

## ASYMPTOTIC BEHAVIOUR OF STOCHASTIC QUASI DISSIPATIVE SYSTEMS

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**Abstract.** We prove uniqueness of the invariant measure and the exponential convergence to equilibrium for a stochastic dissipative system whose drift is perturbed by a bounded function.

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### 1. INTRODUCTION

Let us consider the following differential stochastic equation on a Hilbert space  $H$  (with norm  $|\cdot|$  and inner product  $\langle \cdot, \cdot \rangle$ )

$$\begin{cases} dX(t) = (AX(t) + F(X(t)) + G(X(t)))dt + BdW(t), \\ X(0) = x \in H, \end{cases} \quad (1.1)$$

where  $A : D(A) \subset H \rightarrow H$  is self-adjoint and such that  $A \leq -\omega I$  with  $\omega > 0$ ,  $B : H \rightarrow H$  is linear bounded,  $F : D(F) \subset H \rightarrow H$  is  $m$ -dissipative<sup>2</sup> and  $G : H \rightarrow H$  is Lipschitz continuous and bounded. Moreover  $W(t)$  is a cylindrical Wiener process in  $H$ , defined in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

System (1.1) is called *quasi dissipative*; it is called *dissipative* if  $A + F + G$  is dissipative.

Assume that equation (1.1) has a solution  $X(t, x)$ . Then we consider the corresponding *transition semigroup* in  $C_b(H)$  defined by the formula

$$T_t \varphi(x) := \mathbb{E}[\varphi(X(t, x))], \quad x \in H, t > 0, \varphi \in C_b(H), \quad (1.2)$$

where  $\mathbb{E}$  denotes the expectation. Here  $C_b(H)$  is the Banach space of all uniformly continuous and bounded mappings  $\varphi : H \rightarrow \mathbb{R}$  endowed with the norm  $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$ .

Problem (1.1), which arises in several applications as: Reaction-Diffusion equations, Ginzburg–Landau models, Spin Systems, has been considered by several authors, see [12] and references therein and [4].

When system (1.1) is dissipative, one can show see [11], that for any  $x \in H$  there exists the limit

$$\lim_{t \rightarrow +\infty} \mathcal{L}(X(t, x)) = \zeta, \quad (1.3)$$

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<sup>2</sup>  $F$  is called  $m$ -dissipative if  $\langle F(x) - F(y), x - y \rangle \leq 0$  for all  $x, y \in D(F)$  and the range of  $I - F$  is equal to  $H$ .

where  $\mathcal{L}(X(t, x))$  is the law of  $X(t, x)$  and  $\zeta$  is the unique invariant measure of  $T_t$ , that is

$$\int_H T_t \varphi(x) \zeta(dx) = \int_H \varphi(x) \zeta(dx), \quad t \geq 0, \varphi \in C_b(H).$$

As a consequence of (1.3) we have

$$\lim_{t \rightarrow +\infty} T_t \varphi(x) = \int_H \varphi(y) \zeta(dy), \quad x \in H, \varphi \in C_b(H), \tag{1.4}$$

so that the invariant measure  $\zeta$  is ergodic and strongly mixing.

When  $G \neq 0$  system (1.1) is not necessarily dissipative, see Examples 1.2 and 1.3 below, and (1.3) does not hold in general. We notice that for a non dissipative system there is not in general existence and uniqueness of invariant measures.

In this paper we shall prove that, when  $G$  is Lipschitz continuous and bounded and some suitable additional assumptions are fulfilled, there is an invariant measure  $\zeta$  for (1.1) such that

$$\left| T_t \varphi(x) - \int_H \varphi(y) \zeta(dy) \right| \leq \kappa t^{-(1+\gamma)/2} e^{-\omega t} (1 + |x|) \|\varphi\|_0, \quad x \in H, \varphi \in C_b(H), \tag{1.5}$$

where  $\kappa$  and  $\gamma$  are positive constants.

An important consequence of (1.5) is that  $\zeta$  is the unique invariant measure of  $T_t$ .

**Remark 1.1.** (i) If  $G = 0$  estimate (1.5) was proved in [11].

(ii) If  $B$  has bounded inverse then the exponential convergence to equilibrium of  $T_t$  was proved in [8] by a different method.

In the last part of the paper, we give an application of estimate (1.5) to the asymptotic behaviour, as  $t \rightarrow \infty$ , of the following Hamilton–Jacobi equation

$$\begin{cases} D_t u = \frac{1}{2} \text{Tr} [BB^* D^2 u] + \langle Ax + F(x) + G(x), Du \rangle - \frac{1}{2} |B^* Du|^2, & x \in D(A), \\ u(0) = \varphi \in C_b(H), \end{cases} \tag{1.6}$$

where  $B^*$  is the adjoint of  $B$ . We show that

$$\lim_{t \rightarrow +\infty} u(t, x) = -\log \int_H e^{-\varphi(y)} \zeta(dy), \quad x \in H. \tag{1.7}$$

We end this section by giving two examples:

**Example 1.2.** Let us consider the following reaction-diffusion equation on  $[0, \pi]$

$$\begin{cases} dX(t, \xi) = [D_\xi^2 X(t, \xi) + p(X(t, \xi))]dt + dW(t)(\xi), & \xi \in [0, \pi], t > 0, \\ X(t, 0) = X(t, \pi) = 0, & t > 0, \\ X(0, \xi) = x(\xi), & \xi \in [0, \pi], \end{cases} \tag{1.8}$$

where  $p$  is a polynomial having odd degree  $s$  and negative leading coefficient,  $x \in L^2(0, \pi)$  and  $W$  is a cylindrical Wiener process in  $H = L^2(0, \pi)$ .

Set  $B = I$ , and

$$Ax = D_\xi^2, \quad x \in D(A) := H^2([0, \pi]) \cap H_0^1([0, \pi]).$$

As easily checked, system (1.8) is dissipative if and only if  $p'(\xi) - \omega \leq 0$ .

We show now, following [8], that there exist  $F$  dissipative and  $G$  Lipschitz continuous and bounded such that  $p(\xi) = F(\xi) + G(\xi)$ , so that (1.8) is of the form (1.1).

Let  $\xi_1, \xi_2 \in \mathbb{R}$  with  $\xi_1 \leq \xi_2$  be such that  $p(\xi_1) = p(\xi_2) = 0$  and  $p$  is decreasing on  $(-\infty, \xi_1] \cup [\xi_2, +\infty)$ . Then, setting

$$F(\xi) = \begin{cases} p(\xi) & \text{if } \xi \in (-\infty, \xi_1] \cup [\xi_2, +\infty), \\ 0 & \text{if } \xi \in [\xi_1, \xi_2], \end{cases}$$

and

$$G(x) = \begin{cases} 0 & \text{if } \xi \in (-\infty, \xi_1] \cup [\xi_2, +\infty), \\ p(\xi) & \text{if } \xi \in [\xi_1, \xi_2], \end{cases}$$

we see that  $F$  and  $G$  have the required properties.

**Example 1.3.** Consider the following reaction-diffusion equation on  $D := [0, \pi]^3$ .

$$\begin{cases} dX(t, \xi) = [\Delta_\xi X(t, \xi) + p(X(t, \xi))]dt + (-\Delta_\xi)^{-\delta/2} dW(t)(\xi), & \xi \in D, t > 0, \\ X(t, \xi) = 0, & t > 0, \xi \in \partial D, \\ X(0, \xi) = x(\xi), & \xi \in D, \end{cases} \quad (1.9)$$

where  $p$  is a polynomial having odd degree  $s$  and negative leading coefficient,  $\delta > 0$ ,  $x \in L^2(D)$  and  $W$  is a cylindrical Wiener process in  $H = L^2(D)$ .

Set

$$Ax = \Delta_\xi, \quad x \in D(A) := H^2(D) \cap H_0^1(D).$$

Now, choosing  $F$  and  $G$  as in Example 1.2, we can write system (1.9) in the form (1.1) with  $F$  dissipative and  $G$  Lipschitz continuous and bounded.

## 2. HYPOTHESES AND PRELIMINARIES

In this section we recall some known results about problem (1.1).

Concerning the operators  $A$  and  $B$  we shall assume

**Hypothesis 2.1.**

(i)  $A$  is a self-adjoint operator in  $H$  such that

$$\langle Ax, x \rangle \leq -\omega|x|^2, \quad x \in H, \quad (2.1)$$

for some  $\omega > 0$ . Moreover  $A^{-1}$  is compact.

(ii)  $B \in L(H)$ <sup>3</sup> and for any  $t > 0$  the linear operator  $Q_t$ , defined as

$$Q_t x = \int_0^t e^{sA} B B^* e^{sA^*} x ds, \quad x \in H,$$

is of trace class.

(iii) There exists  $\rho \in (0, \frac{1}{2})$  such that

$$\int_0^{+\infty} s^{-2\rho} \text{Tr} [e^{sA} B B^* e^{sA^*}] ds < +\infty. \quad (2.2)$$

<sup>3</sup> $L(H)$  represents the Banach algebra of all linear bounded operators from  $H$  into  $H$  endowed with the sup norm.

We set

$$Q_t x = \int_0^t e^{sA} B B^* e^{sA^*} x ds, \quad x \in H, \quad t \geq 0. \tag{2.3}$$

By Hypothesis 2.1 it follows that  $Q_t$  is of trace class and that the *stochastic convolution*

$$W_A(t) = \int_0^t e^{sA} B B^* e^{sA^*} dW(s), \quad t \geq 0,$$

is a Gaussian random variable with mean 0, covariance operator  $Q_t$  and continuous paths, see [12].

We notice that the assumption that  $A$  is self-adjoint is not essential, it could be replaced by  $A$  variational.

Concerning  $F$  we need two groups of assumptions, Hypotheses 2.2 and 2.3 below.

**Hypothesis 2.2.**  $F$  is  $m$ -dissipative and  $D(F) = K$ , where  $K$  is a reflexive Banach space continuously and densely embedded in  $H$ .

We denote by  $A_K$  and  $F_K$  the parts of  $A$  and  $F$  in  $K$ :

$$D(A_K) = \{x \in D(A) \cap K : Ax \in K\}, \quad A_K x = Ax, \quad x \in D(A_K),$$

$$D(F_K) = \{x \in K : F(x) \in K\}, \quad F_K(x) = F(x), \quad x \in D(F_K).$$

The second group of assumptions is:

**Hypothesis 2.3.**

(i)  $A_K : D(A_K) \subset K \rightarrow K$  generates a strongly continuous semigroup  $e^{tA_K}$  in  $K$ . Moreover, there exists  $\omega_1 > 0$  such that

$$\|e^{tA_K}\|_{L(K)} \leq e^{-\omega_1 t}, \quad t \geq 0. \tag{2.4}$$

(ii)  $F_K$  is  $m$ -dissipative in  $K$ .

(iii)  $F$  maps bounded subsets of  $K$  into bounded subsets of  $H$ .

(iv)  $W_A(t)$  takes values on  $D(F_K)$  and

$$\sup_{t \geq 0} \mathbb{E} (\|W_A(t)\|_K^2 + \|F(W_A(t))\|_K^2) < +\infty. \tag{2.5}$$

We shall need two different notions of solutions of problem (1.1), *mild* solutions and *generalized* solutions.

Let  $x \in K$  and  $T > 0$ . We say that  $X(\cdot) = X(\cdot, x)$  is a *mild solution* of (1.1) on  $[0, T]$  if

(i)  $X(\cdot) \in C_W([0, T]; H)$ <sup>4</sup>.

(ii)  $X(t, x) \in K$ ,  $\mathbb{P}$ -a.s. for all  $t \in [0, T]$  and

$$X(t) = e^{tA} x + \int_0^t e^{(t-s)A} (F + G)(X(s)) ds + W_A(t), \quad \mathbb{P} - \text{a.s.} \tag{2.6}$$

We say that  $X(\cdot) = X(\cdot, x)$  is a *generalized solution* of (1.1) on  $[0, T]$  if assumption (i) holds and there exists a sequence  $\{x_n\} \subset K$  convergent to  $x$  in  $H$ , such that for any  $n \in \mathbb{N}$  there exists a mild solution  $X(\cdot, x_n)$  to (1.1) on  $[0, T]$  and

$$\lim_{t \rightarrow +\infty} X(\cdot, x_n) = X(\cdot, x) \quad \text{in } C_W([0, T]; H).$$

The following result is proved in [11].

<sup>4</sup> $C_W([0, T]; H)$  is the Banach space of all continuous mappings  $[0, T] \rightarrow L^2(\Omega, H)$  which are adapted to  $W(t)$ , endowed with the norm:  $\|X\|_{C_W([0, T]; H)} = \sup_{t \in [0, T]} [\mathbb{E} (|X(t)|^2)]^{1/2}$ .

**Theorem 2.4.** *Assume that Hypotheses 2.1, 2.2, and 2.3, hold and that  $G$  is Lipschitz continuous.*

- (i) *If  $x \in K$ , problem (1.1) has a unique mild solution  $X(\cdot, x)$ .*
- (ii) *If  $x \in H$ , problem (1.1) has a unique generalized solution  $X(\cdot, x)$ .*

By the Markov property  $T_t$  is a semigroup of linear bounded operators in  $C_b(H)$ . However, it is not strongly continuous in general. Following [3] we define the infinitesimal generator  $S$  of  $T_t$  through its Laplace transform  $F(\lambda)$

$$F(\lambda)f(x) := \int_0^{+\infty} e^{-\lambda t} T_t f(x) dt, \quad f \in C_b(H), \lambda > 0.$$

It is easy to check that  $F(\lambda)$  maps  $C_b(H)$  into itself for all  $\lambda > 0$  and that  $F(\lambda)$  is a pseudo-resolvent. Consequently, there exists a unique closed operator  $S$  in  $C_b(H)$  such that its resolvent  $R(\lambda, S)$  is given by

$$R(\lambda, S) = (\lambda - S)^{-1} = F(\lambda), \quad \lambda > 0.$$

$S$  is called the *infinitesimal generator* of  $T_t$  on  $C_b(H)$ .

**Example 2.5.** Let us consider equation (1.8). Define  $A$  and  $F$  as in Example 1.2. Then  $A$  is self-adjoint and its spectrum  $\sigma(A)$  is given by

$$\sigma(A) = \{-\pi^2 k^2 : k \in \mathbb{N}\}.$$

Moreover

$$Q_t = -\frac{1}{2} A^{-1}(I - e^{2tA}), \quad t \geq 0.$$

Consequently

$$\text{Tr } Q_t = \sum_{k=1}^{\infty} \frac{1 - e^{-tk^2}}{k^2} < +\infty,$$

and Assumptions 2.1(i, ii) are fulfilled. Also 2.1(iii) holds provided  $\rho < 1/4$ .

Finally, the domain of  $F$  is given by  $K = L^{2s}(0, \pi)$  and we have

$$D(A_K) = H^{2,2s}([0, \pi]) \cap H_0^{1,2s}([0, \pi]), \quad D(F_K) = L^{2s^2}([0, \pi]).$$

Now, it is not difficult to see [12] that also Hypotheses 2.2, and 2.3, are fulfilled and so Theorem 2.4 applies.

**Example 2.6.** Let us consider equation (1.9) and define  $A$ ,  $F$  and  $B$  as in Example 1.3. Then  $A$  is self-adjoint and its spectrum  $\sigma(A)$  is given by

$$\sigma(A) = \{-\pi^2(k_1^2 + k_2^2 + k_3^2) : (k_1, k_2, k_3) \in \mathbb{N}^3\}.$$

Moreover

$$Q_t = \frac{1}{2} (-A)^{-1-\delta}(I - e^{2tA}), \quad t \geq 0.$$

Consequently

$$\text{Tr } Q_t = \sum_{k=1}^{\infty} \frac{1 - e^{-t(k_1^2 + k_2^2 + k_3^2)}}{(k_1^2 + k_2^2 + k_3^2)^{1+\delta}} < +\infty,$$

provided  $\delta > 1/2$ . Therefore Hypotheses 2.1(i, ii) is fulfilled. Moreover, choosing  $\rho < \min\{1/2, 2(\delta - 1/2)\}$ , also Hypothesis 2.1(iii) holds.

Finally, the domain of  $F$  is given by  $K = L^{2s}(D)$  and we have

$$D(A_K) = H^{2,2s}(D) \cap H_0^{1,2s}(D), \quad D(F_K) = L^{2s^2}(D).$$

Then, it is easy to see that Hypotheses 2.2, and 2.3, are fulfilled. Therefore, if  $\delta > 1/2$ , Theorem 2.4 applies.

### 3. THE MAIN RESULT

In this section we shall assume that Hypotheses 2.1, 2.2, and 2.3, hold.

#### 3.1. Existence of an invariant measure

We first prove two estimates.

**Lemma 3.1.** *Assume that Hypotheses 2.1, 2.2, and 2.3, hold.*

(i) *Let  $x \in H$  and let  $X(t, x)$  be the generalized solution of (1.1). Then there exists  $\kappa_1 > 0$  such that*

$$\mathbb{E}|X(t, x)|^2 \leq \kappa_1(1 + e^{-2\omega t}|x|^2), \quad t \geq 0. \tag{3.1}$$

(ii) *Let  $x \in K$  and let  $X(t, x)$  be the mild solution of (1.1). Then there exists a constant  $\kappa_2 > 0$  such that*

$$\mathbb{E}|F(X(t, x))|^2 \leq \kappa_1(1 + e^{-2\omega t}|x|_K^2), \quad t \geq 0. \tag{3.2}$$

*Proof.* Let us prove (i). Setting  $Z(t) = X(t, x) - W_A(t)$ , equation (1.1) becomes

$$\begin{cases} \frac{d}{dt} Z(t) = AZ(t) + F(Z(t) + W_A(t)) + G(Z(t) + W_A(t)), \\ Z(0) = x. \end{cases} \tag{3.3}$$

Multiplying the first equation in (3.3) by  $Z(t)$  and taking into account (2.1) and the dissipativity of  $F$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |Z(t)|^2 &\leq -\omega|Z(t)|^2 + \langle F(Z(t) + W_A(t)) - F(W_A(t)), Z(t) \rangle + \langle F(W_A), Z(t) \rangle + \|G\|_0|Z(t)| \\ &\leq -\omega|Z(t)|^2 + \langle F(W_A), Z(t) \rangle + \|G\|_0|Z(t)| \\ &\leq -\frac{\omega}{2} |Z(t)|^2 + c_1(|F(W_A)|^2 + 1), \end{aligned}$$

where  $c_1$  is a suitable constant.

By the Gronwall lemma it follows that

$$|Z(t)|^2 \leq e^{-\omega t}|x|^2 + 2c_1 \int_0^t e^{-2\omega(t-s)}(|F(W_A(s))|^2 + 1)ds,$$

and finally, for some positive constant  $c_2$

$$|X(t, x)|^2 \leq c_2 e^{-2\omega t}|x|^2 + c_2 \left( \int_0^t e^{-2\omega(t-s)}(|F(W_A(s))|^2 + 1)ds + |W_A(t)|^2 \right). \tag{3.4}$$

Now (3.1) follows taking expectation and recalling (2.5).

Let us now prove (ii). We have

$$\frac{d^+}{dt} |Z(t)|_K \leq -\omega_1|Z(t)|_K + \|G\|_0 + \langle F(Z(t) + W_A(t)), \nu_t \rangle,$$

where  $\frac{d^+}{dt}$  represents the right derivative and  $\nu_t$  belongs to the subdifferential of  $|Z(t)|_K$ , see [17]. Since  $F$  is dissipative in  $K$  (by Hypothesis 2.3(ii)), we obtain

$$\begin{aligned} \frac{d^+}{dt} |Z(t)|_K &\leq -\omega_1 |Z(t)|_K + \|G\|_0 + \langle F(Z(t) + W_A(t)) - F(W_A(t)), \nu_t \rangle + \langle F(W_A(t)), \nu_t \rangle \\ &\leq -\omega_1 |Z(t)|_K + \|G\|_0 + |F(W_A(t))|_K. \end{aligned}$$

Now the conclusion follows using the Gronwall lemma and taking into account (2.5).  $\square$

We are now in position to prove the existence of an invariant measure for  $T_t$ . To this purpose let us first recall that there exists a positive constant  $\kappa_\rho$  such that

$$|(-A)^\rho e^{tA} x| \leq \kappa_\rho t^{-\rho} e^{-\omega t} |x|, \quad x \in H, \quad t > 0,$$

where  $\rho$  was defined in (2.2).

**Proposition 3.2.** *Assume that Hypotheses 2.1, 2.2, and 2.3, hold. Then there exists an invariant measure  $\zeta$  for  $T_t$ . Moreover*

$$\int_H |x|^2 \zeta(dx) < +\infty. \quad (3.5)$$

*Proof.* Let  $x \in K$  and let  $X(t, x)$  be the mild solution of (1.1). Then we have

$$\begin{aligned} (-A)^\rho X(t, x) &= (-A)^\rho e^{tA} x + \int_0^t (-A)^\rho e^{(t-s)A} (F + G)(X(s, x)) ds \\ &\quad + (-A)^\rho W_A(t), \quad t \geq 0, \quad x \in H. \end{aligned} \quad (3.6)$$

Moreover

$$\begin{aligned} \mathbb{E} [ |(-A)^\rho W_A(t)|^2 ] &= \int_0^t \text{Tr} \left[ (-A)^\rho e^{sA} B B^* (-A^*)^\rho e^{sA^*} \right] ds \\ &\leq 2\kappa_\rho^2 \int_0^t t^{-2\rho} \text{Tr} [ e^{sA} B B^* e^{sA^*} ] ds. \end{aligned}$$

Therefore, in view of Hypothesis 2.1(iii), there exists  $c_\rho > 0$  such that

$$\mathbb{E} [ |(-A)^\rho W_A(t)|^2 ] \leq c_\rho, \quad t \geq 0. \quad (3.7)$$

From (3.6) and (3.7) it follows that there exists a constant  $c_{1,\rho}(x)$  such that

$$\mathbb{E} |(-A)^\rho X(t, x)| \leq t^{-\rho} c_{1,\rho}(x), \quad t \geq 0.$$

We can now show that the set of the laws of  $X(t, x)$ ,  $\{\mathcal{L}(X(t, x))\}_{t \geq 1}$ , is tight. For any  $R > 0$ , denote by  $B_R^\rho$  the ball  $B_R^\rho := \{y \in H : |y|_{D((-A)^\rho)} < R\}$ . Then we have

$$\begin{aligned} \mathcal{L}(X(t, x))((B_R^\rho)^c) &= \int_{|y|_{D((-A)^\rho} \geq R} \mathcal{L}(X(t, x))(dy) \\ &\leq \frac{1}{R} \int_H |y|_{D((-A)^\rho)} \mathcal{L}(X(t, x))(dy) = \mathbb{E} |(-A)^\rho X(t, x)| \leq \frac{1}{R} c_{1,\rho}(x). \end{aligned}$$

This implies tightness of  $\{\mathcal{L}(X(t, x))\}_{t \geq 1}$ , because the embedding  $D((-A)^\rho) \subset H$  is compact by Hypothesis 2.1(i). Therefore, from the Krylov–Bogoliubov theorem it follows that there exists an invariant measure  $\zeta$  for  $T_t$ .

Let us prove now (3.5). By (3.1) we have in fact, integrating with respect to  $\zeta$  and taking into account the invariance of  $\zeta$

$$\int_H |x|^2 \zeta(dx) \leq \kappa_1 (1 + e^{-2\omega t} \int_H |x|^2 \zeta(dx)).$$

Choosing  $t_0 > 0$  such that  $\kappa_1 e^{-2\omega t_0} < 1$  we have

$$\int_H |x|^2 \zeta(dx) \leq \frac{\kappa_1}{1 - \kappa_1 e^{-2\omega t_0}},$$

and (3.5) is proved. □

### 3.2. Strong Feller property

Together with problem (1.1) it is useful to consider the following dissipative problem, obtained by setting  $G = 0$  in (1.1)

$$\begin{cases} dY(t) = (AY(t) + F(Y(t)))dt + BdW(t), \\ Y(0) = x \in H. \end{cases} \tag{3.8}$$

Again by Theorem 2.4 problem (3.8) has a unique generalized solution  $Y(t, x)$  for any  $x \in H$ . We denote by  $P_t$  the corresponding transition semigroup

$$P_t \varphi(x) := \mathbb{E}[\varphi(Y(t, x))], \quad x \in H, \varphi \in C_b(H), \tag{3.9}$$

and by  $N$  its infinitesimal generator in  $C_b(H)$ , defined through its resolvent.

We also need to introduce the *Yosida approximations* of  $F$ . For any  $\alpha > 0$  we set

$$F_\alpha(x) := \frac{1}{\alpha} (J_\alpha(x) - x), \quad x \in H,$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \quad x \in H, \alpha > 0.$$

$F_\alpha$  is Lipschitz continuous, but not differentiable in general. Therefore we introduce a further regularization, as in [10], by setting

$$F_{\alpha, \beta}(x) := \int_H e^{\beta C} F_\alpha(e^{\beta C} x + y) N_{\frac{1}{2} C^{-1}(e^{2\beta C} - 1)}(dy), \quad \alpha, \beta > 0, \tag{3.10}$$

where  $C : D(C) \subset H \rightarrow H$  is a self-adjoint negative definite operator such that  $C^{-1}$  is of trace class and  $N_{\frac{1}{2} C^{-1}(e^{2\beta C} - 1)}$  is the Gaussian measure with mean 0 and covariance operator  $\frac{1}{2} C^{-1}(e^{2\beta C} - 1)$ .

$F_{\alpha, \beta}$  is dissipative and of class  $C^\infty$  with bounded derivatives of all orders. Moreover, as  $\alpha, \beta \rightarrow 0$ ,  $F_{\alpha, \beta} \rightarrow F$  pointwise, see [12].

Now we consider, for any  $\alpha > 0, \beta > 0$ , the approximating problems

$$\begin{cases} dX_{\alpha, \beta}(t) = [AX_{\alpha, \beta}(t) + (F_{\alpha, \beta} + G)(X_{\alpha, \beta}(t))]dt + BdW(t), \\ X_{\alpha, \beta}(0) = x \in H, \end{cases} \tag{3.11}$$



and

$$\begin{cases} dY_{\alpha,\beta}(t) = (AY_{\alpha,\beta}(t) + F_{\alpha,\beta}(Y_{\alpha,\beta}(t)))dt + BdW(t), \\ Y_{\alpha,\beta}(0) = x \in H. \end{cases} \quad (3.12)$$

We denote by  $X_{\alpha,\beta}(t, x)$  and  $Y_{\alpha,\beta}(t, x)$  the mild solutions of (3.11) and (3.12) respectively. It is easy to check, arguing as in [7], that for any  $x \in H$  and  $T > 0$  we have,

$$\lim_{\alpha \rightarrow 0, \beta \rightarrow 0} X_{\alpha,\beta}(\cdot, x) = X(\cdot, x) \quad \text{in } C_W(0, T; H), \quad (3.13)$$

and

$$\lim_{\alpha \rightarrow 0, \beta \rightarrow 0} Y_{\alpha,\beta}(\cdot, x) = Y(\cdot, x) \quad \text{in } C_W(0, T; H),$$

where  $X(\cdot, x)$  and  $Y(\cdot, x)$  are the generalized solutions of (1.1) and (3.8) respectively.

We shall denote by  $T_t^{\alpha,\beta}$  and  $P_t^{\alpha,\beta}$  the transition semigroups

$$T_t^{\alpha,\beta}\varphi(x) = \mathbb{E}[\varphi(X_{\alpha,\beta}(t, x))], \quad t \geq 0, \quad \varphi \in C_b(H), \quad (3.14)$$

$$P_t^{\alpha,\beta}\varphi(x) = \mathbb{E}[\varphi(Y_{\alpha,\beta}(t, x))], \quad t \geq 0, \quad \varphi \in C_b(H), \quad (3.15)$$

respectively, and by  $S_{\alpha,\beta}$  and  $N_{\alpha,\beta}$  their infinitesimal generators in  $C_b(H)$ .

Since  $F_{\alpha,\beta}$  is regular, it is easy to see that  $Y_{\alpha,\beta}(t, x)$  is Gateaux differentiable with respect to  $x$  and setting  $\eta_{\alpha,\beta}^h(t, x) = DY_{\alpha,\beta}(t, x)$  we have

$$\begin{cases} D_t \eta_{\alpha,\beta}^h(t, x) = A\eta_{\alpha,\beta}^h(t, x) + DF_{\alpha,\beta}(X_{\alpha,\beta}(t, x))\eta_{\alpha,\beta}^h(t, x), \\ \eta_{\alpha,\beta}^h(0, x) = h. \end{cases} \quad (3.16)$$

The following result will be useful later.

**Proposition 3.3.** *Assume that Hypotheses 2.1, 2.2 and 2.3 hold. Let  $h \in H$  and let  $\eta_{\alpha,\beta}^h$  be the solution of (3.16). Then for any  $\gamma \in [0, 1]$  we have*

$$\int_0^T |(-A)^{\gamma/2} \eta_{\alpha,\beta}^h(t, x)|^2 dt \leq 2^{-\gamma} T^{1-\gamma} |h|^2, \quad T > 0. \quad (3.17)$$

*Proof.* We first notice that, since  $F_{\alpha,\beta}$  is dissipative, we have

$$\langle DF_{\alpha,\beta}(y)z, z \rangle \leq 0, \quad y, z \in H.$$

Therefore, multiplying the first equation in (3.16) by  $\eta_{\alpha,\beta}^h(t, x)$ , yields

$$\frac{1}{2} \frac{d}{dt} |\eta_{\alpha,\beta}^h(t, x)|^2 \leq \langle A\eta_{\alpha,\beta}^h(t, x), \eta_{\alpha,\beta}^h(t, x) \rangle = -|(-A)^{1/2} \eta_{\alpha,\beta}^h(t, x)|^2, \quad (3.18)$$

for  $x \in H$ , and  $t \geq 0$ . It follows

$$\frac{1}{2} \frac{d}{dt} |\eta_{\alpha,\beta}^h(t, x)|^2 \leq -\omega |\eta_{\alpha,\beta}^h(t, x)|^2, \quad x \in H, \quad t \geq 0,$$

which yields, by a standard comparison result,

$$|\eta_{\alpha,\beta}^h(t, x)|^2 \leq e^{-2\omega t} |h|^2, \quad x \in H, \quad t \geq 0. \tag{3.19}$$

Now, by (3.18) we find for any  $T > 0$

$$\frac{1}{2} |\eta_{\alpha,\beta}^h(T, x)|^2 + \int_0^T |(-A)^{1/2} \eta_{\alpha,\beta}^h(t, x)|^2 dt \leq \frac{1}{2} |h|^2,$$

and consequently

$$\int_0^T |(-A)^{1/2} \eta_{\alpha,\beta}^h(t, x)|^2 dt \leq \frac{1}{2} |h|^2. \tag{3.20}$$

By using the well known interpolatory estimate

$$|(-A)^{\gamma/2} x| \leq |x|^{1-\gamma} |(-A)^{1/2} x|^\gamma, \quad x \in D((-A)^{1/2}), \quad \gamma \in [0, 1],$$

we find, using the Hölder inequality,

$$\begin{aligned} \int_0^T |(-A)^{\gamma/2} \eta_{\alpha,\beta}^h(t, x)|^2 dt &\leq \int_0^T |\eta_{\alpha,\beta}^h(t, x)|^{2(1-\gamma)} |(-A)^{1/2} \eta_{\alpha,\beta}^h(t, x)|^{2\gamma} dt \\ &\leq \left( \int_0^T |(-A)^{1/2} \eta_{\alpha,\beta}^h(t, x)|^2 dt \right)^\gamma \left( \int_0^T |\eta_{\alpha,\beta}^h(t, x)|^2 dt \right)^{1-\gamma}. \end{aligned}$$

Now (3.17) follows from (3.19) and (3.20). □

**Corollary 3.4.** *Assume that Hypotheses 2.1, 2.2, and 2.3 hold. Let  $\varphi \in C_b^1(H)^5$  and let  $\alpha > 0, \beta > 0$ . Then we have*

$$|DP_t^{\alpha,\beta} \varphi(x)| \leq e^{-\omega t} \|\varphi\|_1, \quad x \in H. \tag{3.21}$$

*Proof.* Let  $\varphi \in C_b^1(H)$ ,  $t \geq 0$  and  $x \in H$ . Then by (3.15) we have for any  $h \in H$ ,

$$\langle DP_t^{\alpha,\beta} \varphi(x), h \rangle = \mathbb{E} [\langle D\varphi(X_{\alpha,\beta}(t, x)), \eta_{\alpha,\beta}^h(t, x) \rangle ],$$

and so the conclusion follows from (3.19) and the arbitrariness of  $h$ . □

We give now an infinite dimensional generalization of the Bismut–Elworthy formula [1, 15], which we shall use later. For this we need another assumption on  $B$ .

**Hypothesis 3.5.** *We have  $\text{Ker } B = \{0\}$ . Moreover there is  $\gamma \in [0, 1)$  and a linear operator  $V \in L(H)$  such that  $B^{-1} = VA^{\gamma/2}$ .*

Notice that Hypothesis 3.5 is obviously fulfilled in the Example 2.5, whereas it is fulfilled in the Example 2.6 provided  $1/2 < \delta < 1$ .

The following result was proved in [9] when  $B = 1$  and in [3] in the case of general reaction-diffusion systems.

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<sup>5</sup> $C_b^1(H)$  is the space of all mappings  $\varphi : H \rightarrow \mathbb{R}$  which are uniformly continuous and bounded together with their first derivatives. For any  $C_b^1(H)$  we set  $\|\varphi\|_1 = \|\varphi\|_0 + \sup_{x \in H} |D\varphi(x)|$ .

**Proposition 3.6.** *Assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold. Let  $\alpha > 0, \beta > 0$ . Then for any  $\varphi \in C_b(H)$  and any  $t > 0$  we have  $P_t^{\alpha, \beta} \varphi \in C_b^1(H)$  and, for any  $h \in H$ ,*

$$\langle DP_t^{\alpha, \beta} \varphi(x), h \rangle = \frac{1}{t} \mathbb{E} \left[ \varphi(Y_{\alpha, \beta}(t, x)) \int_0^t \langle B^{-1} \eta_{\alpha, \beta}^h(s, x), dW(s) \rangle \right], \quad x \in H, \quad (3.22)$$

where  $Y_{\alpha, \beta}(t, x)$  and  $\eta_{\alpha, \beta}^h(s, x)$  are the mild solutions of (3.8) and (3.16) respectively.

**Corollary 3.7.** *Assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold. Then there exists a constant  $k_\gamma > 0$  such that for any  $\varphi \in C_b(H)$ ,  $t > 0$ ,  $\alpha > 0, \beta > 0$  and for any  $h \in H$ , we have*

$$|DP_t^{\alpha, \beta} \varphi(x)| \leq k_\gamma t^{-(1+\gamma)/2} e^{-\omega t} \|\varphi\|_0, \quad x \in H, t > 0. \quad (3.23)$$

*Proof.* By (3.22), using the Hölder inequality, we have

$$|\langle DP_t^{\alpha, \beta} \varphi(x), h \rangle|^2 \leq \frac{1}{t^2} \|\varphi\|_0^2 \|V\|^2 \int_0^t \mathbb{E} |(-A)^{-\gamma/2} \eta_{\alpha, \beta}^h(s, x)|^2 ds. \quad (3.24)$$

Now, by Proposition 3.3 it follows that

$$|\langle DP_t^{\alpha, \beta} \varphi(x), h \rangle|^2 \leq \frac{2^{-\gamma}}{t^{1+\gamma}} \|V\|^2 \|\varphi\|_0^2 |h|^2,$$

and so, by the arbitrariness of  $h$ , we find

$$|DP_t^{\alpha, \beta} \varphi(x)| \leq 2^{-\gamma/2} t^{-(1+\gamma)/2} \|V\|^2 \|\varphi\|_0^2, \quad x \in H, t > 0. \quad (3.25)$$

Now let  $t \geq \varepsilon > 0$ . Then  $DP_t^{\alpha, \beta} \varphi = DP_{t-\varepsilon}^{\alpha, \beta} P_\varepsilon^{\alpha, \beta} \varphi$ . By Proposition 3.6 we have  $P_\varepsilon^{\alpha, \beta} \varphi \in C_b^1(H)$ , so that, in view of (3.21),

$$|DP_t^{\alpha, \beta} \varphi(x)| \leq e^{-\omega(t-\varepsilon)} \|P_\varepsilon^{\alpha, \beta} \varphi\|_1, \quad x \in H, t \geq \varepsilon.$$

Finally, by (3.25),

$$|DP_t^{\alpha, \beta} \varphi(x)| \leq 2^{-\gamma/2} \varepsilon^{-(1+\gamma)/2} e^{-\omega(t-\varepsilon)} \|V\|^2 \|\varphi\|_0, \quad x \in H, t \geq \varepsilon. \quad (3.26)$$

The conclusion follows by (3.25), and (3.26).  $\square$

### 3.3. A perturbation result

Here we assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold and that  $G$  is Lipschitz continuous and bounded. Moreover we fix  $\alpha > 0$ ,  $\beta > 0$  and  $\varphi \in C_b(H)$

The goal of this subsection is to obtain an estimate for  $DT_t^{\alpha, \beta} \varphi$  independent of  $\alpha$  and  $\beta$ , similar to the estimate (3.23) proved in Corollary 3.7 (where is essential the factor  $e^{-\omega t}$ ). This estimate will be needed to prove (1.5), see next section.

To this purpose we cannot follow the method used to prove Corollary 3.7. Assume in fact, to simplify, that  $G \in C_b^1(H; H)$ . Then, by proceeding as before we arrive at the following estimate

$$|DT_t^{\alpha, \beta} \varphi(x)| \leq k_\gamma t^{-(1+\gamma)/2} e^{(\|F\|_1 - \omega)t} \|\varphi\|_0, \quad \varphi \in C_b(H), x \in H, t \geq 0. \quad (3.27)$$

If  $\|F\|_1 - \omega > 0$  we cannot conclude that  $|DT_t^{\alpha, \beta} \varphi(x)|$  vanishes as  $t \rightarrow +\infty$ , which is essential to prove (1.5).

For this reason we use a different method by considering the following Kolmogorov equation (that was introduced in a different setting in [11]).

$$\begin{cases} D_t v_{\alpha,\beta}(t, x) = N_{\alpha,\beta} v_{\alpha,\beta}(t, x) + \langle G(x), Dv_{\alpha,\beta}(t, x) \rangle, & t > 0, x \in H, \\ v_{\alpha,\beta}(0, x) = \varphi(x), & \varphi \in C_b(H), x \in H, \end{cases} \tag{3.28}$$

which can be written in the following integral form

$$v_{\alpha,\beta}(t, \cdot) = P_t^{\alpha,\beta} \varphi + \int_0^t P_{t-s}^{\alpha,\beta} (\langle G(\cdot), Dv_{\alpha,\beta}(s, \cdot) \rangle) ds. \tag{3.29}$$

We shall solve equation (3.29) by a fixed point argument in the space  $Z_T$  consisting of the set of all mappings  $u : [0, T] \times H \rightarrow \mathbb{R}$  such that

- (i)  $u \in B_b([0, T] \times H)$ <sup>6</sup>.
- (ii)  $u(t, \cdot) \in C_b^1(H)$  for all  $t > 0$ .
- (iii)  $\sup_{t \in (0, T]} t^{(1+\gamma)/2} \|u(t, \cdot)\|_1 < +\infty$ .
- (iv) for all  $x \in H$ ,  $Du(\cdot, x)$  is measurable.

Then we shall show that

$$v_{\alpha,\beta}(t, x) = T_t^{\alpha,\beta} \varphi(x), \quad x \in H, t \geq 0. \tag{3.30}$$

It is easy to check that  $Z_T$ , endowed with the norm

$$\|u\|_{Z_T} := \|u\|_0 + \sup_{t \in (0, T]} t^{(1+\gamma)/2} \|u(t, \cdot)\|_1,$$

is a Banach space.

**Proposition 3.8.** *Assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold and that  $G$  is Lipschitz continuous and bounded. Then for any  $\varphi \in C_b(H)$  there is a unique solution  $v_{\alpha,\beta} \in Z_T$  of equation (3.29). Moreover*

$$v_{\alpha,\beta}(t, x) = T_t^{\alpha,\beta} \varphi(x), \quad x \in H, t \geq 0.$$

*Proof.* We write equation (3.29) as

$$v_{\alpha,\beta} = P_t^{\alpha,\beta} \varphi + \gamma(v_{\alpha,\beta}),$$

where

$$\gamma(v)(t, \cdot) = \int_0^t P_{t-s}^{\alpha,\beta} (\langle G(\cdot), Dv(s, \cdot) \rangle) ds, \quad t \geq 0, v \in Z_T.$$

We first notice that  $P_t^{\alpha,\beta} \varphi$  belongs to  $Z_T$  in view of (3.22). Now we are going to show that  $\gamma$  is a contraction on  $Z_T$ , provided  $T$  is sufficiently small. Then the conclusion will follow by a standard fixed point argument. We have in fact

$$|\gamma(v)(t, x)| \leq \|G\|_0 \int_0^t |Dv(s, \cdot)(x)| ds \leq 2t^{(1+\gamma)/2} \|v\|_{Z_T}, \quad x \in H, t \geq 0,$$

---

<sup>6</sup> $B_b([0, T] \times H)$  is the space of all mappings  $u : [0, T] \times H \rightarrow \mathbb{R}$  which are Borel and bounded.

and, since

$$D\gamma(v)(t, x) = \int_0^t D[P_{t-s}^{\alpha, \beta}(\langle G(\cdot), Dv(s, \cdot)(x) \rangle)] ds, \quad x \in H, t \geq 0,$$

we have

$$\begin{aligned} t^{(1+\gamma)/2} |D\gamma(v)(t, x)| &\leq t^{(1+\gamma)/2} \int_0^t \|D[P_{t-s}^{\alpha, \beta}(\langle G(\cdot), Dv(s, \cdot) \rangle)]\|_0 ds \\ &\leq t^{(1+\gamma)/2} \|G\|_0 \int_0^t (t-s)^{-(1+\gamma)/2} s^{-(1+\gamma)/2} ds \|v\|_{Z_T} \\ &= t^{(1+\gamma)/2} \|G\|_0 \int_0^1 (1-s)^{-(1+\gamma)/2} s^{-(1+\gamma)/2} ds \|v\|_{Z_T}. \end{aligned}$$

Thus  $\gamma$  maps  $Z_T$  into  $Z_T$  and it is a contraction, provided  $T$  is sufficiently small, as required.

The last statement follows by a standard argument, see [7].  $\square$

**Corollary 3.9.** *Assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold and that  $G$  is Lipschitz continuous and bounded. Then there exists  $c_{1,\gamma} > 0$  such that for any  $\varphi \in C_b(H)$  and any  $t > 0$  we have  $T_t^{\alpha, \beta} \varphi \in C_b^1(H)$  and*

$$|DT_t^{\alpha, \beta} \varphi(x)| \leq c_{1,\gamma} t^{-(1+\gamma)/2} e^{-\omega t} \|\varphi\|_0, \quad x \in H, t > 0. \quad (3.31)$$

*Proof.* Let  $\varphi \in C_b(H)$ , then by (3.29) we have

$$DT_t^{\alpha, \beta} \varphi = DP_t^{\alpha, \beta} \varphi + \int_0^t DP_{t-s}^{\alpha, \beta}(\langle G(\cdot), DT_s^{\alpha, \beta} \varphi \rangle) ds.$$

Taking into account (3.23) it follows that for any  $x \in H$

$$|DT_t^{\alpha, \beta} \varphi(x)| \leq \kappa_\gamma t^{-(1+\gamma)/2} e^{-\omega t} \|\varphi\|_0 + ds \kappa_\gamma \|G\|_0 \int_0^t e^{-\omega(t-s)} (t-s)^{-(1+\gamma)/2} \|DT_s^{\alpha, \beta} \varphi\|_0 ds.$$

Then, setting

$$g(t) = e^{\omega t} \|DT_t^{\alpha, \beta} \varphi\|_0, \quad t \geq 0,$$

we have

$$g(t) \leq \kappa_\gamma t^{-(1+\gamma)/2} \|\varphi\|_0 + \kappa_\gamma \|G\|_0 \int_0^t (t-s)^{-(1+\gamma)/2} g(s) ds.$$

By a well known generalization of the Gronwall lemma there exists  $c_\gamma > 0$  such that

$$g(t) \leq ct^{-(1+\gamma)/2}, \quad t > 0.$$

Thus the conclusion follows.  $\square$

**Proposition 3.10.** *Assume that Hypotheses 2.1, 2.2, 2.3 and 3.5 hold and that  $G$  is Lipschitz continuous and bounded. Then there exists  $c > 0$  such that for any  $x, y \in H$ , and any  $\varphi \in C_b(H)$  we have*

$$|T_t \varphi(x) - T_t \varphi(y)| \leq ct^{-(1+\gamma)} e^{-\omega t} \|\varphi\|_0 |x - y|. \quad (3.32)$$

*Proof.* By (3.13) it follows that

$$\lim_{\alpha, \beta \rightarrow 0} T_t^{\alpha, \beta} \varphi(x) = T_t \varphi(x), \quad \varphi \in C_b(H), \quad x \in H.$$

On the other hand, by (3.31) it follows that for any  $x, y \in H$ , and any  $\varphi \in C_b(H)$  we have

$$|T_t^{\alpha, \beta} \varphi(x) - T_t^{\alpha, \beta} \varphi(y)| \leq ct^{-(1+\gamma)} e^{-\omega t} \|\varphi\|_0 |x - y|.$$

Therefore the conclusion follows letting  $\alpha$  and  $\beta$  tend to zero. □

**Remark 3.11.** By Proposition 3.10 it follows that  $T_t$  is strong Feller. This result was proved by Cerrai [4] for general reaction-diffusion systems.

### 3.4. Asymptotic behaviour of $T_t$

We can prove now the main result of the paper.

**Theorem 3.12.** *Assume that Hypotheses 2.1, 2.2, 2.3, and 3.5, hold and that  $G$  is Lipschitz continuous and bounded. Then there exists  $\kappa > 0$  such that for any  $\varphi \in C_b(H)$  we have*

$$\left| T_t \varphi(x) - \int_H \varphi(y) \zeta(dy) \right| \leq \kappa t^{-(1+\gamma)/2} e^{-\omega t} (1 + |x|) \|\varphi\|_0, \quad x \in H. \tag{3.33}$$

Moreover  $\zeta$  is the unique invariant measure for  $T_t$ .

*Proof.* Let  $\varphi \in C_b(H)$ . Then, taking into account the invariance of  $\zeta$ , we obtain

$$\left| T_t \varphi(x) - \int_H \varphi(y) \zeta(dy) \right| = \left| \int_H [T_t \varphi(x) - T_t \varphi(y)] \zeta(dy) \right|.$$

Now by (3.32) we find

$$\left| T_t \varphi(x) - \int_H \varphi(y) \zeta(dy) \right| \leq ct^{-(1+\gamma)/2} e^{-\omega t} \int_H |x - y| \zeta(dy) \|\varphi\|_0.$$

But

$$\int_H |x - y| \zeta(dy) \leq |x| + \int_H |y| \zeta(dy) < +\infty$$

in virtue of (3.5). Therefore (3.33) is proved.

Finally, let  $\eta$  be another invariant measure for  $T_t$ . Then by (3.33) we find,

$$\lim_{t \rightarrow +\infty} \int_H T_t \varphi(x) \eta(dx) = \int_H \varphi(x) d\eta(dx) = \int_H \varphi(x) d\zeta(dx),$$

so that  $\eta = \zeta$ . □

## 4. APPLICATION TO HAMILTON–JACOBI EQUATIONS

We are here concerned with the following Hamilton–Jacobi equation

$$\begin{cases} D_t u = \frac{1}{2} \operatorname{Tr} [BB^* D^2 u] + \langle Ax + F(x) + G(x), Du \rangle - \frac{1}{2} |B^* Du|^2, & x \in D(A) \\ u(0) = \varphi \in C_b(H) \end{cases} \quad (4.1)$$

where  $A, B, F$  and  $G$  fulfill Hypotheses 2.1, 2.2, 2.3, and 3.5 and  $G$  is Lipschitz continuous and bounded.

Equation (4.1) is related with the following optimal control problem:

minimize

$$J(x, z) := \mathbb{E} \left( \int_0^T \left[ \frac{1}{2} |z(t)|^2 \right] dt + \varphi(X(T, x; z)) \right), \quad (4.2)$$

over all  $z \in L^2_W(0, T; L^2(\Omega, H))$  (the Hilbert space of all square integrable processes adapted to  $W$  defined on  $[0, T]$  and with values in  $H$ ), subject to the state equation

$$\begin{cases} dX = (AX + F(X) + G(X) + z(t))dt + BdW_t, & t \geq 0, \\ X(0) = x \in H. \end{cases} \quad (4.3)$$

In fact, as proved in [5], if  $\varphi \in C_b^1(H)$  and  $T > 0$ , equation (4.1) has a mild solution  $u$  and there is a unique optimal control  $z^*$  related to the *optimal state*  $X^*$  by the *feedback* formula

$$z^*(t) = -Du(T - t, X^*(t, x)), \quad t \in [0, T], \quad (4.4)$$

where  $X^*$  is the solution of the *closed loop equation*

$$\begin{cases} dX = (AX + F(X) + G(X) - Du(T - t, X))dt + Q^{1/2}dW(t), & t \in [0, T] \\ X(0) = x \in H. \end{cases} \quad (4.5)$$

Finally, the *optimal cost* is given by

$$J^*(x) = u(T, x), \quad x \in H. \quad (4.6)$$

In [6] the control problem with infinite horizon but with a discount factor in the cost functional, is also solved.

In this section we want to show the existence of the limit of the solution  $u(t, x)$  as  $t \rightarrow +\infty$ . This result can be used to study the infinite horizon problem without discount factor.

First we find an explicit solution of (4.1) by exploiting the special form of the Hamiltonian and using the well known *Hopf transform*. We notice that, in different situations, this method was used in [13].

Namely we set  $u(t, x) = -\log v(t, x)$ ,  $t \geq 0$ ,  $x \in H$ , so that (4.1) becomes

$$\begin{cases} D_t v = \frac{1}{2} \operatorname{Tr} [BB^* D^2 u] + \langle Ax + F(x) + G(x), Du \rangle, & x \in D(A) \\ v(0) = e^{-\varphi}. \end{cases} \quad (4.7)$$

Now the solution  $v$  of (4.7) can be expressed in terms of the transition semigroup  $T_t$  introduced in Section 3 as

$$v(t, x) = T_t[e^{-\varphi}](x), \quad t \geq 0, \quad x \in H. \quad (4.8)$$

Finally by Theorem 3.12 we find the following result:

**Theorem 4.1.** *Assume that Hypotheses 2.1, 2.2, 2.3, and 3.5 hold and that  $G$  is Lipschitz continuous and bounded. Let  $u(t, x)$  be the solution of the Hamilton–Jacobi equation (4.1). Then for any  $\varphi \in C_b(H)$  we have*

$$\lim_{t \rightarrow +\infty} u(t, x) = -\log \int_H e^{-\varphi(y)} \zeta(dy), \quad x \in H, \quad (4.9)$$

where  $\zeta$  is the unique invariant measure for  $T_t$ .

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