

ON THE STRUCTURE OF LINEAR RECURRENT ERROR-CONTROL CODES

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Abstract. We are extending to linear recurrent codes, *i.e.*, to time-varying convolutional codes, most of the classic structural properties of fixed convolutional codes. We are also proposing a new connection between fixed convolutional codes and linear block codes. These results are obtained thanks to a module-theoretic framework which has been previously developed for linear control.

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1. INTRODUCTION

This paper is devoted to various aspects of convolutional codes which are with linear block codes the most popular class of error-control codes (see, *e.g.* [2,4,30,31]). It is based on the well known ties between convolutional codes and linear automatic control (see, *e.g.* [21–23,25–27,33,34,38]).

Our aim is twofold:

- we are extending to linear recurrent codes, *i.e.*, to time-varying convolutional codes, most of the classic structural properties of fixed, *i.e.*, time-invariant, convolutional codes (see, *e.g.* [6,26,34,37]). Although Shannon’s channel coding theorem has been extended to time-varying convolutional codes (see [42]) and not to fixed ones, those time-varying codes were much less utilized in practice than the time-invariant ones (see, nevertheless [26]);
- the connection between fixed convolutional codes and special types of linear block codes, like cyclic codes, which has been the subject of many investigations (see, *e.g.* [31,37,40] and the references therein), is here approached from a new perspective. Our main theorem states that for an arbitrary block code there exists a convolutional code such that all its *frames* (see Sect. 4.2 for a precise definition) are isomorphic to this block code. This leads to families of convolutional codes and to a feedback decoding procedure, which seem to be novel.

The relationship between those two rather independent subjects is a module-theoretic approach to linear control [9,11,12,15,18–20]³, which has been quite useful in practice [16–19,36]⁴. We are utilising some elementary

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³One should also cite some recent works of different natures on the connection between coding and control (see, *e.g.* [7,24,32,38–40]).

⁴See also [13] and the references therein.

notions of difference algebra [5], homological algebra [41], and non-commutative algebra [29, 35], which is most natural in the time-varying case (see, e.g. [9, 11, 12, 15]).

In the first part we define, following [26], *transducers*, i.e., input-output systems, and study their main properties: state-variable representation, controllability, observability, transfer matrices, input-output inversion. In particular, an *encoder* is a right invertible transducer. The second part is devoted to codes. A code, here, is an equivalence class between encoders having the same output. We derive syndrome formers, dual codes, parity check matrices, polynomial and basic encoders, and Forney’s theory in a manner which is often very short thanks to our algebraic setting. We end with the connection with block codes.

The following topics will be developed in future publications:

- constructions of cyclic-like convolutional codes, i.e., convolutional codes which thanks to the results of Section 4 will also benefit from the properties of some types of cyclic codes;
- turbo-codes [1]. They are often given by two convolutional encoders in parallel with an interleaver, and are known to be related to time-varying convolutional codes;
- non-linear tree codes, which correspond to non-linear encoders, i.e., to right invertible non-linear input-output systems [8] (see Sect. 2.6.2);
- cryptography is already known to be related to error-control codes (see, e.g. [45]). Encrypters will be associated to invertible square input-output systems (see Sect. 2.6.2).

2. LINEAR RECURRENT TRANSDUCERS

2.1. Algebraic preliminaries

2.1.1. Difference fields

A *difference field* [5] is a commutative field F , equipped with a *transformation* $\delta : F \rightarrow F$, i.e., a monomorphism. Here δ should be understood as the *delay operator* of one unit of time. A *constant* is an element $c \in F$, such that $c\delta = c$ (mappings are written on the right). The *subfield of constants* of F is the subfield of all constant elements of F . A *field of constants* is a difference field which coincide with its subfield of constants. The *inversive closure* $F^{\mathfrak{A}}$ [5] of F , which is unique up to isomorphism, is the smallest difference overfield of F such that δ is an automorphism. The difference field F is said to be *inversive* if, and only if, $F = F^{\mathfrak{A}}$.

Example 2.1. Let $\mathbb{F}(t)$ be the field of rational functions in the indeterminate t over the field \mathbb{F} , a finite field for instance. With the \mathbb{F} -automorphism $\delta : \mathbb{F}(t) \rightarrow \mathbb{F}(t)$, $t \mapsto t - 1$, $\mathbb{F}(t)$ becomes an inversive difference field, where the subfield of constants is \mathbb{F} .

2.1.2. A principal right ideal ring

The set of polynomials of the form

$$\sum_{\text{finie}} \delta^s a_s \tag{2.1}$$

$a_s \in F$, is a *principal right ideal ring* $F[\delta]$. It is commutative if, and only if, F is a field of constants.

2.2. Input-output system

A *linear system* is a finitely generated right $F[\delta]$ -module, where F is an inversive difference field⁵. A *linear recurrent transducer*, or a *time-varying convolutional transducer*, or a *linear input-output system*, \mathcal{T} is a system with the following properties:

- there is an *input*, i.e., a finite subset $\mathbf{u} = (u_1, \dots, u_k)$ of \mathcal{T} , such that the quotient module $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is torsion. The input will be assumed to be *independent*, i.e., the module $\text{span}_{F[\delta]}(\mathbf{u})$ is free, of rank k ;

⁵This assumption on F being inversive will simplify several further developments. It does not seem to bring any limitation from a practical viewpoint (see, e.g. [26]).

- there is an *output*, i.e., a finite subset $\mathbf{y} = (y_1, \dots, y_n)$ of \mathcal{T} ;
- \mathcal{T} is *causal* (cf. [11]), or *nonanticipative*, i.e., the *semi-linear* [3] mapping $\delta : \mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u}) \rightarrow \mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is injective.

Example 2.2. The transducer $y\delta = u$, i.e., $y(t - 1) = u(t)$, where $k = n = 1$, should obviously be viewed as non-causal. It is also non-causal in our abstract setting. As a matter of fact the quotient module $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is a 1-dimensional F -vector space spanned by an element corresponding to $u(t + 1)$, which is mapped to 0 by δ .

When F is a field of constants, a linear recurrent transducer is called a *fixed*, or *time-invariant*, *convolutional transducer*.

2.3. State-variable representation

When viewed as a F -vector space, the finitely generated torsion module $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is of finite dimension, m . Take a basis $\xi = (\xi_1, \dots, \xi_m)$. The next lemma is clear.

Lemma 2.3. $\xi\delta$ is also a basis.

Corollary 2.4. $\xi = \xi\delta A$, $A \in F^{m \times m}$, $\det(A) \neq 0$.

Take in \mathcal{T} a m -tuple $\eta = (\eta_1, \dots, \eta_m)$ the image of which in $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is ξ . Then Corollary 2.4 yields a *generalised state-variable representation* of the transducer \mathcal{T}

$$\eta = \eta\delta A + \sum_{\mu=0}^{\nu} \mathbf{u}\delta^{\mu} \bar{B}_{\mu} \tag{2.2}$$

$$\mathbf{y} = \xi \bar{C} + \sum_{\text{finite}} \mathbf{u}\delta^{\iota} \bar{D}_{\iota} \tag{2.3}$$

$\bar{B}_{\mu} \in F^{k \times m}$, $\bar{C} \in F^{m \times n}$, $\bar{D}_{\iota} \in F^{k \times n}$. Let ξ' be another basis of $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$. Thus $\xi' = \xi P$, $P \in F^{m \times m}$, $\det(P) \neq 0$. Take a m -tuple $\eta' = (\eta'_1, \dots, \eta'_m)$ in \mathcal{T} the image of which in $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is ξ' . Then

$$\eta' = \eta + \sum_{\text{finite}} \mathbf{u}\delta^{\iota} Q_{\iota} \tag{2.4}$$

$Q \in F^{k \times m}$. Note that (2.4) is input-dependent. If, in (2.2), $\nu \geq 2$ and $\bar{B}_{\nu} \neq 0$, set

$$\eta = \tilde{\eta} - \mathbf{u}\delta^{\nu-1} (\bar{B}_{\nu} A^{-1} \delta^{-1}).$$

It yields

$$\tilde{\eta} = \tilde{\eta}\delta A + \sum_{\mu=0}^{\nu-1} \mathbf{u}\delta^{\mu} \tilde{B}_{\mu}.$$

If $\bar{B}_0 \neq 0$, setting

$$\tilde{\eta} = \bar{\eta} + \mathbf{u} \bar{B}_0$$

yields

$$\bar{\eta} = \bar{\eta}\delta + \sum_{\mu=1}^{\nu-1} \mathbf{u}\delta^{\mu} \bar{B}_{\mu}.$$

We have proved the following theorem which is a time-varying generalisation of [11].

Theorem 2.5. *A causal linear recurrent transducer may be given the Kalman state-variable representation*

$$\mathbf{x} = \mathbf{x}\delta A + \mathbf{u}\delta B \tag{2.5}$$

$$\mathbf{y} = \mathbf{x} C + \sum_{\text{finite}} \mathbf{u}\delta^\iota D_\iota \tag{2.6}$$

where $\mathbf{x} = (x_1, \dots, x_m)$, $m = \dim_F(\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u}))$, $A \in F^{m \times m}$, $\det A \neq 0$, $B \in F^{k \times m}$, $C \in F^{m \times n}$, $D_\iota \in F^{k \times m}$.

Remark 2.6. Setting $\mathbf{x} = \bar{\mathbf{x}} - \mathbf{u} (BA^{-1}\delta^{-1})$ yields $\bar{\mathbf{x}} = \bar{\mathbf{x}}\delta A + \mathbf{u} (BA^{-1}\delta^{-1})$ which might also be interesting in some applications.

2.4. Controllability and observability

2.4.1. Controllability

The transducer \mathcal{T} is called *controllable* if, and only if, the module \mathcal{T} free. The next result is an extension to (2.5) of the classic Kalman controllability criterion (compare with [43]):

Proposition 2.7. *The transducer \mathcal{T} is controllable if, and only if, the matrix*

$$(B, B\delta A, \dots, B(\delta A)^{m-1})$$

is of rank m .

Proof. It is easy to check that $\text{rk}(B, B\delta A, \dots, B(\delta A)^{m-1}) < m$ is equivalent to the existence of a nontrivial torsion submodule of \mathcal{T} . □

2.4.2. Observability

The transducer \mathcal{T} is called *observable* if, and only if, the modules \mathcal{T} and $\text{span}_{F[\delta]}(\mathbf{u}, \mathbf{y})$ coincide. The next result is an extension to (2.5, 2.6) of the classic Kalman observability criterion (compare with [43]):

Proposition 2.8. *The transducer \mathcal{T} is observable if, and only if, the matrix*

$$({}^t C, {}^t C \delta {}^t A^{-1}, \dots, {}^t C (\delta {}^t A^{-1})^{m-1})$$

where ${}^t \bullet$ indicates the transpose matrix, is of rank m .

Proof. Utilize $\mathbf{x}\delta = \mathbf{x} A^{-1} - \mathbf{u}\delta B A^{-1}$ for expressing $\mathbf{y}\delta^\iota$, $\iota = 1, \dots, m - 1$, as F -linear combinations of the components of \mathbf{x} and $\mathbf{u}\delta^\kappa$, $\kappa \geq 0$. □

Remark 2.9. By utilizing the inverse $A\delta^{-1}$ of δA^{-1} , Proposition 2.8 becomes

$$\text{rk} ({}^t C, {}^t C A\delta^{-1}, \dots, {}^t C (A\delta^{-1})^{m-1}) = m.$$

2.5. Transfer matrices

2.5.1. Definition

Let $F(\delta)$ be the quotient field of $F[\delta]$ which is a right Ore ring. The right $F(\delta)$ -vector space $\hat{\mathcal{T}} = \mathcal{T} \otimes_{F[\delta]} F(\delta)$ is called the *transfer vector space* of \mathcal{T} [12]. The $F[\delta]$ -linear mapping $\mathcal{T} \rightarrow \hat{\mathcal{T}}$, $\tau \mapsto \hat{\tau} = \tau \otimes 1$, is the (*formal*) *Laplace transform* [12]. Its kernel is the torsion submodule of \mathcal{T} . It is thus injective if, and only if, the module \mathcal{T} is free. As \mathbf{u} is independent, $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_k)$ is a basis of $\hat{\mathcal{T}}$. It yields

$$\hat{\mathbf{y}} = (\hat{y}_1, \dots, \hat{y}_n) = \hat{\mathbf{u}} G \tag{2.7}$$

where $G \in F(\delta)^{m \times n}$ is the *rational transfer matrix*, or the *rational generating matrix*, of the transducer (compare with [28]). When $k = n = 1$, G is called a *rational transfer*, or *generating function*.

Remark 2.10. Note that the dimension of $\hat{\mathcal{T}}$ is equal to the rank of \mathcal{T} .

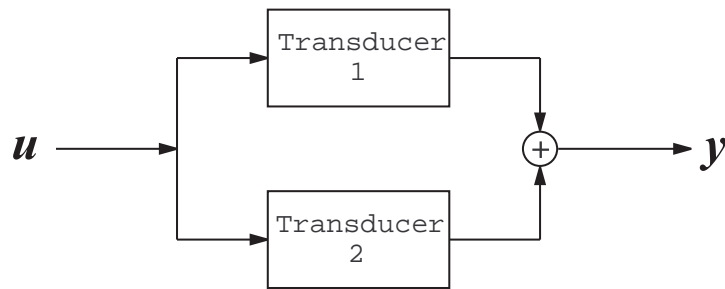
Any element of $F(\delta)$ may be written as a Laurent series $\sum_{\nu \geq \nu_0} \delta^\nu a_\nu$, $a_\nu \in F$, $\nu_0 \in \mathbb{Z}$. It is said to be *causal* if, and only if, $\nu_0 \geq 0$. The matrix G is said to be *causal* if, and only if, all its entries are causal.

Theorem 2.11. *Any causal linear recurrent transducer possesses a rational causal transfer matrix. Conversely, any rational causal matrix is the transfer matrix of a causal linear recurrent transducer, which is controllable and observable.*

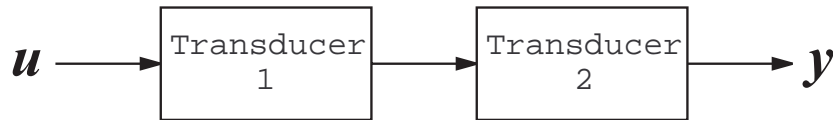
Proof. The first part is an immediate consequence of the definition of causality in Section 2.2 and of the input-output relation (2.7). For the second part, utilize the right coprime factorization $G = ND^{-1}$, $N \in F[\delta]^{k \times n}$, $D \in F[\delta]^{n \times n}$, where D is invertible (see [12]). The transfer matrix of the transducer $\mathbf{y}D = \mathbf{u}N$, which is both controllable and observable (see [12]), is G . □

2.5.2. *Interconnection*

Let $h_v : \Sigma \rightarrow \mathcal{S}_v$, $v \in \Upsilon$, be a morphism of systems, *i.e.*, of finitely generated right $F[\delta]$ -modules. The corresponding fibered sum is a *system interconnection* (*cf.* [14]). The parallel interconnection



and the series interconnection



are particular instances of system interconnections. The proof of the following result is straightforward.

Proposition 2.12. *The transfer matrix of the parallel (resp. series) interconnection of linear recurrent transducers is the sum (resp. product) of the transfer matrices.*

Remark 2.13. Interconnections as simple as those in Proposition 2.12 may lead to a lost of controllability or observability⁶ which is not readable *via* transfer matrices [14].

2.6. **Input-output inversion**

2.6.1. *General results*

The *output rank* [8] of the transducer \mathcal{T} is $\varrho = \text{rk}(\text{span}_{F[\delta]}(\mathbf{y}))$. Obviously $0 \leq \varrho \leq \min(k, n)$. The transducer \mathcal{T} is said to be *right invertible* (*resp. left invertible*) if, and only if, $\varrho = k$ (*resp.* $\varrho = n$).

⁶The continuous-time examples and the results in [14] (see also the references therein) may trivially be adapted to our discrete-time context.

Proposition 2.14. \mathcal{T} is right invertible, if and only if, the quotient module $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{y})$ is torsion.

Proof. $\text{rk}(\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{y})) = \text{rk}(\mathcal{T}) - \varrho$. Since $\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{u})$ is torsion, $\text{rk}(\mathcal{T}) = \text{rk}(\text{span}_{F[\delta]}(\mathbf{u})) = k$. Thus $\text{rk}(\mathcal{T}/\text{span}_{F[\delta]}(\mathbf{y})) = 0$ if, and only if, $\varrho = k$. □

In a more down to earth language, Lemma 2.14 means that \mathbf{u} may be obtained from \mathbf{y} thanks to difference equations. The example $y = u\delta$, where $k = n = 1$, shows that the right inverse transducer is not generally causal. Left invertibility means that the components of \mathbf{y} are $F[\delta]$ -linearly independent.

The next proposition is an immediate consequence of Remark 2.10.

Proposition 2.15. The linear recurrent transducer \mathcal{T} is right (resp. left) invertible if, and only if, its transfer matrix is right (resp. left) invertible.

Corollary 2.16. If the linear recurrent transducer \mathcal{T} is right (resp. left) invertible, then $n \geq k$ (resp. $n \leq k$).

If $k = n$, the transducer is said to be *square*. Then right and left invertibilities coincide. An *invertible* square transducer is right and left invertible.

2.6.2. Encoders

A linear recurrent transducer, which is right invertible, is called a *linear recurrent encoder*, or a (*time-varying*) *convolutional encoder*. If F is a field of constants, it is called a (*fixed*) *convolutional encoder*⁷. A square encoder is called a *linear recurrent encrypter*.

2.7. Some useful constructions

2.7.1. Blocking

For any integer $\Omega > 1$, $F[\delta^\Omega] \subset F[\delta]$. Thus any right $F[\delta]$ -module \mathbf{M} may also be viewed as a right $F[\delta^\Omega]$ -module \mathbf{M}_Ω called the Ω^{th} -*blocking*, or Ω^{th} -*interleaving*, module.

Lemma 2.17. $\text{rk}(\mathbf{M}_\Omega) = \Omega \text{rk}(\mathbf{M})$.

Proof. If ξ_1, \dots, ξ_ℓ are $F[\delta]$ -linearly independent elements in \mathbf{M} , then $\xi_1, \xi_1\delta, \dots, \xi_1\delta^{\Omega-1}, \dots, \xi_\ell, \xi_\ell\delta, \dots, \xi_\ell\delta^{\Omega-1}$ are $F[\delta^\Omega]$ -linearly independent. □

The Ω^{th} -blocking transducer, or Ω^{th} -interleaving transducer, \mathcal{T}_Ω of \mathcal{T} is the linear recurrent transducer defined by

- its module is the Ω^{th} -blocking module \mathcal{T}_Ω ;
- its input and output are respectively $(\mathbf{u}, \mathbf{u}\delta, \dots, \mathbf{u}\delta^{\Omega-1})$ and $(\mathbf{y}, \mathbf{y}\delta, \dots, \mathbf{y}\delta^{\Omega-1})$.

The next result is clear:

Proposition 2.18. If \mathcal{T} is controllable (resp. observable, right invertible, left invertible), then \mathcal{T}_Ω is also controllable (resp. observable, right invertible, left invertible).

2.7.2. Puncturing

Puncturing a linear recurrent transducer \mathcal{T} means taking a linear recurrent transducer \mathcal{T}_P defined by the same module, the same input and by an output which is a subset of \mathbf{y} . The next result is clear:

Proposition 2.19. If \mathcal{T} is controllable (resp. left invertible), then \mathcal{T}_P is also controllable (resp. left invertible). If \mathcal{T} is observable (resp. right invertible), then \mathcal{T}_P is not necessarily observable (resp. right invertible).

⁷Even if F is a finite field, there exists several definitions of convolutional encoders in the existing literature.

3. SOME PROPERTIES OF LINEAR RECURRENT CODES

3.1. Equivalence of encoders and codes

3.1.1. Equivalence

Two linear recurrent encoders with inputs $\mathbf{u} = (u_1, \dots, u_k)$, $\mathbf{u}' = (u_1, \dots, u_{k'})$ and outputs $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{y}' = (y'_1, \dots, y'_{n'})$ are said to be *equivalent* if, and only if, the following conditions are satisfied:

- (1) $n = n'$;
- (2) there exists $\sigma \in S_n$, where S_n is the symmetric group over $\{1, \dots, n\}$, such that the mapping $y_\iota \mapsto y'_{\iota\sigma}$, $\iota = 1, \dots, n$, defines an isomorphism between the modules $\text{span}_{F[\delta]}(\mathbf{y})$ and $\text{span}_{F[\delta]}(\mathbf{y}')$.

Proposition 3.1. *The inputs of two equivalent linear recurrent encoders possess the same number of components.*

Proof. Let ϱ and ϱ' be the output ranks of the encoders \mathcal{T} and \mathcal{T}' . The right invertibility of \mathcal{T} and \mathcal{T}' implies $\varrho = k$ and $\varrho' = k'$. The equivalence of \mathcal{T} and \mathcal{T}' implies $\varrho = \varrho'$. \square

3.1.2. Codes

A *linear recurrent code*, or a (*time-varying*) *convolutional code* is an equivalence class between linear recurrent encoders. From Proposition 3.1, we know already two integers $k, n, 0 < k \leq n$ which are attached to the code, which is therefore called a (n, k) linear recurrent code. Its *rate* is $\frac{k}{n}$. By a slight abuse of language, $\text{span}_{F[\delta]}(\mathbf{y})$ is sometimes called a linear recurrent code, or a (*time-varying*) convolutional code. When F is a finite field of constants, a linear recurrent code is called a (*fixed*) *convolutional code*. A code is said to be *free*, or *controllable*, if, and only if, the module $\text{span}_{F[\delta]}(\mathbf{y})$ is free.

3.2. Syndrome formers

Let \mathcal{F}_n be the free right $F[\delta]$ -module, with basis $\bar{y}_1, \dots, \bar{y}_n$. The mapping $\bar{y}_\iota \mapsto y_\iota, \iota = 1, \dots, n$, defines an epimorphism $\mathcal{F}_n \rightarrow \text{span}_{F[\delta]}(\mathbf{y})$ and the short exact sequence

$$0 \rightarrow \mathcal{F}_{n-k} \rightarrow \mathcal{F}_n \rightarrow \text{span}_{F[\delta]}(\mathbf{y}) \rightarrow 0 \tag{3.1}$$

where \mathcal{F}_{n-k} a free right $F[\delta]$ -module of rank $n - k$. A *syndrome former* of the code is a presentation matrix of $\text{span}_{F[\delta]}(\mathbf{y})$, which corresponds here to the monomorphism $\mathcal{F}_{n-k} \rightarrow \mathcal{F}_n$.

The sequence (3.1) splits, *i.e.*, $\mathcal{F}_n \simeq \mathcal{F}_{n-k} \oplus \text{span}_{F[\delta]}(\mathbf{y})$, if, and only if, the code is free.

3.3. Some properties of free codes

From now on and until the end of the paper codes are assumed to be free⁸. When F is a finite field of constants, a (*fixed*) convolutional code may be defined as a certain $F[\delta]$ -submodule of the $F[\delta]$ -module $\mathcal{L} = \{\sum_{v \geq 0} \delta^v a_{1v}, \dots, \sum_{v \geq 0} \delta^v a_{nv}\}$ of n -tuple of formal power series. The relationship with our approach⁹ is given the $F[\delta]$ -module $\text{Hom}(\text{span}_{F[\delta]}(\mathbf{y}), \mathcal{L})$ of $F[\delta]$ -module morphisms $\Phi = (\phi_1, \dots, \phi_n) : \text{span}_{F[\delta]}(\mathbf{y}) \rightarrow \mathcal{L}, (y_1, \dots, y_n) \mapsto (y_1\phi, \dots, y_n\phi)$ (compare with [38]).

3.3.1. Dual codes and parity check matrices

The image of \mathcal{F}_{n-k} in \mathcal{F}_n is called the *dual code*. A syndrome former of the dual code is called a *parity check matrix* of the code.

Remark 3.2. When F is a finite field of constants, the dual code of a convolutional code is usually defined as for block codes *via* an orthogonality relation. We leave to the reader to construct explicitly the relationship with our definition.

⁸When F is a finite field of constants, a (*fixed*) convolutional code is often defined as a vector subspace of $F(\delta)^{1 \times n}$ (see, *e.g.* [26, 34]). With respect to this transfer matrix setting the freeness may always be assumed.

⁹This is more generally the relationship (see [10]) between our module-theoretic setting and Willems' *behavioral approach* [44].

3.3.2. Polynomial and basic encoders

A controllable and observable encoder \mathcal{E} is said to be *polynomial* if, and only if, \mathbf{u} is a basis of the free module \mathcal{E} . The next property is an immediate consequence of Theorem 2.11:

Proposition 3.3. *A controllable and observable encoder is polynomial if, and only if, the entries of its transfer matrix are polynomial, i.e., belong to $F[\delta]$.*

The polynomial encoder \mathcal{E} is said to be *basic* if, and only if, $\mathcal{E} = \text{span}_{F[\delta]}(\mathbf{y})$. By taking for \mathbf{u} any basis of the free module $\text{span}_{F[\delta]}(\mathbf{y})$ we obtain the

Proposition 3.4. *Any free code admits a basic encoder.*

3.3.3. Systematic encoders

Proposition 3.5. *Any free code admits a systematic encoder, i.e., an encoder where k components of the output are identical to the k components of the input.*

Proof. The result is clear if $k = n$; \mathbf{y} is a basis of $\text{span}_{F[\delta]}(\mathbf{y})$ and can be taken as an input. Assume that the result holds for $n = n_0 \geq k$. Take $n = n_0 + 1$. Since the components of \mathbf{y} are $F[\delta]$ -linearly dependent we may write

$$y_1\gamma_1 + \cdots + y_{n_0+1}\gamma_{n_0+1} = 0 \quad (3.2)$$

where $\gamma_1, \dots, \gamma_{n_0+1} \in F[\delta]$ are right coprime. At least one of the coefficients γ_ι , $\iota = 1, \dots, n_0 + 1$, γ_{n_0+1} for instance, when expressed as a sum (2.1), is such that $a_0 \neq 0$. Apply the assumption to the code spanned by y_1, \dots, y_{n_0} and utilise the causal relation $y_{n_0+1} = -(y_1\gamma_1 + \cdots + y_{n_0}\gamma_{n_0})\gamma_{n_0+1}^{-1}$. \square

3.3.4. Non-catastrophic encoders

The ring of Laurent polynomials $F[\delta, \delta^{-1}]$ is the localized ring of $F[\delta]$ by the multiplicative monoid $\{\delta^s \mid s \geq 0\}$, which satisfies the right Ore condition. The corresponding localized right $F[\delta, \delta^{-1}]$ -module $\mathcal{E} \otimes_{F[\delta]} F[\delta, \delta^{-1}]$ of $\text{span}_{F[\delta]}(\mathbf{u})$ is free, if \mathcal{E} is controllable. The canonical mapping $\mathcal{E} \rightarrow \mathcal{E} \otimes_{F[\delta]} F[\delta, \delta^{-1}]$, $v \mapsto v \otimes 1$, being injective, \mathcal{E} may be considered as a subset of $\mathcal{E} \otimes_{F[\delta]} F[\delta, \delta^{-1}]$. A controllable encoder is said to be *non-catastrophic* if, and only if, \mathbf{u} belongs to $\text{span}_{F[\delta]}(\mathbf{y}) \otimes_{F[\delta]} F[\delta, \delta^{-1}]$. The next result is an immediate consequence of Proposition 3.4.

Proposition 3.6. *Any free code admits a non-catastrophic encoder.*

3.4. Forney's theorem

3.4.1. An important filtration

Define a *filtration* of $F[\delta]$ by setting $\mathbf{F}_\alpha = \{P\delta^\alpha\}$, $\alpha \geq 0$, $P \in F[\delta]$. Thus $F[\delta] = \mathbf{F}_0 \supset \mathbf{F}_1 \supset \dots$. The corresponding filtration for the free module $\text{span}_{F[\delta]}(\mathbf{y})$ is obtained by setting $\mathbf{C}_\alpha = \text{span}_{F[\delta]}(\mathbf{y})\mathbf{F}_\alpha$. Thus $\text{span}_{F[\delta]}(\mathbf{y}) = \mathbf{C}_0 \supset \mathbf{C}_1 \supset \dots$. Any element $x \in \text{span}_{F[\delta]}(\mathbf{y})$ may be written uniquely as a finite sum

$$x = \sum_{\alpha=\nu}^{\mu} \xi_\alpha \delta^\alpha \quad (3.3)$$

where $\xi_\alpha \delta^\alpha$ is *homogeneous*, of *weight* α (ξ_0 is homogeneous of weight 0). The element x is said to be of *order* ν (resp. *degree* μ) if, and only if, $\xi_\nu \neq 0$ (resp. $\xi_\mu \neq 0$). It is homogeneous if, and only if, $\nu = \mu$. The next results are clear.

Lemma 3.7. *The semi-linear linear mapping $\delta^\ell : \mathbf{C}_\alpha \rightarrow \mathbf{C}_{\alpha+\ell}$, $\ell > 0$, is bijective.*

Corollary 3.8. *For any homogeneous element $x_{\alpha+\ell}$ of order $\alpha + \ell$ there exists a homogeneous element x_α of order α such that $x_\alpha \delta^\ell = x_{\alpha+\ell}$.*

Lemma 3.9. *Homogeneous elements of order ν are $F[\delta]$ -linearly independent if, and only if, they are F -linearly independent.*

Corollary 3.10. *The F -vector space $\mathbf{C}_\alpha/\mathbf{C}_{\alpha+1}$ is of dimension k .*

3.4.2. *The result*

Let ε_1 be the highest degree of the components of \mathbf{y} , when written as in (3.3). Let V_1 be the ϖ_1 -dimensional F -vector space spanned by the corresponding homogeneous elements. Choose according to Corollary 3.8 homogeneous elements u_1, \dots, u_{ϖ_1} , of degree 0, such that $V_1 = \text{span}(u_1\delta^{\varepsilon_1}, \dots, u_{\varpi_1}\delta^{\varepsilon_1})$. Let $\varepsilon_2 < \varepsilon_1$ be the first integer such that $u_1\delta^{\varepsilon_2}, \dots, u_{\varpi_1}\delta^{\varepsilon_2}$ does not span the F -vector space spanned by the homogeneous components of order ε_2 in \mathbf{y} . Complete then u_1, \dots, u_{ϖ_1} as above. We obtain a basis $\mathbf{u} = (u_1, \dots, u_m)$ and a corresponding polynomial transfer matrix with lines of degrees¹⁰ $e_1 \leq e_2 \leq \dots \leq e_k$.

We must show that the above basic encoder is *minimal*, i.e., that the degrees $f_1 \leq f_2 \leq \dots \leq f_k$ of the lines of any polynomial generating matrix verify $e_\iota \leq f_\iota$, $\iota = 1, \dots, k$. The next lemma, which is obvious, demonstrates that this result holds true if $k = 1$.

Lemma 3.11. *Take a free $F[\delta]$ -module M of rank 1. Two bases b and b' are related by $b = b'\gamma$, $\gamma \in F$, $\gamma \neq 0$. Let $N \supseteq M$ be another free $F[\delta]$ -module of rank 1. Then, for any basis c of N , $b = b\pi$, $\pi \in F[\delta]$.*

By considering the quotient module $\text{span}_{F[\delta]}(\mathbf{y})/\text{span}_{F[\delta]}(u_1)$, which is free of rank $k - 1$, we obtain the minimality for any $k \geq 2$, assuming that it holds true for $k - 1$.

We have proved:

Theorem 3.12. *For any free linear recurrent code, there exists a basic encoder, called minimal, such that the degrees of the lines of its transfer matrix are $e_1 \leq e_2 \leq \dots \leq e_k$. The degrees $f_1 \leq f_2 \leq \dots \leq f_k$ of the lines of a transfer matrix of any equivalent polynomial encoder verify $e_\kappa \leq f_\kappa$, $\kappa = 1, \dots, k$.*

A corresponding input is called a *Forney input*.

4. A CONNECTION BETWEEN CONVOLUTIONAL AND BLOCK CODES

From now on F is a finite field \mathbb{F}_q of constants. We will therefore be working with free (fixed) linear convolutional codes.

4.1. Sliding block codes

A *sliding presentation* of a free (n, k) linear convolutional code is given by a submodule C of rank k of a free $\mathbb{F}_q[\delta]$ -module E of rank n such that the quotient module E/C is free¹¹. The *sliding (linear) block code of order Ω* of a given sliding presentation is given by the \mathbb{F}_q -vector subspace $C/C\delta^\Omega$ of the \mathbb{F}_q -vector space $E/E\delta^\Omega$. It is obviously a $(n\Omega, k\Omega)$ block code.

Theorem 4.1. *For integers n, k, Ω , $1 \leq k < n$, $\Omega \geq 1$, there exists a free (n, k) convolutional code with a sliding presentation such that its sliding block code of order Ω is an arbitrary $(n\Omega, k\Omega)$ block code.*

Proof. Take an arbitrary $(n\Omega, k\Omega)$ block code defined by a $k\Omega$ -dimensional subspace U of a $n\Omega$ -dimensional of a \mathbb{F}_q -vector space Y . For any integer $\nu \geq 0$, set $Y\delta^{\nu\Omega} = \{y\delta^{\nu\Omega} \mid y \in E\}$. Define the free $\mathbb{F}_q[\delta^\Omega]$ -modules $U = \bigoplus_{\nu \geq 0} U\delta^{\nu\Omega} \subset Y = \bigoplus_{\nu \geq 0} Y\delta^{\nu\Omega}$. Consider now the free $\mathbb{F}_q[\delta]$ -modules $\mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} U \subset \mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} Y$.

Any basis $\underline{\mathbf{u}} = (\underline{u}_1, \dots, \underline{u}_{k\Omega})$ of U may be viewed as a basis of U . By considering a systematic presentation of the block code, we may complete some basis $\underline{\mathbf{u}}$ as a basis $\underline{\mathbf{y}} = (\underline{y}_1, \dots, \underline{y}_{n\Omega})$ of Y ; $\underline{\mathbf{y}}$ may also be viewed as a basis of Y . Take a partition of $\underline{\mathbf{u}}$ consisting of k disjoint sets of Ω elements. Complete it as a partition of $\underline{\mathbf{y}}$ of n disjoint sets of Ω elements. For the subsets $\{(z_\nu^1, \dots, z_\nu^\Omega) \mid \nu = 1, \dots, n$ of the partition, define the submodule $P = \text{span}_{\mathbb{F}_q[\delta]}(\{z_\nu^\kappa\delta - z_{\nu+1}^\kappa \mid \nu = 1, \dots, n; \kappa = 1, \dots, \Omega - 1\})$ of $\mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} U \subset \mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} Y$.

¹⁰The degree of a line is the maximum degree of its entries.

¹¹This is an immediate consequence of the splitting property of the short exact sequence (3.1).

Lemma 4.2. *The quotient $\mathbb{F}_q[\delta]$ -module $E = \mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} Y/P$ is free of rank n . The canonical image C of $\mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} U$ into $\mathbb{F}_q[\delta] \otimes_{\mathbb{F}_q[\delta^\Omega]} Y/P$ is free of rank k .*

Proof. A basis of E is $\{z'_\iota \mid \iota = 1, \dots, n\}$. □

The solution is given by the sliding presentation $C \subset E$. □

4.2. Sketch of a feedback decoding procedure

The $\mathbb{F}_q[\delta, \delta^{-1}]$ -modules $\mathbb{F}_q[\delta, \delta^{-1}] \otimes_{\mathbb{F}_q[\delta]} C$ and $\mathbb{F}_q[\delta, \delta^{-1}] \otimes_{\mathbb{F}_q[\delta]} E$ may be seen as $\mathbb{F}_q[\delta]$ -modules \bar{C} and \bar{E} . For any integer $\alpha \geq 0$, the $(\alpha + 1)^{\text{th}}$ frame¹² is the block code is given by the \mathbb{F}_q -vector subspace $\bar{C}\delta^{(-\alpha-\Omega)}/\bar{C}\delta^{-\alpha}$ of the \mathbb{F}_q -vector space $\bar{E}\delta^{(-\alpha-\Omega)}/\bar{E}\delta^{-\alpha}$. It is clear that all those frames are isomorphic to the sliding block code of order Ω .

A decoding procedure of any frame will of course take advantage of the nature of the sliding block code. Comparing the results for the β^{th} and $(\beta + \ell)^{\text{th}}$, $\beta, \ell \geq 1$, permits some checking if $\ell < \Omega$. This *feedback* type decoding of the convolutional code may be enriched by some *concatenations* (see, e.g. [2, 4, 30]) of the frames.

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¹²This mathematical definition of frames, where the previous codewords are subtracted, differs from the usual meaning in the literature (see, e.g. [2]). Note however that this subtraction of previous codewords is asserted in [2] to be of utmost importance in feedback decoding.

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