

CONTROL OF TRANSONIC SHOCK POSITIONS *

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Abstract. We wish to show how the shock position in a nozzle could be controlled. Optimal control theory and algorithm is applied to the transonic equation. The difficulty is that the derivative with respect to the shock position involves a Dirac mass. The one dimensional case is solved, the two dimensional one is analyzed .

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INTRODUCTION

Sensitivity of the position of the shocks with respect to the parameters of the flow is the problem we would like to investigate here. There are many important applications such as the fluttering of wings. Up to now fluttering is investigated by solving the nonlinear transonic (or Euler) equations at each time step (see [6]), but the sensitivity analysis which follows will make it possible to use the linearized transonic equation instead, thereby reducing greatly the computing time and possibly improving the accuracy.

Godlewski *et al.* [2] have studied the same for the shock tube flow problem and solved it completely for Burger's equation when the sensitivity is with respect to initial data. Giles [5] showed that the adjoint equation of the Euler equation is well posed and continuous across the shock, but to our knowledge no control of the shock position has been tried with Giles' conditions.

The problem is complex; a partial solution was given in [7] but a condition was lacking and the system proposed was incomplete. On the other hand it was shown that the derivative of the transonic potential has a shock (so the derivative velocity has a Dirac) and that it was perfectly computable by automatic differentiation. So we were in the awkward position of having a computer based answer and no understanding.

In this article we present the missing condition for the one dimensional case and an application for the control of the position of a shock in a transonic nozzle.

1. THE TRANSONIC POTENTIAL EQUATION

Consider the Euler equations for compressible perfect isentropic gas in a domain Ω :

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad \partial_t \rho u + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma = 0.$$

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For stationary flows, the following holds

$$\nabla \cdot (\rho u \otimes u) = \rho u \nabla u = \nabla \left(\frac{\rho u^2}{2} \right) - \rho u \times \nabla \times u$$

so that there are irrotational solutions to the stationary Euler equations which then reduce to

$$\nabla \cdot (\rho u) = 0, \quad \nabla \left(\frac{\rho u^2}{2} + \rho^\gamma \right) = 0, \quad \nabla \times u = 0.$$

The second equation gives an algebraic relation between ρ and u^2 :

$$\rho^{\gamma-1} = K - \frac{u^2}{2},$$

where the constant K is fixed by the boundary conditions. The third equation tells us that u derives from a potential ϕ , *i.e.* $u = \nabla \phi$. The first equation, which determines ϕ , is known as the transonic equation. After renormalization, $u \rightarrow u/\sqrt{2K}$, the transonic equation in a domain Ω of boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, reads:

$$\nabla \cdot (\rho u) = 0, \quad u = \nabla \phi, \quad \rho = (1 - |u|^2)^\beta \text{ in } \Omega, \quad \rho \frac{\partial \phi}{\partial n} |_{\Gamma_1} = g, \quad \phi |_{\Gamma_2} = \phi_\Gamma$$

with $\gamma = 1.4$, $\beta = 1/(\gamma - 1) = 2.5$ in air. In short it is:

$$\nabla \cdot ((1 - |\nabla \phi|^2)^\beta \nabla \phi) = 0. \tag{1}$$

Boundary conditions for nozzle flow on $\Gamma = \partial\Omega$ are of two exclusive kinds

$$\frac{\partial \phi}{\partial n} |_{\Gamma_n} = u_n \quad \phi = \phi_d \text{ on } \Gamma_d = \Gamma \setminus \Gamma_n \tag{2}$$

Γ_n must contain the lateral walls of the nozzle Γ_w where the flow is tangent to the walls and so $u_n = 0$. It can also contain either Γ_i is the inflow boundary and/or Γ_o the outflow one. If $\Gamma_n = \Gamma$ then it is necessary to add a compatibility relation on the data and a condition on ϕ

$$\int_\Gamma (1 - |\nabla \phi|^2)^\beta \nabla \phi \cdot n = 0 \quad \int_\Omega \phi = 0 \tag{3}$$

because (1) involves $\nabla \phi$ only and so ϕ would be defined up to a constant only.

An entropy inequality must be added for well posedness (see Glowinski [8] and Nečas [3]):

$$\Delta \phi > -\infty.$$

It is automatically satisfied when u is continuous and also when u is discontinuous with a decreasing jump in the direction of the flow u .

The variational formulation of the problem is

$$\int_\Omega (1 - |\nabla \phi|^2)^\beta \nabla \phi \cdot \nabla w = \int_{\Gamma_n} g w \quad \forall w \in V; \quad \phi - \phi_d \in V \tag{4}$$

where V is the subspace of $H^1(\Omega)$ of functions which are zero on Γ_d or satisfy (3) if Γ_d is empty. The flux g is related to u_n by

$$g = (1 - u_n^2)^\beta u_n.$$

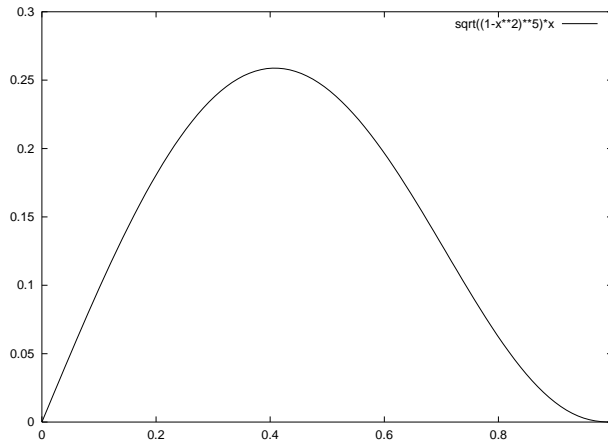


FIGURE 1. Plot of $u \rightarrow (1 - u^2)^{2.5}u$.

2. THE ONE DIMENSIONAL CASE

Consider an horizontal nozzle of length L and slowly varying cross section $S(x)$ where x is the horizontal coordinate. Following Landau–Lifschitz [4], we assume that ϕ is quasi constant in the vertical coordinate y and integrate (4) vertically; it leads to

$$\int_0^L \left(1 - \left|\frac{\partial\phi}{\partial x}\right|^2\right)^\beta \frac{\partial\phi}{\partial x} \cdot \frac{\partial w}{\partial x} S(x) dx = 0 \quad \forall w \in H_0^1(]0, L[). \tag{5}$$

Taking $w(x) = x$ in $(0, b)$ and $w(x) = b(L - x)/(L - b)$ in (b, L) in (5) gives, with $u = \partial\phi/\partial x$,

$$(L - b) \int_0^b (1 - u^2)^\beta u S(x) dx = b \int_b^L (1 - u^2)^\beta u S(x) dx \tag{6}$$

which in turn implies that, for some constant K ,

$$(1 - u^2)^\beta u = \frac{K}{S(x)} \quad \forall x \in]0, L[. \tag{7}$$

In Figure 1 it is seen that there are two or no roots to (7). Denote these u^b, u^p with $u^b \leq u^p$.

Assume that the boundary conditions are such that u_i is on the u^p side of the curve, *i.e.* supersonic entry flow and u_o is on the subsonic side of the curve. If both u_i and u_o are given with the flux conservation condition

$$S(0)u_i(1 - u_i^2)^\beta = S(L)u_o(1 - u_o^2)^\beta$$

then the position of the shock can be anywhere, the problem is ill posed in the sense that there are infinitely many solutions.

If u_i and $\phi_d(L)$ are given, let us construct a solution with a discontinuity (shock) of u at say $x = b$. Using the fact that u is the derivative of ϕ we obtain

$$K = (1 - u_i^2)^\beta u_i S(0) \quad \phi(x) = \begin{cases} \phi(0) + \int_0^x u^p(x) dx & x \leq b \\ \phi(b) + \int_b^x u^b(x) dx & b < x \leq L \end{cases} \tag{8}$$

and

$$\phi_d(L) = \phi(0) + \int_0^b u^p(x)dx + \int_b^L u^b(x)dx.$$

Still b is not fixed! some integral condition like (3) seems to be necessary in this case too, or equivalently $\phi(0) = 0$. If that is so then b is fixed by the equation:

$$\int_0^b u^p(x)dx + \int_b^L u^b(x)dx = \phi_d(L). \tag{9}$$

2.1. Differentiation

Assume now that ϕ_d is function of a scalar parameter a . To evaluate the derivative of the shock position b with respect to a we differentiate (9). It gives

$$\frac{db}{da} = -\frac{\phi'_d(L)}{[u]} \text{ with the notation } [u] = u^b - u^p. \tag{10}$$

The derivative of u with respect to a is a Dirac function at the shock position, with weight $[u]^{-1} = 1/(u^p - u^b)$:

$$\frac{du}{da} = -\phi'_d(L) \frac{\delta(x - b)}{[u]}. \tag{11}$$

Naturally the derivative of ϕ is a Heavyside function

$$\frac{d\phi}{da} = -\phi'_d(L) \frac{H(x - b)}{[u]}. \tag{12}$$

3. THE TWO DIMENSIONAL CASE

Now consider a two dimensional nozzle flow and the transonic equation for ϕ . Assume that u_i is given at the inflow boundary and that $\phi_d(y, a)$ is given at the outflow boundary, where a is a scalar parameter. All derivatives denoted with a prime will be with respect to a .

We consider the case where there is a shock at $\Sigma = \{\vec{b}(y)\}_{y \in Y}$; it was shown in [7] that the derivative ϕ' is not the classical solution of the linearized transonic equation:

$$\int_{\Omega} \rho' \nabla \phi \cdot \nabla w + \int_{\Omega} \rho \nabla \phi' \cdot \nabla w \equiv \int_{\Omega} \left(\rho \left(1 - \frac{2\beta u \otimes u}{1 - |u|^2} \right) \nabla \phi' \right) \cdot \nabla w = 0 \quad \forall w \in V \tag{13}$$

because ϕ' is discontinuous.

Lemma 1. *Let $f(x, y)$ be a piecewise differentiable function, continuous but with discontinuous derivatives at the shock $\Sigma = \{\vec{b}(\tau)\}_{\tau \in T}$. Assume that f and b depend on a scalar parameter a , then the jump across Σ of f' , the derivative with respect to a is*

$$[f']_{\Sigma} = -[\nabla f]_{\Sigma} \cdot b'.$$

Proof. We may express that ∇f has a shock by writing, for some smooth extension \tilde{f}^- of f^- beyond the shock on any curve $x(s)$

$$f(x(s)) = \tilde{f}^-(x(s)) + (s - s_\Sigma)[\nabla f] \cdot \dot{x}(s_\Sigma)H(s - s_\Sigma)$$

where $s \rightarrow x(s)$ is a curve which crosses the shock at $x(s_\Sigma)$ and \dot{x} its s-derivative .

By differentiating this equation with respect to a , Heavyside functions are found on both sides and give

$$[f']|_{x(s_\Sigma)} = -\frac{ds_\Sigma}{da}[\nabla f \cdot \dot{x}(s_\Sigma)]_{x(s_\Sigma)} = [\nabla f \cdot \dot{x}(s_\Sigma)]_{x(s_\Sigma)}b' \cdot \dot{x}(s_\Sigma).$$

□

Applying this result to ϕ gives

$$[\phi'] = -u \cdot b'.$$

Applying this result to ρu , which is continuous across the shock gives

$$[\rho'u + \rho u'] \equiv \left[\rho \left(1 - \frac{2\beta u \otimes u}{1 - |u|^2} \right) u' \right]_\Sigma = -b' \cdot \nabla(\rho u) = 0 \tag{14}$$

because ρu is continuous across the chock.

So to differentiate the transonic equation we must account for the discontinuity of ϕ' , giving

$$\int_\Omega \rho \left(I - \frac{2\beta u \otimes u}{1 - |u|^2} \right) \nabla \phi' \cdot \nabla w + \int_\Sigma cb' \cdot n_\Sigma \frac{\partial w}{\partial n} = 0 \quad \forall w \in V \tag{15}$$

for some c function of ϕ . Note however that the integral on Σ is not well defined because w may not have the required regularity, but Bernardi in [1] showed that it is possible to give a meaning to such an equation for $w \in H^2$ and $\phi' \in H^{-2}$, that its solution is indeed discontinuous and also that its standard finite element approximation converges (\sqrt{h} for linear triangular element, even if the mesh is not Σ -conforming).

In the strong sense it would be interpreted as

$$\begin{aligned} \nabla \cdot \left(\rho \left(I - \frac{2\beta u \otimes u}{1 - |u|^2} \right) \nabla \phi' \right) &= 0 \text{ in } \Omega^- \cup \Omega^+, \\ \left[\rho \left(1 - \frac{2\beta u^2}{1 - |u|^2} \right) \frac{\partial \phi'}{\partial n_\Sigma} \right]_\Sigma &= 0 \quad [\phi']_\Sigma = -[u] \cdot n_\Sigma b' \cdot n_\Sigma. \end{aligned} \tag{16}$$

The problem, already pointed out in [7] is that an equation seems to be missing to determine $b' \cdot n_\sigma$ (or $cb' \cdot n_\sigma$).

In view of the one dimensional case the difficulty seems to be connected to the boundary conditions.

3.1. Boundary conditions revisited

The matrix $I - \frac{2\beta}{1-|u|^2}u \otimes u$ is positive definite at all x where $|u| < u_c \equiv (2\beta + 1)^{-1}$, the speed of sound, otherwise it has a positive and a negative eigenvalue. Therefore (1) is elliptic at subsonic speed and hyperbolic otherwise.

In our example the flow enters the nozzle at supersonic speed, so it makes sense to give *two* boundary conditions on Γ_i ; then ϕ is completely fixed up to the shock. Across the shock, Σ , $\rho u \cdot n_\Sigma$ is continuous so $\frac{\partial \phi}{\partial n}$ is known on Σ and then we will be able to compute ϕ in the subsonic region, after the shock, with ϕ given on Γ_o .

We summarize this in the following:

Conjecture 1. When Ω is a diverging nozzle with supersonic entry flow, u_i and subsonic exit flow u_o (1) is well posed with

$$\frac{\partial \phi}{\partial n}|_{\Gamma_i} = u_i \quad \phi|_{\Gamma_i} = 0 \quad \phi|_{\Gamma_o} = \phi_d.$$

3.2. Differentiation

If this picture is correct then a change in ϕ_d has no effect on the flow upstream of the shock. Consequently, because of the continuity (14) there is an homogeneous Neumann condition on Σ and so the method is

Step 1. Solve

$$\nabla \cdot \left(\rho \left(I - \frac{2\beta u \otimes u}{1 - |u|^2} \right) \nabla \phi' \right) = 0 \text{ in } \Omega^+ \text{ with } \frac{\partial \phi'}{\partial n}|_{\Sigma} = 0 \quad \phi'|_{\Gamma_o} = \phi'_d. \tag{17}$$

Step 2. Set $b' = -\vec{n}_{\Sigma} \frac{\phi'_d}{[u]}|_{\Sigma}$.

The method is however difficult numerically because it requires a triangulation of Ω^+ . Notice that when ϕ'_d is constant then $\phi' = \phi'_d$ is the solution in Ω^+ therefore, even in the multi-dimensional case, the displacement of the shock is as in one dimension given by

$$b' = -\frac{\phi'_d}{[u]}. \tag{18}$$

4. NUMERICAL SIMULATION

To close the system we apply on each streamline the same argument as in the one dimensional case, neglecting the change in streamline position due to the y -derivative of ϕ_d .

4.1. Verification of the hypothesis

Using `freefem+` (<http://www.freefem.org>) we computed the solution of the transonic equation in a symmetric expanding nozzle of equation

$$\Gamma_w = \left\{ (y(x), x) : y(x) = 1 + \frac{1}{8}(3x^2 - 2x^3), x \in (0, 1) \right\}.$$

As the equation is nonlinear we used a Newton algorithm with a small under-relaxation parameter; convergence is obtained with 50 iterations.

We performed two computations with $u_i = 0.5$, one with $a = 0.9$ and the other one with $a = 1$. The level curves of $\frac{\partial \phi}{\partial x}$ for these are reported in Figure 2. In Figure 3 we have a plot of $x \rightarrow \phi(x)|_{\Gamma_w}$. It shows that relation (10) is well verified. It also shows that indeed there is almost no change upstream of the shock and that the approximation of quasi one dimensional flow is reasonable even for this nozzle.

4.2. Control

Now we search for an outflow boundary condition ϕ_d which brings the shock at a given position. In this example we wish to bend the shock top forward, and move the top part by $(0.1 + 0.03) L$ and the bottom part by $(0.1 - 0.03) L$, so we apply the boundary condition $\phi_d(y) = 0.4L(1.1 + 0.2y)$ instead of $\phi_d = 0.4L$ for which the shock was almost at $x = 67/215 * (0.95 + \cos(t))$, $y = 123/215 * \sin(t)$ (shown as a yellow circular arc left of the shock in Fig. 4). The mean displacement is somewhat correct but the bending has succeeded partially only

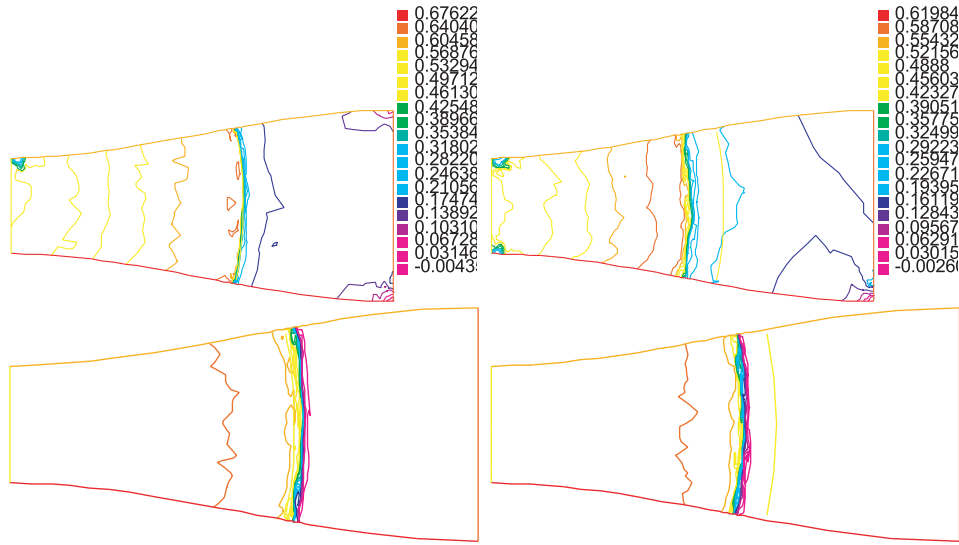


FIGURE 2. Top: level lines of $\frac{\partial \phi}{\partial x}$ for $\phi|_{\Gamma_i} = 0$, $\phi|_{\Gamma_o} = a$ with $a = 1$ (left) and $a = 0.9$ (right). The yellow line indicates the position of the shock for $a = 1$ in the figure where $a = 0.9$. Bottom: same but with a changing Neuman condition on the inflow boundary (going from -0.68 for the left picture to -0.63 for the right one) and a Dirichlet condition on ϕ on the outflow boundary.

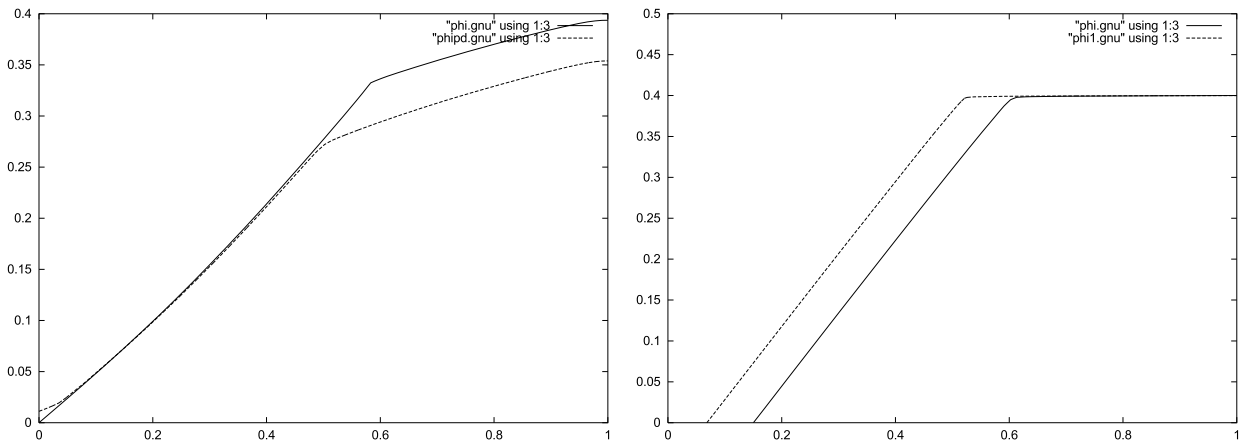


FIGURE 3. Left: plot of $x \rightarrow \phi(x)|_{\Gamma_w}$ for $a = 1$ (top curve) and $a = 0.9$ (bottom curve). Notice that (10) is rather well verified and that the curves are fairly straight, suggesting that the flow is quasi one dimensional. Right: same but the inflow Neuman condition changes and the Dirichlet one on the right is fixed. The velocity after the shock is less than 0.1 so ϕ is almost constant.

because a second shock appears at the top right corner of the nozzle. For this problem we have also solved (17) (with a penalty term upstream of the shock so as to avoid the construction of Ω^+) with $\phi'_d = y$ and its level lines is also shown in Figure 4; these show that ϕ' is almost constant near Σ and hence that the shock cannot be bent very much by y -gradient additions to ϕ_a .

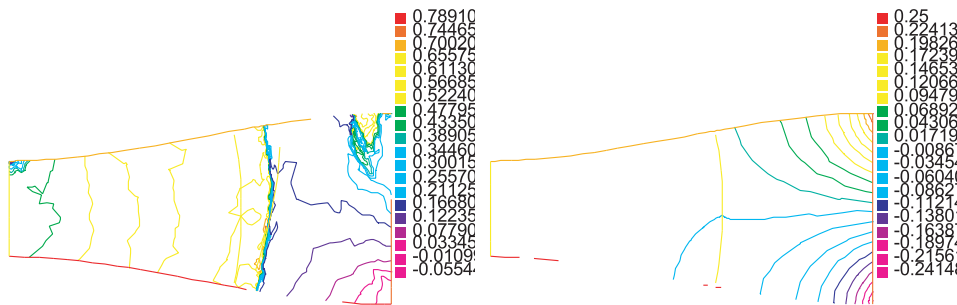


FIGURE 4. An attempt at moving the shock forward and bend it, has partially succeeded. On the left the level lines of $\frac{\partial \phi}{\partial x}$ when $\phi_d = 0.4 * (1.1 + 0.2 * y)$. The old position is at the circular yellow arc, the target position is the bent yellow circular arc. A second shock begins to appear at the top right corner. The derivative ϕ' appears to be quasi-constant near the shock, proving also that the shock can be translated horizontally but cannot be bent very much by changes in ϕ_d .

Another test was done where the Neuman condition g on the inflow boundary is changes while the potential is prescribed to a fixed value on the outflow boundary. The values where chosen to give a similar shock displacement as in the first test. Here it is the opposite, the derivative ϕ'_g of ϕ with respect g is zero downstream of the shock. We could not compute it because the operator of the equation for ϕ'_g is hyperbolic and it requires a dedicated software.

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