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SIGN CHANGING SOLUTIONS FOR ELLIPTIC EQUATIONS WITH CRITICAL GROWTH IN CYLINDER TYPE DOMAINS *

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Abstract. We prove the existence of positive and of nodal solutions for $-\Delta u = |u|^{p-2}u + \mu|u|^{q-2}u$, $u \in \mathrm{H}_0^1(\Omega)$, where $\mu > 0$ and $2 < q < p = 2N(N-2)$, for a class of open subsets Ω of $\mathbb{R}^N$ lying between two infinite cylinders.

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Introduction

We are concerned with the existence of nonzero solutions for the nonlinear second order elliptic equation

$$-\Delta u = |u|^{p-2}u + \mu|u|^{q-2}u, \qquad u \in \mathrm{H}_0^1(\Omega), \tag{P}$$

where Ω is a smooth unbounded domain of $\mathbb{R}^N$ with $N \geq 3$, $\mu \in \mathbb{R}^+$, $2 < q < p$ and p is the critical Sobolev exponent $p = 2^* = 2N/(N-2)$. Without loss of generality we assume that $0 \in \Omega$.

In the case where Ω is bounded, the proof of the existence of positive and of nodal (sign changing) solutions for (P) or similar equations goes back to the work in [3, 4, 10]. In the case where Ω is unbounded and p is subcritical ($p < 2^*$), we refer for example to [5, 12]. On the other hand, motivated by the work in [1, 2, 5, 7], in [8] the authors prove the existence of a positive solution for a class of unbounded domains, concerning the (somewhat simpler) equation $-\Delta u = \lambda u + |u|^{p-2}u$, where λ is positive and small (see also [9] for a related result).

The present work complements the quoted results. Following [5, 8], we fix a number $1 \leq \ell \leq N-1$ and write $\mathbb{R}^N = \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$, $z = (t, y) \in \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$. For a given subset $A \subset \mathbb{R}^{N-\ell}$ we denote $A_\delta = \{y \in \mathbb{R}^{N-\ell} : \operatorname{dist}(y, A) < \delta\}$ and $\widehat{A} = \mathbb{R}^\ell \times A$. Also, for $t \in \mathbb{R}^\ell$ we let $\Omega^t = \{y \in \mathbb{R}^{N-\ell} : (t, y) \in \Omega\}$. We shall consider both situations (H) and $(\mathrm{H})_0$ below:

(H) there exist two nonempty bounded open sets $F \subset G \subset \mathbb{R}^{N-\ell}$ such that F is a Lipschitz domain and $\widehat{F} \subset \Omega \subset \widehat{G}$. Moreover, for each $\delta > 0$ there is $R > 0$ such that $\Omega^t \subset F_\delta$ for all $|t| \geq R$;

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$(\mathrm{H})_0$ there exists an open bounded set $G \subset \mathbb{R}^{N-\ell}$ such that $\Omega \subset \widehat{G}$ and moreover for each $\delta > 0$ there is $R > 0$ such that $\Omega^t \subset B_{\mathbb{R}^{N-\ell}}(0, \delta)$ for all $|t| \geq R$.

We have denoted by $B_{\mathbb{R}^{N-\ell}}(0, \delta)$ the open ball in $\mathbb{R}^{N-\ell}$ centered at the origin with radius $\delta > 0$. The case $(\mathrm{H})_0$ can be seen as a limit case of (H), with $F = \{0\}$. We prove the following:

Theorem 1. *Consider problem* (P) *with* $2 < q < p = 2^*$ *and assume either* (H) *or* $(\mathrm{H})_0$. *Then, for every* $\mu > 0$, *the problem admits a positive (and a negative) solution of least energy.*

In order to prove the existence of nodal solutions in case (H), we impose further restrictions on Ω, namely that Ω approaches $\widehat{F}$ "smoothly and slowly".

$(\mathrm{H})'$ Assume (H) and that Ω is of class $\mathrm{C}^{1,1}$ in such a way that the local charts as well as their inverses have uniformly bounded Lipschitz constants. Moreover, there exist constants $m > 0$ and $0 < a_1 < a_0$ such that $\left(1 + \frac{a}{|t|^m}\right) F \subset \Omega^t$ for every $a \in [a_1, a_0]$ and every $|t|$ large.

Theorem 2. *Consider problem* (P) *with* $2 < q < p = 2^*$ *and assume either* $(\mathrm{H})'$ *or* $(\mathrm{H})_0$. *In case* $(\mathrm{H})_0$ *holds, assume moreover that* $q > (N+2)/(N-2)$. *Then, for every* $\mu > 0$, *the problem admits a sign changing solution.*

In Theorem 2 the conclusion is that (P) has a pair of sign changing solutions, since the nonlinearity is odd. In case $(\mathrm{H})_0$, the extra restriction on q is merely needed in lower dimensions ($N = 3, 4, 5$), since $(N+2)/(N-2) \geq 2$ for $N \geq 6$. In fact, Theorem 2 still holds if $q = (N+2)/(N-2)$ provided μ is sufficiently large (see the remark which follows the proof of Prop. 2.5).

The proof of our main theorems is given in Section 2 (see Props. 1.1 and 1.4); it relies on the concentration-compactness principle at infinity and on some ideas of [4, 8]. Section 3 provides technical estimates which are needed in the proof of Theorem 2. We also give further information on the decay properties of the solutions found in Theorems 1 and 2.

1. Concentration-compactness

It is well known that the solutions of (P) correspond to critical points of the energy functional (for simplicity of notations, we take $\mu = 1$ in (P)):

$$I(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p} \int |u|^p - \frac{1}{q} \int |u|^q, \qquad u \in \mathrm{H}_0^1(\Omega),$$

where the integrals are taken over the domain Ω. We recall $2 < q < p = 2^*$. It follows from assumptions (H) or $(\mathrm{H})_0$ that we can choose the norm $||u|| := \left(\int |\nabla u|^2\right)^{1/2}$ in $\mathrm{H}_0^1(\Omega)$. Let

$$c_0 := \inf\{I(u) : \ u \in \mathrm{H}_0^1(\Omega), \ u \neq 0 \text{ and } I'(u)u = 0\} \cdot \tag{1.1}$$

It is also clear that $c_0 > 0$ and that every nonzero critical point u of I is such that $I(u) \geq c_0$. The following result proves Theorem 1.

Proposition 1.1. *Under assumptions* (H) *or* $(\mathrm{H})_0$, *the infimum in* (1.1) *is attained in a critical point of* I.

Proof. 1. We shall omit what concerns standard arguments (*cf.* [3, 4]). We first recall that there exists a Palais–Smale sequence $(u_n) \subset \mathrm{H}_0^1(\Omega)$ at level c_0, namely

$$I(u_n) \to c_0 \qquad \text{and} \qquad I'(u_n) \to 0. \tag{1.2}$$

Since moreover $c_0 > 0$, equation (1.2) implies that $\liminf ||u_n|| > 0$. This sequence is bounded and, up to a subsequence, $u_n \rightharpoonup u$ weakly in $\mathrm{H}_0^1(\Omega)$, $u_n(x) \to u(x)$ a.e. and $I'(u) = 0$, $I(u) \geq 0$. Since $\liminf ||u_n|| > 0$ and

$I'(u_n)u_n \to 0$, we also have that $\liminf \int |u_n|^p > 0$; indeed, if $\int |u_n|^p \to 0$ along a subsequence, then, since $(\int u_n^2)$ is bounded, by interpolation $\int |u_n|^q \to 0$, whence $||u_n|| \to 0$, as $I'(u_n)u_n \to 0$.

2. Up to subsequences, there exist measures μ and ν on Ω such that $|\nabla(u_n - u)|^2 \rightharpoonup \mu$ and $|u_n - u|^p \rightharpoonup \nu$ weakly in the space $M(\Omega)$ of finite measures in Ω. Clearly, $||\mu|| \geq S||\nu||^{2/p}$, where S is the best constant for the embedding $\mathrm{H}^1(\mathbb{R}^N) \subset \mathrm{L}^p(\mathbb{R}^N)$. By testing $I'(u_n) \to 0$ with $u_n\varphi$ for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and since $I'(u)u\varphi = 0$ we also see that

$$||\mu|| = ||\nu||. \tag{1.3}$$

In particular,

$$\mu \neq 0 \Rightarrow ||\mu|| \geq S^{p/(p-2)} = S^{N/2}. \tag{1.4}$$

3. Define

$$\begin{aligned} \mu_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2, \\ \nu_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^p, \\ \eta_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^q. \end{aligned}$$

Again, it is clear that

$$\mu_\infty \geq S\, \nu_\infty^{2/p}. \tag{1.5}$$

By testing $I'(u_n) \to 0$ with $u_n\psi_R$ $(R > 0)$ where $\psi_R \in \mathrm{C}^\infty(\Omega)$, $0 \leq \psi_R \leq 1$ is such that $\psi_R(x) = 0$ if $|x| \leq R$ and $\psi_R(x) = 1$ if $|x| \geq R + 1$, it follows easily that

$$\mu_\infty = \nu_\infty + \eta_\infty. \tag{1.6}$$

4. We recall from [1, 2, 11] that

$$\begin{aligned} \int |\nabla u_n|^2 &= \int |\nabla u|^2 + ||\mu|| + \mu_\infty + \mathrm{o}(1), \\ \int |u_n|^p &= \int |u|^p + ||\nu|| + \nu_\infty + \mathrm{o}(1), \\ \int |u_n|^q &= \int |u|^q + \eta_\infty + \mathrm{o}(1). \end{aligned}$$

As a consequence, and thanks to (1.2, 1.3) and (1.6), we have that

$$c_0 = I(u) + \left(\frac{1}{2} - \frac{1}{p}\right) ||\mu|| + \left(\frac{1}{2} - \frac{1}{p}\right) \nu_\infty + \left(\frac{1}{2} - \frac{1}{q}\right) \eta_\infty. \tag{1.7}$$

In particular, $c_0 \geq I(u)$. Since $I'(u) = 0$, the proof will be complete once we show that $u \neq 0$. Indeed, in this case we have that $I(u) \geq c_0$, whence $I(u) = c_0$. (Incidentally, Eqs. (1.6) and (1.7) also show that, in fact, $||\mu|| = \mu_\infty = 0$, hence $u_n \to u$ in $\mathrm{H}_0^1(\Omega)$.)

5. We recall from [3] that $c_0 < S^{N/2}/N$. Since (1.7) implies that

$$c_0 \geq \left(\frac{1}{2} - \frac{1}{p}\right) ||\mu|| = \frac{1}{N}\, ||\mu||,$$

we deduce from (1.3, 1.4) that $\mu = \nu = 0$. Thus $u_n \to u$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega)$ and

$$c_0 = I(u) + \left(\frac{1}{2} - \frac{1}{p}\right) \nu_\infty + \left(\frac{1}{2} - \frac{1}{q}\right) \eta_\infty. \tag{1.8}$$

6. Suppose first that $\Omega = \widehat{F}$. Since $\liminf \int |u_n|^p > 0$, by Lemma 2.1 in [8] we may assume that, up to translations, $\int_{B_1(0)} |u_n|^p \geq c$ for some $c > 0$. Since $u_n \to u$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega)$, we conclude that $u \neq 0$ and this proves Proposition 1.1 for the case $\Omega = \widehat{F}$. Moreover, the argument shows that $c_0(\widehat{F_\delta}) \to c_0(\widehat{F})$ as $\delta \to 0$ (see (H) and (1.12) for the notations).

7. We complete the proof in case $(\mathrm{H})_0$ holds. Assume by contradiction that $u = 0$. Then, clearly $\int u_n^2 \to 0$ (see *e.g.* (2.1) in [8]). By interpolation, also $\int |u_n|^q \to 0$. In particular, $\eta_\infty = 0$. Since $c_0 < S^{N/2}/N$, equations (1.5, 1.6) and (1.8) show that then $\nu_\infty = 0$, whence, by the second identity in Step 4, $\int |u_n|^p \to 0$. This contradicts the fact that $\liminf \int |u_n|^p > 0$ and proves Proposition 1.1 under $(\mathrm{H})_0$.

8. At last, we consider the case where (H) holds and $\Omega \neq \widehat{F}$. Again, assume by contradiction that $u = 0$. Let $\delta > 0$ be given and take $R > 0$ according to assumption (H). Let ψ_R be as in Step 3 and denote

$$v_n = u_n \psi_R \in \mathrm{H}^1_0(\widehat{F_\delta}).$$

Since $u_n \to 0$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega)$, clearly we have that

$$I(v_n) = I(u_n) + \mathrm{o}(1) \qquad \text{and} \qquad I'(v_n)v_n = \mathrm{o}(1). \tag{1.9}$$

We claim that

$$I(v_n) + \mathrm{o}(1) \geq c_0(\widehat{F_\delta}). \tag{1.10}$$

Assuming the claim for a moment, it follows from (1.9, 1.10) that

$$c_0 = I(u_n) + \mathrm{o}(1) = I(v_n) + \mathrm{o}(1) \geq c_0(\widehat{F_\delta}).$$

Since $\delta > 0$ is arbitrary, we conclude that $c_0 \geq c_0(\widehat{F})$. On the other hand, since $\widehat{F} \subset \Omega$ and $c_0(\widehat{F})$ is attained (see Step 6 above), we must have that $c_0 < c_0(\widehat{F})$. This contradiction completes the proof.

It remains to prove the inequality in (1.10). For this, we observe that (1.9) together with the fact that $\liminf I(u_n) > 0$ implies that $\liminf ||v_n|| > 0$ and $\liminf \int |v_n|^p > 0$. Now, let

$$w_n = t_n v_n \qquad (t_n > 0)$$

be such that $I'(w_n)w_n = 0$; namely, t_n is given by

$$\frac{t_n^{p-2} \int |v_n|^p + t_n^{q-2} \int |v_n|^q}{\int |\nabla v_n|^2} = 1.$$

Then (t_n) is bounded and, since $I'(v_n)v_n \to 0$, we see that $t_n \to 1$. In particular,

$$I(w_n) = I(v_n) + \mathrm{o}(1). \tag{1.11}$$

Now, by definition, $I(w_n) \geq c_0(\widehat{F_\delta})$ and (1.10) follows from (1.11). □

Using the notation in assumption (H), we denote

$$c_0(\widehat{F}) := \inf\{I(u) : \ u \in \mathrm{H}_0^1(\widehat{F}),\ u \neq 0 \text{ and } I'(u)u = 0\} < S^{N/2}/N. \tag{1.12}$$

We also let

$$c_0^\infty := c_0(\widehat{F}) \quad \text{in case (H)}, \qquad c_0^\infty := S^{N/2}/N \quad \text{in case } \mathrm{(H)}_0. \tag{1.13}$$

We have shown in the proof of Proposition 1.1 that $c_0(\widehat{F})$ is attained by a critical point of the energy functional in $\mathrm{H}_0^1(\widehat{F})$. In fact, the argument above yields the following compactness result.

Proposition 1.2. *Under assumptions* (H) *or* $\mathrm{(H)}_0$*, let* $(u_n) \subset \mathrm{H}_0^1(\Omega)$ *be such that*

$$\limsup I(u_n) < c_0^\infty \qquad \text{and} \qquad I'(u_n)(u_n\psi) \to 0 \tag{1.14}$$

for every $\psi \in \mathrm{C}^\infty(\Omega) \cap W^{1,\infty}(\Omega)$*. Suppose* $u_n \rightharpoonup u$ *weakly in* $\mathrm{H}_0^1(\Omega)$*,* $u_n(x) \to u(x)$ *a.e. and* $I'(u)(u\psi) = 0$ *for such functions* ψ*. Then* $u_n \to u$ *in* $\mathrm{H}_0^1(\Omega)$*.*

Proof. Since $I'(u)u = 0$, we have that $I(u) \geq 0$. Denote $v_n := u_n - u$. By the Brezis–Lieb lemma,

$$I(v_n) = I(u_n) - I(u) + \mathrm{o}(1) < c_0^\infty + \mathrm{o}(1)$$

and

$$I'(v_n)(v_n\psi) = I'(u_n)(u_n\psi) - I'(u)(u\psi) + \mathrm{o}(1) \to 0$$

for every $\psi \in \mathrm{C}^\infty(\Omega) \cap W^{1,\infty}(\Omega)$. Since (v_n) converges weakly to zero, a similar (though easier) argument as in the proof of Proposition 1.1 shows that we cannot have $\limsup I(v_n) > 0$. Thus $I(v_n) \to 0$. Since also $I'(v_n)v_n \to 0$, we conclude that $||v_n|| \to 0$, hence $u_n \to u$ in $\mathrm{H}_0^1(\Omega)$. □

Next we turn to the proof of Theorem 2. Following [4], let

$$c_1 := \inf\{I(u) : \ u \in \mathrm{H}_0^1(\Omega),\ u^\pm \neq 0 \text{ and } I'(u^\pm)u^\pm = 0\} \geq c_0 > 0, \tag{1.15}$$

where we denote $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The following proposition will be proved in Section 3 (*cf.* Props. 2.4 and 2.5).

Proposition 1.3. *Assume* $\mathrm{(H)}'$ *or* $\mathrm{(H)}_0$ *holds; in the latter case, we also assume that* $q > (N+2)/(N-2)$*. Then*

$$c_1 < c_0 + c_0^\infty.$$

Our final result completes the proof of Theorem 2.

Proposition 1.4. *Assume* $\mathrm{(H)}'$ *or* $\mathrm{(H)}_0$ *holds; in the latter case, we also assume that* $q > (N+2)/(N-2)$*. Then the infimum in* (1.15) *is attained in a critical point of* I*.*

Proof. It is known (*cf.* [4]) that there exists a Palais–Smale sequence at level c_1, namely

$$I(u_n) \to c_1 \qquad \text{and} \qquad I'(u_n) \to 0,$$

with the additional property that

$$I(u_n^{\pm}) \geq c_0 + \mathrm{o}(1) \tag{1.16}$$

(so that, in fact, $c_1 \geq 2c_0$). As in Step 1 in the proof of Proposition 1.1, modulo a subsequence, (u_n) converges weakly in $\mathrm{H}_0^1(\Omega)$ and pointwise a.e. to a critical point u of I. Observe that $I'(u_n) \to 0$ implies that

$$I'(u_n^{\pm})(u_n^{\pm}\psi) = I'(u_n)(u_n^{\pm}\psi) \to 0 \tag{1.17}$$

for every $\psi \in \mathrm{C}^\infty(\Omega) \cap W^{1,\infty}(\Omega)$. Similarly, $I'(u^{\pm})(u^{\pm}\psi) = 0$. Since moreover $I(u_n) = I(u_n^+) + I(u_n^-) = c_1 + \mathrm{o}(1)$, we deduce from (1.16) and Proposition 1.3 that

$$\limsup I(u_n^{\pm}) < c_0^{\infty}. \tag{1.18}$$

It follows from (1.17, 1.18) and Proposition 1.2 that $u_n^{\pm} \to u^{\pm}$ in $\mathrm{H}_0^1(\Omega)$. Hence $u_n \to u$ in $\mathrm{H}_0^1(\Omega)$, $I(u) = c_1$ and $I(u^{\pm}) \geq c_0 > 0$. This finishes the proof. □

2. Decay and energy estimates

This section is devoted to general equations of the form

$$-\Delta u - \lambda u = g(u), \qquad u \in \mathrm{H}_0^1(\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open set with $\mathrm{C}^{1,1}$ boundary and g satisfies (recall that $p = 2^* = 2N/(N-2)$)

$$|g(s)| \leq C\ (|s| + |s|^{p-1}), \qquad \forall s \in \mathbb{R}. \tag{2.2}$$

Under assumption (2.2), it follows from the Brezis–Kato estimates and classical elliptic regularity theory that the solutions of (2.1) lie in $\mathrm{C}^2(\Omega) \cap \mathrm{L}^\infty(\Omega) \cap \mathrm{C}(\overline{\Omega})$. In view of the applications that we have in mind (*cf.* assumptions (H)-(H)$_0$), we let $\mathbb{R}^N = \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$ with $1 \leq \ell < N$ and accordingly write $(t, y) \in \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$ for any point $(t, y) \in \mathbb{R}^N$.

Proposition 2.1. *Let $\Omega = \mathbb{R}^\ell \times F$ where $F \subset \mathbb{R}^{N-\ell}$ is a $\mathrm{C}^{1,1}$ domain and let $g \in \mathrm{C}^1(\mathbb{R})$ satisfy (2.2), $g(0) = 0$ and $g'(s) = o(s^\varepsilon)$ near 0, for some $\varepsilon > 0$. Let u be a solution of*

$$-\Delta u - \lambda u = g(u), \qquad u \in \mathrm{H}_0^1(\Omega), \tag{2.3}$$

where $\lambda < \lambda_1$ and λ_1 is the first eigenvalue of $(-\Delta, \mathrm{H}_0^1(F))$. Then

$$|u(t, y)| + |\nabla_t u(t, y)| \leq \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}, \qquad \forall (t, y) \in \Omega, \tag{2.4}$$

where φ is a positive eigenfunction associated to λ_1. Also, there exists a constant $C > 0$ such that

$$|\nabla u(t, y)| \leq C\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}, \qquad \forall (t, y) \in \Omega.$$

Proof. 1. Since $u \in \mathrm{L}^\infty(\Omega)$, we have from (2.2) that $|g(u(x))| \leq c|u(x)|$ for every $x \in \Omega$. By elliptic regularity theory (Th. 9.15 of [6]), there exists $c > 0$ such that, for all $\alpha \geq 2$,

$$||u||_{W^{2,\alpha}(B_1(0)\times F)} \leq c\ ||u||_{\mathrm{L}^\alpha(B_2(0)\times F)}.$$

Due to invariance by translations,

$$||u||_{W^{2,\alpha}(B_1(t)\times F)} \le c\,||u||_{\mathrm{L}^{\alpha}(B_2(t)\times F)} \qquad \forall t \in \mathbb{R}^{\ell}. \tag{2.5}$$

In particular,

$$u(t,y) \to 0 \text{ as } |t| \to +\infty, \text{ uniformly for } y \in F \tag{2.6}$$

and

$$|\nabla u(t,y)| \to 0 \text{ as } |t| \to +\infty, \text{ uniformly for } y \in F. \tag{2.7}$$

2. Suppose $\mu \in]\lambda, \lambda_1[$ is fixed and let

$$\Psi(t) := \alpha \mathrm{e}^{-\sqrt{1+(\lambda_1-\mu)|t|^2}} \in \mathrm{H}^1(\mathbb{R}^{\ell}),$$

where α will be chosen later. An easy computation shows that

$$-\Delta\Psi + (\lambda_1-\mu)\Psi = (\lambda_1-\mu)\,\Psi\,((\ell-1)\theta^{-1/2} + \theta^{-1} + \theta^{-3/2}) \tag{2.8}$$

where $\theta(t) := 1 + (\lambda_1-\mu)|t|^2$. In particular,

$$-\Delta\Psi + (\lambda_1-\mu)\Psi \ge \frac{\alpha(\lambda_1-\mu)}{1+(\lambda_1-\mu)|t|^2}\mathrm{e}^{-\sqrt{1+(\lambda_1-\mu)|t|^2}} =: h(t).$$

Let φ be a positive eigenfunction associated to λ_1 and

$$z(t,y) := \varphi(y)\Psi(t).$$

The function z satisfies

$$-\Delta z - \mu z \ge \varphi(y)h(t).$$

Hence, for $w := z - u$, we have

$$-\Delta w - \mu w \ge \varphi(y)h(t) + (\mu-\lambda)u - g(u) =: k(t,y). \tag{2.9}$$

Since $g(0) = 0 = g'(0)$, it follows from (2.6) that if $u(t,y) \ge 0$, then

$$(\mu-\lambda)u - g(u) \ge 0$$

if $|t| > R$, where R is chosen large; hence also $k(t,y) \ge 0$. In summary,

$$w < 0 \Rightarrow -\Delta w - \mu w \ge 0, \tag{2.10}$$

if $|t| > R$. Since $\partial z/\partial\nu = h\;\partial\varphi/\partial\nu < 0$ (ν stands for the outward normal to $\partial\Omega$), we can fix α so large that $w \ge 0$ for $|t| \le R$. Let $\omega := \{x \in \Omega : w(x) < 0\}$. Since

$$w^-(x) = 0 \qquad \forall x \in \partial\omega,$$

by multiplying (2.9) by w^- and integrating, it follows from (2.10) that $\omega = \emptyset$. Therefore $u \le z$. In the same way we can prove that $-u \le z$, and so

$$|u(t,y)| \le \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\mu)|t|^2}}, \qquad \forall (t,y) \in \Omega; \tag{2.11}$$

the constant α has been incorporated into the function φ.

3. We now improve the previous estimate. Since $g'(s) = o(s^\varepsilon)$, there exists $C > 0$ such that

$$|g(u(t,y))| \le C|u(t,y)|^{1+\varepsilon}, \qquad \forall (t,y) \in \Omega. \tag{2.12}$$

We fix $\mu \in]\lambda, \lambda_1[$, sufficiently close to λ, so that

$$\gamma := (1+\varepsilon)\sqrt{\lambda_1 - \mu} > \sqrt{\lambda_1 - \lambda}.$$

Combining (2.11) and (2.12),

$$|g(u(t,y))| \le C\varphi(y)^{1+\varepsilon}\mathrm{e}^{-\gamma|t|}, \qquad \forall (t,y) \in \Omega. \tag{2.13}$$

Let $z(t,y) := \varphi(y)\Psi(t)$, where Ψ is like in Step 2, with μ replaced by λ. For $w := z - u$, we have

$$-\Delta w - \lambda w \ge \frac{\alpha(\lambda_1 - \lambda)}{1 + (\lambda_1 - \lambda)|t|^2}\varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} - g(u(t,y)) =: p(t,y).$$

Since $\gamma > \sqrt{\lambda_1 - \lambda}$, it follows from (2.13) that $p(t,y) \ge 0$ if $|t|$ is large. Choosing α sufficiently large leads to $p \ge 0$ in Ω. We conclude from the maximum principle, as before, that $u \le z$ in Ω and in the same way, $|u| \le z$ in Ω.

4. To finish the proof we use the decay of u. Specifically, the derivatives $v = \partial u/\partial t_i$, for $i = 1, \dots, \ell$, satisfy

$$-\Delta v - \lambda v = g'(u)v \qquad \text{and} \;\; v \in \mathrm{H}_0^1(\Omega).$$

The argument in Steps 2 and 3 above proves an analogous decay for v. The main point in the final argument is that if $\mu \in]\lambda, \lambda_1[$ is sufficiently close to λ then

$$\frac{\alpha(\lambda_1 - \lambda)}{1 + (\lambda_1 - \lambda)|t|^2}\varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} - C\varphi^\varepsilon(y)\mathrm{e}^{-\varepsilon\sqrt{1+(\lambda_1-\lambda)|t|^2}} \times \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\mu)|t|^2}}$$

is positive for $|t|$ large. The final assertion in the statement of Proposition 2.1 follows from (2.5). ☐

We now consider the setting analyzed in Section 2. Again, we denote by $\lambda_1 = \lambda_1(F)$ the first eigenvalue of $(-\Delta, \mathrm{H}_0^1(F))$.

Proposition 2.2. *Suppose Ω is a domain satisfying assumption* (H) *and moreover that Ω is of class $\mathrm{C}^{1,1}$ in such a way that the local charts as well as their inverses have uniformly bounded Lipschitz constants. Let $g \in \mathrm{C}^1(\mathbb{R})$ be as in Proposition* 2.1 *and u be a solution of*

$$-\Delta u - \lambda u = g(u), \qquad u \in \mathrm{H}_0^1(\Omega),$$

with $\lambda < \lambda_1$. Then, for each $\overline{\lambda} \in]\lambda, \lambda_1[$, there exists a constant $C > 0$ such that

$$|u(t,y)| + |\nabla u(t,y)| \le C\mathrm{e}^{-\sqrt{1+(\overline{\lambda}-\lambda)|t|^2}}, \qquad \forall (t,y) \in \Omega.$$

Proof. The proof is similar to that of Proposition 2.1, so we just stress the differences. Thanks to our assumption on Ω, the constant c in (2.5) can be taken uniformly bounded, hence (2.6) still holds. Now, fix $\delta > 0$ in such a way that $\lambda < \lambda_1(F_\delta) < \lambda_1$. Running through the argument in Step 2 of the proof of Proposition 2.1 we see that, similarly to (2.11),

$$|u(t,y)| \le \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1(F_\delta)-\mu)|t|^2}}, \qquad \forall (t,y) \in \Omega,\ |t| \ge R,$$

provided $R > 0$ is sufficiently large; here, $\mu \in]\lambda, \lambda_1(F_\delta)[$ and φ is an eigenfunction associated to $\lambda_1(F_\delta)$. Arguing as in Step 3 of the quoted proof, the previous estimate for u can be improved to

$$|u(t,y)| \le \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1(F_\delta)-\lambda)|t|^2}}, \qquad \forall (t,y) \in \Omega,\ |t| \ge R.$$

This clearly implies that we can choose $C > 0$ such that

$$|u(t,y)| \le C\mathrm{e}^{-\sqrt{1+(\lambda_1(F_\delta)-\lambda)|t|^2}}, \qquad \forall (t,y) \in \Omega. \tag{2.14}$$

A similar decay estimate for the derivatives of u follows from (2.5) and (2.14). Since $\lambda_1(F_\delta)$ can be chosen arbitrarily close to λ_1 (see Lem. 2.3 of [8]), this proves the proposition. ☐

Going back to Proposition 2.1, it may be interesting to observe that the asymptotic estimates can be sharpened as follows:

Proposition 2.3. *Under the assumptions of Proposition* 2.1*, let* u *be a solution of problem* (2.3)*. Then:*

(a) *the conclusion of Proposition* 2.1 *still holds with* $\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}$ *replaced by* $\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}\ |t|^{-\frac{\ell-1}{2}}$*;*

(b) (Hopf lemma) *If* u *is positive and* $\eta < \lambda$ *then* $u(t,y) \ge \widetilde{\varphi}(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\eta)|t|^2}}$ *for every* $(t,y) \in \Omega$*, for some positive eigenfunction* $\widetilde{\varphi}$ *associated to* λ_1*.*

Proof. (a) We improve the estimate (2.4) by repeating the argument with

$$\Psi(t) := \mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}\ |t|^{-\frac{\ell-1}{2}}.$$

Indeed,

$$-\Delta\Psi + (\lambda_1-\lambda)\Psi = \Psi\ \left((\lambda_1-\lambda)\theta^{-1} + (\lambda_1-\lambda)\theta^{-3/2} + \frac{\ell-1}{2}\,\frac{\ell-3}{2}\,\frac{1}{|t|^2}\right),$$

a computation that can be easily checked using (2.8); here, of course, $\theta(t) := 1 + (\lambda_1-\lambda)|t|^2$. As a consequence, for sufficiently large $|t|$ we have that

$$-\Delta\Psi + (\lambda_1-\lambda)\Psi \ge \frac{1}{2}\,\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}\ |t|^{-\frac{\ell+3}{2}} =: h(t).$$

Due to the assumptions on g, for the function on $w := \alpha\varphi\Psi - u$, with α a fixed positive number, we have

$$-\Delta w - \lambda w \ge \alpha h(t)\varphi(y) - A\varphi^{1+\varepsilon}(y)\mathrm{e}^{-(1+\varepsilon)\sqrt{1+(\lambda_1-\lambda)|t|^2}}.$$

The right hand member above is positive for sufficiently large $|t|$. Using the maximum principle, we conclude, as in (2.11), that

$$|u(t,y)| \le \alpha\varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}\ |t|^{-\frac{\ell-1}{2}}, \qquad \forall (t,y) \in \Omega. \tag{2.15}$$

Finally, as in Step 4 of the quoted proof, a similar estimate for the derivatives of u follows from (2.4, 2.15) and the fact that

$$\frac{\alpha}{2}\varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}|t|^{-\frac{\ell+3}{2}} - C\varphi^{\varepsilon}(y)\mathrm{e}^{-\varepsilon\sqrt{1+(\lambda_1-\lambda)|t|^2}}|t|^{-\varepsilon\frac{\ell-1}{2}} \times \varphi(y)\mathrm{e}^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}$$

is positive for $|t|$ large.

(b) Here we let $\Psi(t) := \mathrm{e}^{-\sqrt{1+(\lambda_1-\eta)|t|^2}}$. Fix any $\mu \in]\eta, \lambda[$. Similarly to (2.8), we can check that

$$h(t) := -\Delta\Psi + (\lambda_1 - \mu)\Psi \leq 0 \quad \text{ for every } |t| \geq R$$

with R sufficiently large. Since $u(t,y) \to 0$ as $|t| \to \infty$ and since $g(0) = 0 = g'(0)$ we can choose R in such a way that also $(\mu-\lambda)u - g(u) \leq 0$ for $|t| \geq R$. Letting $z := \varphi\Psi$, we can fix a small $\alpha > 0$ so that $w := \alpha z - u \leq 0$ if $|t| \leq R$; this is possible because $u \in \mathrm{C}^1(\overline{\Omega})$, $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$ (outward normal derivative). In summary, we have that (compare with (2.9))

$$-\Delta w - \mu w = \alpha\varphi h + (\mu - \lambda)u - g(u) =: k(t,y)$$

and $k(t,y) \leq 0$ for $|t| \geq R$, while $w \leq 0$ for $|t| \leq R$. Using the maximum principle as in the proof of Proposition 2.1 we conclude that $w \leq 0$ for all (t,y). □

We end this section with the proof of Proposition 1.3, which is contained in Propositions 2.4 and 2.5 below. We will refer to the functional I introduced at the beginning of Section 2 as well as to the quantities c_0, c_0^∞ and c_1 defined in (1.1, 1.13) and (1.15), respectively.

Proposition 2.4. *Assume* (H)′ *holds. Then* $c_1 < c_0 + c_0^\infty$.

Proof. 1. We know that c_0 is attained by a positive function $v \in \mathrm{H}_0^1(\Omega)$ and c_0^∞ is attained by some positive function $\psi \in \mathrm{H}_0^1(\mathbb{R}^\ell \times F)$ (*cf.* Prop. 1.1). Let $m > 0$ and $0 < a_1 < a_0$ be given by assumption (H)′ and denote $A := a_0/a_1 > 1$. Fix a large number M such that $M > 2A$ and

$$\frac{a_1}{a_0} < \left(\frac{M-A}{M+A}\right)^m \cdot \tag{2.16}$$

Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\rho(s) = 1$ for $|s| \leq 1$ and $\rho(s) = 0$ for $|s| \geq A$. We define ρ_R and η_R in $\mathbb{R}^\ell$ by $\rho_R = \rho(|t|/R)$ and $\eta_R(t) = \rho_R(t - MR\mathbf{e}_1) = \rho(|\frac{t}{R} - M\mathbf{e}_1|)$. We also let

$$v_R(t,y) := v(t,y)\rho_R(t)$$

and

$$\psi_R(t,y) := \lambda_R^{-N/p}\, \psi\left(\frac{t - MR\mathbf{e}_1}{\lambda_R}, \frac{y}{\lambda_R}\right)\eta_R(t),$$

where

$$\lambda_R := 1 + \frac{a_0}{(M+A)^m R^m} \cdot \tag{2.17}$$

We observe that v_R and ψ_R have disjoint supports. Moreover, both functions belong to $\mathrm{H}_0^1(\Omega)$ if R is sufficiently large. Indeed, suppose $(t,y) \in \partial\Omega$ and let us show that $\psi_R(t,y) = 0$. We may already assume that $|t - MR\mathbf{e}_1| \leq AR$. In particular,

$$(M-A)R \leq |t| \leq (M+A)R. \tag{2.18}$$

Now, to prove the claim it is sufficient to show that $(\frac{t-MR\mathbf{e}_1}{\lambda_R}, \frac{y}{\lambda_R}) \notin \widehat{F}$, *i.e.* that $\frac{y}{\lambda_R} \notin F$. Observing that

$$y = \left(1 + \frac{a}{|t|^m}\right) \frac{y}{\lambda_R}$$

where, according to (2.16–2.18),

$$a := a_0 \left(\frac{|t|}{(M+A)R}\right)^m \in [a_1, a_0],$$

the conclusion follows from (H)$'$ and the fact that $(t, y) \notin \Omega$.

2. Thanks to Proposition 2.2 (with $\lambda = 0$), we know that $|v(t,y)| + |\nabla v(t,y)| = \mathrm{O}(\mathrm{e}^{-\delta|t|})$ and similarly for ψ. Here and henceforth δ denotes various positive constants. It then follows easily that $I(v_R) \to I(v)$ and $I(\psi_R) \to I(\psi)$ as $R \to \infty$ and also that

$$I(v_R) = I(v) + \mathrm{O}(\mathrm{e}^{-\delta R}), \qquad I(\psi_R) = I(\psi) + \mathrm{O}(\mathrm{e}^{-\delta R}). \tag{2.19}$$

In fact, the second estimate can be improved, observing that

$$\int \psi_R^p = \int \psi^p \rho_R^p = \int \psi^p + \int \psi^p(\rho_R^p - 1) = \int \psi^p + \mathrm{O}(\mathrm{e}^{-\delta R})$$

and similarly $\int |\nabla \psi_R|^2 = \int |\nabla \psi|^2 + \mathrm{O}(\mathrm{e}^{-\delta R})$, while

$$\int \psi_R^q = \lambda_R^{N(1-\frac{q}{p})} \int \psi^q + \mathrm{O}(\mathrm{e}^{-\delta R})$$

so that

$$\begin{aligned} I(\psi_R) &= I(\psi) + \left(1 - \lambda_R^{N(1-\frac{q}{p})}\right) \int \psi^q + \mathrm{O}(\mathrm{e}^{-\delta R}) \\ &\le I(\psi) - N\left(1 - \frac{q}{p}\right) \frac{a_0}{(M+A)^m R^m} \int \psi^q + \mathrm{O}(\mathrm{e}^{-\delta R}), \end{aligned}$$

whence, for every sufficiently large R,

$$I(\psi_R) < I(\psi). \tag{2.20}$$

3. Clearly, as in (2.19, 2.20), for large R and uniformly for $\tau_1, \tau_2 \in [1/2, 2]$, we have that

$$\begin{aligned} I(\tau_1 v_R - \tau_2 \psi_R) &= I(\tau_1 v_R) + I(\tau_2 \psi_R) < I(\tau_1 v) + I(\tau_2 \psi) \\ &\le \sup_{s\ge 0} I(sv) + \sup_{s \ge 0} I(s\psi) = c_0 + c_0^\infty. \end{aligned}$$

The last equality above is a direct consequence of the definitions of c_0 and c_0^∞, by standard arguments (*cf.* [3, 4, 11]). In summary, there exists R_0 such that

$$\sup_{1/2 \le \tau_1, \tau_2 \le 2} I(\tau_1 v_R - \tau_2 \psi_R) < c_0 + c_0^\infty, \qquad \forall R \ge R_0. \tag{2.21}$$

4. Thanks to (2.21), to complete the proof it remains to show that there exist $\tau_1, \tau_2 \in [1/2, 2]$ and $R \ge R_0$ such that $w := \tau_1 v_R - \tau_2 \psi_R$ satisfies $I'(w^\pm)w^\pm = 0$. Since v_R and ψ_R have disjoint supports, this amounts to prove

that there exist $\tau_1, \tau_2 \in [1/2, 2]$ and $R \geq R_0$ such that

$$I'(\tau_1 v_R)v_R = 0 \qquad \text{and} \qquad I'(\tau_2 \psi_R)\psi_R = 0. \tag{2.22}$$

Now, we have that $I'(v_R/2)v_R \to I'(v/2)v > 0$ and $I'(2v_R)v_R \to I'(2v)v < 0$ as $R \to \infty$ and similarly for ψ. Hence (2.22) follows by applying the intermediate value theorem. □

Proposition 2.5. *Assume* $(\mathrm{H})_0$ *holds and moreover that* $q > (N+2)/(N-2)$. *Then* $c_1 < c_0 + c_0^\infty = c_0 + S^{N/2}/N$.

Proof. Let $U(x) = c_N/(1+|x|^2)^{(N-2)/2}$ be the Talenti instanton, normalized in such a way that $\int |U|^p = \int |\nabla U|^2 = S^{N/2}$ (*i.e.* $c_N = (N(N-2))^{(N-2)/4}$). Let $U_\varepsilon(x) = \varepsilon^{-N/p} U(x/\varepsilon)$ be its rescaling, so that also $\int |U_\varepsilon|^p = \int |\nabla U_\varepsilon|^2 = S^{N/2}$. The following argument is similar to that in [12], except that we cut down the least energy solution and also U_ε and estimate the error in doing so, instead of computing the interference between their energies.

Recall that, without loss of generality, we are assuming that $0 \in \Omega$. By Proposition 1.1, we know that c_0 is achieved by a positive function $v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{C}^1(\Omega)$. Let ρ, $\eta : \mathbb{R} \to \mathbb{R}$ be smooth functions such that $\rho(s) = 1$ for $|s| \leq 1$, $\rho(s) = 0$ for $|s| \geq 2$, $\eta(s) = 0$ for $|s| \leq 2$ and $\eta(s) = 1$ for $|s| \geq 3$. We define ρ_ε and $\eta_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ by $\rho_\varepsilon(x) = \rho(|x|/\sqrt{\varepsilon})$ and $\eta_\varepsilon(x) = \eta(|x|/\sqrt{\varepsilon})$. We also define

$$u_\varepsilon := U_\varepsilon\, \rho_\varepsilon \qquad \text{and} \qquad v_\varepsilon := v\, \eta_\varepsilon.$$

It is clear that u_ε and v_ε have disjoint supports and that they both belong to $\mathrm{H}_0^1(\Omega)$. We can estimate

$$\begin{aligned}
\int |\nabla v_\varepsilon|^2 &\leq \int |\nabla v|^2 + 2\left(\int_{2\varepsilon^{1/2} \leq |x| \leq 3\varepsilon^{1/2}} (|\nabla v|^2 \eta_\varepsilon^2 + v^2 |\nabla \eta_\varepsilon|^2)\right) \\
&\leq \int |\nabla v|^2 + \mathrm{O}(\varepsilon^{N/2}) + \mathrm{O}(\varepsilon^{(N-2)/2}) \\
&= \int |\nabla v|^2 + \mathrm{O}(\varepsilon^{(N-2)/2}),
\end{aligned}$$

while

$$\int v_\varepsilon^p = \int v^p + \int v^p(\eta_\varepsilon^p - 1) \geq \int v^p - \int_{|x| \leq 3\varepsilon^{1/2}} v^p \geq \int v^p + \mathrm{O}(\varepsilon^{N/2})$$

and similarly for $\int v_\varepsilon^q$, so that

$$I(v_\varepsilon) \leq I(v) + \mathrm{O}(\varepsilon^{(N-2)/2}). \tag{2.23}$$

As for u_ε,

$$\begin{aligned}
\int |\nabla u_\varepsilon|^2 &\leq \int |\nabla U_\varepsilon|^2 + 2\left(\int |\nabla U_\varepsilon|^2 \rho_\varepsilon^2 + U_\varepsilon^2 |\nabla \rho_\varepsilon|^2\right) \\
&\leq S^{N/2} + \mathrm{O}(\varepsilon^{(N-2)/2}),
\end{aligned}$$

while, denoting by $c > 0$ some constant which is independent of ε,

$$\int u_\varepsilon^p \geq S^{N/2} + \mathrm{O}(\varepsilon^{N/2}) \quad \text{and} \quad \int u_\varepsilon^q \geq c\, \varepsilon^{N(1-\frac{q}{p})},$$

as can be checked directly, using the explicit expression of U_ε. In summary,

$$I(u_\varepsilon) \leq \left(\frac{1}{2} - \frac{1}{p}\right) S^{N/2} + \mathrm{O}(\varepsilon^{\frac{N-2}{2}}) - c\, \varepsilon^{N(1-\frac{q}{p})}. \tag{2.24}$$

Combining (2.23) and (2.24) yields

$$I(u_\varepsilon) + I(v_\varepsilon) \le c_0 + \frac{S^{N/2}}{N} + c_1\, \varepsilon^{\frac{N-2}{2}} - c_2\, \varepsilon^{N(1-\frac{q}{p})}, \tag{2.25}$$

for some positive constants c_1 and c_2. In particular,

$$I(u_\varepsilon) + I(v_\varepsilon) < c_0 + \frac{S^{N/2}}{N} \tag{2.26}$$

if ε is sufficiently small since, by assumption, $\frac{N-2}{2} > N(1-\frac{q}{p})$; indeed, this condition is equivalent to $q > p-1 = (N+2)/(N-2)$. From (2.26) we can end the proof of Proposition 2.5 with similar arguments as in Steps 3 and 4 in the proof of Proposition 2.4. □

Remark 2.6. As observed at the beginning of Section 2, for simplicity of notations we have assumed that $\mu = 1$ in problem (P). In the general case, (2.25) reads as

$$I(u_\varepsilon) + I(v_\varepsilon) \le c_0 + \frac{S^{N/2}}{N} + c_1\, \varepsilon^{\frac{N-2}{2}} - \mu\, c_2\, \varepsilon^{N(1-\frac{q}{p})}.$$

Thus one still has (2.26) in case $q = (N+2)/(N-2)$ provided μ is sufficiently large.

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