q + |\xi''|^q).$$

If

$$w_k \to w_0 \textit{ in measure in } \Omega'$$

and

$$(v_k, \eta_k) \rightharpoonup (v_0, \eta_0) \textit{ weakly in } L_q(\Omega'; \mathbf{R}^{nm} \times \mathbf{R}^{nm}),\ \mathbf{A}(v_k,\eta_k) = 0 \ \textit{ in } \Omega'$$

then

$$\int_{\Omega'} F(x, w_0(x), v_0(x), \eta_0(x))\mathrm{d}x \le \lim_{k\to\infty}\ \inf \int_{\Omega'} F(x, w_k(x), v_k(x), \eta_k(x))\mathrm{d}x.$$

We shall need a few additional notions.

Definition 2.4. A continuous function $h : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ is said to be **A**-*quasiaffine* if h and $-h$ are **A**-quasiconvex.

Definition 2.5. A continuous function $F : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ is said to be **A**-*polyconvex* if

$$F(\xi',\xi'') = \varphi(h_1(\xi',\xi''), \dots, h_{r_0}(\xi',\xi''))$$

where the functions h_r, $r = 1, \dots, r_0$, are **A**-quasiaffine and the function $\varphi : \mathbf{R}^{r_0} \to \mathbf{R}$ is convex.

Definition 2.6. Given a bounded below continuous function $F_0 : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$ we define the $\mathbf{A}$-*polyconvex envelope* of F_0 at $(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ as

$$\mathcal{P}F_0(\xi', \xi'') = \sup\Big\{F(\xi', \xi'') \ \big|\ F \text{ is } \mathbf{A}\text{-polyconvex},\ F(z', z'') \le F_0(z', z'') \text{ for all } (z', z'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}\Big\}.$$

Definition 2.7. A function $F : \mathbf{R}^{nm} \to \mathbf{R}$ is said to be *isotropic* if there exists a function $\varphi : \mathbf{R} \to \mathbf{R}$ such that for all $z \in \mathbf{R}^{nm}$ $\qquad F(z) = \varphi(|z|)$.

Now we are able to formulate the main results of this paper.

Theorem 2.1. *Let the hypotheses* H1–H4 *hold. Then the functional*

$$(v, \eta) \to \int_\Omega Q\mathcal{F}_0(v(x) + g(x), \eta(x) + f(x))\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $\mathcal{V} \times \mathcal{N}$ for every fixed pair $(g, f) \in L_2(\Omega; \mathbf{R}^{nm}) \times L_2(\Omega; \mathbf{R}^{nm})$ and

$$wcl\, Z(g, f) = \left\{(v, \eta) \in \mathcal{V} \times \mathcal{N} \ \Big|\ \int_\Omega Q\mathcal{F}_0(v(x) + g(x), \eta(x) + f(x))\mathrm{d}x = 0\right\}.$$

Theorem 2.2. *Let the hypotheses* H1–H4 *hold and let the functions F_s, $s = 1, \ldots, s_0$, be isotropic. Then for every $(\xi', \xi'') \in \Lambda$*

$$\begin{aligned} Q\mathcal{F}_0(\xi', \xi'') &= \mathcal{P}\mathcal{F}_0(\xi', \xi'') \\ &= \inf_{\sigma \in S,\, \sigma = \sigma(x_1)} \ \inf_{v \in \mathcal{V}^\#,\, v = v(x_1)} \ \inf_{\eta \in \mathcal{N}^\#,\, \eta = \eta(x_1)} \int_K \left\{\sum_{s=1}^{s_0} \sigma_s(x_1)[F_s(v(x_1) + R\xi') + F_s^*(\eta(x_1) + R\xi'')]\right\}\mathrm{d}x \end{aligned}$$

where $R\xi' = (R\xi'^1, \ldots, R\xi'^m)$, $R\xi'' = (R\xi''^1, \ldots, R\xi''^m)$ and $R \in SO(n)$ is such that the vectors $R\xi'^1, \ldots, R\xi'^m$ are parallel to $e_1 = (1, 0, \ldots, 0)$ and the vectors $R\xi''^1, \ldots, R\xi''^m$ are orthogonal to e_1.

3. Auxiliary results

Throughout the paper the constants whose precise values are not important we shall denote by c, if necessary, we shall write, for instance, $c(n, \Omega)$ to indicate that this particular constant depends only on n and Ω. For a measurable set $E \subset \mathbf{R}^n$ by $|E|$ we shall denote the Lebesgue measure of E, the characteristic function of E we shall denote by χ_E.

Lemma 1. *For every fixed $f \in L_2(\Omega; \mathbf{R}^{nm})$ and $\sigma \in S$*

$$\begin{aligned} \int_\Omega \sum_{s=1}^{s_0} \sigma_s(x) F_s^*(f(x))\mathrm{d}x &= \int_\Omega \left\{\sum_{s=1}^{s_0} \sigma_s(x) \sup_{z_s \in \mathbf{R}^{nm}} [\langle z_s, f(x)\rangle - F_s(z_s)]\right\}\mathrm{d}x \\ &= \sup_{\alpha \in L_2(\Omega; \mathbf{R}^{nm})} \int_\Omega \left\{\sum_{s=1}^{s_0} \sigma_s(x)[\langle \alpha(x), f(x)\rangle - F_s(\alpha(x))]\right\}\mathrm{d}x. \end{aligned} \tag{3.1}$$

Proof. Let us denote by $\beta_s(x)$ the maximizer of the expression

$$\langle \beta, f(x)\rangle - F_s(\beta)$$

over $\beta \in \mathbf{R}^{nm}$. By virtue of H1–H4, the element $\beta_s(x)$ is uniquely defined and

$$|\beta_s(x)| \leq c_1(|f(x)|+1). \tag{3.2}$$

Let $\varepsilon > 0$ be given. There exists a closed set $D \subset \Omega$ such that $|\Omega \setminus D| < \varepsilon$ and f is continuous on D. The values $|f(x)|$ on D are bounded too. We want to show that $\beta_s(\cdot)$ is continuous on D. Let a sequence $\{x_k\} \subset D$ converges to some $x_0 \in D$. Without loss of generality we can assume that the sequence $\{\beta_s(x_k)\}$ converges to an element $\beta_0 \in \mathbf{R}^{nm}$. Elements $\beta_s(x_k)$ satisfy the Euler equation

$$F_s'(\beta_s(x_k)) = f(x_k),\ k = 1, 2, \ldots$$

We can pass to the limit $k \to \infty$ in these relationships (hypothesis H1) what gives

$$F_s'(\beta_0) = f(x_0).$$

But $F_s'(\beta_s(x_0)) = f(x_0)$ and from H4 it follows immediately that $\beta_0 = \beta_s(x_0)$. Therefore, β_s is continuous on D. From this, from arbitrariness of $\varepsilon > 0$ and from the estimate (3.2) it follows that β_s is measurable on Ω and that $\beta_s \in L_2(\Omega; \mathbf{R}^{nm})$. Clearly, the function α_0, defined as

$$\alpha_0(x) = \sum_{s=1}^{s_0} \sigma_s(x)\beta_s(x),\ x \in \Omega,$$

belongs to $L_2(\Omega; \mathbf{R}^{nm})$, and, by construction of α_0,

$$\int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x)[\langle \alpha_0(x), f(x)\rangle - F_s(\alpha_0(x))] \right\} \mathrm{d}x = \int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s^*(f(x)) \right\} \mathrm{d}x.$$

On the other hand, the functions σ_s are nonnegative and for every $\alpha \in L_2(\Omega; \mathbf{R}^{nm})$

$$\int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x)[\langle \alpha(x), f(x)\rangle - F_s(\alpha(x))] \right\} \mathrm{d}x \leq \int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x) \sup_{z_s \in \mathbf{R}^{nm}} [\langle z_s, f(x)\rangle - F_s(z_s)] \right\} \mathrm{d}x$$

what concludes the proof. □

Lemma 3.2. *Let $g, f \in L_2(\Omega; \mathbf{R}^{nm})$ be fixed. Then*

$$\begin{aligned} \inf_{\sigma \in S} \sup_{\alpha \in L_2(\Omega; \mathbf{R}^{nm})} \int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s(g(x)) - \sum_{s=1}^{s_0} \sigma_s(x) F_s(\alpha(x)) + \langle \alpha(x), f(x)\rangle - \langle g(x), f(x)\rangle \right\} \mathrm{d}x \\ = \int_\Omega \left\{ \inf_{\theta \in S_0} \left[\sum_{s=1}^{s_0} \theta_s F_s(g(x)) + \sum_{s=1}^{s_0} \theta_s F_s^*(f(x)) - \langle g(x), f(x)\rangle \right] \right\} \mathrm{d}x. \end{aligned} \tag{3.3}$$

Proof. By virtue of Lemma 3.1, we can bring the supremum over α inside the integral. Thus, the expression in the left hand side of (3.3) is equal to

$$\inf_{\sigma \in S} \int_\Omega \left\{ \sum_{s=1}^{s_0} \sigma_s(x)[F_s(g(x)) + F_s^*(f(x))] - \langle g(x), f(x)\rangle \right\} \mathrm{d}x. \tag{3.4}$$

Let $\Omega' \subset \Omega$ be the set of all Lebesgue points for all functions $g(\cdot)$, $f(\cdot)$, $F_s(g(\cdot))$, $F_s^*(f(\cdot))$, $s = 1, \dots, s_0$. Clearly $|\Omega'| = |\Omega|$. Define sets $E_0, E_1, \dots, E_{s_0}$ as

$$
\begin{aligned}
&E_0 = \emptyset,\\
&E_s = \Big\{x \in \Omega' \setminus (E_0 \bigcup E_1 \bigcup \cdots \bigcup E_{s-1}) \mid F_s(g(x)) + F_s^*(f(x))\\
&\le \min_{l=s+1,\dots,s_0} [F_l(g(x)) + F_l^*(f(x))]\Big\},\ s = 1, \dots, s_0 - 1,\\
&E_{s_0} = \Omega' \setminus \bigcup_{s=1}^{s_0-1} E_s.
\end{aligned}
$$

By construction, the sets E_s are measurable, $E_{s_1} \bigcap E_{s_2} = \emptyset$ if $s_1 \neq s_2$, $\bigcup\limits_{s=1}^{s_0} E_s = \Omega'$ and the inner infimum over $\theta \in S_0$ in the right hand side of (3.3) for a.e. $x \in \Omega$ is attained at $\theta = \theta(x)$,

$$
\begin{aligned}
&\theta(x) = (\chi_{E_1}(x), \dots, \chi_{E_{s_0}}(x)),\ x \in \Omega',\\
&\theta(x) = (1, 0, \dots, 0),\ x \in \Omega \setminus \Omega'.
\end{aligned}
$$

The function σ^0, defined as $\sigma^0(x) = \theta(x)$, $x \in \Omega$, belongs to S, hence, the right hand side in (3.3) is greater than or equal to the left hand side in (3.3). The inverse inequality is obvious. □

We recall that the expression in square brackets in the right hand side of (3.3) is equal to $\mathcal{F}(\theta, g(x), f(x))$, but the integrand in the right hand side of (3.3) is equal to $\mathcal{F}_0(g(x), f(x))$. In turn, the integrand in (3.4) is equal to $\mathcal{F}(\sigma(\cdot), g(\cdot), f(\cdot))$. Thus, we have the following:

Corollary 3.1. *For every fixed $g, f \in L_2(\Omega; \mathbf{R}^{nm})$*

$$
\int\limits_\Omega \mathcal{F}_0(g(x), f(x))\mathrm{d}x = \inf_{\sigma \in S} \int\limits_\Omega \mathcal{F}(\sigma(x), g(x), f(x))\mathrm{d}x.
$$

Corollary 3.2. *For every $(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$*

$$
Q\mathcal{F}_0(\xi', \xi'') = \inf_{\sigma \in S} \inf_{v \in \mathcal{V}^\#} \inf_{\eta \in \mathcal{N}^\#} \int_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x)[F_s(v(x) + \xi') + F_s^*(\eta(x) + \xi'') - \langle \xi', \xi'' \rangle \right\} \mathrm{d}x.
$$

Let us introduce, for a given $\sigma \in S$, the functional

$$
J(\sigma, \cdot, \cdot) : L_2(\Omega; \mathbf{R}^{nm}) \times L_2(\Omega; \mathbf{R}^{nm}) \to \mathbf{R}, \quad J(\sigma, g, f) = \int_\Omega \mathcal{F}(\sigma(x), g(x), f(x))\mathrm{d}x.
$$

Lemma 3.3. *The functional $J(\sigma, \cdot, \cdot)$ is continuous and Gateaux differentiable.*

Proof. The proof follows immediately from hypotheses H1–H4. □

Lemma 3.4. *For every fixed $\sigma \in S$, $g, f \in L_2(\Omega; \mathbf{R}^{nm})$ the functional*

$$
\begin{aligned}
&J(\sigma, g + \cdot, f + \cdot) : \mathcal{V} \times \mathcal{N} \to \mathbf{R},\\
&J(\sigma, g + v, f + \eta) = \int\limits_\Omega \mathcal{F}(\sigma(x), g(x) + v(x), f(x) + \eta(x))\mathrm{d}x
\end{aligned}
$$

is continuous, Gateaux differentiable, strictly convex and

$$J(\sigma, g+v, f+\eta) \geq 0 \text{ for all } (v,\eta) \in \mathcal{V}\times\mathcal{N},$$
$$J(\sigma, g+v, f+\eta) \geq c(\nu_1, g, f)(\|v\|^2 + \|\eta\|^2 - 1) \text{ for all } (v,\eta) \in \mathcal{V}\times\mathcal{N}$$

for some positive constant $c(\nu_1, g, f)$.

Proof. The statements of Lemma are straight consequences from hypotheses H1–H4, inequalities (2.2) and the fact that $\mathcal{N}$ is the orthogonal complement of $\mathcal{V}$. □

Lemma 3.5. *For every fixed* $\sigma \in S$, $g, f \in L_2(\Omega; \mathbf{R}^{nm})$ *the functional* $J(\sigma, g+\cdot, f+\cdot)$ *attains its minimum over* $(v,\eta) \in \mathcal{V}\times\mathcal{N}$ *on a unique pair* $(v(\sigma), \eta(\sigma))$ *defined by the relationship (2.4) and* $J(\sigma, g+v(\sigma), f+\eta(\sigma)) = 0$.

Proof. The existence of an unique minimizer for the functional $J(\sigma, g+\cdot, f+\cdot)$ on $\mathcal{V}\times\mathcal{N}$ is a straight consequence from Lemma 3.4 and the reflexivity of Lebesgue spaces L_p for $1 < p < \infty$. Let the pair $(v(\sigma), \eta(\sigma)) \in \mathcal{V}\times\mathcal{N}$ satisfies (2.4). For every $\xi'' \in \mathbf{R}^{nm}$ and a.e. $x \in \Omega$ the equation

$$\sum_{s=1}^{s_0} \sigma_s(x) F_s'(z) = \xi''$$

with respect to $z \in \mathbf{R}^{nm}$ has an unique solution. From this, the definition of conjugate functions, the properties of $\sigma \in S$ (σ represents a s_0-tuple of characteristic functions of pairwise disjoint sets) and (2.4) it follows

$$\sum_{s=1}^{s_0} \sigma_s(x) F_s^*(f(x) + \eta(\sigma)(x)) = \langle g(x) + v(\sigma)(x), f(x) + \eta(\sigma)(x)\rangle \text{ a.e. } x \in \Omega.$$

This and the analytical expressions (2.1) for $\mathcal{F}(\theta, \xi', \xi'')$ give that

$$\mathcal{F}(\sigma(x), g(x) + v(\sigma)(x), f(x) + \eta(\sigma)(x)) = 0 \text{ a.e. } x \in \Omega.$$

Since $J(\sigma, g+v, f+\eta) \geq 0$ for all $(v,\eta) \in \mathcal{V}\times\mathcal{N}$, then $(v(\sigma), \eta(\sigma))$ is the minimizer of $J(\sigma, g+\cdot, f+\cdot)$ on $\mathcal{V}\times\mathcal{N}$. □

Lemma 3.6. *Let the sequences*

$$\{\varepsilon_k\} \subset \mathbf{R},\ \{\sigma^k\} \subset S,\ \{(v_k, \eta_k)\} \subset \mathcal{V}\times\mathcal{N},\ \{(a_k, b_k)\} \in L_2(\Omega; \mathbf{R}^{nm}) \times L_2(\Omega; \mathbf{R}^{nm})$$

be such that

$$\varepsilon_k > 0,\ \varepsilon_k \to 0 \text{ as } k \to \infty, \quad \|(a_k, b_k)\| \to 0 \text{ as } k \to \infty, \quad J(\sigma_k, g + a_k + v_k, f + b_k + \eta_k) < \varepsilon_k,\ k = 1, 2, \dots$$

Then

(i) $\|(v_k - v(\sigma^k), \eta_k - \eta(\sigma^k)\| \to 0$ *as* $k \to \infty$;

(ii) there exists a subsequence, still denoted by $\{(v_k, \eta_k)\}$, *such that*

$$(v_k, \eta_k) \rightharpoonup (v_0, \eta_0) \text{ weakly in } \mathcal{V}\times\mathcal{N} \text{ as } k \to \infty$$
and
$$(v_0, \eta_0) \in wcl\, Z(g, f).$$

Proof. Recall that by definition and Lemma 3.5

$$0 = J(\sigma^k, g + v(\sigma^k), f + \eta(\sigma^k)) \leq J(\sigma^k, g + v, f + \eta) \text{ for all } (v, \eta) \in \mathcal{V}\times\mathcal{N}.$$

Since the pair $(v(\sigma^k), \eta(\sigma^k))$ is the minimizer, then the Gateaux derivative $J'(\sigma^k, g + v(\sigma^k), f + \eta(\sigma^k))$ as an element of $L_2(\Omega; \mathbf{R}^{nm}) \times L_2(\Omega; \mathbf{R}^{nm})$ is orthogonal to $\mathcal{V} \times \mathcal{N}$. Hence,

$$\begin{aligned}&J(\sigma^k, g + a_k + v_k, f + b_k + \eta_k) - J(\sigma^k, g + v(\sigma^k), f + \eta(\sigma^k))\\ &-\langle J'(\sigma^k, g + v(\sigma^k), f + \eta(\sigma^k)), (v_k - v(\sigma^k), \eta_k - \eta(\sigma^k))\rangle < \varepsilon_k,\ k = 1, 2, \dots,\end{aligned}$$

or, what is the same,

$$\begin{aligned}\int_\Omega \Bigg\{ \sum_{s=1}^{s_0} \sigma_s^k(x)\big[F_s(g(x) &+ a_k(x) + v_k(x)) - F_s(g(x) + v(\sigma^k)(x))\\ &- \langle F_s'(g(x) + v(\sigma^k)(x)), v_k(x) - v(\sigma^k)(x)\rangle\\ &+ F_s^*(f(x) + b_k(x) + \eta_k(x)) - F_s^*(f(x) + \eta(\sigma^k)(x))\\ &- \langle F_s^{*\prime}(f(x) + \eta(\sigma^k)(x)), \eta_k(x) - \eta(\sigma^k)(x)\rangle\big]\\ &- \langle g(x), \eta_k(x) - \eta(\sigma^k)(x)\rangle - \langle f(x), v_k(x) - v(\sigma^k)(x)\rangle \Bigg\} \mathrm{d}x\\ &< \varepsilon_k,\\ &\quad k = 1, 2, \dots\end{aligned} \tag{3.5}$$

Since F_s' and $F_s^{*\prime}$ are continuous, then H4 implies

$$F_s(z+\xi) - F_s(z) - \langle F_s'(z), \xi\rangle \geq \nu_4|\xi|^2/2, \quad F_s^*(z+\xi) - F_s^*(z) - \langle F_s^{*\prime}(z), \xi\rangle \geq \nu_4|\xi|^2/2 \quad \text{for all } z, \xi \in \mathbf{R}^{nm}. \tag{3.6}$$

By virtue of Lemma 3.4, the set $Z(g, f)$ is bounded, hence, from (3.5) and (3.6) it follows

$$\int_\Omega \left\{|v_k(x) + a_k(x) - v(\sigma^k)(x)|^2 + |\eta_k(x) + b_k(x) - \eta(\sigma^k)(x)|^2\right\} \mathrm{d}x \leq 8\varepsilon_k/\nu_4 + c(\nu_3, \nu_4, \|g\|, \|f\|)(\|a_k\|^2 + \|b_k\|^2)$$

what together with the assumptions of lemma give the statements of lemma. □

Corollary 3.3. *For every fixed $g, f \in L_2(\Omega; \mathbf{R}^{nm})$ the set $Z(g, f)$ is bounded,*

$$Z(g, f) = \left\{(v, \eta) \in \mathcal{V} \times \mathcal{N} \mid J(\sigma, g + v, f + \eta) = 0,\ \sigma \in S\right\}$$

and the closure $cl\, Z(g, f)$ of the set $Z(g, f)$ in the strong topology is equal to

$$cl\, Z(g, f) = \left\{(v, \eta) \in \mathcal{V} \times \mathcal{N} \mid J(\sigma, g + v, f + \eta) = 0,\ \sigma \in \overline{co}\, S\right\} \cdot$$

Proof. The boundedness of $Z(g, f)$ is a straight consequence from Lemma 3.4, and the representation for $Z(g, f)$ follows from Lemma 3.5.

Let $\{\sigma^k\} \subset S$ and let the sequence $\{v(\sigma^k), \eta(\sigma^k)\}$ converges strongly to an element (v_0, η_0). Without loss of generality we can assume that the sequence $\{\sigma^k\}$ converges weak -$*$ in $L_\infty(\Omega; \mathbf{R}^{s_0})$ to an element $\sigma_0 \in \overline{co}\, S$, and also that the sequence $\{v(\sigma^k), \eta(\sigma^k)\}$ converges almost uniformly in Ω. This and the hypotheses H1-H4 ensure that we can pass to the limit

$$0 = J(\sigma^k, g + v(\sigma^k), f + \eta(\sigma^k)) \to J(\sigma_0, g + v_0, f + \eta_0) \text{ as } k \to \infty.$$

On the other hand, let us suppose that $J(\sigma_0, g + v_o, f + \eta_0) = 0$ for some triple $(\sigma_0, v_0, \eta_0) \in \overline{co}\, S \times \mathcal{V} \times \mathcal{N}$. Since the integrand in J depends on σ in an affine way, then there exists a sequence $\{\sigma^k\} \subset S$ such that

$$\begin{aligned}&\sigma^k \rightharpoonup \sigma_0 \text{ weak} - * \text{as}\, k \to \infty,\\ &J(\sigma^k, g + v_0, f + \eta_0) \leq 1/k,\ k = 1, 2, \dots\end{aligned}$$

From this and Lemma 3.6 it follows that the sequence $\{v(\sigma^k), \eta(\sigma^k)\}$ converges strongly to (v_0, η_0) as $k \to \infty$. $\square$

4. Proof of Theorem 2.1

Throughout this section the pair $(g, f) \in L_2(\Omega; \mathbf{R}^{nm}) \times L_2(\Omega; \mathbf{R}^{nm})$ is fixed.

The function $\mathcal{F}_0$,

$$\mathcal{F}_0(\xi', \xi'') = \inf_{\theta \in S} \left\{ \sum_{s=1}^{s_0} \theta_s [F_s(\xi') + F_s^*(\xi'') - \langle \xi', \xi'' \rangle \right\}$$

is continuous and satisfies inequalities (2.2). From this, the definition of spaces $\mathcal{V}$ and $\mathcal{N}$, Corollary 3.2 and Young's inequality we get that for all $\xi', \xi'' \in \mathbf{R}^{nm}$

$$\begin{aligned} Q\mathcal{F}_0(\xi', \xi'') &= \inf_{\sigma \in S} \inf_{v \in \mathcal{V}^\#} \inf_{\eta \in \mathcal{N}^\#} \int_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x) [F_s(v(x) + \xi') + F_s^*(\eta(x) + \xi'')] - \langle v(x) + \xi', \eta(x) + \xi'' \rangle \right\} \mathrm{d}x \\ &\geq \inf_{v \in \mathcal{V}^\#} \inf_{\eta \in \mathcal{N}^\#} \int_K \left\{ \nu_1 [|v(x) + \xi'|^2 + |\eta(x) + \xi''|^2] - \langle \xi', \xi'' \rangle \right\} \mathrm{d}x \\ &\geq \nu_1 (|\xi'|^2 + |\xi''|^2) - \langle \xi', \xi'' \rangle. \end{aligned} \tag{4.1}$$

$$0 \leq Q\mathcal{F}_0(\xi', \xi'') \leq \mathcal{F}_0(\xi', \xi'') \leq (1 + \nu_2)(|\xi'|^2 + |\xi''|^2 + 2). \tag{4.2}$$

By Proposition 2.1, the function $Q\mathcal{F}_0$ is convex with respect to the characteristic cone Λ defined by (2.8). The cone Λ contains all basis vectors in $\mathbf{R}^{nm} \times \mathbf{R}^{nm}$, hence, $Q\mathcal{F}_0$ is separately convex what gives that $Q\mathcal{F}_0$ is locally Lipschitz, see Dacorogna [2] or Ball *et al.* [1].

The continuity of $Q\mathcal{F}_0$ and estimates (4.1) and (4.2) are sufficient for that the mapping

$$(x, z, \xi', \xi'') \to \tilde{\mathcal{F}}(x, z, \xi', \xi'') = Q\mathcal{F}_0(z' + \xi', z'' + \xi''), \quad z = (z', z'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$$

withy $w(x) = (g(x), f(x))$ satisfies all assumptions of Proposition 2.2. This gives that the functional

$$(v, \eta) \to \int_\Omega Q\mathcal{F}_0(g(x) + v(x), f(x) + \eta(x)) \mathrm{d}x \tag{4.3}$$

is sequentially weakly lower semicontinuous on $\mathcal{V} \times \mathcal{N}$.

As an immediate consequence we have that the set

$$QZ(g, f) = \left\{ (v, \eta) \in \mathcal{V} \times \mathcal{N} \mid \int_\Omega Q\mathcal{F}_0(g(x) + v(x), f(x) + \eta(x)) \mathrm{d}x = 0 \right\}$$

is bounded (estimates (4.1)) and weakly closed.

Since

$$0 \leq Q\mathcal{F}_0(\xi', \xi'') \leq \mathcal{F}(\theta, \xi', \xi''),$$

then, by virtue of Corollary 3.3,

$$Z(g, f) \subset QZ(g, f),$$

and, as a straight consequence

$$wcl\, Z(g, f) \subset QZ(g, f).$$

It remains to show that

$$QZ(g, f) \subset wcl\, Z(g, f).$$

Let $(v_0, \eta_0) \in QZ(g, f)$. We want to show that there exists a sequence $\{(\sigma^k, v(\sigma^k), \eta(\sigma^k))\} \subset S \times \mathcal{V} \times \mathcal{N}$ such that

$$(v(\sigma^k), \eta(\sigma^k)) \rightharpoonup (v_0, \eta_0) \text{ weakly as } k \to \infty.$$

It is well known, see, for instance, Zhikov *et al.* [10], that the space $\mathcal{N}$ has the representation (we recall that Ω is homeomorphic to the unit ball)

$$\mathcal{N} = \left\{ \eta \in L_2(\Omega; \mathbf{R}^{nm}) \mid \eta = (\eta^1, \ldots, \eta^m),\ \eta^j = \sum_{l=1}^{l_0} T_l \nabla u_{jl},\ u_{jl} \in W_2^1(\Omega),\ l = 1, \ldots, l_0;\ j = 1, \ldots, m \right\} \quad (4.4)$$

and there exists a constant $c(n, \Omega)$ such that the functions u_{jl} in (4.4) can be chosen so that

$$\|u_{jl}\|_{W_2^1(\Omega)} \le c(n, \Omega)\|\eta\|,\ l = 1, \ldots, l_0;\ j = 1, \ldots, m. \quad (4.5)$$

Here matrices T_l are the same as in the definition of the space $\mathcal{N}^{\#}$ by (2.6). The representation (4.4) and estimate (4.5) ensure that $\mathcal{N}$ contains a dense subset of piecewise constant elements. Clearly, the same property has the space $\mathcal{V}$.

Let $\varepsilon > 0$ be given. The estimates (4.1, 4.2) and continuity of $Q\mathcal{F}_0$ ensure that there exist piecewise constant elements $(v_\varepsilon, \eta_\varepsilon) \in \mathcal{V} \times \mathcal{N}$ and $(g_\varepsilon, f_\varepsilon) \in L_2(\Omega; \mathbf{R}^{nm})$ such that

$$\|(v_\varepsilon, \eta_\varepsilon) - (v_0, \eta_0)\| < \varepsilon/8, \quad \|(g_\varepsilon, f_\varepsilon) - (g, f)\| < \varepsilon/8, \quad \int_\Omega Q\mathcal{F}_0(v_\varepsilon(x) + g_\varepsilon(x), \eta_\varepsilon(x) + f_\varepsilon(x))\mathrm{d}x < \varepsilon/8. \quad (4.6)$$

In addition, the elements $v_\varepsilon, \eta_\varepsilon, g_\varepsilon, f_\varepsilon$ can be chosen so that there exists a partition

$$\Omega = \Omega_0 \cup E_1 \cup \cdots \cup E_{r_0},\ |\Omega_0| < \varepsilon,$$

such that $\{E_r\}$ are pairwise disjoint cubes,

$$E_r = x^r + \tau_r K,\ r = 1, \ldots, r_0,$$

that in every E_r the functions $v_\varepsilon, \eta_\varepsilon, g_\varepsilon, f_\varepsilon$ are constant, say

$$\begin{gathered} v_\varepsilon(x) + g_\varepsilon(x) = a_r \in \mathbf{R}^{nm},\ \text{if } x \in E_r, \\ \eta_\varepsilon(x) + f_\varepsilon(x) = b_r \in \mathbf{R}^{nm},\ \text{if } x \in E_r, \\ r = 1, \ldots, r_0, \end{gathered}$$

and that for all $\sigma \in S$

$$\int_{\Omega_0} \big|\mathcal{F}(\sigma(x), v_\varepsilon(x) + g_\varepsilon(x), \eta_\varepsilon(x) + f_\varepsilon(x))\big|\mathrm{d}x + \int_{\Omega_0} \big|Q\mathcal{F}_0(v_\varepsilon(x) + g_\varepsilon(x), \eta_\varepsilon(x) + f_\varepsilon(x))\big|\mathrm{d}x < \varepsilon/8. \quad (4.7)$$

The estimates (4.6) and (4.7) give

$$0 \le \sum_{r=1}^{r_0} |E_r| Q\mathcal{F}_0(a_r, b_r) < \varepsilon/4. \quad (4.8)$$

Denote by $\mathcal{V}^{\#}(E_r)$ and $\mathcal{N}^{\#}(E_r)$ the spaces defined by (2.6) with E_r instead of K, $r = 1, \ldots, r_0$. Then, after an obvious transform of co-ordinates, from Corollary 3.2 it follows

$$|E_r| Q\mathcal{F}_0(a_r, b_r) = \inf_{v \in \mathcal{V}^{\#}(E_r)} \inf_{\eta \in \mathcal{N}^{\#}(E_r)} \inf_{\sigma \in S} \int_{E_r} \mathcal{F}(\sigma(x), v(x) + a_r, \eta(x) + a_r)\mathrm{d}x.$$

By continuity of $\mathcal{F}$ and by estimates (2.2), there exist piecewise constant elements

$$(\sigma^r, v_r, \eta_r) \in S \times \mathcal{V}^{\#}(E_r) \times \mathcal{N}^{\#}(E_r)$$

such that

$$|E_r| Q\mathcal{F}_0(a_r, b_r) \geq \int_{E_r} \mathcal{F}(\sigma^r(x), v_r(x) + a_r, \eta_r(x) + b_r)\mathrm{d}x - \delta/2, \quad \delta = \varepsilon/(2r_0 + 1).$$

Denote by w_j^r and w_{jl}^r, $l = 1, \dots, l_0$; $j = 1, \dots, m$, the functions from the representation (2.6) for v_r and η_r respectively (more precisely, analogues of (2.6) with E_r instead of K), and let us extend these functions *via* E_r-periodicity to the whole $\mathbf{R}^n$. Then, for any integer $k = 1, 2, \dots$

$$\begin{aligned} \int_{E_r} \mathcal{F}\Bigg(\sigma^r(kx), \Big(\frac{1}{k}\nabla w_1^r(kx) + a_r^1, \dots, \frac{1}{k}\nabla w_m^r(kx) + a_r^m\Big), \\ \Big(\frac{1}{k}\sum_{l=1}^{l_0} T_l \nabla w_{1l}^r(kx) + b_r^1, \dots, \frac{1}{k}\sum_{l=1}^{l_0} T_l \nabla w_{ml}^r(kx) + b_r^m\Big)\Bigg)\mathrm{d}x \\ \leq |E_r| Q\mathcal{F}_0(a_r, b_r) + \delta/2. \end{aligned} \tag{4.9}$$

Now, by means of appropriate cut-off functions, which are equal to zero near the boundary of E_r, we obtain the existence of elements

$$(v_{rk}, \eta_{rk}) \in \mathcal{V}^{\#}(E_r) \times \mathcal{N}^{\#}(E_r),\ k = 1, 2, \dots,$$

which are equal to zero near the boundary of E_r, such that for k large enough, $k \geq c(\delta, v_r, \eta_r, a_r, b_r)$,

$$\int_{E_r} \mathcal{F}(\sigma^r(kx), v_{rk}(x) + a_r, \eta_{rk}(x) + b_r)\mathrm{d}x \leq |E_r| Q\mathcal{F}_0(a_r, b_r) + \delta \tag{4.10}$$

and

$$(v_{rk}, \eta_{rk}) \rightharpoonup 0 \text{ weakly as } k \to \infty.$$

This procedure and the estimates (4.7)-(4.10) give that for

$$\sigma^k = \sigma^k(x) = \begin{cases} \sum_{r=1}^{r_0} \chi_{E_r}(x)\sigma^r(kx),\ x \in \Omega \setminus \Omega_0. \\ (1, 0, \dots, 0)\ x \in \Omega_0, \end{cases}$$

$$(v_k, \eta_k) = (v_k(x), \eta_k(x)) = \sum_{r=1}^{r_0} \chi_{E_r}(x)\big(v_{rk}(x), \eta_{rk}(x)\big),\ x \in \Omega,$$

and for k large enough

$$\begin{aligned} &\int_\Omega \mathcal{F}\big(\sigma^k(x), v_\varepsilon(x) + g_\varepsilon(x) + v_k(x), \eta_\varepsilon(x) + f_\varepsilon(x) + \eta_k(x)\big)\mathrm{d}x \\ &\quad < \int_{\Omega_0} Q\mathcal{F}_0\big(v_\varepsilon(x) + g_\varepsilon(x), \eta_\varepsilon(x) + f_\varepsilon(x)\big)\mathrm{d}x + 1/2\varepsilon + \delta r_0 \\ &\quad < 2\varepsilon. \end{aligned}$$

In addition, $\{(v_k, \eta_k)\} \subset \mathcal{V} \times \mathcal{N}$ and $(v_k, \eta_k) \rightharpoonup 0$ weakly as $k \to \infty$.

After an appropriate diagonal process with $\varepsilon \to 0$ and $k \to \infty$ we have a new sequence

$$\{(\sigma^k, v_k, \eta_k)\} \subset S \times \mathcal{V} \times \mathcal{N}$$

such that

$$\begin{aligned}&(v_k, \eta_k) \rightharpoonup (v_0, \eta_0) \text{ weakly as } k \to \infty,\\ &J(\sigma^k, v_k + g + (g_k - g), \eta_k + f + (f_k - f)) \to 0 \text{ as } k \to \infty,\\ &(g_k, f_k) = (g_\varepsilon + v_\varepsilon - g - v_0, f_\varepsilon + \eta_\varepsilon - f - \eta_0) \to 0 \text{ as } k \to \infty.\end{aligned}$$

These convergences and Lemma 3.6 give that $(v_0, \eta_0) \in wcl, Z(g, f)$, what completes the proof. □

5. Evaluation of $Q\mathcal{F}_0$ on the characteristic cone Λ

In the first part of this section we shall give the proof of Theorem 2.2.

Recall that the characteristic cone Λ is

$$\Lambda = \bigcup_{e\in\mathbf{R}^n,\,|e|=1} \Big\{(\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm} \mid \xi' = (\alpha_1 e, \dots, \alpha_m e), \\ \alpha_j \in \mathbf{R},\, j = 1, \dots, m;\ \xi'' = (\xi''^1, \dots, \xi''^m),\ \langle \xi''^j, e\rangle = 0,\, j = 1, \dots, m\Big\}.$$

From Proposition 2.1 it follows immediately that every **A**-quasiaffine function h is affine with respect to Λ, *i.e.*

$$\begin{gathered}h\big(\lambda(\xi', \xi'') + (1-\lambda)(z', z'')\big) = \lambda h(\xi', \xi'') + (1-\lambda) h(z', z'')\\ \text{whenever } (\xi', \xi'') - (z', z'') \in \Lambda \text{ and } 0 \le \lambda \le 1.\end{gathered} \tag{5.1}$$

The linear hull of Λ is equal to $\mathbf{R}^{nm} \times \mathbf{R}^{nm}$, therefore, from Murat [7] it follows that the function h is a polynom of degree less than or equal to n. After that, a simple observation gives that h has the representation

$$\begin{aligned}h(\xi', \xi'') &= c_0 + \langle a, \xi'\rangle + \langle b, \xi''\rangle + \gamma\langle \xi', \xi''\rangle + \sum_\alpha c_\alpha X_\alpha(\xi') \text{ if } n \ge 3,\\ h(\xi', \xi'') &= c_0 + \langle a, \xi'\rangle + \langle b, \xi''\rangle + \gamma\langle \xi', \xi''\rangle + \sum_\alpha c_\alpha X_\alpha(\xi')\\ &\quad + \sum_\alpha d_\alpha Y_\alpha(\xi'') \text{ if } n = 2,\end{aligned} \tag{5.2}$$

where $a, b \in \mathbf{R}^{nm}$; $c_0, \gamma, c_\alpha, d_\alpha$ are arbitrary constants, X_α are minors of the $n \times m$-matrix X constructed from $\xi' = (\xi'^1, \dots, \xi'^m)$ with ξ'^j as columns, Y_α are minors of the $n \times m$-matrix Y constructed from $\xi'' = (\xi''^1, \dots, \xi''^m)$ with ξ''^j as columns.

Indeed, let us show, for instance, that, if $n \ge 3$, then h is affine with respect to ξ''. Denote by h_{ij} $n \times n$-matrices, $i, j = 1, \dots, m$, which corresponds to the second derivative of h with respect to ξ''^i and ξ''^j from the representation $\xi'' = (\xi''^1, \dots, \xi''^m)$. Clearly, h_{ii} are symmetric matrices and $h_{ji} = h_{ij}^t$ where h_{ij}^t is the transpose to h_{ij} matrix. From (5.1) it follows immediately that

$$\sum_{i,j=1}^m \langle h_{ij} z^i, z^j\rangle = 0 \text{ whenever } (0, (z^1, \dots, z^m)) \in \Lambda. \tag{5.3}$$

If one chooses $z^2 = \dots = z^n = 0$ then there are not restrictions on $z^1 \in \mathbf{R}^n$ and from (5.3) it follows that $h_{11} = 0$ (h_{ii} are symmetric). Exactly in the same way we get $h_{ii} = 0$, $i = 1, \dots, m$. In the second step we choose $z^3 = \dots = z^m = 0$. (5.3) gives

$$\langle h_{12} z^1, z^2\rangle = 0 \text{ for all } z^1, z^2 \in \mathbf{R}^n,$$

because for $n \geq 3$ every element of the form

$$(0, (z^1, z^2, 0, \ldots, 0))$$

belongs to Λ. Thus, $h_{ij} = 0,\ i, j = 1, \ldots, m$, and, as a consequence, h is affine with respect to ξ'' if $n \geq 3$.

Other properties of h can be shown in an analogous way, we only point out that for a fixed ξ'' the function $\xi' \to h(\xi', \xi'')$ must have the same properties as quasiaffine functions for the standard variational case, *i.e.*, with the operator $(\mathrm{curl})^{\mathrm{m}}$.

Now, as we know the type of all **A**-quasiaffine functions, exactly in the same way as in Dacorogna [2] one can show that for a given continuous function $\Phi : \mathbf{R}^{nm} \times \mathbf{R}^{nm}$ its **A**-polyconvex envelope is

$$\begin{aligned} \mathcal{P}\Phi(\xi'.\xi'') = \sup \Big\{ \inf_{\zeta', \zeta'' \in \mathbf{R}^{nm}} \Big[&\langle z', \xi' - \zeta' \rangle + \langle z'', \xi'' - \zeta' \rangle + \gamma \langle \xi', \xi'' \rangle - \gamma \langle \zeta', \zeta'' \rangle + \sum_\alpha c_\alpha \big(X_\alpha(\xi') - X_\alpha(\zeta') \big) \\ &+ \sum_\alpha d_\alpha \big(Y_\alpha(\xi'') - Y_\alpha(\zeta'') \big) + \Phi(\zeta', \zeta'') \Big] \Big| \, z', z'' \in \mathbf{R}^{nm}, \\ &\gamma,\ c_\alpha,\ d_\alpha \in \mathbf{R},\ d_\alpha = 0 \text{ if } n > 2 \Big\}. \end{aligned} \tag{5.4}$$

We want to estimate $\mathcal{PF}_0$ from below on Λ. Clearly, one will get such an estimate by choosing in the representation (5.4) for $\mathcal{PF}_0$ some parameters in a special way, say $\gamma = -1,\ c_\alpha = d_\alpha = 0$. Such a choice gives, for $(\xi', \xi'') \in \Lambda$,

$$\begin{aligned} \mathcal{F}_0(\xi', \xi'') \geq \mathcal{PF}_0(\xi', \xi'') \geq \sup_{z', z'' \in \mathbf{R}^{nm}} \inf_{\zeta', \zeta'' \in \mathbf{R}^{nm}} \Big\{ &\langle z', \xi' \rangle + \langle z'', \xi'' \rangle \\ &+ \min_s [F_s(\zeta') + F_s^*(\zeta'')] - \langle z', \zeta' \rangle - \langle z'', \zeta'' \rangle \Big\}. \end{aligned} \tag{5.5}$$

By bringing the minimum over s outsaid the braces and by using the definition of conjugate functions (recall that F_s are convex and $\langle \xi', \xi'' \rangle = 0$), we get

$$\begin{aligned} \mathcal{PF}_0(\xi', \xi'') &\geq \sup_{z', z'' \in \mathbf{R}^{nm}} \min_s \Big\{ -F_s^*(z') - F_s(z'') + \langle z', \xi' \rangle + \langle z'', \xi'' \rangle \Big\} \\ &= \inf_{\theta \in \overline{co} S_0} \sup_{z', z'' \in \mathbf{R}^{nm}} \Big\{ -\sum_{s=1}^{s_0} \theta_s \Big[F_s^*(z') + F_s(z'') \Big] + \langle z', \xi' \rangle + \langle z'', \xi'' \rangle \Big\} \\ &= \inf_{\sigma \in S\, \sigma = \sigma(x_1)} \sup_{z', z'' \in \mathbf{R}^{nm}} \int_0^1 \Big\{ -\sum_{s=1}^{s_0} \sigma_s(x_1) \Big[F_s^*(z') + F_s(z'') \Big] + \langle z', \xi' \rangle + \langle z'', \xi'' \rangle \Big\} \mathrm{d}x_1 \\ &= \inf_{\sigma \in S\, \sigma = \sigma(x_1)} \inf_{\varphi \in L_2^0} \inf_{\psi \in L_2^0} \int_0^1 \Big\{ \sum_{s=1}^{s_0} \sigma_s(x_1) \Big[F_s(\varphi(x_1) + \xi') + F_s^*(\psi(x_1) + \xi'') \Big] \Big\} \mathrm{d}x_1 \end{aligned} \tag{5.6}$$

where by L_2^0 is denoted the subspace of $L_2((0,1); \mathbf{R}^{nm})$ of functions with the zero mean value.

The functions F_s and F_s^* are isotropic and $(\xi', \xi'') \in \Lambda$, therefore, by introducing a rotation $R \in SO(n)$ such that the vectors $R\xi'^j,\ j = 1, \ldots, m$, are parallel to $e_1 = (1, 0, \ldots, 0)$ and the vectors $R\xi''^j, j = 1, \ldots, m$, are orthogonal to e_1, we can rewrite (5.6) as

$$\mathcal{PF}_0(\xi', \xi'') \geq \inf_{\sigma \in S\, \sigma = \sigma(x_1)} \inf_{\varphi \in L_2^0} \inf_{\psi \in L_2^0} \int_K \Big\{ \sum_{s=1}^{s_0} \sigma_s(x_1) \big[F_s(\varphi(x_1) + R\xi') + F_s^*(\psi(x_1) + R\xi'') \big] \Big\} \mathrm{d}x. \tag{5.7}$$

Since F_s and F_s^* are isotropic, then from Euler equations it follows that the inner infimum over (φ, ψ) in (5.7) is attained on elements $(\varphi(\sigma), \psi(\sigma))$ such that for a.e. $x_1 \in (0,1)$ the vectors $\varphi(\sigma)(x_1)$ and $\psi(\sigma)(x_1)$ are parallel to $R\xi'$ and $R\xi''$ respectively. In turn, the special structure of elements of Λ (clearly $(R\xi', R\xi'') \in \Lambda$) gives that the elements $\varphi(\sigma)$ and $\psi(\sigma)$ coincide with some $v \in \mathcal{V}^{\#},\ v = v(x_1)$, and $\eta \in \mathcal{N}^{\#},\ \eta = \eta(x_1)$, respectively. Thus, we have, for $(\xi', \xi'') \in \Lambda$,

$$\begin{aligned}\mathcal{PF}_0(\xi', \xi'') &\geq \inf_{\sigma \in S\ \sigma=\sigma(x_1)} \inf_{v \in \mathcal{V}^{\#}\ v=v(x_1)} \inf_{\eta \in \mathcal{N}^{\#}\ \eta=\eta(x_1)} \int\limits_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x_1) \big[F_s(v(x_1) + R\xi') + F_s^*(\eta(x_1) + R\xi'') \big] \right\} \mathrm{d}x \\ &\geq Q\mathcal{F}_0(\xi', \xi''). \end{aligned} \tag{5.8}$$

From definitions (2.2–2.6), Jensen's inequality and properties of $\mathcal{F}_0$ it follows that $\mathcal{PF}_0$ is nonnegative, **A**-quasiconvex and that $\mathcal{PF}_0(\xi', \xi'') \leq \mathcal{F}_0(\xi', \xi'')$ for all $\xi', \xi'' \in \mathbf{R}^{nm}$. These properties ensure that

$$\mathcal{PF}_0(\xi', \xi'') \leq Q\mathcal{F}_0(\xi', \xi'') \text{ for all } (\xi', \xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}.$$

From this and estimate (5.8) follows the statement of Theorem 2.2.

In the remaining part of this section we shall give an estimate on Λ for homogenized functions.

Definition 5.1. A function $F_\sigma : \mathbf{R}^{nm} \to \mathbf{R}$ is said to be *homogenized function* (corresponding to a chosen $\sigma \in S$) if for every $\xi' \in \mathbf{R}^{nm}$

$$F_\sigma(\xi') = \inf_{v \in \mathcal{V}^{\#}} \int_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s(v(x) + \xi') \right\} \mathrm{d}x. \tag{5.9}$$

The corresponding conjugate function F_σ^* has the representation

$$\begin{aligned} F_\sigma^*(\xi'') &= \sup_{z \in \mathbf{R}^{nm}} \left[\langle z, \xi'' \rangle - \inf_{v \in \mathcal{V}^{\#}} \int\limits_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s(v(x) + z) \right\} \mathrm{d}x \right] \\ &= \sup_{z \in \mathbf{R}^{nm}} \inf_{\eta \in \mathcal{N}^{\#} \oplus \mathbf{R}^{nm}} \left[\langle z, \xi'' \rangle + \int\limits_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s^*(\eta(x)) - \langle \eta(x), z \rangle \right\} \mathrm{d}x \right] \\ &= \inf_{\eta \in \mathcal{N}^{\#}} \int\limits_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x) F_s^*(\eta(x) + \xi'') \right\} \mathrm{d}x. \end{aligned} \tag{5.10}$$

Clearly, F_σ and F_σ^* are continuous convex functions and they satisfy hypothesis H2. Indeed, by analogous constructions as in (5.10) we get that $(F_\sigma^*)^* = F_\sigma$, hence, they both are convex. The estimates from below in H2 for F_σ and F_σ^* follow from the estimates for F_s as F_s^*, from (5.9) and (5.10) and from Jensen's inequality. The estimates from above for F_σ and F_σ^* follow immediately from (5.9) and (5.10) with $v = \eta = 0$. Finally, the continuity of F_σ and F_σ^* follows from H2 and convexity of F_σ and F_σ^*.

Definition 5.2. A continuous function $F^1 : \mathbf{R}^{nm} \to \mathbf{R}$ is said to be *rank-one laminate* if there exist a $\sigma \in S$, $\sigma = \sigma(x_1)$), and $R \in SO(n)$ such that for all $\xi' \in \mathbf{R}^{nm}$

$$F^1(\xi') = \inf_{v \in \mathcal{V}^{\#}} \int_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x_1) F_s(v(x) + R\xi') \right\} \mathrm{d}x.$$

The same argument as above gives

$$F^{1^*}(\xi'') = \inf_{\eta\in\mathcal{N}^\#} \int_K \left\{ \sum_{s=1}^{s_0} \sigma_s(x_1) F_s^*(\eta(x) + R\xi'') \right\} \mathrm{d}x$$

and that F^1 and F^{1^*} are continuous and convex, and satisfy H2.

Introduce the space

$$\mathcal{C}_2 = \left\{ F : \mathbf{R}^{nm} \to \mathbf{R} \,\middle|\, F \text{ is continuous and } \|F\|_2 = \sup_{z\in\mathbf{R}^{nm}} \frac{|F(z)|}{1+|z|^2} < \infty \right\} \cdot$$

By construction, $\mathcal{C}_2$ is a Banach space with the norm $\|\cdot\|_2$.

Theorem 5.1. *Let the hypotheses* H1–H4 *hold and let the functions F_s and F_s^*, $s = 1,\ldots,s_0$, be isotropic. Then for every fixed homogenized function F_σ there exists a function F^1, which belongs to the closure in $\mathcal{C}_2$ of the set of all rank-one laminates, such that for every $(\xi',\xi'')\in\Lambda$*

$$F_\sigma(\xi') + F_\sigma^*(\xi') \geq \min_{R\in SO(n)} \left[F^1(R\xi') + F^{1^*}(R\xi'') \right].$$

Proof. In the first step we shall prove the statement of this theorem for a piecewise constant $\sigma \in S$.

Let $(\xi',\xi'')\in\Lambda$ and let

$$K = E_0 \cup E_1 \cup \cdots \cup E_{r_0},\ |E_0| = 0,\ r_0 = N^n,$$

be a partition of K by pairwise disjoint cubes E_r,

$$E_r = \left\{ x\in K \ \mid\ x_{r_i} < x_i < x_{r_i} + 1/N,\ i = 1,\ldots,n \right\},\ r = 1,\ldots,r_0,$$

and let $\sigma\in S$ be constant , say σ^r, in every E_r. From (5.9, 5.10) and Jensen's inequality we get

$$F_\sigma(\xi') + F_\sigma^*(\xi'') \geq \sum_{r=1}^{r_0} |E_r| \sum_{s=1}^{s_0} \sigma_s^r \left[F_s(v_r + \xi') + F_s^*(\eta_r + \xi'') \right] = J_1,$$

where, for $f\in L_2(K;\mathbf{R}^{nm})$,

$$f_r = \frac{1}{|E_r|} \int_{E_r} f(x)\mathrm{d}x,$$

and v_r, η_r are the corresponding mean values in E_r of the minimizers in the right hand side of (5.9) and (5.10) respectively.

By construction,

$$\sum_{r=1}^{r_0} |E_r| v_r = 0,\ \sum_{r=1}^{r_0} |E_r| \eta_r = 0. \tag{5.11}$$

Let $R_0\in SO(n)$ be such that all components of $R_0\xi'$ are parallel to $e_1 = (1,0,\ldots,0)$ and all components of $R_0\xi''$ are orthogonal to e_1. Let a_r and b_r be projections of v_r and η_r respectively on the subspace generated by (ξ',ξ''), *i.e.*

$$\begin{aligned} a_r &= (\langle v_r^1, e\rangle e, \ldots \langle v_r^m, e\rangle e),\\ b_r &= (b_r^1 - \langle b_r^1, e\rangle e, \ldots, b_r^m - \langle b_r^m, e\rangle e),\\ r &= 1,\ldots,r_0,\quad e = R_0^{-1} e_1. \end{aligned}$$

Since F_s, F_s^* are isotropic, then

$$F_s(v_r+\xi') \geq F_s(a_r+\xi'), \quad F_s^*(\eta_r+\xi'') \geq F_s^*(b_r+\xi'')$$

and

$$\begin{aligned} J_1 &\geq \sum_{r=1}^{r_0} |E_r| \sum_{s=1}^{s_0} \sigma_s^r \big[F_s(R_0(a_r+\xi')) + F_s^*(R_0(b_r+\xi''))\big] \\ &= \int_0^1 \left\{ \sum_{s=1}^{s_0} \sigma_s^0(x_1)\big[F_s(\varphi(x_1)+R_0\xi') + F_s^*(\psi(x_1)+R_0\xi'')\big] \right\} \mathrm{d}x_1, \end{aligned}$$

where σ^0, φ, ψ are piecewise constant (in intervals $I_r = (r-1)N^n < x_1 < rN^{-r},\ r=1,\dots,r_0$) functions,

$$\begin{aligned} &\sigma^0(x_1) = \sigma^r \quad \text{if } x_1 \in I_r, \\ &\varphi(x_1) = R_0 a_r \ \text{ if } x_1 \in I_r, \\ &\psi(x_1) = R_0 b_r \ \text{ if } x_1 \in I_r, \\ &r = 1,\dots,r_0. \end{aligned}$$

By construction and by virtue of (5.11), $\sigma^0 \in S$, the functions φ and ψ have zero mean value and can be treated as elements of $\mathcal{V}^\#$ and $\mathcal{N}^\#$ respectively. That gives

$$\begin{aligned} F_\sigma(\xi') + F_\sigma^*(\xi'') \geq \inf_{v\in\mathcal{V}^\#} \inf_{\eta\in\mathcal{N}^\#} \int_K \Bigg\{ \sum_{s=1}^{s_0} \sigma_s^0(x_1)\big[F_s(v(x)+R_0\xi') \\ + F_s^*(\eta(x)+R_0\xi'')\big] \Bigg\} \mathrm{d}x = F^1(R_0\xi') + F^{1^*}(R_0\xi''), \end{aligned} \tag{5.12}$$

where the rank-one laminate F^1 is defined by $\sigma^0 \in S$, $\sigma^0 = \sigma^0(x_1)$, and $R = 0$.

In the second step we shall consider arbitrary $\sigma \in S$.

Let $\sigma^0 \in S$ be chosen. For every fixed ξ', ξ'' the values $F_\sigma(\xi')$ and $F_\sigma^*(\xi'')$ are continuous with respect to the convergence

$$\sigma \to \sigma^0 \text{ in measure.}$$

Therefore, there exists a sequence $\{\sigma^k\} \subset S$ of piecewise functions (analogous to σ in the first step), which converges to σ^0 in measure, such that for every ξ', ξ''

$$\begin{aligned} &F_{\sigma^k}(\xi') \to F_{\sigma^0}(\xi') \ \text{ as } k\to\infty, \\ &F_{\sigma^k}^*(\xi'') \to F_{\sigma^0}^*(\xi'') \ \text{ as } k\to\infty. \end{aligned}$$

Fix a pair $(\xi',\xi'') \in \Lambda$ and let R_0 be the corresponding matrix from $SO(n)$ from the first step.

According to the first step, for each $k = 1,2,\dots,$, there exists a rank-one laminate F_k^1 such, that F_k^1 does not depend on the choice of $(\xi',\xi'') \in \Lambda$ and that

$$F_{\sigma^k}(\xi') + F_{\sigma^k}(\xi'') \geq F_k^1(R_0\xi') + F_k^{1^*}(R_0\xi''). \tag{5.13}$$

In the left hand side of (5.13) we can pass to the limit as $k \to \infty$. It remains to show that it is possible to do it in the right hand side of (5.13) too.

We have shown above that all functions F_σ and F_σ^* with $\sigma \in S$ are convex and satisfy H2. Clearly, the functions F_k^1, $F_k^{1^*}$ have the same property. Estimates from hypothesis H2 and convexity of F_k^1 and $F_k^{1^*}$ ensure that these functions are equi-locally Lipschitz, *i.e.* for every $N > 0$ there exists a constant $L > 0$ such that for all $k = 1,2,\dots$,

$$\begin{aligned} &|F_k^1(\xi') - F_k^1(z')| \leq L|\xi'-z'|, \quad |F_k^{1^*}(\xi'') - F_k^{1^*}(z'')| \leq L|z''-\xi''| \\ &\qquad \text{whenever } |\xi'|+|\xi''|+|z'|+|z''| \leq N. \end{aligned}$$

These properties together are sufficient for that the sequences $\{F_k^1\}$ and $\{F_k^{1*}\}$ are precompact in the space $\mathcal{C}_2$. Without loss of generality we can assume that there exists a functions $F^1 : \mathbf{R}^{nm} \to \mathbf{R}$, $F^* : \mathbf{R}^{nm} \to \mathbf{R}$ such that

$$\|F_k^1 - F^1\|_2 \to 0 \text{ as } k \to \infty,$$
$$\|F_k^{1*} - F^*\|_2 \to 0 \text{ as } k \to \infty.$$

From these convergences and inequalities from H2 it follows immediately that

$$F^* = F^{1*}$$

and that (after the passage to the limit as $k \to \infty$ in (5.13))

$$F_{\sigma^0}(\xi') + F^*_{\sigma^0}(\xi'') \geq F^1(R_0\xi') + F^{1*}(R_0\xi''). \tag{5.14}$$

□

6. Infinite number of functions

In this section we shall extend results of previous sections to the case of infinite number of functions F.

Let M be a set of functions $F : \mathbf{R}^{nm} \to \mathbf{R}$, which satisfies the following hypotheses:

H6. Every $F \in M$, together with its adjoint function F^*, satisfies hypotheses H1–H5.

H7. For every $\varepsilon > 0$ there exists a finite subset $M_\varepsilon = \{F_1, \dots, F_{s_\varepsilon}\} \subset M$ such that for every $F \in M$ there exists a function $F_s \in M_\varepsilon$ such that

$$\begin{aligned} &\|F - F_s\|_2 + \|F^* - F_s^*\|_2 < \varepsilon, \\ &\|F' - F_s'\|_1 + \|F^{*\prime} - F_s^{*\prime}\|_1 < \varepsilon, \end{aligned} \tag{6.1}$$

where the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are defined as

$$\|F\|_2 = \sup_{z \in \mathbf{R}^{nm}} \{|F(z)|/(1 + |z|^2)\},$$
$$\|F'\|_1 = \sup_{z \in \mathbf{R}^{nm}} \{|F'(z)|/(1 + |z|)\}.$$

Denote by $\mathcal{M}$ the set of all measurable on $\Omega \times \mathbf{R}^{nm}$ functions $\Phi = \Phi(x, \xi')$ such that

(i) Φ together with its derivative $\Phi'_{\xi'}$ is measurable in $x \in \Omega$ and continuous in $\xi' \in \mathbf{R}^{nm}$;

(ii) for a.e. $x \in \Omega \quad \Phi(x, \cdot) \in M$;

(iii) the mapping $x \to \Phi(x, \cdot) \in M$ is measurable, *i.e.* for every $\delta > 0$ there exists a closed set $D \subset \Omega$ with $|\Omega \setminus D| < \delta$ such that for every $\delta' > 0$ there exists $\tau > 0$ with the property

$$\|\Phi(x^1, \cdot) - \Phi(x^2, \cdot)\|_2 + \|\Phi^*(x^1, \cdot) - \Phi^*(x^2, \cdot)\|_2 < \delta',$$
$$\|\Phi'_{\xi'}(x^1, \cdot) - \Phi'_{\xi'}(x^2, \cdot)\|_1 + \|\Phi^{*\prime}_{\xi'}(x^1, \cdot) - \Phi^{*\prime}_{\xi'}(x^2, \cdot)\|_1 < \delta'$$

whenever $|x^1 - x^2| < \tau$ and x^1, $x^2 \in D$.

Here and in sequal, by definition,

$$\Phi^*(x, \xi'') = \sup_{z \in \mathbf{R}^{nm}} \Big\{ \langle \xi'', z \rangle - \Phi(x, z) \Big\}.$$

We are interested, for given $g, f \in L_2(\Omega; \mathbf{R}^{nm})$, in the weak closure of the set

$$Z(g, f) = \Big\{ (v, \eta) \in \mathcal{V} \times \mathcal{N} \;\big|\; \Phi'_{\xi'}(x, v(x) + g(x)) = \eta(x) + f(x) \text{ a.e. } x \in \Omega,\ \Phi \in \mathcal{M} \Big\}.$$

Denote, for given $\varepsilon > 0$,

$$\mathcal{M}_\varepsilon = \Big\{\Phi \in \mathcal{M} \mid \Phi(x,\cdot) \in M_\varepsilon \text{ a.e. } x \in \Omega\Big\},$$
$$Z_\varepsilon(g,f) = \Big\{(v,\eta) \in \mathcal{V} \times \mathcal{N} \mid \Phi'_{\xi'}(x, v(x)+g(x)) = \eta(x) + f(x) \text{ a.e. } x \in \Omega,\ \Phi \in \mathcal{M}_\varepsilon\Big\}.$$

From properties of $\Phi \in \mathcal{M}$ it follows immediately that every $\Phi \in \mathcal{M}_\varepsilon$ has the representation

$$\Phi(x,\xi') = \sum_{s=1}^{s_\varepsilon} \sigma_s(x) F_s(\xi')$$

with some $\sigma \in S_\varepsilon$ and $F_s \in M_\varepsilon$, where

$$S_\varepsilon = \Big\{\sigma \in L_\infty(\mathbf{R}^n; \mathbf{R}^{s_\varepsilon}) \mid \sigma = (\sigma_1, \dots, \sigma_{s_\varepsilon}),\ \sigma_j(x) = 0 \text{ or } 1,\ j = 1, \dots, s_\varepsilon,$$
$$\sigma_1(x) + \dots + \sigma_{s_\varepsilon}(x) = 1 \text{ a.e. } x \in \mathbf{R}^n\Big\}.$$

Clearly, to the sets $\mathcal{M}_\varepsilon$ and $Z_\varepsilon(g,f)$ can be applied all results from the previous sections.

In the first step we want to show that the set $Z(g,f)$ can be approximated in the strong topology by means of sets $Z_\varepsilon(g,f)$.

Let $\varepsilon_0 > 0$ be given and let $\Phi \in \mathcal{M}$. By virtue of H6, H7 and the measurability of the mapping $x \to \Phi(x,\cdot)$, for every $0 < \varepsilon < \varepsilon_0$ there exist a closed subset $D \subset \Omega$ and a function $\Phi_\varepsilon \in \mathcal{M}_{\varepsilon_0}$ such that

$$\begin{aligned} &|\Omega \setminus D| < \varepsilon, \\ &\|\Phi'_{\xi'}(x,\cdot) - \Phi'_{\varepsilon\xi'}(x,\cdot)\|_1 < \varepsilon_0. \end{aligned} \tag{6.2}$$

Let

$$\begin{aligned} \Phi'_{\xi'}(x, v_0(x)+g(x)) &= \eta_0(x) + f(x) \text{ a.e. } x \in \Omega, \\ \Phi'_{\varepsilon\xi'}(x, v_\varepsilon(x)+g(x)) &= \eta_\varepsilon(x) + f(x) \text{ a.e. } x \in \Omega. \end{aligned} \tag{6.3}$$

From (6.3) and monotonicity properties of $\Phi'_{\xi'}$ we have

$$\begin{aligned} \nu_4\|v_0 - v_\varepsilon\|^2 \le{}& \Big|\int_\Omega \langle \Phi'_{\xi'}(x, v_0(x)+g(x)) - \Phi'_{\varepsilon\xi'}(x, v_0(x)+g(x)), v_0(x) - v_\varepsilon(x)\rangle \mathrm{d}x\Big| \\ &+ 2\varepsilon_0 \int_D \big(1 + |v_0(x)| + |g(x)|\big)|v_0(x) - v_\varepsilon(x)|\mathrm{d}x \\ &+ 2\nu_3 \int_{\Omega\setminus D} \big(1 + |v_0(x)| + |g(x)|\big)|v_0(x) - v_\varepsilon(x)|\mathrm{d}x. \end{aligned} \tag{6.4}$$

From the hypotheses H6, H3 and H4 and from (6.4) with $\varepsilon \to 0$ (*i.e.* $|\Omega \setminus D| \to 0$) we get that for $\varepsilon > 0$ small enough

$$\|v_0 - v_\varepsilon\| \le c(\nu_4, \nu_3, g, f)\varepsilon_0.$$

The same procedure we can repeat to the equations

$$\begin{aligned} {\Phi^*_{\xi'}}'(x, \eta_0(x) + f(x)) &= v_0(x) + g(x) \text{ a.e. } x \in \Omega, \\ {\Phi^*_{\varepsilon\xi'}}'(x, \eta_\varepsilon(x) + f(x)) &= v_\varepsilon(x) + g(x) \text{ a.e. } x \in \Omega, \end{aligned}$$

what gives an analogous estimate

$$\|\eta_0 - \eta_\varepsilon\| \le c(\nu_4, \nu_3, g, f)\varepsilon_0$$

for $\varepsilon > 0$ small enough.

Here we had used the relationship

$$(F')^{-1} = F^{*\prime}.$$

Thus, we have established that for every $\delta_0 > 0$ there exist $\varepsilon_0 > 0$ and a corresponding finite set $M_{\varepsilon_0} \subset M$ such that

$$\sup_{(v',\eta')\in Z(g,f)} \inf_{(v,\eta)\in Z_{\varepsilon_0}(g,f)} \|(v',\eta') - (v,\eta)\| < \delta_0,$$

i.e. the sets $Z_\varepsilon(g,f)$ approximate the set $Z(g,f)$ in the strong topology and

$$cl\, Z(g,f) = cl \bigcup_{\varepsilon>0} Z_\varepsilon(g,f),$$
$$wcl\, Z(g,f) = wcl \bigcup_{\varepsilon>0} Z_\varepsilon(g,f).$$

In the second step we shall show the approximability of the corresponding **A**-quasiconvex envelope.

Denote, for $(\xi',\xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$,

$$\mathcal{F}(\xi',\xi'') = \inf\{F(\xi') + F^*(\xi'') - \langle \xi',\xi''\rangle \mid F \in M\},$$
$$\mathcal{F}_\varepsilon(\xi',\xi'') = \inf\{F(\xi') + F^*(\xi'') - \langle \xi',\xi''\rangle \mid F \in M_\varepsilon\},$$

and denote by $Q\mathcal{F}$ and $Q\mathcal{F}_\varepsilon$ the corresponding **A**-quasiconvex envelopes.

By definition of the sets M_ε, for all $(\xi',\xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$

$$|\mathcal{F}(\xi',\xi'') - \mathcal{F}_\varepsilon(\xi',\xi'')| < 2\varepsilon(1 + |\xi'|^2 + |\xi''|^2),\ Q\mathcal{F}_\varepsilon(\xi',\xi'') \geq Q\mathcal{F}(\xi',\xi'').$$

Let $\delta > 0$ be given and let, for a fixed (ξ',ξ''),

$$Q\mathcal{F}(\xi',\xi'') \geq \int_K \mathcal{F}\big(v_\delta(x) + \xi', \eta_\delta(x) + \xi''\big)\mathrm{d}x - \delta,$$

where $(v_\delta,\eta_\delta) \in \mathcal{V}^\# \times \mathcal{N}^\#$. Then

$$\begin{aligned} Q\mathcal{F}_\varepsilon(\xi',\xi'') &\leq \int_K \mathcal{F}_\varepsilon\big(v_\delta(x) + \xi', \eta_\delta(x) + \xi''\big)\mathrm{d}x \\ &\leq \int_K \mathcal{F}\big(v_\delta(x) + \xi', \eta_\delta(x) + \xi''\big)\mathrm{d}x + \int_K \Big\{\mathcal{F}_\varepsilon\big(v_\delta(x) + \xi', \eta_\delta(x) + \xi''\big) \\ &\quad - \mathcal{F}\big(v_\delta(x) + \xi', \eta_\delta(x) + \xi''\big)\Big\}\mathrm{d}x \\ &\quad \to Q\mathcal{F}(\xi',\xi'') + \delta \text{ as } \varepsilon \to 0. \end{aligned} \tag{6.5}$$

This way, for every $(\xi',\xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$,

$$Q\mathcal{F}_\varepsilon(\xi',\xi'') \to Q\mathcal{F}(\xi',\xi'') \text{ as } \varepsilon \to 0.$$

Moreover, to $\mathcal{F}_\varepsilon$ we can apply results of Sections 3, 4, which give the estimate

$$\|(v_\delta,\eta_\delta)\| \leq c(\nu_3,\nu_4)(1 + |\xi'| + |\xi''| + \delta).$$

This estimate, together with (6.5), ensure the uniform (in the norm $\|\cdot\|_2$) estimate for all $(\xi',\xi'') \in \mathbf{R}^{nm} \times \mathbf{R}^{nm}$

$$|Q\mathcal{F}(\xi',\xi'') - Q\mathcal{F}_\varepsilon(\xi',\xi'')| \leq \varepsilon c(\nu_3,\nu_4)(1 + |\xi'|^2 + |\xi''|^2). \tag{6.6}$$

Let $(v_0, \eta_0) \in \mathcal{V} \times \mathcal{N}$ be such that

$$\int_\Omega Q\mathcal{F}\big(v_0(x) + g(x), \eta_0(x) + f(x)\big)\mathrm{d}x = 0.$$

From here and the estimate (6.6) we have

$$\int_\Omega Q\mathcal{F}_\varepsilon\big(v_0(x) + g(x), \eta_0(x) + f(x)\big)\mathrm{d}x = d_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{6.7}$$

The function $\mathcal{F}_\varepsilon$ is defined as minimum over a finite subset M_ε

$$\mathcal{F}_\varepsilon(\xi', \xi'') = \min_{F \in M_\varepsilon} \Big\{F(\xi') + F^*(\xi'') - \langle \xi', \xi'' \rangle\Big\},$$

hence, by results of Section 3,

$$Q\mathcal{F}_\varepsilon(\xi', \xi'') = \inf_{\sigma \in S_\varepsilon} \inf_{v \in \mathcal{V}^\#} \inf_{\eta \in \mathcal{N}^\#} \int\limits_K \left\{ \sum_{s=1}^{s_\varepsilon} \sigma_s(x)\big[F_s(v(x) + \xi') + F_s^*(\eta(x) + \xi'')\big] - \langle \xi', \xi'' \rangle \right\} \mathrm{d}x,$$

where the set S_ε corresponds to the set M_ε according the hypothesis H7. Therefore, exactly in the same way as in the proof of Theorem 2.1 in Section 4, from (6.7) we have the existence of sequences

$$\begin{aligned} &\{\sigma^k\} \subset S_\varepsilon, \ \ \big\{(v_{\varepsilon k}, \eta_{\varepsilon k})\big\} \subset \mathcal{V} \times \mathcal{N}, \\ &\{g_k\} \subset L_2(\Omega; \mathbf{R}^{nm}), \ \ \{f_k\} \subset L_2(\Omega; \mathbf{R}^{nm}) \end{aligned}$$

such that

$$\begin{aligned} \int\limits_\Omega \Bigg\{ &\sum_{s=1}^{s_\varepsilon} \sigma_s^k(x)\big[F_s(v_{\varepsilon k}(x) + v_0(x) + g(x) + g_k(x)) + F_s^*(\eta_{\varepsilon k}(x) + \eta_0(x) + f(x) + f_k(x))\big] \\ &- \langle v_{\varepsilon k}(x) + v_0(x) + g(x) + g_k(x), \eta_{\varepsilon k}(x) + \eta_0(x) + f(x) + f_k(x) \rangle \Bigg\} \mathrm{d}x \le 2d_\varepsilon, \\ &\|g_k\| + \|f_k\| \to 0 \text{ as } k \to \infty, \\ &(v_{\varepsilon k}, \eta_{\varepsilon k}) \rightharpoonup 0 \text{ weakly as } k \to \infty. \end{aligned}$$

Further, in the same way as in the proof of Lemma 3.6 we obtain the existence of a pair

$$(v_\varepsilon, \eta_\varepsilon) \in Z_\varepsilon(g, f)$$

such that for k large enough

$$\begin{aligned} &\|(v_\varepsilon, \eta_\varepsilon) - (v_0, \eta_0) - (v_{\varepsilon k}, \eta_{\varepsilon k})\| \le c(\nu_4) d_\varepsilon, \\ &(v_{\varepsilon k}, \eta_{\varepsilon k}) \rightharpoonup 0 \text{ weakly as } k \to \infty. \end{aligned}$$

Now, after an appropriate diagonal process we have the existence of a sequence

$$\begin{aligned} &\{(v_r, \eta_r)\} \subset Z_{1/r}(g, f) \subset \mathcal{V} \times \mathcal{N}, \\ &(v_r, \eta_r) \rightharpoonup (v_0, \eta_0) \text{ as } r \to \infty, \end{aligned}$$

which gives

$$Z_0 = \left\{ (v, \eta) \in \mathcal{V} \times \mathcal{N} \ \Big| \ \int_\Omega Q\mathcal{F}\big(v(x) + g(x), \eta(x) + f(x)\big)\mathrm{d}x = 0 \right\} \subset wcl\, Z(g, f).$$

The inverse inclusion follows immediately from the inequality

$$Q\mathcal{F}(\xi',\xi'') \le \mathcal{F}(x,\xi',\xi''),\ \mathcal{F}\in\mathcal{M},$$

and from that the mapping

$$(v,\eta) \to \int_\Omega Q\mathcal{F}\big(v(x)+g(x),\eta(x)+f(x)\big)\mathrm{d}x$$

is sequentially weakly lower semicontinuous.

This way, we have proved the following result.

Theorem 6.1. *Let the hypotheses* H6-H7 *hold and let the set $\mathcal{M}$ satisfies* (i–iii). *Then for every fixed pair $(g,f)\in L_2(\Omega;\mathbf{R}^{nm})\times L_2(\Omega;\mathbf{R}^{nm})$*

$$wcl\, Z(g,f) = \left\{(v,\eta)\in\mathcal{V}\times\mathcal{N}\ \middle|\ \int_\Omega Q\mathcal{F}\big(v(x)+g(x),\eta(x)+f(x)\big)\mathrm{d}x = 0\right\},$$

where $Q\mathcal{F}$ is the $\mathbf{A}$-quasiconvex envelope, associated with the operator $\mathbf{A}=(\mathrm{curl},\mathrm{div})^m$, for the function

$$\mathcal{F}(\xi',\xi'') = \inf_{F\in M}\left\{F(\xi')+F^*(\xi'')-\langle\xi',\xi''\rangle\right\}.$$

Corollary 6.1. *Let the assumptions of Theorem 6.1 hold and let, in addition, all functions $F\subset M$ are isotropic. Then for every $(\xi',\xi'')\in\Lambda$, Λ being the characteristic cone for the operator $\mathbf{A}=(\mathrm{curl},\mathrm{div})^m$,*

$$\begin{aligned}Q\mathcal{F}(\xi',\xi'') &= \mathcal{P}\mathcal{F}(\xi',\xi'') = \sup_{z',z''\in\mathbf{R}^{nm}}\ \inf_{F\in M}\left\{-F^*(z')-F(z'')+\langle z',\xi'\rangle+\langle z'',\xi''\rangle\right\}\\ &= \inf_{\varepsilon>0}\ \inf_{\sigma\in S_\varepsilon\ \sigma=\sigma(x_1)}\ \inf_{v\in\mathcal{V}^\#\ v=v(x_1)}\ \inf_{\eta\in\mathcal{N}^\#\ \eta=\eta(x_1)}\int_K\left\{\sum_{s=1}^{s_\varepsilon}\sigma_s(x_1)[F_s(v(x_1)\right.\\ &\qquad \left.+R\xi')+F_s^*(\eta(x_1)+R\xi'')\right\}\mathrm{d}x,\end{aligned}\tag{6.8}$$

where $\mathcal{P}\mathcal{F}$ is $\mathbf{A}$-polyconvex envelope of $\mathcal{F}$ and $R\in SO(n)$ is such that $R\xi'^1,\dots,R\xi'^m$ are parallel to $e_1=(1,0,\dots,0)$ and $R\xi''^1,\dots,R\xi''^m$ are orthogonal to e_1.

Proof. By Theorem 2.2 and by construction

$$\begin{aligned}Q\mathcal{F}_\varepsilon(\xi',\xi'') &= \mathcal{P}\mathcal{F}_\varepsilon(\xi',\xi'')\\ &= \inf_{\sigma\in S_\varepsilon\ \sigma=\sigma(x_1)}\ \inf_{v\in\mathcal{V}^\#\ v=v(x_1)}\ \inf_{\eta\in\mathcal{N}^\#\ \eta=\eta(x_1)}\int_K\left\{\sum_{s=1}^{s_\varepsilon}\sigma_s(x_1)[F_s(v(x_1)+R\xi')+F_s^*(\eta(x_1)+R\xi'')]\right\}\mathrm{d}x,\end{aligned}$$

provided $(\xi',\xi'')\in\Lambda$.

Clearly, the left hand side and the right hand side in this relationship give the corresponding terms in (6.8) as $\varepsilon \to 0$. In turn, exactly in the same way as in Section 5 we have the estimate (on Λ)

$$\begin{aligned}
\mathcal{PF}(\xi',\xi'') &\geq \sup_{z',z''\in\mathbf{R}^{nm}} \inf_{\zeta',\zeta''\in\mathbf{R}^{nm}} \Big\{ \langle z',\xi'\rangle + \langle z'',\xi''\rangle \\
&\quad + \inf_{F\in M} [F(\zeta') + F^*(\zeta'')] - \langle z',\zeta'\rangle - \langle z'',\zeta''\rangle \Big\} \\
&= \sup_{z',z''\in\mathbf{R}^{nm}} \lim_{\varepsilon\to 0} \inf_{F\in M_\varepsilon} \Big\{ -F^*(z') - F(z'') + \langle z',\xi'\rangle + \langle z'',\xi''\rangle \Big\} \\
&= \lim_{\varepsilon\to 0} \sup_{z',z''\in\mathbf{R}^{nm}} \inf_{F\in M_\varepsilon} \Big\{ -F^*(z') - F(z'') + \langle z',\xi'\rangle + \langle z'',\xi''\rangle \Big\} \\
&\geq \lim_{\varepsilon\to 0} \inf_{\sigma\in S_\varepsilon} \inf_{\sigma=\sigma(x_1)} \inf_{v\in\mathcal{V}^\#} \inf_{v=v(x_1)} \inf_{\eta\in\mathcal{N}^\#} \inf_{\eta=\eta(x_1)} \int_K \Big\{ \sum_{s=1}^{s_\varepsilon} \sigma_s(x_1)[F_s(v(x_1)+\xi') + F_s^*(\eta(x_1)+\xi'')] \Big\} \mathrm{d}x
\end{aligned}$$

by virtue of H6, H7. These estimates are sufficient for validity of (6.8). □

Finally, we want to show that $Q\mathcal{F} \subset C^1$. *i.e.* that the function $Q\mathcal{F}$ is continuously differentiable.

Let us recall, in a slightly reformulated form, the necessary results from Ball *et al.* [1] and Miettinen and Raitums [6].

Proposition 6.1 (Ball *et al.* [1]). *Let the function $G : \mathbf{R}^N \to \mathbf{R}$ is separately convex and let for every $z_0 \in \mathbf{R}^N$ there exists an element $a_0 \in \mathbf{R}^N$ such that*

$$G(z_0 + z) - G(z_0) - \langle a_0, z\rangle \ \leq \ o(|z|). \tag{6.9}$$

Then $G \in C^1$.

Proposition 6.2 (Miettinen and Raitums [6]). *Let the family $\{G(\beta,\cdot,\cdot)\}$ of functions $G(\beta,\cdot,\cdot) : \mathbf{R}^{nm} \times \mathbf{R}^{nm} \to \mathbf{R}$, $\beta \in B \subset L_\infty(\mathbf{R}^n;\mathbf{R}^N)$, be such that*

(i) *$G(\beta,\cdot,\cdot) \in C^1$ and $G(\beta(\cdot), z', z'')$ is measurable;*

(ii) *there exist a constant ν_6 and a continuous increasing function $\gamma_1 : \mathbf{R} \to \mathbf{R}$ with $\gamma_1(0) = 0$ such that*

$$\begin{aligned}
&0 \leq G(\beta(x), z', z'') \leq \nu_6(1 + |z'|^2 + |z''|^2), \\
&|G'(\beta(x), z', z'')| \leq \nu_6(1 + |z'| + |z''|), \\
&|G'(\beta(x), \xi' + z', \xi'' + z'') - G'(\beta(x), \xi', \xi'')| \\
&\leq \nu_6(1 + |\xi'| + |\xi''|)\gamma_1(|z'| + |z''|);
\end{aligned}$$

(iii) *if $\beta \in B$ then, for every integer r, the function $x \to \beta(rx)$ belongs to B too.*

Then the function

$$G(\xi',\xi'') = \inf_{\beta\in B} \inf_{v\in\mathcal{V}^\#} \inf_{\eta\in\mathcal{N}^\#} \int_K G(\beta(x), v(x) + \xi', \eta(x) + \xi'')\mathrm{d}x$$

belongs to C^1.

Proposition 6.3 (Miettinen and Raitums [6]). *Let the family $\{G(\alpha,\cdot)\}$ of functions $G(\alpha,\cdot) : \mathbf{R}^N \to \mathbf{R}$, $\alpha \in A$, A being a set of parameters of a general nature, be such that*

(i) *$G(\alpha, z) \geq 0$;*

(ii) *there exist a constant ν_7 and a continuous increasing function $\gamma_2 : \mathbf{R} \to \mathbf{R}$ with $\gamma_2(0) = 0$ such that for every $(\alpha, z_0) \in A \times \mathbf{R}^N$ there exists an element $a(\alpha, z_0) \in \mathbf{R}^N$ with the properties*

$$\begin{aligned}
&|a(\alpha, z_0)| \leq \nu_7(1 + |z_0|), \\
&G(\alpha, z_0 + z) - G(\alpha, z_0) - \langle a(\alpha, z_0), z\rangle \leq (1 + |z_0|)\gamma_2(|z|)|z|.
\end{aligned} \tag{6.10}$$

Then the function G_0,

$$G_0(z) = \inf_{\alpha \in A} G(\alpha, z),$$

has the same properties (6.10) with elements $a(z_0) \in \mathbf{R}^N$ *instead of* $a(\alpha_0, z_0)$.

From Proposition 6.2, Corollary 3.3 and Hypotheses H6, H7 it follows immediately that the functions $Q\mathcal{F}_\varepsilon$, $0 < \varepsilon$, are continuously differentiable on $\mathbf{R}^{nm} \times \mathbf{R}^{nm}$.

Exactly in the same way as in the proofs of Lemma 1 and Lemma 2 in Miettinen and Raitums [6] we get that the family $\{Q\mathcal{F}_\varepsilon\}$ satisfies assumptions of Proposition 6.3. Clearly,

$$Q\mathcal{F}(\xi', \xi'') = \inf_{\varepsilon > 0} Q\mathcal{F}_\varepsilon(\xi', \xi''),$$

which, together with Proposition 6.3, gives that the function $Q\mathcal{F}$ satisfies (6.9). Since $Q\mathcal{F}$ is convex with respect to Λ, then $Q\mathcal{F}$ is separately convex too. From here and Proposition 6.1 it follows immediately that $Q\mathcal{F} \in C^1$. Thus, we have proved the following result.

Theorem 6.2. *Let the hypotheses* H6 *and* H7 *hold. Then* $Q\mathcal{F} \in C^1$.

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