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UNIVALENT σ-HARMONIC MAPPINGS: APPLICATIONS TO COMPOSITES

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Abstract. This paper is part of a larger project initiated with [2]. The final aim of the present paper is to give bounds for the homogenized (or effective) conductivity in two dimensional linear conductivity. The main focus is therefore the periodic setting. We prove new variational principles that are shown to be of interest in finding bounds on the homogenized conductivity. Our results unify previous approaches by the second author and make transparent the central role of quasiconformal mappings in all the two dimensional G-closure problems in conductivity.

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1. Introduction

This paper is part of a larger project initiated with [2]. The underlying main theme, is the study of a branch of homogenization studying fine properties of so-called composite materials.

We will first introduce some notations and preliminary definitions and then explain the main focus of the paper. We shall denote by $\mathcal{M}^s$ the class of real, two by two symmetric matrices and by $\mathcal{M}^s_K$, $K \geq 1$, the subclass of matrices $\sigma = \{\sigma_{ij}\} \in \mathcal{M}^s$ satisfying the uniform ellipticity condition

$$K^{-1}|\xi|^2 \leq \sigma_{ij}\xi_i\xi_j \leq K|\xi|^2 \quad \text{for} \quad \text{every} \quad \xi \in R^2. \tag{1.1}$$

Let Ω be an open set in R^2, we shall refer to any given $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ as a *conductivity* and a mapping $U \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ will be said to be *σ-harmonic* if its components u_1 and u_2 are weak solutions to the divergence form elliptic equation

$$\text{div } (\sigma \nabla u_i) = 0 \quad \text{in} \quad \Omega, \quad i = 1, 2. \tag{1.2}$$

We shall mainly consider $\Omega = R^2$. We set $Q = (0,1) \times (0,1)$ and we shall deal with functions which are 1-periodic with respect to each of its variables x_1 and x_2, which we will call *Q-periodic*, or for short, *periodic*.

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This will be indicated by the subscript $\sharp$ in the relevant function spaces. For instance

$$L^\infty_\sharp(R^2,\mathcal{M}^s_K)\equiv\{\sigma\in L^\infty(R^2,\mathcal{M}^s_K)\quad|$$
$$\sigma(x_1+m,x_2+n)=\sigma(x_1,x_2)\quad\text{for}\quad\text{a.e.}\quad(x_1,x_2)\in R^2,\quad\forall m,n\in Z\},$$
$$W^{1,2}_\sharp(R^2,R^2)\equiv\{U\in W^{1,2}_{\rm loc}(R^2,R^2)\quad|$$
$$U(x_1+m,x_2+n)=U(x_1,x_2)\quad\text{for}\quad\text{a.e.}\quad(x_1,x_2)\in R^2,\quad\forall m,n\in Z\}\cdot$$

It is also convenient to define, for any two by two matrix A,

$$W^{1,2}_{\sharp,A}(R^2,R^2)\equiv\{U\in W^{1,2}_{\rm loc}(R^2,R^2)\mid U-Ax\in W^{1,2}_\sharp(R^2,R^2)\}\cdot\tag{1.3}$$

We are especially interested in boundary conditions of periodic type because of their central role in homogenization and in particular in the so-called G-closure problems. Indeed let us recall the following very basic facts in homogenization theory. Let $\sigma\in L^\infty_\sharp(R^2,\mathcal{M}^s_K)$ be given and let Ω be a bounded open and simply connected set with Lipschitz boundary. Let $f\in W^{-1,2}(\Omega,R)$ and, for $\epsilon>0$, set $\sigma_\epsilon(x)=\sigma(\frac{x}{\epsilon})$. Consider the problem

$$\begin{cases}-\text{div}(\sigma_\epsilon(x)\nabla u_\epsilon(x))=f\quad\text{in}\quad\Omega\\ u_\epsilon\in W^{1,2}_0(\Omega,R).\end{cases}$$

Then $u_\epsilon\rightharpoonup u_0$ in $W^{1,2}(\Omega,R)$ where u_0 solves the following (homogenized) problem:

$$\begin{cases}-\text{div}(\sigma_{\rm hom}\nabla u_0(x))=f\quad\text{in}\quad\Omega\\ u_0\in W^{1,2}_0(\Omega,R).\end{cases}$$

The result is well known [46] and [47]. (See [11] for an introduction to the subject.)

The new (constant) matrix $\sigma_{\rm hom}$, called *homogenized conductivity*, belongs to $\mathcal{M}^s_K$ and it is determined by the following rule

$$\forall\xi\in R^2,\ \langle\sigma_{\rm hom}\xi,\xi\rangle=\inf_{u-\langle\xi,x\rangle\in W^{1,2}_\sharp(R^2,R)}\int_Q\langle\sigma(y)\nabla u(y),\nabla u(y)\rangle\mathrm{d}y.\tag{1.4}$$

It is important to change the definition (1.4) to an equivalent but more convenient one. To this end, we denote the set of real two by two matrices by $\mathcal{M}$ and define $\sigma_{\rm hom}$ as the unique constant (and symmetric) matrix satisfying the following rule

$$\forall A\in\mathcal{M},\ \text{tr}(A\sigma_{\rm hom}A^T)=\inf_{U\in W^{1,2}_{\sharp,A}(R^2,R^2)}\int_Q\text{tr}\left[DU(y)\sigma(y)DU(y)^T\right]\mathrm{d}y.\tag{1.5}$$

Note that the infimum in (1.5) is taken on a class of vector fields rather than functions. We use the notation D (rather than ∇) to denote the gradient of vector valued mappings. Our convention is that for $F=(f,g)$,

$$DF=\begin{pmatrix}f_{x_1}&f_{x_2}\\ g_{x_1}&g_{x_2}\end{pmatrix}.$$

By the linearity of the Euler–Lagrange equations associated to the variational principle (1.4), one concludes that (1.5) and (1.4) are indeed equivalent. Given $A\in\mathcal{M}$, denote by U^A a solution (unique up to an additive constant vector) of

$$\begin{cases}\text{Div}\left[\sigma(y)(DU^A(y))^T\right]=0\quad\text{in}\quad R^2\\ U^A\in W^{1,2}_{\sharp,A}(R^2,R^2),\end{cases}\tag{1.6}$$

where for any matrix B, Div$B$ is the vector whose i-th component is the divergence of the vector whose components form the i-th column of B.

Note that, by (1.6), U^A is a solution of the Euler–Lagrange equations associated to (1.5).

A topic of great interest in material science and in optimal design is the so-called G-closure problem. The simplest non trivial example is the so-called *two-phase problem* which we now describe. Assume that

$$\sigma(x) = (K\chi_E(x) + K^{-1}(1 - \chi_E(x)))I$$

where E is a measurable subset of Q. The G-closure problem in this case can be roughly described as follows. The given data are the conductivities in each phase (KI and $K^{-1}I$) and the *volume fractions* p, $1-p$ with $p \in [0,1]$. The unknown is the set E, called the *microgeometry*. As E varies, so does the homogenized matrix $\sigma_{\rm hom}$. The goal is to characterize the exact range of $\sigma_{\rm hom}$ as the measurable set E varies in the family of *all* possible measurable subsets of Q satisfying the constraint $|E| = p$ (more precisely its closure in the space of symmetric matrices equipped with its natural norm).

The study of the two-phase problem has been initiated by Hashin and Shtrikman [23] and it has been completely solved only about twenty years later by Tartar and Murat [40, 49] and by Cherkaev and Lurie [30]. However, many interesting G-closure problems are still open. For instance the *three-phase problem* in which σ takes three distinct values with prescribed volume fractions, has attracted considerable attention.

Periodicity is a very convenient setting. However it is not necessary. On the contrary, the more general setting of arbitrary sequences of matrices satisfying (1.1), has inspired very deep progress in this field. The general plan to establish bounds on the effective conductivity is very clearly outlined by Tartar in his fundamental paper [49] and also in the less known [48]. The idea is to consider any known differential constraint on the fields coming into play. Transform this constraint (using the theory of compensated compactness developed by Murat and Tartar [50] and [39]) into a compactness property for suitable weakly convergent sequences of fields hence establishing necessary conditions that any effective conductivity (or more generally any H-limit in the language of Murat and Tartar [49]) must satisfy. This approach has been tremendously successful. In two dimensions, and restricting attention to periodic boundary conditions, the essence of the method, is to use what (in the slightly different context of multi-well problems) are called the "minor relations". In other words one takes advantage of the fundamental fact that given any $A \in \mathcal{M}$ and any $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$, in addition to the obvious constraint $\int_Q DU(x)\mathrm{d}x = A$ one has

$$\int_Q \det DU(x)\mathrm{d}x = \det A. \tag{1.7}$$

The latter is often expressed by saying that $A \to \det A$ is a *null-Lagrangian* on the space $W^{1,2}_{\sharp,A}(R^2, R^2)$. The equality (1.7) is a special instance of a much more general phenomenon leading to the notion of *quasiconvexity*: a real valued function F on the space of two by two matrices is quasiconvex if for any matrix A one has:

$$U \in W^{1,2}_{\sharp,A}(R^2, R^2) \Rightarrow \int_Q F(DU)\mathrm{d}x \geq F(A).$$

By Jensen's inequality, convex functions have this property and if the target space of U has dimension one, the set of quasiconvex functions, reduces itself to the set of convex functions [51]. However, if both the domain and the target space have dimension greater than one, there exist quasiconvex functions which are not convex as shown for instance by (1.7). The compensated compactness developed by Murat and Tartar [40,50] is the natural mathematical tool to find bounds on homogenized coefficients by using the existence of these functions. Due to its elegance, simplicity and generality, the method has been a tremendous source of stimulus and results in material sciences, optimal design as well as in their connections to certain branches of the calculus of variations.

In its general form, the compensated compactness method works with any quasiconvex function, not just null-Lagrangians. However, as observed by Milton [35], many different quasiconvex functions may lead to the same bound. Therefore the method faces another difficulty, namely that very little is known about the set of quasiconvex functions. In practice, in two dimensional linear conductivity, all the bounds obtained with this approach select only the determinant (or not relevant modifications of it) in the (unknown) class of all quasiconvex functions and use it as efficiently as possible. This is what we will call the *conventional translation method*. Use of different quasiconvex functions (*unconventional translation method*) is in principle possible but, at present, no other efficient candidates are available, at least in dimension two.

In recent years, however, there has been some progress in finding bounds for the G-closure problems. This has been made by using (apparently) different approaches. In the first of such results [42], use has been made of the fact (proved in [10]) that suitably chosen σ-harmonic mappings U are sense preserving ($\det DU \geq 0$ a.e.).

In another piece of work [43], use has been made of a recent fundamental advance in the theory of quasiconformal mappings due to Astala [6].

In these papers the strategy is inspired by the compensated compactness approach and it is similar to it. The difference consists in using constraints on the variable $U \in W^{1,2}_{\sharp,A}$ which are valid *only* for those fields U which, in addition, are (in our language) σ-harmonic mappings.

The additional difficulty in this approach is that this constraint must depend only on the *a priori* information on the pertinent G-closure problem.

The first goal achieved in this paper is to show that the papers [42] and [43] have indeed a very clear common theme. In Section 3, we establish a new variational principle which defines σ_{hom} in an alternative way. This is a refinement of a variational principle of Astala and Miettinen [7]. We will compare the latter with our new one in some detail in Section 3 (see the discussion after Cor. 3.2). Ours has the important advantage that the minimizers are quasiconformal independently of any assumption on the conductivity matrix!

The crucial ingredient which we will need in our analysis is the following.

Theorem A (Alessandrini–Nesi [2], Ths. 2 and 5)**.** *Set* $\det A > 0$*, and let* U^A *be a solution to (1.6). Then*

(i) U^A *is univalent, that is, a homeomorphism of* R^2 *onto itself;*

(ii) U^A *is sense preserving, that is,*

$$\det DU^A > 0 \text{ a.e.} \tag{1.8}$$

The quasiconformality of the minimizers of the *new* variational principle proved in Section 3 (see also the statements of Ths. 3.1 and 3.2 later in this Introduction), is indeed a consequence of Theorem A. This rather surprising connection is developed in the next Section 2, see, in particular, Propositions 2.1, 2.2. It is worth noticing here that, instead, the quasiconformality of the minimizer is not guaranteed in general for the original variational principle (1.5). In fact, it has been shown in [3] that for generic conductivities (in the sense of Baire category) the minimizer of (1.5) is *not* quasiconformal.

In turn, the fact that, for the new variational principle, the results of Section 2 imply the quasiconformality of the minimizers explains why studying the properties of quasiconformal mappings is central to any two dimensional G-closure problem in linear conductivity.

Thus, this circle of ideas unifies and expands the work in [42] and [43] under a common scheme, and, especially, it explains why the area distortion theorem of Astala [6] is a natural ingredient in this context.

To state in more detail the main results of the paper we need some further background. We recall that, given $f \in W^{1,2}_{\text{loc}}(\Omega, R^2)$, the *dilatation quotient* for f is defined for almost every $x \in \Omega$ as

$$\mathcal{D}_f(x) = \frac{\max_{|\xi|=1} |\partial_\xi f(x)|}{\min_{|\xi|=1} |\partial_\xi f(x)|} \tag{1.9}$$

where ∂_ξ denotes directional derivative in the direction ξ, and that, for a given $K \geq 1$, a non constant $f \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ is said to be a (sense preserving) K-*quasiregular* mapping if

$$\mathcal{D}_f(x) \leq K, \quad \text{and} \quad \det Df > 0, \text{ for almost every } x \in \Omega, \tag{1.10}$$

where Df denotes the Jacobian matrix of f. A mapping f will be said K-*quasiconformal* if in addition it is injective. We also recall that equivalent conditions to (1.10) are given by

$$\operatorname{tr}(DfDf^T) \le (K + K^{-1})\det Df \quad \text{almost everywhere in } \Omega, \tag{1.11}$$

or else

$$|f_{\bar z}| \le \frac{K-1}{K+1}|f_z| \text{ almost everywhere in } \Omega, \tag{1.12}$$

where the standard identification $z = x_1 + ix_2$ is used. See, as a basic reference for quasiregular mappings in the plane, Lehto and Virtanen [29].

The connections between σ-harmonic and quasiregular mappings are many-sided.

First of all, the components u_1 and u_2 of a σ-harmonic mapping U are also the components of quasiregular mappings. In fact, to each σ-harmonic function u (that is a solution to (1.2)) we can associate in a natural fashion, a new function, the so-called *stream function* $\tilde u$ which generalizes the harmonic conjugate, and which is defined as follows. If Ω is simply connected and u is σ- harmonic, then there exists, and it is uniquely determined up to an additive constant, a function $\tilde u \in W^{1,2}_{\mathrm{loc}}(\Omega, R)$ such that

$$\nabla \tilde u = J\sigma\nabla u \quad \text{a.e.} \quad x \in \Omega \tag{1.13}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Such a stream function associated to u is $\frac{\sigma}{\det\sigma}$-harmonic *i.e.* we have, in the weak sense

$$\operatorname{div}\left(\frac{\sigma}{\det\sigma}\nabla\tilde u\right) = 0 \ \text{ in } \ \Omega$$

and is such that the mapping $f = u + i\tilde u$ turns out to be K-quasiregular. Another (non trivially) equivalent but very useful way to express the quasiregularity of a map f is the following. A mapping $f \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ is K-quasiregular , if and only if there exists $G \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, with $\det G(x) = 1$ almost everywhere, such that

$$Df(x)^T Df(x) = G(x)\det Df(x) \text{ a.e.} \tag{1.14}$$

These facts, which can be traced back to the functional analytic approach for two-dimensional elliptic equations due to Bers and Nirenberg [13] (see also Bers *et al.* [12], Chap. II.6, and Vekua [52]), have been the starting point for the geometric study of σ-harmonic functions in [1].

We now describe in more detail the results of the present paper restricting attention to periodic σ-harmonic mappings. The first result, proved in Section 2, says that, given any sense preserving, periodic σ-harmonic mapping $U = (u_1, u_2)$, there exists a family of mappings obtained as a suitable combination of U and $\tilde U = (\tilde u_1, \tilde u_2)$ which is K-quasiconformal. We refer to Section 2, Propositions 2.1 and 2.2.

In Section 3, we apply the results of Section 2 to issues in homogenization. We prove various new variational principles. All of them enjoy the following properties. First, they *define* the effective conductivity. Second the associated minimizers are quasiconformal. This fact does not depend on σ in any way and shows that, in two dimensional conductivity, the effective conductivity can be defined in terms of quasiconformal mappings only and therefore the latter are the natural tool to tackle the corresponding G-closure problem.

The main results of Section 3 are Theorem 3.1, Theorem 3.2 and the Corollaries 3.1 and 3.2. We state immediately the theorems for the reader's convenience.

Given $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, we set

$$d_m = \operatorname{ess}\inf_{x\in Q} \sqrt{\det\sigma}. \tag{1.15}$$

Theorem 3.1. *Let $K > 1$ be given, if $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, then the homogenized conductivity $\sigma_{\rm hom}$ satisfies*

$$\det \sigma_{\rm hom} \geq d_m^2 \tag{1.16}$$

and, for every $\lambda \in (-d_m, d_m)$ and every $A \in \mathcal{M}$

$$\frac{{\rm tr}(A\sigma_{\rm hom}A^T) - 2\lambda \det A}{\det \sigma_{\rm hom} - \lambda^2} = \inf_{\phi \in W^{1,2}_{\sharp,A}(R^2;R^2)} \frac{1}{|Q|}\int_Q \frac{{\rm tr}[D\phi(y)\sigma(y)D\phi(y)^T] - 2\lambda \det D\phi(y)}{\det \sigma(y) - \lambda^2}{\rm d}y. \tag{1.17}$$

Moreover the minimizer of (1.17) is uniquely determined up to an additive constant vector and is given by

$$\phi_{U^{B_\lambda},\lambda} = \lambda U^{B_\lambda} + J\tilde{U}^{B_\lambda}, \tag{1.18}$$

where U^{B_λ} is the solution to (1.5) when A is replaced with

$$B_\lambda = \frac{-\lambda A + {\rm Adj}(A\sigma_{\rm hom})}{\det \sigma_{\rm hom} - \lambda^2}\cdot \tag{1.19}$$

Here Adj$A$ *stands for the adjugate of the matrix A.*

The variational principle (1.17) is similar to that proved in [7]. We will later see the advantages of the new one.

To state the next result it is convenient to make the following definitions.

Let $S \in \mathcal{M}^s$ be positive definite, set $s = \sqrt{\det S}$ and let $\lambda \geq 0$. We define a set of *quasiconformal matrices* and a corresponding function space as follows:

$$m(S,\lambda) \equiv \begin{cases} \{M \in \mathcal{M}\colon\ (s^2+\lambda^2)\det M > \lambda {\rm tr}(M{\rm Adj}(S)M^T)\} \text{ if } \lambda \in [0,s), \\ \{M \in \mathcal{M}\colon\ (s^2+\lambda^2)\det M = \lambda {\rm tr}(M{\rm Adj}(S)M^T)\} \text{ if } \lambda = s, \\ \mathcal{M} \text{ if } \lambda > s. \end{cases} \tag{1.20}$$

$$W(A,\sigma,\lambda) \equiv \{\phi \in W^{1,2}_{\sharp,A}(R^2,R^2):\ {\rm Adj}D\phi(x) \in m(\sigma(x),\lambda) \text{ a.e. } x \in Q\}\cdot \tag{1.21}$$

It is easy to check that $S^{-\frac{1}{2}} \in m(S,\lambda)$ for every $\lambda \geq 0$, hence

$$\forall \lambda \geq 0,\ m(S,\lambda) \neq \emptyset. \tag{1.22}$$

In Corollary 3.1 we shall show that, under the assumptions of Theorem 3.1, for every $\lambda \in [0, d_m)$ and for every $A \in m(\sigma_{\rm hom}, \lambda)$, the minimizers of (1.17) are not only quasiregular but also quasiconformal.

We continue to adopt the notation of Theorem 3.1. We set

$$Q_{d_m} = \left\{x \in Q:\ \sqrt{\det \sigma(x)} = d_m\right\} \tag{1.23}$$

and consider the following space:

$$\mathcal{B}(A,\sigma,d_m) \equiv \{\phi \in W^{1,2}_{\sharp,A}(R^2,R^2):\ {\rm Adj}D\phi(x) \in m(\sigma(x),d_m) \text{ a.e. } x \in Q_m\}. \tag{1.24}$$

This space is easily described in words. It is the subspace of $W^{1,2}_{\sharp,A}(R^2,R^2)$ given by those mappings which in the set Q_m satisfy the first order Beltrami equation (1.14), with the matrix G defined as follows

$$G = \left(\frac{\sigma}{\sqrt{\det \sigma}}\right)^{-1}\cdot$$

We remark that the set $\mathcal{B}(A,\sigma,d_m)$ is a *closed linear* subspace of $W^{1,2}_{\sharp,A}(R^2,R^2)$.

In the G-closure problems one is particularly interested in the case when the eigenvalues of σ take a finite number of values. On the other hand if $Q=Q_{d_m}$ the G-closure is well-known. It is convenient to avoid this special case assuming that $|Q_{d_m}|<1$. The assumptions of the next theorem are therefore very natural.

Theorem 3.2. *Let $K>1$ be given. For $\sigma\in L^\infty_\sharp(R^2,\mathcal{M}^s_K)$, set d_m and Q_{d_m} according to (1.15) and (1.23) respectively. Assume that $|Q_{d_m}|>0$ and*

$$\operatorname{ess}\inf_{Q\setminus Q_{d_m}}\sqrt{\det\sigma}>d_m. \tag{1.25}$$

Then

$$\det\sigma_{\rm hom}>d_m^2 \tag{1.26}$$

and the homogenized conductivity $\sigma_{\rm hom}$ satisfies the following variational principle. For every $A\in m(\sigma_{\rm hom},d_m)$,

$$\begin{aligned}\frac{\operatorname{tr}(A\sigma_{\rm hom}A^T)-2d_m\det A}{\det\sigma_{\rm hom}-d_m^2} &= \inf_{\phi\in\mathcal{B}(A,\sigma,d_m)}\int_Q\left\{\chi_{Q_{d_m}}(y)\frac{\operatorname{tr}[D\phi(y)\sigma(y)D\phi(y)^T]}{2\det\sigma(y)}\right.\\ &\quad\left.+(1-\chi_{Q_{d_m}}(y))\frac{\operatorname{tr}[D\phi(y)\sigma(y)D\phi(y)^T]-2d_m\det D\phi(y)}{\det\sigma(y)-d_m^2}\right\}\mathrm{d}y. \end{aligned}\tag{1.27}$$

Moreover the minimizer of (1.27) is unique up to a constant vector. It is given by

$$\phi_{U^{B_{d_m}},d_m}=\lambda U^{B_{d_m}}+J\tilde{U}^{B_{d_m}}, \tag{1.28}$$

where $U^{B_{d_m}}$ is the solution to (1.6) when A is replaced with

$$B_{d_m}=\frac{-d_mA+\operatorname{Adj}(A\sigma_{\rm hom})}{\det\sigma_{\rm hom}-d_m^2}\,. \tag{1.29}$$

The minimizer of (1.27) given by (1.28) can be thought of as a constrained minimizer, to emphasize that it might be different from the minimum of the same functional on the whole space $W^{1,2}_{\sharp,A}(R^2,R^2)$.

In Corollary 3.2 we shall show that, under the assumptions of Theorem 3.2, the minimizer of (1.27) is quasiconformal.

We conclude Section 3 with some examples illustrating the use of Theorems 3.1 and 3.2 in the two-phase problem.

In Section 4, we give several examples and applications to G-closure problems. In particular, Theorem 4.1 gives new bounds which are a generalization of those proved in [42] and [43]. It must be said that these new bounds are always at least as tight as those that one would obtain by all the other known bounds and reduce to the standard one when the choice $\phi(x)\equiv Ax$ is made in either the variational principles (1.17) or (1.27) or (4.1).

2. Quasiconformal mappings generated by σ-harmonic mappings

Given a simply connected open set Ω, $K\ge1$ and $\sigma\in L^\infty(\Omega,\mathcal{M}^s_K)$, any σ-harmonic *function* u generates a K-quasiregular mapping $f=u+i\tilde{u}$ via conjugation with its stream function $\tilde{u}$ (see the Introduction and [2], Sect. 2 for more details). In this section we show that any σ-harmonic *mapping* U which is sense preserving (*i.e.* such that $\det DU\ge0$ almost everywhere), generates a one parameter family of quasiregular mappings. In particular one element of this family is exactly K-quasiregular. It is important to recall from [2], that typically, locally univalent σ-harmonic mappings can be taken to be sense preserving. Here, by *locally univalent* we mean locally one-to-one. We will see later in the Example 2.1, that the classical way to generate K-quasiregular mappings from a given σ-harmonic function u is a special case of our construction.

We will also see in Section 3, that the family of mappings introduced in the present section, has a central role in questions concerning homogenized coefficients. We denote by

$$\sigma_1(x) \leq \sigma_2(x) \tag{2.1}$$

the eigenvalues of σ, at the point $x \in \Omega$. Let $U = (u_1, u_2) \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ be σ-harmonic and let $\tilde{U} = (\tilde{u_1}, \tilde{u_2})$ be the mapping whose components are the stream functions of u_1 and u_2 respectively. For $\lambda > 0$ we define

$$\phi_{U,\lambda} = \lambda U + J\tilde{U} \tag{2.2}$$

$$z_\lambda(x) = \max\left(1, \min\left(\frac{\sigma_2(x)}{\lambda}, \frac{\lambda}{\sigma_1(x)}\right)\right) \tag{2.3}$$

and

$$k_\lambda(x) = \max\left(\frac{\sigma_2(x)}{\lambda}, \frac{\lambda}{\sigma_1(x)}\right), \; K_\lambda = ||k_\lambda(x)||_{L^\infty(\Omega)}. \tag{2.4}$$

We denote by $\mathcal{D}_{U,\lambda}(x)$ the dilatation quotient of $\phi_{U,\lambda}$, see (1.9).

Proposition 2.1. *Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and let $U \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ be σ-harmonic.*

i) *If $\det DU \geq 0$ almost everywhere in Ω, then, for any $\lambda > 0$, $\phi_{U,\lambda}$ is K_λ-quasiregular. More precisely*

$$z_\lambda(x) \leq \mathcal{D}_{U,\lambda}(x) \leq k_\lambda(x) \text{ almost everywhere} \tag{2.5}$$

and hence

$$||\mathcal{D}_{U,\lambda}||_{L^\infty(\Omega)} \leq K_\lambda. \tag{2.6}$$

In particular $\phi_{U,1}$ is K-quasiregular.

ii) *If, in addition, U is locally univalent, then, for every $\lambda > 0$, $\phi_{U,\lambda}$ is locally univalent.*

We will write $\mathcal{M}_+$ for the two by two real matrices with positive determinant.

Proposition 2.2. *Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ and let $A \in \mathcal{M}_+$. If $U^A \in W^{1,2}_{\sharp,A}(R^2, R^2)$ (see (1.3)) is a σ-harmonic mapping, then, for every $\lambda > 0$, $\phi_{U^A,\lambda} = \lambda U^A + J\tilde{U}^A$ is, in addition, an homeomorphism of R^2 onto itself and therefore a K_λ-quasiconformal mapping.*

Remark 2.1. An immediate corollary of Propositions 2.1 and 2.2 is that under the assumptions of Proposition 2.2, $\phi_{U^A,1}$ is actually K-quasiconformal.

Remark 2.2. At the end of the section (Ex. 2.3) we show that one can construct an example in which $||\mathcal{D}_{U,\lambda}||_{L^\infty(\Omega)} = K > 1$ for every $\lambda > 0$. Therefore, in general, one cannot choose λ so that ϕ_λ is conformal and, moreover, the inequality in (2.6) cannot be improved for $\lambda = 1$.

In the sequel tr$F$ and Adj$F$ denote the trace and the adjugate of a matrix F respectively. We recall that, by definition, for two by two matrices $\mathrm{Adj}F = JFJ^T$ and, also that $F\mathrm{Adj}F^T = (\det F)\mathrm{I}$. The proof of Proposition 2.1 is based upon a simple algebraic fact.

Lemma 2.1. *Let $A \in \mathcal{M}$ and $S \in \mathcal{M}^s$. For $\lambda \in R$, set*

$$B = \lambda A + \mathrm{Adj}(AS). \tag{2.7}$$

Then

$$(\det S - \lambda^2)A = -\lambda B + \mathrm{Adj}(BS), \tag{2.8}$$

$$\det B = \det A(\det S + \lambda^2) + \lambda \mathrm{tr}(ASA^T), \tag{2.9}$$

$$(\det S - \lambda^2)^2 \det A = \det B(\det S + \lambda^2) - \lambda \mathrm{tr}(BSB^T). \tag{2.10}$$

Proof of Lemma 2.1. To verify (2.8), we compute its right hand side according to definition (2.7):

$$\begin{aligned} -\lambda B + \mathrm{Adj}(BS) &= -\lambda^2 A - \lambda \mathrm{Adj}(AS) + \mathrm{Adj}(\lambda AS) + \mathrm{Adj}[\mathrm{Adj}(AS)S] \\ &= -\lambda^2 A - \lambda \mathrm{Adj}(AS) + \lambda \mathrm{Adj}(AS) + (AS)\mathrm{Adj}S = -\lambda^2 A + A \det S. \end{aligned}$$

This proves (2.8). To prove (2.9) and (2.10), we use the identity

$$\det(F + G) = \det F + \det G + \mathrm{tr}[F(\mathrm{Adj}G)^T]$$

which holds for any pair of two by two matrices F and G. The calculation is omitted. □

Proof of i) of Proposition 2.1. We fix $x \in \Omega$. The strategy is to apply Lemma 2.1 with the following choices: $S = \sigma$ and $A = DU$, with U a σ-harmonic and sense-preserving mapping. By (1.13) $D\tilde{U} = DU\sigma J^T$ and therefore $\mathrm{Adj}(DU\sigma) = JD\tilde{U}$ which, by (2.2), implies

$$D\phi_{U,\lambda} = \lambda DU + JD\tilde{U} = \lambda DU + \mathrm{Adj}(DU\sigma).$$

We set $B = D\phi_{U,\lambda}$, take $\lambda > 0$ and then apply Lemma 2.1. Since U is sense preserving, (2.10) implies

$$\det D\phi_{U,\lambda}(\det \sigma + \lambda^2) - \lambda \mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T) \geq 0. \tag{2.11}$$

Set $0 \leq a_\lambda^1 \leq a_\lambda^2$ to be the singular values of $D\phi_{U,\lambda}$ (*i.e.* the eigenvalues of the square root of the matrix $D\phi_{U,\lambda}^T D\phi_{U,\lambda}$).

Using (1.10), we see that $\phi_{U,\lambda}$ is L-quasiregular if and only if

$$\mathcal{D}_{U,\lambda} = \frac{a_\lambda^2}{a_\lambda^1} \leq L. \tag{2.12}$$

We write (2.11) in these new variables. We have,

$$a_\lambda^1 a_\lambda^2(\det \sigma + \lambda^2) = \det D\phi_{U,\lambda}(\det \sigma + \lambda^2) \geq \lambda \mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T) \geq \lambda(\sigma_2(a_\lambda^1)^2 + \sigma_1(a_\lambda^2)^2). \tag{2.13}$$

The latter inequality, follows by the so-called Von-Neumann theorem (saying that the $\mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T)$ is minimized when σ and $D\phi_{U,\lambda}^T D\phi_{U,\lambda}$ are simultaneously diagonal and their eigenvalues are ordered with opposite monotonicity). Recalling (2.1) and (2.12), we regard (2.13) as an inequality in the variable $\mathcal{D}_{U,\lambda}$ and we obtain

$$z_\lambda(x) \leq \mathcal{D}_{U,\lambda}(x) \leq k_\lambda(x) \tag{2.14}$$

with k_λ defined in (2.4). This implies (2.6). Moreover, since $K^{-1} \le \sigma_1 \le \sigma_2 \le K$,

$$k_\lambda(x) \le K \max\left(\frac{1}{\lambda}, \lambda\right), \tag{2.15}$$

hence $\phi_{U,1}$ is K-quasiregular.

In order to proceed to the proof of ii) of Proposition 2.1, we will need results from [2], we summarize some of them here.

Let U and $\tilde{U}$ be as above. For any fixed $\xi \in R^2$ with $|\,\xi\,| = 1$, we set

$$u = \langle \xi, U\rangle = \xi_1 u_1 + \xi_2 u_2, \quad \tilde{u} = \langle \xi, \tilde{U}\rangle = \xi_1 \tilde{u}_1 + \xi_2 \tilde{u}_2, \quad f = u + i\tilde{u}. \tag{2.16}$$

Theorem B. *The following properties are equivalent.*

(i) u has no geometric critical points for every ξ, $|\,\xi\,| = 1$;

(ii) f is locally univalent for every ξ, $|\,\xi\,| = 1$;

(iii) U is locally univalent;

(iv) $\tilde{U}$ is locally univalent.

See [2] Theorem 3 for a proof. The notion of *geometric critical point*, which appears above, has been introduced in [1], Definition 2.3, see also [2], Section 2, where this notion has been used systematically. For the present purposes it suffices to know that when a σ-harmonic function u is smooth, its geometric critical points coincide with the usual critical points.

We shall also make use of the following generalization of a classical theorem of Radó.

Theorem C. *Let $\Omega \subset R^2$ be a bounded simply connected open set, whose boundary $\partial\Omega$ is a simple closed curve. Let $\Phi = (\phi_1, \phi_2)$, $\Phi : \partial\Omega \to R^2$ be a homeomorphism of $\partial\Omega$ onto a convex closed curve Γ and let D be the bounded convex domain bounded by Γ.*

Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and let $U \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2) \cap C(\bar{\Omega}, R^2)$ be the σ-harmonic mapping whose components are the solutions of the Dirichlet problems

$$\begin{cases} \operatorname{div}(\sigma \nabla u_i) = 0 & \text{in} \quad \Omega \\ u_i = \phi_i & \text{on} \quad \partial\Omega, \quad i = 1, 2. \end{cases}$$

Then U is a homeomorphism of $\bar{\Omega}$ onto $\bar{D}$.

For a proof and for bibliographical remarks, see [2], Theorem 4.

Proof of ii) of Proposition 2.1. Let us, temporarily, assume in addition that $\sigma \in C^\infty(\Omega, \mathcal{M}^s)$. By (iii)⇒(i) of Theorem B we have that $\det DU > 0$ everywhere. Consequently by (2.10), we obtain

$$\det D\phi_{U,\lambda} = \det(\lambda DU + \mathrm{Adj}(DU\sigma)) = (\det\sigma + \lambda^2)\det DU + \mathrm{tr}(DU\sigma DU^T) > 0 \text{ everywhere.}$$

Now we remove the smoothness assumption. Let $\{\sigma_m\}$ be a sequence of mollified conductivities, such that, for any $p \ge 1$, $\sigma_m \to \sigma$ in L^p_{loc}. Fix $x^0 \in \Omega$. Since U is locally univalent, there exists $\rho > 0$ such that setting $G = U^{-1}(B(U(x^0), \rho))$, U is one-to-one on a neighborhood of G.

Let U_m be the σ_m-harmonic mapping in G such that $U_m|_{\partial G} = U|_{\partial G}$. By Theorem C we obtain that each U_m is univalent in G, moreover, we have $U_m \to U$ in $W^{1,2}_{\mathrm{loc}}(G, R^2)$. Consequently

$$\phi_{U_m,\lambda} \to \phi_{U,\lambda} \text{ in } W^{1,2}_{\mathrm{loc}}(G, R^2).$$

By the above arguments, for every m, $\phi_{U_m,\lambda}$ is locally univalent. By [1] Lemma 1 (which can be viewed as an argument of stability of the degree for sequences of mappings, see also [2], Lem. 1), and by the equivalence (i) ⇔ (iii) of Theorem B, we obtain ii). □

Proof of Proposition 2.2. By Theorem A, $\forall A \in \mathcal{M}_+$, U^A is sense preserving. Hence, by Proposition 2.1, $\phi_{U^A,\lambda}$ is quasiregular. Therefore it is enough to show that $\forall A \in \mathcal{M}_+$, $\phi_{U^A,\lambda}$ is an homeomorphism. Let us outline the strategy of the proof first. We will show that there exists $F_\lambda \in \mathcal{M}_+$ and $C_\lambda \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda})$ such that $\phi_{U^A,\lambda} \in W^{1,2}_{\sharp,F_\lambda}(R^2, R^2)$ and it is a C_λ-harmonic mapping. Then, by Theorem A, it is an homeomorphism.

We fix $A \in \mathcal{M}_+$ and $\lambda > 0$ and, for short, we write $\phi_\lambda = \phi_{U^A,\lambda}$. We set

$$G_\lambda(x) = \begin{cases} \dfrac{D\phi_\lambda^T(x) D\phi_\lambda(x)}{\det D\phi_\lambda(x)} & \text{if} \quad \det D\phi_\lambda(x) \neq 0 \\ I & \text{if} \quad \det D\phi_\lambda(x) = 0. \end{cases} \tag{2.17}$$

By Proposition 2.1, G_λ defines a measurable field of matrices which is symmetric with $\det G_\lambda = 1$ almost everywhere. Moreover, by (2.6), one has

$$K_\lambda^{-1} I \leq G_\lambda \leq K_\lambda I$$

almost everywhere. In other words $G_\lambda \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda})$. Note, for future reference that this clearly implies

$$G_\lambda^{-1} \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda}). \tag{2.18}$$

By construction ϕ_λ satisfies the Beltrami equation

$$D\phi_\lambda^T D\phi_\lambda = G_\lambda \det D\phi_\lambda \tag{2.19}$$

which we rewrite as

$$G_\lambda^{-1} D\phi_\lambda^T = \mathrm{Adj} D\phi_\lambda^T$$

which implies

$$\mathrm{Div}(G_\lambda^{-1} D\phi_\lambda^T) = 0. \tag{2.20}$$

So far we have never used the fact that $U^A \in W^{1,2}_{\sharp,A}$. Now we need this assumption. Indeed, we observe that $\tilde{U^A}$ can be decomposed as a sum of an affine term and a periodic one. More precisely, one has

$$\tilde{U^A} \in W^{1,2}_{\sharp,B}, \; B = J^T \int_Q \mathrm{Adj}(DU^A \sigma).$$

In the language of homogenization (see the Introduction), we have

$$A\sigma_{\mathrm{hom}} = JB, \tag{2.21}$$

where σ_{hom} is the homogenized conductivity see (1.5). The only property that is needed here is that σ_{hom} is a constant, symmetric and positive definite matrix.

Therefore setting $C_\lambda = G_\lambda^{-1}$ and using (2.18) and (2.21), one has that ϕ_λ is C_λ-harmonic and also $\phi_\lambda - F_\lambda x \in W^{1,2}_\sharp(R^2, R^2)$, where $F_\lambda = \lambda A + JB$. In other words, ϕ_λ is a solution to (1.6) when σ and A are replaced by C_λ and F_λ respectively.

In view of Theorem A, to conclude the proof we need to show that $F_\lambda \in \mathcal{M}_+$. Indeed, using (2.9) and the formulas below it, one easily checks that

$$\det F_\lambda = (\lambda^2 + \det \sigma_{\mathrm{hom}}) \det A + \lambda \mathrm{tr}(A\sigma_{\mathrm{hom}} A^T) > 0$$

because both terms in the sum are such. □

We conclude this section with three examples.

Example 2.1. The classical way to construct the quasiregular map $f = u + i\tilde{u}$ from a given σ-harmonic function u and the new way of generating quasiregular mappings explained in this section are actually related in a simple fashion. Indeed, given the σ-harmonic function u, set $U = (u, 0)$ and $\phi_1 = U + J\tilde{U}$. Then it is easy to see that $\phi_{U,1} = (u, \tilde{u})$ so that $\phi_{U,1} = f$ up to the identification between R^2 and C. Clearly, U satisfies $\det DU \geq 0$, in fact $\det DU = 0$ almost everywhere. Therefore, the classical way can be seen as a very special case of the new one.

Example 2.2. If σ has constant determinant ($\det \sigma = d^2$) almost everywhere on a measurable set E, by (2.11) for any σ-harmonic mapping U,

$$\det D\phi_{U,\lambda}(\det \sigma + \lambda^2) - \lambda \mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T) \geq 0, \text{ almost everywhere in } E.$$

In particular, since $\forall B \in \mathcal{M}$

$$2\sqrt{\det \sigma}\det B - \mathrm{tr}(B\sigma B^T) \leq 0,$$

one has that

$$2d \det D\phi_{U,d} = \mathrm{tr}(D\phi_{U,d}\sigma D\phi_{U,d}^T), \text{ almost everywhere in } E.$$

Therefore $\phi_{U,d}$ is K-quasiregular on E and satisfies the Beltrami equation

$$D\phi_{U,d}^T D\phi_{U,d} = \left(\frac{\sigma}{\sqrt{\det \sigma}}\right)^{-1} \det D\phi_{U,d}, \text{ almost everywhere in } E.$$

If σ is the identity on E, $\phi_{U,d} = \phi_{U,1}$ is therefore holomorphic in E (*i.e.* the Cauchy–Riemann system is verified almost everywhere in E).

Example 2.3. This example justifies Remark 2.2. Set Ω to be a ball centered at the origin and of radius one. Define for $x \neq 0$,

$$\sigma(x) = K^{-1}n \otimes n + Kt \otimes t, \; n = \frac{x}{|x|}, \; t = Jn; \; U = x|x|^{K-1}.$$

One can check that U is σ-harmonic in Ω and it is univalent. A calculation shows that

$$DU(x) = |x|^{K-1}(Kn \otimes n + t \otimes n), \; DU(x)\sigma(x) = |x|^{K-1}(n \otimes n + Kt \otimes t)$$

and hence

$$D\phi_{U,\lambda}(x) = (\lambda + 1)|x|^{K-1}(Kn \otimes n + t \otimes t).$$

Therefore, it is readily verified that the dilatation is K for any λ.

3. Proofs of Theorems 3.1, 3.2

In this section we prove Theorems 3.1 and 3.2 stated in the Introduction. We also prove the corresponding Corollaries 3.1 and 3.2. It is very convenient to review the definition and the basic properties of the orthogonal splitting of the set (denoted by $\mathcal{M}$) of two by two matrices into their conformal and anticonformal parts. For $M \in \mathcal{M}$, we write

$$M_+ = \tfrac{1}{2}(M + \mathrm{Adj}M) \quad M_- = \tfrac{1}{2}(M - \mathrm{Adj}M) \tag{3.1}$$

and set as usual

$$\mathcal{H}^+ \equiv \{M \in \mathcal{M} : \; M - \mathrm{Adj}M = 0\}, \; \mathcal{H}^- \equiv \{M \in \mathcal{M} : \; M + \mathrm{Adj}M = 0\} \cdot \tag{3.2}$$

Then one easily checks that $\forall M \in \mathcal{M}$, $M = M_+ + M_-$ and that the decomposition is unique and orthogonal in the sense that

$$A \in \mathcal{H}^+,\ B \in \mathcal{H}^- \ \Rightarrow\ \mathrm{tr}(AB^T) = 0. \tag{3.3}$$

It is convenient to introduce the inner product structure as follows

$$A \cdot B = \mathrm{tr}(AB^T). \tag{3.4}$$

We write $\forall M \in \mathcal{M}$, $|M|^2 = M \cdot M$. Then $\forall M \in \mathcal{M}$

$$\begin{array}{l} |M|^2 = M \cdot M = M_+ \cdot M_+ + M_- \cdot M_- = |M_+|^2 + |M_-|^2, \\ 2\det M = M_+ \cdot M_+ - M_- \cdot M_- = |M_+|^2 - |M_-|^2, \\ \mathrm{Adj}M = M_+ - M_-. \end{array} \tag{3.5}$$

It is convenient to split the proofs of the theorems of the present section presenting first some preliminary results.

Lemma 3.1. *For $F, H \in \mathcal{M}$, $\lambda \in R$ and $S \in \mathcal{M}^s$ and positive definite, we define*

$$f(F, S, \lambda) = \mathrm{tr}(FSF^T) + 2\lambda \det F \tag{3.6}$$

and

$$f^*(H, S, \lambda) = \sup_{F \in \mathcal{M}} [2H \cdot F - f(F, S, \lambda)]. \tag{3.7}$$

Part 1). *As a function of the first variable, f is strictly convex if and only if $\lambda^2 < \det S$; it is convex but not strictly convex if and only if $\lambda^2 = \det S$.*

Part 2). *The explicit expression of f^* is given by*

$$f^*(H, S, \lambda) =$$

$$\begin{cases} \dfrac{\mathrm{tr}(H\mathrm{Adj}SH^T) - 2\lambda \det H}{\det S - \lambda^2} & \text{if} \quad |\lambda| < \sqrt{\det S} \\ \dfrac{\mathrm{tr}[H\mathrm{Adj}SH^T]}{2\det S} & \text{if} \quad \lambda = \sqrt{\det S} \quad \text{and} \quad HS^{-\frac{1}{2}} \in \mathcal{H}^+ \\ \dfrac{\mathrm{tr}[H\mathrm{Adj}SH^T]}{2\det S} & \text{if} \quad \lambda = -\sqrt{\det S} \quad \text{and} \quad HS^{-\frac{1}{2}} \in \mathcal{H}^- \\ +\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

Proof of Lemma 3.1 Part 1). Let us fix $\lambda \in R$ and $S \in \mathcal{M}^s$ and positive definite. We compute the gradient and the Hessian of the function $F \to p(F) = 2H \cdot F - f(F, S, \lambda)$:

$$Dp(F) = 2(H - FS - \lambda \mathrm{AdjF}), \tag{3.9}$$

$$Hp(F) = -2 \begin{pmatrix} S & \lambda J^T \\ \lambda J & S \end{pmatrix}, \tag{3.10}$$

where the four by four Hessian matrix of p has been written in two by two block form. It is easy to see that $H(p)$ is positive definite if and only if $\det S > \lambda^2$ and positive semidefinite if and only if $\det S = \lambda^2$. This proves Part 1).

To prove Part 2), consider

$$p(F) = 2H \cdot F - \mathrm{tr}(FSF^T) - 2\frac{\lambda}{\sqrt{\det S}} \det\left(FS^{\frac{1}{2}}\right).$$

Setting

$$A = FS^{\frac{1}{2}},\ s = \sqrt{\det S}$$

one has

$$p(F) = \tilde{p}(A) = 2\mathrm{tr}(HS^{-\frac{1}{2}}A^T) - |A|^2 - 2\ \frac{\lambda}{s} \det A.$$

We split A and $HS^{-\frac{1}{2}}$ into their conformal and anticonformal parts and use the properties reviewed earlier in this section. It is convenient to set

$$B = HS^{-\frac{1}{2}}. \tag{3.11}$$

Using this formalism, equations (3.1–3.4) and (3.5), one has

$$\begin{aligned} p(F) = \tilde{p}(A) &= 2B_+ \cdot A_+ + 2B_- \cdot A_- - |A_+|^2 - |A_-|^2 - 2\frac{\lambda}{s}(\det A_+ + \det A_-) \\ &= 2B_+ \cdot A_+ + 2B_- \cdot A_- - |A_+|^2 - |A_-|^2 - \frac{\lambda}{s}(|A_+|^2 - |A_-|^2), \end{aligned}$$

hence

$$\tilde{p}(A) = 2B_+ \cdot A_+ - |A_+|^2\left(1 + \frac{\lambda}{s}\right) + 2B_- \cdot A_- - |A_-|^2\left(1 - \frac{\lambda}{s}\right)\cdot \tag{3.12}$$

Now we consider several cases.

Case 1: ($|\lambda| < s$). By (3.12), $\tilde{p}(A)$ is strictly convex and it is maximized at the unique stationary point A^{opt} which satisfies

$$A_+^{\mathrm{opt}} = \frac{s}{s+\lambda}B_+,\ A_-^{\mathrm{opt}} = \frac{s}{s-\lambda}B_-. \tag{3.13}$$

The value of the maximum is displayed in (3.8), first line.

Case 2: ($\lambda = s$). By (3.12), $\tilde{p}(A)$ is still convex but not strictly convex. Indeed

$$\tilde{p}(A) = 2B_+ \cdot A_+ - 2|A_+|^2 + 2B_- \cdot A_-. \tag{3.14}$$

From (3.14) one sees that necessary and sufficient condition for the supremum of $\tilde{p}$ to be finite is $B_- \cdot A_- = 0$ for all A. This holds if and only if $B_- = 0$, which, by (3.1) and (3.2) is equivalent to $B \in \mathcal{H}^+$. In view of (3.11), the latter is equivalent to $HS^{-1/2} \in \mathcal{H}^+$. The optimal A in this case is not unique, but each of them satisfies

$$A_+^{\mathrm{opt}} = \frac{1}{2}B_+.$$

The anticonformal part of A^{opt} is arbitrary. The value of $\tilde{p}$ at the optimal A's is instead uniquely determined and displayed in (3.8), second line.

Case 3: ($\lambda > s$). By assumption $\lambda = s(1+2\alpha^2)$ for some $\alpha \neq 0$. By (3.12),

$$\tilde{p}(A) = 2B_+ \cdot A_+ - 2|A_+|^2(1+\alpha^2) + 2B_- \cdot A_- + 2|A_-|^2\alpha^2. \tag{3.15}$$

Therefore (3.15) is unbounded in the anticonformal part, consequently $f^* = +\infty$.

The remaining cases, $\lambda = -s$ and $\lambda < -s$, are similar to Case 2 and Case 3 respectively and will be omitted. □

Lemma 3.2. *Let $S \in \mathcal{M}^s$ and positive definite and set $s = \sqrt{\det S}$. For $\lambda \geq 0$, $H \in m(S,\lambda)$ (see (1.20)) and f as in (3.6), we define*

$$f^{*,+}(H,S,\lambda) = \sup_{F \in \mathcal{M}_+} [2H \cdot F - f(F,S,\lambda)]. \tag{3.16}$$

Then

$$f^{*,+}(H,S,\lambda) = \begin{cases} \dfrac{\operatorname{tr}(H\operatorname{Adj}SH^T) - 2\lambda \det H}{s^2 - \lambda^2} & \text{if} \quad 0 \leq \lambda < s \\ \dfrac{\operatorname{tr}[H\operatorname{Adj}SH^T]}{2s^2} & \text{if} \quad \lambda = s \text{ and } HS^{-\frac{1}{2}} \in \mathcal{H}^+ \\ \dfrac{||H\operatorname{Adj}SH^t||}{s^2} & \text{if} \quad \lambda > s. \end{cases} \tag{3.17}$$

The symbol $||\cdot||$ denotes the operator norm which for positive semidefinite symmetric matrices is the same as the largest eigenvalue.

Proof of Lemma 3.2. Let us first rephrase the hypothesis $H \in m(S,\lambda)$ in terms of the new variable B (see (3.11)). One can easily check that

$$\begin{array}{l} H \in m(S,\lambda) \Leftrightarrow (s-\lambda)^2|B_+|^2 - (s+\lambda)^2|B_-|^2 > 0 \quad \text{if} \quad \lambda \in [0,s) \\ H \in m(S,s) \Leftrightarrow B_- = 0. \end{array} \tag{3.18}$$

It is convenient to treat separately the following three cases.

Case 1: ($0 \leq \lambda < s$). The optimal A for the unconstrained problem (relative to Case 1 of Lem. 3.1), satisfies (3.13). It follows that it satisfies

$$\det A^{\text{opt}} = \det A_+^{\text{opt}} + \det A_-^{\text{opt}} = \left(\frac{s}{s+\lambda}\right)^2 |B_+|^2 - \left(\frac{s}{s-\lambda}\right)^2 |B_-|^2.$$

Therefore, by (3.18), $A^{\text{opt}} \in \mathcal{M}_+$ if and only if $H \in m(S,\lambda)$. This shows that A^{opt} as defined in (3.13) is also optimal for the constrained problem because the constraint is automatically satisfied. Note that the hypothesis $H \in m(S,\lambda)$ is not only sufficient but also necessary to have $\det A^{\text{opt}} > 0$.

Case 2: ($\lambda = s$). By (3.18), we have $B_+ = 0$. Hence by (3.12), we obtain

$$\tilde{p}(A) = 2B_+ \cdot A_+ - 2|A_+|^2. \tag{3.19}$$

The function in (3.19) ought to be maximized over the open set $\mathcal{M}_+$. If the supremum is achieved at an interior point, then, this point must be stationary and then one easily verifies the corresponding value of $\tilde{p}$ is given by (3.17), second line. If the supremum were only achieved at some $\tilde{A}$ belonging to the closure of $\mathcal{M}_+$, then $\det \tilde{A} = 0$ which can be written as

$$|\tilde{A}_+|^2 = |\tilde{A}_-|^2. \tag{3.20}$$

However, equation (3.19) does not depend on A_-. Therefore the constraint (3.20) can be always satisfied and therefore it is irrelevant and the supremum of $\tilde{p}$ is not (strictly) increased by it.

Case 3: $(s < \lambda)$. First we note that on $\mathcal{M}_+$, the function $F \to f(F,S,\lambda)$ is positive, hence bounded below and it tends to $+\infty$ when $||F||$ tends to $+\infty$. Therefore $f^{*,+}$ is finite. Moreover, by continuity the supremum is the same as the maximum taken over the closure of the set $\mathcal{M}_+$.

Such a maximum is either attained at a stationary point of the function $F \to 2H \cdot F - f(F,S,\lambda)$, or it is attained on the set of matrices with zero determinant.

The first set of values is given by

$$\frac{\mathrm{tr}(H\mathrm{Adj}SH^T) - 2\lambda \det H}{s^2 - \lambda^2}$$

as one immediately obtains using the calculations of Case 1. The second set of values is given by

$$\frac{||H\mathrm{Adj}SH^t||}{s^2}.$$

It is easy to check that for $\lambda > s$ and for any $H \in \mathcal{M}$, the latter is greater or equal to the former. □

Lemma 3.3. *Under the assumptions of Theorem 3.1,* $\det \sigma_{\mathrm{hom}} \geq d_m^2$*. Moreover setting* Q_{d_m} *as in (1.23) we have*

$$|Q_{d_m}| < 1 \Rightarrow \sqrt{\det \sigma_{\mathrm{hom}}} > d_m, \tag{3.21}$$

$$|Q_{d_m}| = 1 \Rightarrow \sqrt{\det \sigma_{\mathrm{hom}}} = d_m.$$

Proof of Lemma 3.3. Our starting point is formula (1.6). We set $A \in \mathcal{M}_+$. Applying (1.7) to $\tilde{U}^A$ and recalling that $|Q| = 1$, one has

$$\int_Q \det D\tilde{U}^A = \det A \det \sigma_{\mathrm{hom}} = \int_Q \det \sigma(y) \det DU^A(y)\mathrm{d}y = \int_{Q_{d_m}} d_m^2 \det DU^A(y)\mathrm{d}y + \int_{Q\backslash Q_{d_m}} \det \sigma(y) \det DU^A(y)\mathrm{d}y.$$

Therefore applying (1.7) to U^A,

$$\det A(\det \sigma_{\mathrm{hom}} - d_m^2) = \int_{Q\backslash Q_{d_m}} (\det \sigma(y) - d_m^2) \det DU^A(y)\mathrm{d}y.$$

By (1.8, 1.23) and (1.15), the integrand of the right hand side is strictly positive almost everywhere. Therefore, the right hand side is nonnegative and hence $\det \sigma(y) - d_m^2 \geq 0$. Now there are two cases. If $|Q_{d_m}| = 1$, then obviously $\det \sigma_{\mathrm{hom}} = d_m^2$. Otherwise, the measure of $Q\backslash Q_{d_m}$ is strictly positive, hence the right hand side in the above formula is strictly positive and therefore so is its left hand side. □

To state the next result, recall (1.18) and (1.21).

Lemma 3.4. *Let* $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$*,* $\lambda \in [0, d_m]$ *and* $A \in m(\sigma_{\mathrm{hom}}, \lambda)$ *be given and let* $\phi_{U^{B_\lambda},\lambda}$ *be defined by (1.18). Then:*

i) $\phi_{U^{B_\lambda},\lambda} \in W(A,\sigma,\lambda)$ *and*

ii)

$$\begin{aligned}\int_Q \frac{\mathrm{tr}(D\phi_{U^{B_\lambda},\lambda}(y)\sigma(y)D\phi_{U^{B_\lambda},\lambda}(y)^T) - 2\lambda \det D\phi_{U^{B_\lambda},\lambda}(y)}{\det \sigma - \lambda^2}\mathrm{d}y &= \mathrm{tr}(B_\lambda \sigma_{\mathrm{hom}} B_\lambda^T) + 2\lambda \det B_\lambda \\ &= \frac{\mathrm{tr}(A\sigma_{\mathrm{hom}}A^T) - 2\lambda \det A}{\det \sigma_{\mathrm{hom}} - \lambda^2}.\end{aligned} \tag{3.22}$$

In particular, for any choice of $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\text{hom}}, \lambda)$,

$$W(A, \sigma, \lambda) \neq \emptyset. \tag{3.23}$$

Proof of Lemma 3.4. We will check that for any choice of $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\text{hom}}, \lambda)$, the mapping $\phi_{U^{B_\lambda},\lambda} \in W(A, \sigma, \lambda)$. By (1.18) and (1.13), we have

$$D\phi_{U^{B_\lambda},\lambda} = \lambda DU^{B_\lambda} + DJ\tilde{U}^{B_\lambda} = \lambda DU^{B_\lambda} + \text{Adj}(DU^{B_\lambda}\sigma)$$

and by (2.10)

$$(\det \sigma - \lambda^2)^2 \det DU^{B_\lambda} = \det D\phi_{U^{B_\lambda},\lambda} \left(\det \sigma + \lambda^2\right) - \lambda \text{tr} \left(D\phi_{U^{B_\lambda},\lambda} \sigma D\phi^T_{U^{B_\lambda},\lambda}\right). \tag{3.24}$$

By (1.8), $\det B_\lambda > 0 \Rightarrow \det DU^{B_\lambda} > 0$ almost everywhere and, by (1.19) and (2.10),

$$\det B_\lambda > 0 \Leftrightarrow \det A(\det \sigma_{\text{hom}} + \lambda^2) - \lambda \text{tr}(A\sigma_{\text{hom}}A^T) > 0 \Leftrightarrow A \in m(\sigma_{\text{hom}}, \lambda).$$

Finally it easy to check that, by construction, $\int_Q D\phi_{U^{B_\lambda},\lambda} = A$. Using (3.24) it follows that $\phi_{U^{B_\lambda},\lambda} \in W(A, \sigma, \lambda)$. (Note that for $\lambda = d_m$, both sides of (3.24) vanish consistently with the definition of $W(A, \sigma, d_m)$ given in (1.21, 1.20).) This establishes part i). Part ii) is a calculation following the same lines and it is omitted. Finally (3.23) is an immediate consequence of part i). □

Proof of Theorem 3.1. A possible proof of (1.17) follows the argument in [7]. Here we give a different and conceptually more direct proof. First of all we note that, for fixed λ, the variational principle (1.17) has a unique minimizer up to an additive constant. Indeed, by Lemma 3.1 applied with $S = \text{Adj}\sigma$, $\forall \lambda \in (-d_m, d_m)$ the function

$$A \to \frac{\text{tr}(A\sigma(y)A^T) - 2\lambda \det A}{\det \sigma(y) - \lambda^2}$$

is the polar of a strictly convex function and therefore itself a convex function, moreover the strict convexity is uniform with respect to x.

Next we show that the minimizers satisfy (1.18) for some matrix A. Indeed the minimizers satisfy the following Euler-Lagrange equations

$$\begin{cases} \text{Div}\left[\frac{[D\Psi_A(y)\sigma(y)]^T - \lambda \text{Adj}[D\Psi_A(y)]^T}{\det \sigma(y) - \lambda^2}\right] = 0 \quad \text{in} \quad R^2 \\ \Psi_A \in W^{1,2}_{\sharp,A}(R^2, R^2), \end{cases} \tag{3.25}$$

in the weak sense. In two dimensions, (3.25) is equivalent to the following conditions. There exists $B_0 \in \mathcal{M}$ and there exist $U_\sharp \in W^{1,2}_\sharp(R^2, R^2)$ such that

$$\frac{D\Psi_A(y)\sigma(y) - \lambda \text{Adj}(D\Psi_A(y))}{\det \sigma(y) - \lambda^2} = \text{Adj}(DU_\sharp + B_0), \text{ almost everywhere} \tag{3.26}$$

or equivalently, setting $U_{B_0} = U_\sharp + B_0 x$ and taking Adj to both sides of (3.26), we have

$$DU_{B_0}(y) = \frac{-\lambda D\Psi_A(y) + \text{Adj}(D\Psi_A(y)\sigma(y))}{\det \sigma(y) - \lambda^2}, \text{ almost everywhere.} \tag{3.27}$$

Using Lemma 2.1, and the definition of $D\Psi_A$, (3.27) can be written as follows. There exists $B_0 \in \mathcal{M}$ and there exist $U_{B_0} \in W^{1,2}_{\sharp,B_0}(R^2,R^2)$ which satisfies

$$\lambda DU_{B_0}(x) + \mathrm{Adj}[DU_{B_0}\sigma(x)] = D\Psi_A(x). \tag{3.28}$$

One can solve (3.28) with respect to U_{B_0} if and only if the left hand side is the differential of a vector field in the suitable space *i.e.* if and only if

$$\begin{cases} \mathrm{Div}[\sigma(x)DU^T_{B_0}(x)] = 0 \quad \text{in} \quad R^2 \\ U_{B_0} \in W^{1,2}_{\sharp,B_0}(R^2,R^2) \end{cases} \tag{3.29}$$

in the weak sense. Note that (3.29) uniquely determines U_{B_0} in terms of B_0 up to an additive constant vector. From (3.29) and the definition (1.6), we see that

$$U_{B_0} = U^{B_0}.$$

Using (3.28) and (3.29) we conclude

$$\Psi_A(x) = \lambda U^{B_0} + J\tilde{U}^{B_0} + \xi, \tag{3.30}$$

for some constant vector ξ.

In view of (3.30) and recalling (1.18) and (1.19), to prove the second part of Theorem 7.1, we are left with showing that $B_0 = B_\lambda$ as defined in (1.19). Indeed, integration of both sides of (3.28) yields

$$A = \int_Q \{\lambda DU^{B_0} + \mathrm{Adj}[DU^{B_0}\sigma(x)]\}\mathrm{d}x = \lambda B_0 + \mathrm{Adj}\int_Q [DU^{B_0}\sigma(x)]\mathrm{d}x.$$

The latter combined with (1.6) and a new integration by parts yields

$$A = \lambda B_0 + \mathrm{Adj}(B_0\sigma_{\mathrm{hom}}). \tag{3.31}$$

We now solve for B_0 in terms of A in (3.31). By (2.8), $B_0 = B_\lambda$ (as defined in (1.19)). Finally, evaluation of the right hand side of (1.17) *via* Lemma 3.4, gives equality with its left hand side. □

Corollary 3.1. *Under the assumptions of Theorem 3.1, if we assume that $A \in m(\sigma_{\mathrm{hom}},\lambda)$, then the minimizers of (1.17) are quasiconformal. Moreover, setting as before $\phi_\lambda = \phi_{U^{B_\lambda},\lambda}$, and recalling (1.9) one has the following inequality*

$$z_\lambda(x) \le \mathcal{D}_{\phi_\lambda}(x) \le k_\lambda(x) \text{ almost everywhere,} \tag{3.32}$$

where z_λ and k_λ are defined in (2.3) and (2.4).

Proof of Corollary 3.1. Indeed, one can check that if $A \in m(\sigma_{\mathrm{hom}},\lambda)$, then $\det B_\lambda > 0$ and hence, by (1.8), $\det DU^{B_\lambda} > 0$ almost everywhere. Therefore an application of Propositions 2.1 and 2.2 to $\phi_{U^{B_\lambda},\lambda}$ yields the thesis. □

Proof of Theorem 3.2. By Lemma 3.3, equation (1.26) holds. By i) of Lemma 3.4 the set $W(A,\sigma,d_m)$ is non empty. Since $W(A,\sigma,d_m) \subset \mathcal{B}(A,\sigma,d_m)$, the latter set is a nonempty linear subspace of $W^{1,2}_{\sharp,A}$.

The uniqueness up to an additive constant vector of the minimizer of (1.27) follows then by the standard theory. Indeed the integrand is strictly convex in the variable involving the gradient uniformly in the x variable (because of the assumption (1.25)).

By ii) of Lemma 3.4 the left hand side of (1.27) is smaller than or equal to its right hand side. Hence, to obtain (1.27) it is enough to prove that the left hand side of (1.27) is greater than or equal to its right hand side. For $F \in \mathcal{M}$ we set

$$g(F) = \mathrm{tr}(F\sigma F^T) + 2d_m \det F \tag{3.33}$$

and then define

$$g^{*,+}(H) = \sup_{F\in\mathcal{M}_+} [2H\cdot F - g(F)],\ H \in \mathcal{M}. \tag{3.34}$$

By (3.34),

$$\forall F \in \mathcal{M}_+,\ \forall H \in \mathcal{M},\ g(F) + g^{*,+}(H) \geq 2H\cdot F. \tag{3.35}$$

The notation reflects the obvious resemblance between $g^{*,+}$ and the polar of the function g usually denoted by g^*.

Choose two arbitrary constant matrices $A \in \mathcal{M}$ and $B \in \mathcal{M}_+$ and set $F = DU$, $H = \mathrm{Adj}DV$ with $U \in W^{1,2}_{\sharp,B}(R^2,R^2)$ (see (1.3)) and satisfying $\det DU > 0$ almost everywhere and $V \in W^{1,2}_{\sharp,A}(R^2,R^2)$. Then use (3.35) and integrate over Q. We obtain

$$\int_Q g(DU)\mathrm{d}x \geq 2\int_Q \mathrm{Adj}(DV)\cdot DU\mathrm{d}x - \int_Q g^{*,+}(\mathrm{Adj}DV)\mathrm{d}x = 2\mathrm{Adj}(A)\cdot B - \int_Q g^{*,+}(\mathrm{Adj}DV)\mathrm{d}x. \tag{3.36}$$

The last equality follows integrating by parts. Now we apply (ii) of Theorem A which implies

$$\forall B \in \mathcal{M}_+,\quad \inf_{\{U\in W^{1,2}_{\sharp,B}(R^2,R^2):\ \det DU>0\ \text{a.e.}\}} \int_Q g(DU) = \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B. \tag{3.37}$$

Using (3.36) and (3.37) we obtain

$$\forall A \in \mathcal{M},\ \forall B \in \mathcal{M}_+,\ \forall\ V \in W^{1,2}_{\sharp,A}(R^2,R^2),$$

$$2\mathrm{Adj}A\cdot B - \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) - 2d_m \det B \leq \int_Q g^{*,+}(\mathrm{Adj}DV)\mathrm{d}x$$

which implies

$$\forall A \in \mathcal{M},\ \sup_{B\in\mathcal{M}_+} \left[2\mathrm{Adj}A\cdot B - \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) - 2d_m \det B\right] \leq \inf_{V\in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q g^{*,+}(\mathrm{Adj}DV)\mathrm{d}x. \tag{3.38}$$

By (1.23, 1.25), Lemma 3.3 and Lemma 3.1, the function $B \to \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B$ is strictly convex and therefore the maximum over $B \in \mathcal{M}$ in the left hand side of (3.38) exists and it is given by the value of the polar function of $\mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B$. Using Lemma 2.1, one checks that the optimal $B \in \mathcal{M}_+ \Leftrightarrow A \in m(\sigma_{\mathrm{hom}}, d_m)$. Therefore in (3.38) the sup over $B \in \mathcal{M}_+$ coincides with that taken over $B \in \mathcal{M}$. The latter is easily calculated using again Lemma 3.1. The previous remarks and (3.38) imply

$$\frac{\mathrm{tr}(\mathrm{Adj}(A)\mathrm{Adj}(\sigma_{\mathrm{hom}})\mathrm{Adj}(A^T)) - 2d_m \det(\mathrm{Adj}A)}{\det\sigma_{\mathrm{hom}} - d_m^2} = \frac{\mathrm{tr}(A\sigma_{\mathrm{hom}}A^T) - 2d_m \det A}{\det\sigma_{\mathrm{hom}} - d_m^2}$$

$$\leq \inf_{V\in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q g^{*,+}(\mathrm{Adj}DV)\mathrm{d}x = \inf_{V\in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q f^{*,+}(\mathrm{Adj}DV(x),\sigma(x),d_m)\mathrm{d}x, \tag{3.39}$$

where $f^{*,+}$ is given by (3.17). Now the plan is to use Lemma 3.2. To this end, we first get an inequality in (3.39) by imposing on V the pointwise constraint

$$\mathrm{Adj}DV(x) \in m(\sigma(x), d_m) \text{ almost everywhere in } Q_m. \tag{3.40}$$

Recall that the latter is equivalent to the requirement $V \in \mathcal{B}(A, \sigma, d_m)$. Hence (3.39) yields

$$\frac{\mathrm{tr}(A\sigma_{\mathrm{hom}}A^T) - 2d_m \det A}{\det \sigma_{\mathrm{hom}} - d_m^2} \leq \inf_{V\in\mathcal{B}(A\sigma,d_m)} \int_Q f^{*,+}(\mathrm{Adj}DV(x), \sigma(x), d_m)\mathrm{d}x, \tag{3.41}$$

Then using Lemma 3.2 and (3.40), we obtain that for all $V \in \mathcal{B}(A, \sigma, d_m)$ and at almost every point in Q one has

$$\begin{aligned} f^{*,+}(\mathrm{Adj}DV(x), \sigma(x), d_m) &= \chi_{Q_{d_m}}(x)\frac{\mathrm{tr}[DV(x)\sigma(x)DV(x)^T]}{2\det\sigma(x)} \\ &\quad +(1-\chi_{Q_{d_m}}(x))\frac{\mathrm{tr}[DV(x)\sigma(x)DV(x)^T] - 2d_m \det DV(x)}{\det\sigma(x) - d_m^2}\cdot \end{aligned} \tag{3.42}$$

By (3.42), the right hand sides of (3.41) and (1.27) are the same and the proof is concluded. □

Corollary 3.2. *Under the conditions of Theorem 3.2, the minimizers of (1.27) are quasiconformal. Moreover, setting as before $\phi_{d_m} = \phi_{U^{B_{d_m}},d_m}$, one has that their dilatation quotient satisfies the following inequality.*

$$z_{d_m}(x) \leq \mathcal{D}_{\phi_{d_m}}(x) \leq k_{d_m}(x) \text{ almost everywhere,}$$

where z_{d_m} and k_{d_m} are defined in (2.3) and (2.4).

Proof. Almost identical to the proof of Corollary 3.1. We omit it. □

We conclude this section with some remarks and examples in order to provide some "*a priori*" motivations of Theorem 3.1 and Theorem 3.2.

Let us begin by trying to explain the progress made with Theorem 3.1, We compare the latter with the more traditional variational principle (1.5). The minimizers of (1.5) are, by definition, σ-harmonic. By Theorem A (see the Introduction), if $\det A > 0$ then these mappings are univalent. The main point, however, is that we prove in [3] that, in general, these mappings are *not* quasiconformal. More precisely, this property depends on the specific conductivity one deals with. In contrast, the new variational principle places no restrictions on the conductivity and yet yields the quasiconformality of the minimizers for a large class of matrices A (see Cor. 3.1). Let us emphasize, that Theorem 3.1 is a slight modification of a similar result obtained by Astala and Miettinen [7] enlarging upon previous work by the second author [43].

We now explain the main difference between the new work and the preceding one. Originally, the second author proved a *variational inequality.* Starting from affine boundary conditions (rather than periodic one) in the classical variational principle (1.5), one obtains an inequality bounding the effective conductivity in terms of the value of the integral on the right hand side of (1.17) evaluated at any quasiconformal mapping in a certain special class called Σ_K (see [43], p. 23, Def. 3.5).

This is a very natural class in the theory of quasiconformal mappings. It was crucial to be able to use this class in the variational inequality because for this class one could make use of a major advance in the field, namely a theorem of Astala [6] on the *area distortion* properties of these mappings. In [43], these results are used (in conjunction with an important contribution of Eremenko and Hamilton [18]) to establish *optimal bounds* in a G-closure problem which was previously open.

Astala and Miettinen [7] proved that the variational inequality was indeed a variational principle. An issue that remained unanswered in the approach of both those papers was that quasiconformal mappings seemed to be somehow introduced by brute force in the problem and only for the special case under consideration namely a mixture of one polycrystal with one isotropic phase in prescribed volume fraction.

The new variational principle resolves satisfactorily this issue. Indeed it shows that, starting with periodic boundary condition, the analogue of the Astala-Miettinen variational principle delivers minimizers that are globally quasiconformal mappings. This depends neither upon the specific choice of the conductivity under consideration nor on the possible optimality of the specific microgeometry for the G-closure problem under consideration. In this respect the new variational principle achieves a geometric rephrasement of any two-dimensional conductivity G-closure problem.

A corresponding statement for the variational principle in [7] is presently unavailable.

The issue of whether it is enough to consider a narrower subclass of quasiconformal mappings in order to address optimality issues in G-closure problem is an interesting but yet rather unexplored one.

We now explain the role of our new variational principle in the context of G-closure problems by considering an example. We go back to the well understood two phase problem defined in the Introduction. Our goal is to show that our variational principle delivers the known optimal bound. Along the way we will give the motivation to Theorem 3.2.

For simplicity, we begin making the assumption that the *composite is isotropic*, *i.e.* we assume that there exist a real number h such that

$$\sigma_{\rm hom} = hI. \tag{3.43}$$

The following is the simplest possible consequence of (1.17). It is obtained by considering an arbitrary matrix A and choosing $\phi \equiv Ax$ as a test field.

Proposition 3.1. *Let $K > 1$ and $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ be given. Assume that the homogenized conductivity $\sigma_{\rm hom}$ is isotropic. Then it satisfies the following family of bounds. For every $\lambda \in (-d_m, d_m)$, for every $A \in \mathcal{M}$ one has*

$$\frac{h|A|^2 - 2\lambda \det A}{h^2 - \lambda^2} \le \int_Q \frac{{\rm tr}[A\sigma(y)A^T] - 2\lambda \det A}{\det \sigma(y) - \lambda^2} {\rm d}y. \tag{3.44}$$

Next we make the following crucial observation.

Proposition 3.2. *In the two phase problem for any choice of $\lambda \in (-d_m, d_m)$ and $A \in \mathcal{M}$, the bound delivered by (3.44) is not "attainable". The same is true under the much weaker assumption that $\sigma(y) = \alpha(y)I$ for some scalar function $\alpha \in L^\infty(R^2; [K^{-1}, K])$ taking a finite (say N) number of values.*

Sketch of the proof

Step 1. Fix $\lambda \in (-d_m, d_m)$. Optimization over the matrix A shows that the tighter bound is obtained either when $A = A^+ \neq 0$ and $A^- = 0$ or conversely when $A = A^- \neq 0$ and $A^+ = 0$.

Step 2. One checks that there is then no loss of generality in choosing $A = I$ and $\lambda \geq 0$.

Step 3. Let $\alpha = \sum_i \alpha_i \chi_i(x) I$. As usual we make the assumptions

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N, \ \int_Q \chi_i(x){\rm d}x = p_i > 0, \ i = 1, 2, \cdots, N, \ \sum_{i=1}^N p_i = 1. \tag{3.45}$$

Assume that for $A = I$ and for some $\lambda \in (-d_m, d_m)$ the bound (3.44) holds as an equality. Then, by (1.18) there exists a particular "microgeometry" $(\chi^0_1, \chi^0_2, \cdots, \chi^0_N)$ and there exists a constant matrix B_λ so that the corresponding σ-harmonic periodic mapping U satisfies $\phi_{U,\lambda} = \lambda U + J\tilde{U}$ By the uniqueness of the infimum of the variational principle (1.17), one has $\phi_{U,\lambda} = x$ identically.

Using Lemma 2.1 we solve the latter for DU and obtain

$$DU(x) = \frac{-\lambda I + \mathrm{Adj}(\sigma(x))}{\det \sigma - \lambda^2} = \frac{1}{\alpha(x)+\lambda} I. \tag{3.46}$$

Step 4. If (3.46) holds, then

$$B = \int_Q DU(x)\mathrm{d}x = bI,\ b = \int_Q \frac{\mathrm{d}x}{\alpha(x)+\lambda} \tag{3.47}$$

and

$$\det B = b^2 = \int_Q \det DU(x)\mathrm{d}x = \int_Q \frac{\mathrm{d}x}{(\alpha(x)+\lambda)^2}\,\cdot \tag{3.48}$$

Then using (3.47, 3.48, 3.45) and Jensen's inequality yield a contradiction.

Step 5. One can treat the case of "approximate solutions" to (3.46) corresponding to a weaker notion of "attainability" (sometimes called "optimality"). In this case one would ask for the existence of a sequence of microgeometry for which the difference between the left and the right hand side of (3.44) can be made smaller and smaller. We will not give details here. □

Let us emphasize that our argument is similar to (and inspired by) the analogue so called "two-gradient" problem in which one asks for necessary and sufficient conditions under which the differential inclusion

$$DU \in \mathcal{K}$$

can be satisfied. The set $\mathcal{K}$ in that case is just the union of two given matrices. Then it is well known that a necessary condition to solve the problem in any reasonably weak sense is that the rank of the difference between the two given matrices be less or equal than one. This result was first obtained by Ball and James [9]. See [38] for an introduction to this subject.

Proposition 3.2 gives a strong motivation to pass to the limit in (3.44) sending λ to d_m. However, there are two potential obstructions to this. The first one is that, if the set Q_{d_m} where $\sigma(y) = d_m I$ has positive measure, then in the right hand side the latter limit is plus infinity *unless* one requires simultaneously

$$\mathrm{tr}[A\sigma(y)A^T] - 2d_m \det A = 0,\ \forall y \in Q_{d_m}. \tag{3.49}$$

Since, in the set Q_{d_m} we have $\sigma(y) = d_m I$, if $|Q_{d_m}| > 0$, equation (3.49) holds if and only if $A^T A = \det A\ I$ *i.e.* if and only if $A \in \mathcal{H}^+$!

Under the latter assumption the bound (3.44) is independent of the specific (nonzero) holomorphic matrix A because $|A|^2 = 2\det A$ and it reads:

$$\frac{1}{h+d_m} \le \int_Q \frac{\mathrm{d}y}{\sigma(y)+d_m}\,\cdot \tag{3.50}$$

This is the the so-called Hashin-Shtrikman bound which is optimal in this particular example. The second potential obstruction is similar but concerns the right hand side of (3.44). This explains why (1.26) is important in the statement of the theorem.

In fact the above calculation can be made without requiring (3.43). The result is similar. In the two-phase problem, the bound is not optimal for $\lambda \in (0, d_m)$ (For $\lambda = 0$ is is optimal at exactly the (harmonic mean,arithmetic mean) point in eigenvalue space.) If we send λ to d_m the bound is non trivial if and only if $A \in \mathcal{H}^+$. In this case the bound does not depend on the specific (nonzero) matrix A and it reads

$$\frac{\mathrm{tr}(\sigma_{\mathrm{hom}}) - 2d_m}{\det \sigma_{\mathrm{hom}} - d_m^2} \leq \int_Q \frac{2}{\sigma(y) + d_m} \mathrm{d}y. \tag{3.51}$$

This is the bound of Tartar and Murat [49] and Lurie and Cherkaev [30]. It defines a curve in eigenvalue space. Any point on this curve is optimal provided the eigenvalues of σ_{hom} satisfy simultaneously the harmonic and arithmetic mean bound. The latter condition, in two dimensions, is automatically satisfied if one uses the upper bound which can be found with the same technique via the usual "duality" argument.

Let us summarize the main points.

First the "best" bound with respect to optimization over λ does not exists. One needs to "push" λ to its "extremal value" $\lambda = d_m$.

Second, in doing so, one needs to satisfy (3.49).

Now we are in the position to explain why a dramatic change must occur if we suppose that in the set Q_{d_m} (assumed to be of positive measure) the conductivity σ is not proportional to the identity and therefore not constant.

In fact, this condition includes many different situations. In any of them in the set Q_{d_m} the conductivity has the form $R^T(x)\mathrm{diag}(\alpha_1, \alpha_2)R(x)$ where $x \to R(x)$ is a measurable field of matrices in $SO(2)$ (which is called a *rotation* of the original *crystal*) and $0 < \alpha_1 < \alpha_2$ are called principal conductivities. This class of conductivities correspond to the situation that, in material science, goes under the name of polycrystalline materials.

Then the corresponding condition (3.49) *cannot be satisfied* (for an arbitrary field of rotations R) by choosing A to be a constant matrix! This gives the exact limitation of what we have called in the introduction the conventional translation method which, as one can check, requires that a *constant* field be optimal in the variational principle (1.17) in the limit as λ tends to d_m.

Theorem 3.2 was motivated by these arguments. We will show in Section 4, Example 4.2 that this approach can deliver optimal bounds in cases where the conventional translation principle does not. Here we conclude with an application of Theorem 3.2.

Example 3.1. We apply Theorem 3.2 to the two-phase problem. Select again $\phi = Ax$. The condition $\phi \in \mathcal{B}(A, \sigma, d_m)$ implies $A \in \mathcal{H}^+$! Moreover the resulting bound is independent of the specific choice of $A \in \mathcal{H}^+$. It is the same as (3.51) and gives the optimal bound.

4. Applications to the G-closure problem

In this section we prove Theorem 4.1 which delivers bounds for two-dimensional G-closure problems. This is an application of Theorems 3.1 and 3.2. This new bound is a generalization of the results in [42, 43] and [7] which unifies these previous work.

We will also give some examples comparing our bounds with the conventional translation method (see the Introduction) and outlining why our progress is interesting. We will then review some basic literature. We conclude this section recalling that the conventional translation method is related to a bound via "polyconvexification" of a suitable function. For this reason our bound is an improvement in the search for quasiconvex functions.

Theorem 4.1. *Let $K > 1$ be given and let $\lambda \geq 0$. For $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, the homogenized conductivity σ_{hom} satisfies the following inequality. For every matrix $A \in \mathcal{M}$ one has*

$$f^{*,+}(A, \sigma_{\mathrm{hom}}, \lambda) \leq \inf_{\phi \in \mathcal{B}(A,\sigma,d_m)} \int_Q f^{*,+}(\mathrm{Adj}D\phi(y), \sigma(y), \lambda)\mathrm{d}y \tag{4.1}$$

where $f^{*,+}$ is defined in (3.17) and $\mathcal{B}(A, \sigma, d_m)$ is defined in (1.24).

Proof. It is very similar to the part of Theorem 3.2 where the inequality "$\leq$" is shown. Let us just outline the necessary changes. Use the same argument with an arbitrary $\lambda \geq 0$. After formula (3.38) the argument has to be changed. By definition, the maximum of the left hand side of (3.38) is given by $f^{*,+}(A, \sigma_{\hom}, \lambda)$. This concludes the proof. □

Remark 4.1. It can be checked that the reverse inequality ("$\geq$") in (4.1) is false in general.

Remark 4.2. Theorem 3.1 asserts that for $\lambda \in [0, d_m)$ equality occurs in (4.1) and moreover the left and the right hand sides of (4.1) agree with the left and the right hand sides of (1.17) respectively. Corollary 3.1 shows that for $A \in m(\sigma_{\hom}, \lambda)$ the minimizers are, in addition, quasiconformal mappings.

Theorem 3.2 asserts that for $\lambda = d_m$ and $A \in m(\sigma_{\hom}, d_m)$ equality occurs in (4.1) and moreover the left hand side of (4.1) is the same as the left hand side of (1.27). Corollary 3.2 shows, in addition, that the minimizers are quasiconformal mappings.

We will see in the next application that also the case $\lambda > d_m$ is very useful.

Example 4.1 (Three-phase problem)**.** Given three real numbers $0 < \sigma_1 < \sigma_2 < \sigma_3$ and three positive numbers p_i with $\sum_i p_i = 1$, assume that $\sigma(x) = \alpha(x)I$ with

$$\alpha(x) = \sum_{i=1}^{3} \chi_i(x)\sigma_i, \quad \int_Q \chi_i = p_i > 0, \; i = 1, 2, 3. \tag{4.2}$$

We now apply Theorem 4.1 with $\lambda = \sigma_1$. To give a bound we restrict attention to test fields of the form $\phi = Ax$. Then one easily checks that $\phi \in \mathcal{B}(A, \sigma, \lambda)$ and $p_1 > 0$ constrain the matrix A to belong to $\mathcal{H}^+$. Then it is easy to check that the bound is independent of the specific matrix $A \in \mathcal{H}^+$ (see Ex. 3.1 for more details) which can therefore set to be the identity. We obtain the bound

$$\begin{aligned} \frac{\operatorname{tr}(\sigma_{\hom}) - 2\sigma_1}{\det \sigma_{\hom} - \sigma_1^2} \leq \int_Q f^{*,+}(I, \sigma(x), \sigma_1) &= \int_Q \left[\chi_1(y)\frac{1}{\sigma_1} + \chi_2(y)\frac{2}{\sigma_1 + \sigma_2} + \chi_3(y)\frac{2}{\sigma_2 + \sigma_3} \right] \mathrm{d}y \\ &= \frac{p_1}{\sigma_1} + \frac{p_2}{\sigma_2} + \frac{2p_3}{\sigma_1 + \sigma_3}, \end{aligned} \tag{4.3}$$

where $f^{*,+}$ is defined by (3.17).

This is the Hashin–Shtrikman–Kohn–Milton [23,36] bound. It is optimal (at least for isotropic composites) *provided one imposes extra conditions* among the parameters in (4.2). However, if p_1 is sufficiently small the bound *is not* optimal.

To explore the latter (suboptimal) regime let us make a preliminary observation. It is known (see [36] and [42]) that, if the composite is isotropic and the parameters in (4.2) are such that the bound (4.3) is suboptimal, then one may assume without loss of generality that $h > \sigma_2$.

We now use Theorem 4.1 with the choice $\lambda = \sigma_2$. If we restrict attention to test fields of the form $\phi = Ax$, then $\phi \in \mathcal{B}(A, \sigma, \lambda)$ and $p_2 > 0$ imply again $A \in \mathcal{H}^+$. Then the bound is again independent of the specific matrix A which can therefore set to be the identity. Restricting attention to isotropic composites we obtain the bound

$$\begin{aligned} f^{*,+}(I, hI, \sigma_2) = \frac{\operatorname{tr}(hI) - 2\sigma_2}{\det(hI) - \sigma_2^2} &= \frac{2}{h + \sigma_2} \leq \int_Q f^{*,+}(I, \sigma(x), \sigma_2) \\ &= \int_Q \left[\chi_1(y)\frac{1}{\sigma_1} + \chi_2(y)\frac{1}{\sigma_2} + \chi_3(y)\frac{2}{\sigma_2 + \sigma_3} \right] \mathrm{d}y = \frac{p_1}{\sigma_1} + \frac{p_2}{\sigma_2} + \frac{2p_3}{\sigma_2 + \sigma_3}, \end{aligned} \tag{4.4}$$

where the first equality is a consequence of the assumption $h > \sigma_2$.

In this particular example, the bound (4.4) is the same as that established in [42]. It is better than the usual Hashin–Shtrikman bound when p_1 is sufficiently small. This bound is not optimal unless $p_1 = 0$ or σ_3 tends to infinity. Nevertheless it is better than any other known bound.

Example 4.2 (Polycrystals)**.** We remove the assumption that σ is (locally) isotropic. For instance assume that σ has the following structure

$$\sigma(x) = \sigma_2\chi(x)I + \sigma_1 R^T(x)\mathrm{diag}(K^{-1}, K)R(x)(1-\chi(x))$$

with $0 < \sigma_1 < \sigma_2$ and $K > 1$.

Recalling (2.1), in this class we have

$$K^{-1}\sigma_1 = \mathrm{ess}\sup_{x\in Q_m}\sigma_1(x) < d_m = \sigma_1 < \mathrm{ess}\inf_{x\in Q_m}\sigma_2(x) = K\sigma_1. \tag{4.5}$$

Applying Theorem 3.1 and Corollary 3.2, a calculation shows that, in the set Q_{d_m}, the dilatation quotient of the minimizers is constant and equal to $K > 1$!

We conclude that the test field *must* be searched for in the class of quasiconformal mappings having prescribed dilatation quotient in the set Q_{d_m}. More precisely, the natural space is a subset of the space of quasiconformal mappings $\phi \in W^{1,2}_{\sharp,A}$ which in addition satisfy

$$D\phi^T D\phi = \left(\frac{\sigma}{\sqrt{\det\sigma}}\right)^{-\frac{1}{2}} \det D\phi \text{ a.e. in } Q_{d_m}, \tag{4.6}$$

i.e. $\phi \in \mathcal{B}(A, \sigma, d_m)$!

The new difficulty now relies in computing an upper bound for the right hand side of (1.27) for such a mapping. However this approach delivers optimal bounds! We refer to [7,43] and [37] for details. In [8] optimal bounds for a much larger class of composites are obtained using further refinements of these ideas. Let us just say that in *all* these cases one has to use in an essential way the fundamental work of Astala [6].

The results of this section should be seen as a continuation of the work of many authors. Let us quote some very relevant literature. The simplest case is the *single-phase polycrystal problem*.

Here the only data is $K > 1$. The conductivity has the form $R^T(x)\mathrm{diag}(K, K^{-1})R(x)$ where $x \to R(x)$ is a measurable field of matrices in $SO(2)$ (which is called a *rotation* of the original *crystal*). Obviously $\sqrt{\det\sigma} = 1$ at almost every point, hence $d_m = 1$ and $|Q_m| = 1$. The corresponding classical literature includes [17,24] and [33] and it is based on the idea of *duality*. In the language we introduced before these are the first papers in the field of composites where the idea of *stream function* shows its power. Next the *two-phase polycrystal problem*. Here the data are two *crystals*, *i.e.* two constant diagonal and positive matrices σ_a and σ_b, (giving the conductivity of the pair of given crystals). To distinguish this example from the first we assume $\det\sigma_a \neq \det\sigma_b$.

The conductivity has the form $\sigma(x) = R^T(x)(\sigma_a\chi_a + \sigma_b\chi_b)R(x)$, where $x \to R(x)$ is a measurable field of matrices in $SO(2)$ and χ_a and χ_b are characteristic functions summing up to one. Phase a is characterized as the set where the principal conductivities of σ and those of σ_a are the same.

If the volume fraction of each phase is not prescribed, the problem is simpler but non trivial. Its study was initiated in [31] and completed later by Francfort and Murat [20] in an interesting paper which, unfortunately, has not been fully appreciated.

Several papers deal with examples using various form of duality including [25,44] and [45]. Some other work partly focusing on the case with prescribed volume fraction, uses duality in conjunction with more refined arguments [19,22] and [41] (Sect. 7.3). More recently, quasiconformal mapping are having an increasing impact on the two-phase polycrystalline problem [7,37,43] and [8].

Finally the *several-phase problem*. Here each phase is isotropic but there is an arbitrary finite number of them. This problem has been considered already by Hashin and Shtrikman (focusing on isotropic composites). Later by Milton [34], by Kohn and Milton [36] (see also [27]) and by Zhikov [53]. Further progress has been made by Lurie and Cherkaev [32], Gibiansky and Cherkaev [16], Cherkaev [15] (see pp. 319-342) and Gibiansky and Sigmund [21]. The best available bounds were established in [42] and the best known microgeometries stem at least for the three-phase problem from a combination of the work in [21] and [15].

We conclude this section recalling the connection between our work and the literature concerning certain problems of optimal design and the more general issue of search for quasiconvex functions.

Following the pioneering work of Kohn and Strang [28] (see also [26]) several authors have used the knowledge of the G-closure in certain specific cases to compute the "quasiconvexification" of certain functions. A typical example is a function $f = \min(f_1, f_2)$ where f_1 and f_2 are quadratic functions.

The quasiconvexification can be computed using results from the G-closure of the two-phase problem. This has been studied in [28] in a limiting case and in a greater generality by Allaire and Francfort [4] and Allaire and Lods [5]. Their analysis, in our language, shows that the conventional translation method gives all the necessary information. However, it turns out that, in this particular case, the conventional translation method corresponds to the "polyconvexification" of f and one proves by other means that this coincides with the "rank-one convexification" and hence with the "quasiconvexification".

The study of the quasiconvexification of a function which is a minimum of several (say three) quadratic functions, reduces again itself to the study of a three-phase G-closure problem. The conventional translation method again delivers the polyconvexification of f. Therefore, since our bounds improve upon it, we are effectively giving a bound on f which is strictly tighter than the bound obtained using the polyconvexification of f. Further investigation on this issue is the subject of ongoing work.

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