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ON THE STRUCTURE OF LAYERS FOR SINGULARLY PERTURBED EQUATIONS IN THE CASE OF UNBOUNDED ENERGY

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Abstract. We consider singular perturbation variational problems depending on a small parameter ε. The right hand side is such that the energy does not remain bounded as $\varepsilon \to 0$. The asymptotic behavior involves internal layers where most of the energy concentrates. Three examples are addressed, with limits elliptic, parabolic and hyperbolic respectively, whereas the problems with $\varepsilon > 0$ are elliptic. In the parabolic and hyperbolic cases, the propagation of singularities appear as an integral property after integrating across the layers.

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1. Introduction

We shall consider several types of singular perturbation problems which may be taken as mathematical model problems to understand the asymptotic behaviour of thin shells as the thickness tends to zero.

Classical abstract singular perturbation theory for symmetric variational problems in real Hilbert spaces is concerned with the following situation:

Let V be a real Hilbert space, and a and b two symmetric forms on V satisfying

$$| a(u,w) | \leq C\|v\|\|w\| \tag{1.1}$$

$$| b(u,w) | \leq C\|u\|\|w\| \tag{1.2}$$

$$a(w,w) \geq 0 \; ; \; b(w,w) \geq 0 \; , \; w \in V \tag{1.3}$$

$$w \in V \; , \; a(w,w) = 0 \Longrightarrow w = 0 \tag{1.4}$$

$$a(w,w) + b(w,w) \geq C\|w\|^2 \; , \; w \in V \tag{1.5}$$

where $\|.\|$ denotes the norm of V and C, c are constants. Let V' be the dual of V. We then consider the variational problems P_ε

$$\begin{cases} u^\varepsilon \in V \\ a(u^\varepsilon, w) + \varepsilon^2 b(u^\varepsilon, w) = \langle f, w \rangle \quad \forall w \in V \end{cases} \tag{1.6}$$

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where $\varepsilon > 0$ is a parameter tending to zero and f is a fixed (*i.e.*, independent of ε) element of V'. It follows from the above hypotheses that P_ε is a Lax–Milgram problem, so that the solution u^ε is well defined.

We note that, according to (1.3, 1.4)

$$\|.\|_{V_a} = a(.,.)^{1/2} \tag{1.7}$$

is a norm on V. We may consider the completion V_a of V with that norm. Clearly

$$V \subset V_a \quad , \quad V' \supset V_a' \tag{1.8}$$

with dense and continuous inclusions. Obviously, $f \in V'$ but it is not necessarily in V_a'. When this happens, the "limit problem" P_0:

$$\begin{cases} u \in V_a \\ a(u,w) = \langle f, w\rangle \quad \forall w \in V_a \end{cases} \tag{1.9}$$

makes sense as a Lax–Milgram problem and u is well defined.

The main abstract classical result in this theory is:

Theorem 1.1. *Under the previous hypotheses, let $f \in V_a'$ and let u^ε and u be the solutions of (1.6) and (1.9) respectively. Then,*

$$u^\varepsilon \to u \quad \text{strongly in } V_a. \tag{1.10}$$

We shall not give here the proof of this theorem, which may be seen for instance in [7] and is analogous to Theorem 2.3 here after. In the case when f is not in V_a', the only general result is:

Proposition 1.2. *Under the previous hypotheses, if $f \notin V_a'$ the energy*

$$\frac{1}{2}\left[a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon)\right] \tag{1.11}$$

of the solution u^ε tends to infinity as $\varepsilon \to 0$.

Proof. If there exists a subsequence such that the energy (1.11) remains bounded, we have

$$a(u^\varepsilon, u^\varepsilon) \le C \quad , \quad b(u^\varepsilon, u^\varepsilon) \le C\varepsilon^{-2} \tag{1.12}$$

so that extracting again a subsequence

$$u^\varepsilon \to u^* \quad \text{weakly in } V_a. \tag{1.13}$$

Let us fix $v \in V$ in (1.6). Using (1.12, 1.13) we pass to the limit:

$$a(u^*, v) = \langle f, v\rangle \tag{1.14}$$

where the left hand side is obviously continuous in the V_a topology, so that the right hand side is too, and this amounts to $f \in V_a'$, whence the conclusion. □

Obviously when $f \in V_a'$, the energy (1.11) remains bounded (as follows immediately from (1.6) with $w = u^\varepsilon$) so that $f \in V_a'$ is a necessary and sufficient condition for the energy (1.11) to remain bounded.

In thin shell theory, the structure of the space V_a and the type of boundary value problem involved in P_0 (1.9) is highly dependent of the geometric properties of the middle surface of the shell and of the kinematic boundary conditions. More precisely, in shell theory hypothesis (1.4) amounts to the geometric rigidity (in the linearized sense) of the middle surface submitted to boundary (fixation) conditions on the boundary or a part of it. The limit problem P_0 (1.9) is of the same type (elliptic, hyperbolic or parabolic) as the points of the middle surface, whereas the sequence of problems P_ε (1.6) are elliptic problems. This implies that boundary conditions are not always adapted to the limit problem, P_0 which is often ill-posed. It follows that V_a is very "large" (in certain cases, said "sensitive" V_a is not contained in distribution spaces, see for instance [9] and [3]) and consequently V_a' is very "small". As a consequence, "usual" loadings f which belongs to V, *i.e.* are admissible for P_ε are not for P_0 as $f \notin V_a'$. The specific behaviour of the asymptotics was considered in several cases (see for instance [5, 6] using formal methods for asymptotic expansions (see [1] for instance). It appears that the asymptotic structure is highly dependent of the loading. Moreover, not only the energy tends to infinity according to Proposition 1.2 but in addition it often concentrates along curves (in particular the characteristics of the limit problem) forming internal (or boundary) layers of high intensity.

The mathematical theory of such problems, including rigorous proof of the convergence to the asymptotic structure is far from well developped. Certain examples with elliptic and parabolic limit problem where considered in [10] and [11]. In the present paper we continue that research which is applied to three model problems with elliptic, parabolic and hyperbolic limits, respectively. They are concerned with an equation (whereas shell theory involves a system) but the total order of the problems P_ε and P_0 as well as the multiplicities of the characteristics are chosen as in shell theory. The methods are analogous to those of [11] which contains model problems of lower orders than those considered here, in the cases of elliptic and parabolic limits. In fact, [11] may be considered as an elementary version of the present paper, but both papers are in fact independent.

In order to choose a diversity of examples of loadings more or less singular (then belonging to V' but not to V_a'), along $x_2 = 0$ in the plane x_1, x_2 for instance, we consider the classical sequence of distribution of x_2 with increasing singularity.

$$...x_2 Y(x_2), \quad Y(x_2), \quad \delta(x_2), \quad \delta'(x_2)... \tag{1.15}$$

where Y and δ denote the Heaviside step function and the Dirac mass. Then loadings will be chosen of the form :

$$...Y(x_2)F(x_1), \quad \delta(x_2)F(x_1), ... \tag{1.16}$$

with F of class L^2 in most cases.

In order to study the phenomenon of formation of a layer along a curve in the solution u^ε as ε tends to zero, we perform a dilatation of the normal coordinate to the curve. The problem P_ε becomes another one, noted $\mathcal{P}_\eta$ in the new coordinate system. The dilatation is obviously anisotropic, so that the asymptotic behaviour of the energies is different in both problems, as boundedness deals with different terms of the expression of the energy. With an appropriate choice of the scaling of the dilatation the "new energy" remains bounded and we have a "limit problem" $\mathcal{P}_0$ of $\mathcal{P}_\eta$. This process, which is specific to each problem and each loading, may be interpreted in terms of the inner and outer matched asymptotic expensions (see [1, 4, 12] for instance). P_ε and $\mathcal{P}_\eta$ are then the expression of the exact $\varepsilon > 0$ problem in the outer and inner variables, respectively whereas P_0 and $\mathcal{P}_0$ are the outer and inner limits respectively.

The three model problems (with elliptic, parabolic and hyperbolic limits, respectively) are addressed in Sections 2, 3 and 5. Sections 4 and 6 are devoted to the phenomenon of propagation of singularities along the characteristics, which appears in the layer as a global property for an expression which involves integration of the solution across the layer. Finally, Section 7 is devoted to some complements, comments and open problems.

Notations are usual. The convention of summation of repeated indices is used.

2. Model problem for P_ε and P_0 elliptic of order 8 and 4 respectively

In this section we consider elliptic problems in the domain

$$\Omega = (0, \pi) \times (-1, +1) \tag{2.1}$$

of the variable $x = (x_1, x_2)$. The bilinear forms are

$$a(u, w) = \int_\Omega \partial_{ij} u \; \partial_{ij} w \; \mathrm{d}x \tag{2.2}$$

$$b(u, w) = \int_\Omega \partial_{ijkh} u \; \partial_{ijkh} w \; \mathrm{d}x. \tag{2.3}$$

When $\varepsilon > 0$, the energy space is

$$V = H_0^4(\Omega). \tag{2.4}$$

The classical formulation of P_ε (1.6) is:

$$(\Delta^2 + \varepsilon^2 \Delta^4) u^\varepsilon = f \quad \text{in } \Omega \tag{2.5}$$

$$u^\varepsilon = \partial_n u^\varepsilon = \partial_n^2 u^\varepsilon = \partial_n^3 u^\varepsilon \quad \text{on } \partial\Omega \tag{2.6}$$

where ∂_n denotes differentiation with respect to the normal.

Classically, using the Poincaré inequality for the function and the derivatives of order 1, 2 and 3, we have:

$$C \|w\|_V^2 \geq a(w, w) + \varepsilon^2 b(w, w) \geq c\varepsilon^2 \|w\|_V^2 \quad w \in V \tag{2.7}$$

so that the coerciveness constant tends to zero at the ratio ε^2. Obviously, $a(w, w)^{1/2}$ is a norm on V and the completion of V with this norm is

$$V_a = H_0^2(\Omega). \tag{2.8}$$

The classical formulation of P_0 (1.9) is

$$\Delta^2 u = f \quad \text{in } \Omega \tag{2.9}$$

$$u = \partial_n u = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

We shall consider two possibilities of right hand sides, chosen in the hierarchy (1.16):

$$a) \quad f = \delta''(x_2) \; F(x_1) \tag{2.11}$$

$$b) \quad f = \delta'''(x_2) \; F(x_1) \tag{2.12}$$

where F is taken in $L^2(0, \pi)$. We have

$$a) \quad \langle f, w \rangle = \int_0^\pi F(x_1) \partial_2^2 w(x_1, 0) \mathrm{d}x_1 \tag{2.13}$$

$$b) \quad \langle f, w \rangle = -\int_0^\pi F(x_1) \partial_2^3 w(x_1, 0) \mathrm{d}x_1 \tag{2.14}$$

and it follows from the trace theorem that in the case a (resp. b) (2.13) (resp. (2.14)) is continuous on H^s if and only if $s > 5/2$ (resp. $7/2$). In both cases we have:

$$f \in H^{-4}(\Omega) = V'; \; f \notin H^{-2}(\Omega) = V'_a \tag{2.15}$$

and we are in the framework of Proposition 1.2.

Let us consider, for $\varepsilon > 0$ the change of variables:

$$x_1 = y_1, \quad x_2 = \eta y_2, \quad \eta = \varepsilon^{1/2} \tag{2.16}$$

and of functions

$$a) \;\; u^\varepsilon(x) = \eta v^\eta(y) \tag{2.17}$$
$$b) \;\; u^\varepsilon(x) = v^\eta(y). \tag{2.18}$$

Denoting

$$D_i = \partial/\partial y_i, \quad i = 1, 2 \tag{2.19}$$

and using the classical relations issued from (2.16):

$$\delta''(x_2) = \eta^{-3}\delta''(y_2); \quad \delta'''(x_2) = \eta^{-4}\delta'''(y_2). \tag{2.20}$$

Problem P_ε (1.6) with (2.2, 2.3) and (2.11) or (2.12) becomes the new problem on $\mathcal{P}_\eta$:

$$\begin{cases} \text{Find } v^\eta \in H_0^4(B_\eta), \quad B_\eta = (0,\pi) \times (-1/\eta, +1/\eta) \\ \text{such that } \forall w \in H_0^4(B_\eta) \\ a_0(v^\eta, w) + \eta^2 a_1(v^\eta, w) + \eta^4 a_2(v^\eta, w) + \eta^6 a_3(v_1^\eta, w) + \eta^8 a_4(v^\eta, w) = \langle \Phi, w \rangle \end{cases} \tag{2.21}$$

where:

$$\begin{cases} a_0(v,w) = \displaystyle\int_{B_0} D_2^2 v \; D_2^2 w \; \mathrm{d}y + \int_{B_0} D_2^4 v \; D_2^4 w \; \mathrm{d}y \\ a_1(v,w) = 2\displaystyle\int_{B_0} D_1 D_2 v D_1 D_2 w \; \mathrm{d}y + 4\int_{B_0} D_2^3 D_1 v D_2^3 D_1 w \; \mathrm{d}y \\ a_2(v,w) = \displaystyle\int_{B_0} D_1^2 v D_1^2 w \; \mathrm{d}y + 6\int_{B_0} D_1^2 D_2^2 v D_1^2 D_2^2 w \; \mathrm{d}y \\ a_3(v,w) = 4\displaystyle\int_{B_0} D_1^3 D_2 v \; D_1^3 D_2 w \; \mathrm{d}y \\ a_4(v,w) = \displaystyle\int_{B_0} D_1^4 v D_1^4 w \; \mathrm{d}y \end{cases} \tag{2.22}$$

and

$$a) \;\; \langle \Phi, w \rangle = \int_0^\pi F(y_1) D_2^2 w(y_1, 0) \mathrm{d}y_1 \tag{2.23}$$

$$b) \;\; \langle \Phi, w \rangle = -\int_0^\pi F(y_1) D_2^3 w(y_1, 0) \mathrm{d}y_1. \tag{2.24}$$

Here and in the sequel, v^η as well as test functions w are considered to be extended with value 0 for $| y_2 |> 1/\eta$, so that they belong to $H_0^4(B_0)$ as well, where obviously

$$B_0 = (0, \pi) \times \mathbb{R}$$

and the integrals in (2.22) may be considered either on B_0 or B_η.

Remark 2.1. The scaling (2.16) was defined in order to have in a_0 (2.22), *i.e.* in the bilinear form at the leading order as $\eta \to 0$, two terms coming from the forms a and b of (1.6, 2.2, 2.3). Analogously the scalings (2.17) and (2.18) of the unknowns were defined in order to have in (2.21) a right hand side of the same order (as $\eta \to 0$) as the leading term of the left hand side. These kind of scalings are classical in formal matched inner and outer asymptotic expansion procedures (see for instance [1, 4]). Moreover, equations (1.6) and (2.21) may be considered as the "outer" and "inner" descriptions of the problem. □

We are now constructing an energy space suited for the "limit problem" of (2.21) as $\eta \to 0$. We observe that the left hand side of (2.21) with $\eta = 0$ only involves derivatives with respect to y_2, of order ≥ 2, so that it "ignores" additive functions of the form

$$\alpha(y_1) + \beta(y_1)y_2. \tag{2.25}$$

The same holds true for the right hand side (see (2.23, 2.24)). Let us now consider the space $H_0^4(-1/\eta, 1/\eta)$ of the variable y_2. Each function v is extended as above, with value zero for $| y_2 |> 1/\eta$. We then construct the space of the equivalence classes $\tilde{v}$ of functions defined up to an additive affine function of y_2. Let us denote by $H_0^4(-1/\eta, 1/\eta)/\mathbb{R}^2$ the above defined space of equivalence classes. We then construct the space of functions of class L^2 of the variable y_1 with values in that space; the new space is obviously noted

$$L^2\left((0,\pi)_{y_1};\ H_0^4(-1/\eta, 1/\eta)/\mathbb{R}^2\right). \tag{2.26}$$

We then consider the union of that spaces for $\eta < 1$, and finally we define its completion

$$\mathcal{V} = \mathcal{C} \bigcup_{\eta<1} L^2\left((0,\pi)_{y_1};\ H_0^4(-1/\eta, 1/\eta)/\mathbb{R}^2\right). \tag{2.27}$$

Here the completion is for the norm:

$$\|\tilde{v}\|_{\mathcal{V}} = a_0(\tilde{v}, \tilde{v})^{1/2} \tag{2.28}$$

which makes sense, as its expression in (2.22) shows that it takes the same value for any element of the equivalence class.

We may now define the limit problem $\mathcal{P}_0$:

$$\begin{cases} \text{Find } \tilde{v} \in \mathcal{V} \text{ such that } \forall \tilde{w} \in \mathcal{V}: \\ a_0(\tilde{v}, \tilde{w}) = \langle \Phi, \tilde{w} \rangle \end{cases} \tag{2.29}$$

where the right hand side is the expression (2.23, 2.24) for any element w of the equivalence class $\tilde{w}$. Obviously

$$| \langle \Phi, \tilde{w} \rangle | \leq C \|F\|_{L^2(0,\pi)} \|\tilde{w}\|_{\mathcal{V}} \tag{2.30}$$

so that (2.29) is a variational problem in the Lax–Milgram framework. Then, the solution $\tilde{v}$ is uniquely defined as an element of $\mathcal{V}$ (and equivalently, as a function it is only defined up to additive functions of the form (2.25)).

Remark 2.2. Given an element v of $H_0^4(B_\eta)$, we may construct its extension with value zero to the strip B_0 and consider it up to additive functions of the form (2.25). The equivalence class constructed in this way will be noted $\tilde{v}$ and is obviously an element of $\mathcal{V}$.

The convergence of the solutions v^η of (2.21) to the solution $\tilde{v}$ of (2.29) then takes the following form:

Theorem 2.3. *Let v^η be the solution of $\mathcal{P}_\eta$ (2.21) and $\tilde{v}^\eta$ the corresponding equivalence class defined according to Remark 2.2. Then*

$$\tilde{v}^\eta \to \tilde{v} \quad \text{strongly in } \mathcal{V} \tag{2.31}$$

where $\tilde{v}$ is the solution of (2.29).

Proof. Let us take in (2.21) $w = v^\eta$. According to the above mentioned property that the right hand side is a continuous functional on $\mathcal{V}$ we obtain the estimates

$$\|\tilde{v}^\eta\|_{\mathcal{V}} = a_0(v^\eta, v^\eta)^{1/2} \le C \tag{2.32}$$

$$a_1(v^\eta, v^\eta)^{1/2} \le C\eta^{-1} \tag{2.33}$$

$$a_2(v^\eta, v^\eta)^{1/2} \le C\eta^{-2} \tag{2.34}$$

$$a_3(v^\eta, v^\eta)^{1/2} \le C\eta^{-3} \tag{2.35}$$

$$a_4(v^\eta, v^\eta)^{1/2} \le C\eta^{-4}. \tag{2.36}$$

It follows from (2.32) that there exist $\tilde{v}^* \in \mathcal{V}$ such that (at lest for a subsequence)

$$\tilde{v}^\eta \to \tilde{v}^* \quad \text{weakly in} \mathcal{V}. \tag{2.37}$$

Let us check that $\tilde{v}^*$ is precisely the solution $\tilde{v}$ of (2.29). Let us fix w belonging to $H_0^4(B_\eta)$ for some $\eta = \eta_1$. After extending it with value zero, it also belongs to $H_0^4(B_\eta)$ with $\eta < \eta_1$ and it may be taken as test function for the corresponding problem $\mathcal{P}_\eta$. From (2.37) and (2.33–2.36) we pass to the limit, so that

$$a_0(\tilde{v}^*, w) = \langle \Phi, w \rangle \tag{2.38}$$

with the fixed w. Obviously, we may write $\tilde{w}$ instead of w in (2.38) (see Rem. 2.2). It follows from (2.27) that such test functions are arbitrary in a dense set of $\mathcal{V}$, so that (2.38) also holds for $\tilde{w} \in \mathcal{V}$ and this proves that $\tilde{v}^*$ is the solution of (2.29).

In order to prove the strong convergence, we write the expression:

$$a_0(v^\eta - \tilde{v}, v^\eta - \tilde{v}) + \eta^2 a_1(v^\eta, v^\eta) + \eta^\varepsilon a_2(v^\eta, v^\eta) + \eta^6 a_3(v^\eta, v^\eta) + \eta^8 a_4(v^\eta, v^\eta) \tag{2.39}$$

which, using (2.21) with $\tilde{w} = v^\eta$ and (2.29) with $w = v^\eta$ and $w = v$ equals:

$$\langle \Phi, v^\eta \rangle + \langle \Phi, \tilde{v} \rangle - 2 \langle \Phi, \tilde{v}^\eta \rangle$$

which tends to zero by virtue of (2.37). In particular, the term a_0 of (2.39) tends to zero and (2.31) is proved. □

Remark 2.4. In order to size the geometric interpretation of the convergence Theorem 2.3, it will prove useful to have an "heuristic picture" of the solution of the limit problem in the variable x (for instance in the case a) (2.11)):

$$\Delta^2 u^0 = F(x_1)\delta''(x_2).$$

It is apparent that fourth order derivatives of u^0 will be singular as $F\delta''(x_2)$ along the line $x_2 = 0$. This corresponds to a jump $F(x_1)$ of $\partial_2 u^0$ at $x_2 = 0$. In the same way, in case b) (2.12), u^0 exhibits a jump of value $F(x_1)$ at $x_2 = 0$.

Let us now solve the limit problem (2.29). Obviously it is an ordinary differential equation in y_2 with parameter y_1. Denoting $\tilde{\tilde{v}} = D_2^2\tilde{v}$, the equation associated with (2.29) is

$$\left(D_2^2 + D_2^6\right)\tilde{\tilde{v}} = \begin{cases} F\ \delta'' & a) \\ F\ \delta''' & b) \end{cases}$$

or

$$\left(1 + D_2^4\right)\tilde{\tilde{v}} = \begin{cases} F\ \delta + \alpha + \beta y_2 & a) \\ F\ \delta' + \alpha + \beta y_2 & b) \end{cases} \tag{2.40}$$

but $\tilde{\tilde{v}} \in L^2(\mathbb{R})$ and it follows from (2.40) with $y_2 > 0$ and $y_2 \leq 0$ that necessarily $\alpha = \beta = 0$. It then follows again from (2.40) that $D_2^3\tilde{\tilde{v}}$ (resp. $D_2^2\tilde{\tilde{v}}$) presents a jump of value F at $y_2 = 0$ in the case a) (resp. b). It is not hard to prove that $\tilde{\tilde{v}}$ is completely determined by (2.40) and the condition $\tilde{\tilde{v}} \in L^2(\mathbb{R})$ as this property implies four conditions (two of decreasing at $+\infty$ at two at $-\infty$) to the solution of a fourth order equation. This leads immediately to:

- *Case a* (2.11): $\tilde{\tilde{v}}$ is even and

$$D_2^3\tilde{\tilde{v}}(0^+) = F/2\ ,\ D_2\tilde{\tilde{v}}(0^+) = 0\ ,\ \tilde{\tilde{v}}(+\infty) = 0$$

- *Case b* (2.12): $\tilde{\tilde{v}}$ is odd and

$$D_2^2\tilde{\tilde{v}}(0^+) = F/2\ ,\ \tilde{\tilde{v}}(0^+) = 0\ ,\ \tilde{\tilde{v}}(+\infty) = 0$$

which determines uniquely $\tilde{\tilde{v}} = D_2^2\tilde{v}$. The very unknown $\tilde{v}$ may then by obtained by integration, and is obviously defined up to arbitrary functions of the form (2.25). Nevertheless, certains "jumps between $-\infty$ and $+\infty$" of $\tilde{v}$, very illuminating in relation with Remark 2.4, are well determined.

Indeed, in case a (2.11), because of the exponential decreasing of $V = D_2^2\tilde{v}$ at $\pm\infty$, the action of this distribution on the test function equal to 1 makes sense. Using (2.40) this gives:

$$\left\langle \tilde{\tilde{v}}, 1 \right\rangle = \left\langle F\delta - D_2^4\tilde{\tilde{v}}, 1 \right\rangle = F - \left\langle \tilde{\tilde{v}}, D_2^4 1 \right\rangle = F$$

so that

$$F = \left\langle \tilde{\tilde{v}}, 1 \right\rangle = \left\langle D_2^2\tilde{v}, 1 \right\rangle = \int_{-\infty}^{+\infty} D_2^2\tilde{v}\mathrm{d}y_2 = D_2\ \tilde{v}|_{-\infty}^{+\infty} \tag{2.41}$$

and F is the jump between $-\infty$ and $+\infty$ of the derivative $D_2\tilde{v}$.

In case b(2.12) we have in the same way:

$$\left\langle \tilde{\tilde{v}}, y_2 \right\rangle = \left\langle F\delta' - D_2^4\tilde{\tilde{v}}, y_2 \right\rangle = -F - \left\langle \tilde{\tilde{v}}, D_2^4 y_2 \right\rangle = -F$$

so that

$$F = -\left\langle \tilde{\tilde{v}}, y_2 \right\rangle = -\left\langle D_2^2\tilde{v}, y_2 \right\rangle = \langle D_2\tilde{v}, 1 \rangle = \int_{-\infty}^{+\infty} D_2\tilde{v}\mathrm{d}y_2 = \tilde{v}|_{-\infty}^{+\infty} \tag{2.42}$$

and F is the jump between $-\infty$ and $+\infty$ of $\tilde{v}$.

Coming back to Remark 2.4, we see that $\tilde{v}$ describes the way where the jump of $\partial_2 u^0(x)$ (resp. $u^0(x)$) is replaced by a smooth function of y_2 (as $y_1 = x_1$ is in fact a parameter). This function is only defined up to a function of the form (2.25), but it may "match" the values of $\partial_2 u^0(0\pm)$ (resp. $u^0(0\pm)$) with those of $\tilde{v}(\pm\infty)$. This is the classical property of formal matched asymptotic expansions (see [1, 4] for instance).

Remark 2.5. We saw in (2.42) that $\tilde{v}$ tends to two different limits as y_2 tends to $+\infty$ and $-\infty$. There is no contradiction between this property and the fact that $\tilde{v}$ belongs to $\mathcal{V}$ defined in (2.27). The functions of $H_0^4(-1/\eta, 1/\eta)/\mathbb{R}^2$ are equal to the same affine function in neighbourhoods of $-\infty$ and $+\infty$, but this property is lost in the completion process involved in the definition of $\mathcal{V}$ as one may check easily (see Sect. 7.2 hereafter for other related topics). The same comment applies to the even more singular behaviour (2.41) where $\tilde{v}$ tends to asymptotes of different slope as y_2 tends to $+\infty$ and $-\infty$.

3. Model problem for P_ε elliptic of order 8 and P_0 parabolic of order 4

In this section we consider problems analogous to the previous ones for $\varepsilon > 0$, but the limit problem is parabolic with characteristics (of order of multiplicity four) $x_2 = \text{const}$. The singular right hand side is applied along the characteristic $x_2 = 0$. The domain Ω is, as before, equation (2.1), but the bilinear forms are:

$$a(u, w) = \int_\Omega \partial_1^2 u \partial_1^2 w \, \mathrm{d}x \tag{3.1}$$

$$b(u, w) = \int_\Omega \partial_{ijkh} u \partial_{ijkh} w \, \mathrm{d}x. \tag{3.2}$$

The energy spaces and its duals are:

$$\begin{aligned}
V &= H_0^4(\Omega)\\
V_a &= L^2\left((-1,+1)_{x_2};\ H_0^2(0,\pi)_{x_1}\right)\\
V' &= H^{-4}(\Omega)\\
V_a' &= L^2\left((-1,+1)_{x_2};\ H^{-2}(0,\pi)_{x_1}\right).
\end{aligned}$$

The classical formulations of P_ε is

$$\left(\partial_1^4 + \varepsilon^2 \Delta^4\right) u^\varepsilon = f \quad \text{in } \Omega \tag{3.3}$$

$$u^\varepsilon = \partial_n u^\varepsilon = \partial_n^2 u^\varepsilon = \partial_n^3 u^\varepsilon = 0 \ \text{ on } \partial\Omega \tag{3.4}$$

whereas P_0 is:

$$\partial_1^4 u = f \ \text{ in } \Omega \tag{3.5}$$

$$u = \partial_1 u = 0 \ \text{ on } \ x_1 = 0 \ \text{ and } x_1 = \pi. \tag{3.6}$$

As for the right hand side, there are four possibilities in the hierarchy (1.15), namely $\delta(x_2)$, $\delta'(x_2)$, $\delta''(x_2)$, $\delta'''(x_2)$ such that, multiplied by functions $F(x_1)$ (in L^2 for instance) give functionals belonging to V' but not belonging to V_a'. We shall write them in the condensated form:

$$f = F(x_1)\delta^{(p)}(x_2)\ ;\ p = 0, 1, 2, 3 \tag{3.7}$$

where $\delta^{(p)} = \partial^p \delta$ and $F \in L^2(0, \pi)$.

We then consider the change of variables

$$x_1 = y_1,\ x_2 = \eta y_2,\ \eta = \varepsilon^{1/4} \tag{3.8}$$

and of functions

$$u^\varepsilon(x) = \eta^{p+1} u^\eta(y),\ p = 0, 1, 2, 3 \tag{3.9}$$

so that, using again the notation (2.19), the problem P_ε becomes $\mathcal{P}_\eta$:

$$\begin{cases} \text{Find } v^\eta \in H_0^4(B_\eta), \quad B_\eta = (0,\pi) \times (-1/\eta, +1/\eta) \\ \text{such that } \forall w \in H_0^4(B_\eta) \\ a_0(v^\eta, w) + \eta^2 a_1(v^\eta, w) + \eta^4 a_2(v^\eta, w) + \eta^6 a_3(v_1^\eta, w) + \eta^8 a_4(v^\eta, w) = \langle \Phi, w \rangle \end{cases} \tag{3.10}$$

where

$$a_0(v, w) = \int_{B_0} D_1^2 v D_1^2 w \ \mathrm{d}y + \int_{B_0} D_2^4 v D_2^4 w \ \mathrm{d}y \tag{3.11}$$

$$a_1(v, w) = 4 \int_{B_0} D_2^3 D_1 v D_2^3 D_1 w \ \mathrm{d}y \tag{3.12}$$

$$a_2(v, w) = 6 \int_{B_0} D_1^2 D_2^2 v D_1^2 D_2^2 w \ \mathrm{d}y \tag{3.13}$$

$$a_3(v, w) = 4 \int_{B_0} D_1^3 D_2 v \ D_1^3 D_2 w \ \mathrm{d}y \tag{3.14}$$

$$a_4(v, w) = \int_{B_0} D_1^4 v D_1^4 w \ \mathrm{d}y \tag{3.15}$$

$$\langle \Phi, w \rangle = (-1)^p \int_0^\pi F(y_1) D_2^p w(y_1, 0) \mathrm{d}y_1, \quad p = 1, 2, 3, 4. \tag{3.16}$$

Obviously, the choice of the change of variables and functions was done according to Remark 2.1; as before, functions are extended to $B_0 = (0,\pi) \times \mathbb{R}$ with value zero for $\mid y_2 \mid > 1/\eta$. Oppositely, the choice of the energy space $\mathcal{V}$ for the limit problem is very different from sect. 2, and even simpler. Indeed, in the present case the limit form a_0 contains derivatives D_1 as well as D_2. Then we shall not take equivalence classes and $\mathcal{V}$ will be defined merely as:

$$\mathcal{V} = \mathcal{C} \bigcup_{\eta < 1} H_0^4(B_\eta) \tag{3.17}$$

where $\mathcal{C}$ denotes completion for the norm (2.28), *i.e.* the energy norm of the limit problem defined by (3.11).

Before going on we need certain properties of $\mathcal{V}$.

Lemma 3.1. *In $\mathcal{V}$ holds true the equivalence of norms*

$$\|.\|_{\mathcal{V}}^2 \simeq \|.\|^2_{L^2(\mathbb{R}_{y_2}; H_0^2(0,\pi))_{y_1}} + \|.\|^2_{L^2((0,\pi) y_1; H^4(\mathbb{R}_{y_2}))}. \tag{3.18}$$

Proof. The square of the norm of $\mathcal{V}$ is obviously less than a constant multiplied by the right hand side of (3.18); let us prove the opposite. Using the Poincaré inequality in $H_0^2(0,\pi)$, its norm is equivalent to the norm of the second derivative in L^2, so that the first term of the right hand side in (3.18) is majorized by the left hand side. It only remains to prove that

$$\|.\|^2_{L^2((0,\pi)_{y_1}; H^4(\mathbb{R}_{y_2}))} \le C \|.\|^2_{\mathcal{V}}. \tag{3.19}$$

From the previous argumentation it follows in particular that

$$\|.\|^2_{L^2((0,\pi)_{y_1};L^2(\mathbb{R}_{y_2}))} \leq C\|.\|^2_{\mathcal{V}}. \tag{3.20}$$

and obviously:

$$\left\|D_2^2 w\right\|^2_{L^2(\mathbb{R}_{y_2};L^2(0,\pi)_{y_1})} \leq \|w\|^2_{\mathcal{V}}. \tag{3.21}$$

Moreover, it is classical even for unbounded intervals (see for instance [8], Th. 2.3, p. 19) that the norms

$$\|w\|^2_{H^2(\mathbb{R}_{y_2})} \quad \text{and} \quad \|w\|^2_{L^2(\mathbb{R}_{y_2})} + \left\|D_2^2 w\right\|^2_{L^2(\mathbb{R}_{y_2})}$$

are equivalent. Then, equations (3.20) and (3.21) imply (3.19). □

Lemma 3.2. *Space $\mathcal{V}$ coincides with*

$$L^2\left(\mathbb{R}_{y_2};\ H_0^2(0,\pi)_{y_1}\right) \bigcap L^2\left((0,\pi)_{y_1};\ H^4(\mathbb{R}_{y_2})\right). \tag{3.22}$$

Proof. Classical techniques of convolution and translation (see for instance [8], p. 14) show that smooth functions with compact support are dense in $\mathcal{V}$. The conclusion then follows from the definition of $\mathcal{V}$ (see (3.17)). □

The limit problem $\mathcal{P}_0$ is then:

$$\begin{cases} \text{Find } v \in \mathcal{V} \text{ such that } \ \forall w \in \mathcal{V} \\ a_0(v,w) = \langle \varphi, w \rangle \end{cases} \tag{3.23}$$

where the right hand side is defined by (3.16). Obviously, $\mathcal{P}_0$ is a variational problem in the lax-Milgram framework, as (3.16) is a continuous functional on $\mathcal{V}$. The solution v is then uniquely defined.

The convergence theorem, which is proved exactly as Theorem 2.3 is:

Theorem 3.3. *Let v^η and v be the solutions of $\mathcal{P}_\eta$ (3.10) and $\mathcal{P}_0$ (3.23) respectively. Then,*

$$v^\eta \to v \ \text{ strongly in } \mathcal{V}. \tag{3.24}$$

Remark 3.4. The completion process to construct $\mathcal{V}$ implies a loss of boundary conditions at $y_1 = 0$ and $y_1 = \pi$. Indeed, for $\eta > 0$, the solution u^η of $\mathcal{P}_\eta$ vanishes as well as its derivatives up to the third order at $y_1 = 0$ and $y_1 = \pi$ (see (3.10) as well as (3.4) for the equivalent problem in x), whereas the solution v of $\mathcal{P}_0$ only satisfies $v = D_1 v = 0$ at $y_1 = 0$ and $y_1 = \pi$ (see Lem. 3.2). This phenomenon is probably associated with an interaction of the layer under study (in the neighbourhood of $x_2 = 0$) and a boundary layer at $x_1 = 0$ and $x_1 = \pi$. Obviously, the dilatation (3.8) is suited for the first one, but the second may give some trouble. This is also the reason why the previous developments work well with $F \in L^2(0,\pi)$ as we took, whereas the limit problem (3.5, 3.6) makes sense with $F \in H^{-2}(0,\pi)$ in (3.7).

The classical formulation of $\mathcal{P}_0$ (3.23) is

$$\left(D_1^4 + D_2^8\right) v = F(y_1)\delta^{(p)}(y_2) \ \text{ in } B_0 \ p = 0,1,2,3 \tag{3.25}$$

$$v = D_1 v = 0 \ \text{ for } \ y_1 = 0 \ \text{ and } \ y_1 = \pi \tag{3.26}$$

$$v \to 0 \ \text{ as } \ \mid y_2 \mid \to \infty. \tag{3.27}$$

Obviously (3.25) and (3.26) follow from the formulation of the variational problem and the properties of $\mathcal{V}$. Condition (3.27) should be understood in the sense of the exponential decreasing of the Fourier components involved in the separation of variables process. Indeed, this problem in the strip B_0 is easily solved by expanding the solution in the basis formed by the eigenfunctions of the bilaplacian in $(0,\pi)$ with Dirichlet boundary conditions (3.26). The coefficients, as functions of y_2, solve an eight order differential equation and (3.27) comes from the finite energy condition. We shall not give here the explicit solution of this problem which is analogous to another one which may be found in [11], Section 3. We shall only point out for ulterior utilization, the fact that the derivatives with respect to y_2 of v of orders from 0 to 7 are continuous at the origin, with the exception of

$$D_2^{8-(p+1)}v$$

which has a jump of value $F(y_1)$ at $y_2 = 0$. This fact follows easily from (3.25).

4. Propagation phenomena in the previous problem

The limit problem in x (3.5, 3.6) is parabolic in x_1, x_2 but in fact x_2 appears as a parameter and the very structure is that of an elliptic problem in x_1 with a parameter. The solution is obviously

$$u = \delta^{(p)}(x_2)U(x_1) \quad p = 0,1,2,3 \tag{4.1}$$

where U is the solution of

$$\partial_1^4 U = F \quad \text{in} \quad (0,\pi) \tag{4.2}$$

$$U(0) = D_1U(0) = U(\pi) = D_1U(\pi) = 0. \tag{4.3}$$

Equation (4.2) (as well as the boundary condition (4.3)) may be seen as a propagation phenomenon of the singularities of a solution of the parabolic problem (3.5, 3.6) along the characteristic $x_2 = 0$. The fourth order equation (4.2) accounts for the multiplicity of order four of the characteristic. Clearly, support (U) is generically larger than support (F) so that a propagation phenomenon holds along the characteristic.

On the other hand, the solution v of the limit problem in y, equations (3.25–3.27) may be considered as an "infinitely dilated in the normal direction" version of the one-dimensional problem (4.2, 4.3). But (3.25) is a parabolic equation with characteristics $y_1 =$ const. of multiplicity eight, so that there are certainly not propagation phenomena in the longitudinal direction $y_1 =$ const. This slightly padoxical situation is explained by the fact that, as we shall see in the sequal, propagation phenomena in (3.25) are not local properties, but global properties obtained after integration in y_2, *i.e.* normally to the layer. This is perfectly consistent with the fact that (4.2, 4.3) is an "infinitely contracted" version of (3.25–3.27).

Let v be the solution of (3.25–3.27). Let us define certain integrals with respect to y_2 inspirated by the distributions $\delta^{(p)}$, $p = 0,1,2,3$ in (4.1). Let

$$\hat{v}(y_1) = (-1)^p \int_{-\infty}^{+\infty} \frac{y_2^p}{p!} v(y_1,y_2)\mathrm{d}y_2 \quad p = 0,1,2,3 \tag{4.4}$$

which make sense because of the exponential decay of v as $\mid y_2 \mid$ tends to infinity. For the same reason we may consider the integral (4.4) as the action of the distribution $v(y_1,.)$ on the test function $y_2^p(p!)^{-1}$:

$$\hat{v}(y_1) = (-1)^p \left\langle v(y_1,y_2), \frac{y_2^p}{p!} \right\rangle \cdot \tag{4.5}$$

Let us consider the sections at $y_1 = $ const. of the various terms in (3.25) as distributions which we apply to the test function y_2^p. We have:

$$\left\langle D_1^4 v, (-1)^p \frac{y_2^p}{p!} \right\rangle = D_1^4 \hat{v}(y_1)$$
$$\left\langle D_2^8 v, (-1)^p \frac{y_2^p}{p!} \right\rangle = \left\langle D_2^{8-p} v, 1 \right\rangle = D_2^{8-p-1} v \Big|_{-\infty}^{+\infty} = 0$$
$$\left\langle F(y_1)\delta^{(p)}(y_2), (-1)^p \frac{y_2^p}{p!} \right\rangle = F(y_1)$$

where the exponential decay of v and its derivatives was used. We then obtain:

$$D_1^4 \hat{v}(y_1) = F(y_1). \tag{4.6}$$

Obviously we also have

$$\hat{v}(0) = D_1 \hat{v}(0) = \hat{v}(\pi) = D_1 \hat{v}(\pi) = 0 \tag{4.7}$$

which is the (global) propagation property of v. We also observe that $\hat{v}(y_1)$ coincides with $U(y_1)$ (see (4.2, 4.3)).

Moreover, let us define:

$$\hat{v}^q(y_1) = (-1)^q \int_{-\infty}^{+\infty} \frac{y_2^q}{q!} v(y_1, y_2) \mathrm{d}y_2 \quad q < p. \tag{4.8}$$

Exactly as before we obtain:

$$D_1^4 \hat{v}^q(y_1) = 0$$
$$\hat{v}^q(0) = D_1 \hat{v}^2(0) = \hat{v}^q(\pi) = D_1 \hat{v}^q(\pi) = 0$$

so that

$$\hat{v}^q(y_1) = 0. \tag{4.9}$$

In other words, the moments of order $q < p$ of $v(y_1, .)$ vanish, whereas the moment of order p, which we called $\hat{v}(y_1)$ is in general different from zero (see (4.6, 4.7)). It is then a classical exercise in distribution theory that as $\eta \to 0$:

$$\frac{1}{\eta^{p+1}} v(y_1, \eta q_2) \longrightarrow \hat{v}(y_1)\delta^{(p)}(y_2) \quad \text{in } \mathcal{D}'(\mathbb{R}_{y_2}). \tag{4.10}$$

On account of the change (3.8, 3.9) this means that, coming back to the initial variables and functions from the limit $v(y)$ of $v^\eta(y)$ (and not from the exact solution v^η) we have a sequence which tends as η tends to zero to the solution of the limit problem in x (4.1–4.3).

5. Model problem for P_ε elliptic of order 8 and P_0 hyperbolic of order 4

We now consider problems analogous to the previous ones for $\varepsilon > 0$, but the limit problem is hyperbolic of order 4, with two families of characteristics $y_1 = $ const. and $y_2 = $ const., each with multiplicity 2. The singular right hand side is applied on the (double) characteristic $x_2 = 0$. The domain Ω is again given by (2.1) and the bilinear forms are

$$a(u,v) = \int_\Omega \partial_1\partial_2 u \partial_1\partial_2 v \, \mathrm{d}y \tag{5.1}$$

$$b(u,v) = \int_\Omega \partial_{ijkh} u \partial_{ijkh} v \, \mathrm{d}y. \tag{5.2}$$

The energy space for $\varepsilon > 0$ and its dual are

$$V = H_0^4(\Omega), \quad V' = H^{-4}(\Omega). \tag{5.3}$$

In order to characterize the space V_a, we note that, denoting $w = \partial_2 v$, $v \in V_a$ implies $\partial_1 w \in L^2(\Omega)$ and using the Poincaré inequality on $(0,\pi)$ for functions $v \in V$ we have

$$w \in L^2\left((-1,+1)_{y_1};\ H_0^1(0,\pi)_{y_2}\right)$$

and applying again the Poincaré inequality on $(-1,+1)$

$$u \in H_0^1\left((-1,+1)_{y_1};\ H_0^1(0,\pi)_{y_2}\right).$$

Moreover, V is dense in that space, so that:

$$V_a = H_0^1\left((-1,+1)_{y_1};\ H_0^1(0,\pi)_{y_2}\right) \tag{5.4}$$

$$V_a' = H^{-1}\left((-1,+1)_{y_1};\ H^{-1}(0,\pi)_{y_2}\right) \tag{5.5}$$

where evidently y_1 and y_2 with their intervals may be exchanged.

The classical formulation of P_ε is:

$$\left(\partial_1^2\partial_2^2 + \varepsilon^2\Delta^4\right) u^\varepsilon = f \quad \text{in } \Omega \tag{5.6}$$

$$u^\varepsilon = \partial_n u = \partial_n^2 u = \partial_n^3 u = 0 \ \text{ on } \partial\Omega \tag{5.7}$$

whereas P_0 is

$$\partial_1^2\partial_2^2 u = f \ \text{ in } \Omega \tag{5.8}$$

$$u = 0 \ \text{ on } \partial\Omega \tag{5.9}$$

which is a hyperbolic problem with non-classical boundary conditions: they look a little as Dirichlet conditions, but there is only one condition for a fourth order equation.

As for the right hand side, there are three possibilities in the hierarchy (1.15), namely $\delta'(x)$, $\delta''(x)$, $\delta'''(x)$ such that their product by sufficiently smooth functions $F(x_1)$ give functionals belonging to V' but not to V_a'. We shall consider

$$f = F(x_1)\delta^{(p)}(x_2); \quad p = 1,2,3 \tag{5.10}$$

with $F \in L^2(0,\pi)$.

In the present case, the change of variables is

$$x_1 = y_1,\ x_2 = \eta y_2,\ \eta = \varepsilon^{1/3} \tag{5.11}$$

whereas a formal study of the solutions of (5.8) leads immediately to a behaviour in $\delta^{(p-2)}$ for u (with $p=1$ a "singularity in $\delta^{(-1)}$" obviously means a jump singularity in Y) which suggest the change on functions

$$u^\varepsilon(x) = \eta^{p-1} v^\eta(y)\ ,\ p = 1,2,3 \tag{5.12}$$

and the problem (1.6) with $\varepsilon > 0$ becomes the new problem $\mathcal{P}_\eta$:

$$\begin{cases} \text{Find } v^\eta \in H_0^4(B_\eta), \quad B_\eta = (0,\pi)\times(-1/\eta,+1/\eta) \\ \text{such that } \forall w \in H_0^4(B_\eta) \\ a_0(v^\eta,w) + \eta^2 a_1(v^\eta,w) + \eta^4 a_2(v^\eta,w) + \eta^6 a_3(v_1^\eta,w) + \eta^8 a_4(v^\eta,w) = \langle \Phi, w\rangle \end{cases} \tag{5.13}$$

where:

$$a_0(v,w) = \int_{B_0} D_1D_2vD_1D_2w\ \mathrm{d}y + \int_{B_0} D_2^4vD_2^4w\ \mathrm{d}y \tag{5.14}$$

$$a_1(v,w) = 4\int_{B_0} D_2^3D_1vD_2^3D_1w\ \mathrm{d}y \tag{5.15}$$

$$a_2(v,w) = 6\int_{B_0} D_1^2D_2^2vD_1^2D_2^2w\ \mathrm{d}y \tag{5.16}$$

$$a_3(v,w) = 4\int_{B_0} D_1^3D_2vD_1^3D_2v\ \mathrm{d}y \tag{5.17}$$

$$a_4(v,w) = \int_{B_0} D_1^4vD_1^4w\ \mathrm{d}y \tag{5.18}$$

$$\langle \Phi, w\rangle = (-1)^p \int_0^\pi F(y_1)D_2^p w(y_1,0)\mathrm{d}y_1,\ \ p = 1,2,3. \tag{5.19}$$

An inspection of the bilinear form a_0 defined in (5.14) shows that it has an "intermediate" character between (2.22) and (3.11). It vanishes on functions depending only on y_2 (which is a little less than on (2.25)) but involve derivatives with respect to both y_1 and y_2 as (3.11). So, the construction of the energy space $\mathcal{V}$ for the limit problem in the variables y recalls a little (2.27) but after taking equivalence classes, leads to partial differential equations in y_1, y_2, as in Section 3.

Let us define the space of functions of y_2:

$$H_0^4(-1/\eta,+1/\eta)/\mathbb{R} \tag{5.20}$$

as the space of the equivalence classes of functions defined up to an additive constant from the functions of $H_0^4(-1/\eta,+1/\eta)$ which we always consider extended with value zero for $\mid y_2 \mid > 0$. We then construct the space of functions of y_1 with values in the previous space

$$L^2\left((0,\pi)_{y_1};\ H_0^4(-1/\eta,+1/\eta)/\mathbb{R}_{y_2}\right). \tag{5.21}$$

Obviously for fixed η the set of equivalence classes obtained from the element $H_0^4(B_\eta)$ by adding arbitrary functions of y_1 is dense in the space (5.21). Finally, we construct the union of the spaces with $\eta < 1$ and we define $\mathcal{V}$ as the completion:

$$\mathcal{V} = \mathcal{C}\bigcup_{\eta<1} L^2\left((0,\pi)_{y_1};\ H_0^4(-1/\eta,1/\eta)/\mathbb{R}_{y_2}\right) \tag{5.22}$$

with the norm

$$\|w\|_{\mathcal{V}}^2 = a_0(w,w) = \int_{B_0} \left[(D_1D_2w)^2 + (D_2^4w)^2\right] \mathrm{d}y. \tag{5.23}$$

It follows from the previous considerations that:

Lemma 5.1. *The set of equivalence classes obtained from the elements of $\bigcup_{\eta<1} H_0^4(B_\eta)$ by adding arbitrary functions of y_1 is dense in $\mathcal{V}$.*

There is also a property analogous to Lemma 3.2, but it involves D_2w instead of w:

Lemma 5.2. *Let $v \in \mathcal{V}$. Then, $\bar{v} = D_2v$ belongs to*

$$L^2\left(\mathbb{R}_{y_2}; H_0^1(0,\pi)_{y_1}\right) \bigcap L^2\left((0,\pi)_{y_1}; H^3(\mathbb{R}_{y_2})\right) \tag{5.24}$$

and the operator D_2 is continuous from $\mathcal{V}$ to (5.24).

Proof. It follows from (5.23) that $w \in \mathcal{V}$ implies that defining $\bar{v} = D_2v$, we have,

$$\bar{v} \in L^2(B_0) \;\text{ and }\; D_2^3\bar{v} \in L^2(B_0) \tag{5.25}$$

with the corresponding inequalities for norms. From Lemma 5.1, we may only consider smooth (in fact of class H^3) function $\bar{w}$ vanishing at $y_1 = 0$ and $y_1 = \pi$. It then follows from the Poincaré inequality on the interval $(0,\pi)$ of y_1 that

$$\bar{w} \in L^2\left(\mathbb{R}_{y_2};\; H_0^1(0,\pi)_{y_1}\right) \tag{5.26}$$

and in particular, after exchanging variables:

$$\bar{w} \in L^2\left((0,\pi)_{y_1};\; L^2(\mathbb{R}_{y_2})\right). \tag{5.27}$$

But from (5.23)

$$D_2^3\bar{w} \in L^2\left((0,\pi)_{y_1};\; L^2(\mathbb{R}_{y_2})\right) \tag{5.28}$$

and using the fact that in $H^3(\mathbb{R}_{y_2})$ the norms

$$\|\bar{w}\|^2_{H^3(\mathbb{R}_{y_2})} \;\text{ and }\; \|\bar{w}\|^2_{L^2(\mathbb{R}_{y_2})} + \left\|D_2^3\bar{w}\right\|^2_{L^2(\mathbb{R}_{y_2})}$$

are equivalent (see [8], Th. 2.3, p. 19), we see that

$$\bar{w} \in L^2\left((0,\pi)_{y_1};\; H^3(\mathbb{R}_{y_2})\right). \tag{5.29}$$

The lemma follows from (5.26) and (5.29) (Note that all the inclusions imply the corresponding inequalities for the norms). □

As for the functional Φ, we have:

Lemma 5.3. *The expression $\langle\Phi, w\rangle$ in (5.19) defines a continuous functional on $\mathcal{V}$.*

Proof. Expression (5.19) vanishes on functions depending only on y_1, so that it defines a functional on the space of equivalence classes. As for the continuity, it follows immediately from Lemma 5.2 by using the trace theorem in $H^3(\mathbb{R}_{y_2})$. □

The limit problem $\mathcal{P}_0$ is then defined by:

$$\begin{cases}\text{Find } \tilde{v} \in \mathcal{V} \ \text{ such that} \forall \tilde{w} \in \mathcal{V} \\ a_0(\tilde{v}, \tilde{w}) = \langle \Phi, \tilde{w} \rangle \end{cases} \tag{5.30}$$

which is clearly in the Lax-Milgram framework. This defines uniquely the solution $\tilde{v}$ as an element of $\mathcal{V}$, but of course, as a function, it is only defined up to an additive function of y_1.

Remark 5.4. The analogous of Remark 2.2 with functions which are constant with respect to y_2 holds true in the present case. Any $v \in H_0^1(B_\eta)$ extended with value zero and considered up to additive functions of y_1 is an element of $\mathcal{V}$.

The convergence theorem, which is proved exactly as Theorem 2.3 is:

Theorem 5.5. *Let v^η be the solution of (5.13). We then construct the corresponding equivalence class $\tilde{v}^\eta$ according to Remark 5.4. We have:*

$$\tilde{v}^\eta \to \tilde{v} \ \text{ strongly in } \mathcal{V} \tag{5.31}$$

where $\tilde{v}$ is the solution of (5.30).

In order to solve explicitly the limit problem (5.30) we obtain immediately the corresponding equation in the distribution sense for any element of the equivalence class $\tilde{v}$:

$$\left(D_1^2 D_2^2 + D_2^8\right) v = F(x_1)\delta^{(p)}(x_2), \ p = 1, 2, 3. \tag{5.32}$$

Denoting

$$D_2 v = \bar{v} \tag{5.33}$$

it may be written

$$D_2 \left(D_1^2 + D_2^6\right) \bar{v} = F(x_1)\delta^{(p)}(x_2), \ p = 1, 2, 3$$

and integrating with respect to y_2 we have

$$\left(D_1^2 + D_2^6\right) \bar{v} = F(y_1)\delta^{(p-1)}(y_2) + \alpha(y_1), \ p = 1, 2, 3 \tag{5.34}$$

where $\alpha(y_1)$ is an unknown function. In fact it vanishes as follows from (see (5.24)):

$$\bar{v} \in L^2\left((0, \pi)_{y_1}; \ H^3(\mathbb{R}_{y_2})\right). \tag{5.35}$$

Indeed, from (5.24) we also have the boundary conditions

$$\bar{v}(0, y_2) = \bar{v}(\pi, y_2) = 0 \tag{5.36}$$

and we develop $v(y_1, y_2)$ taking y_2 as a parameter in the basis φ_n of eigenfunctions of the Dirichlet problem for the Laplacian in the interval $(0, \pi)$:

$$\bar{v} = \varphi_n(y_1)\bar{v}_n(y_2). \tag{5.37}$$

The differential equation for $\bar{v}_n(y_2)$ is

$$(-n^2 + D_2^6)\bar{v}_n(y_2) = F_n \delta^{(p-1)}(x_1) + \alpha_n, \ p = 1, 2, 3. \tag{5.38}$$

On the other hand writing (5.35) under to form

$$\bar{v} \in H^3(\mathbb{R}_{y_2};\ L^2(0,\pi)_{y_1})$$

we see that each component $\bar{v}_n$ belongs to $H^3(\mathbb{R}_{y_2})$. This implies decreasing properties for y_2 tending to $+\infty$ and $-\infty$ which may only be satisfied by a solution of (5.38) with $\alpha_n = 0$.

Finally the problem for $\bar{v}$ becomes

$$(D_1^2 + D_2^6)\bar{v} = F(y_1)\delta^{(p-1)}(y_2), \ p = 1, 2, 3 \tag{5.39}$$

with (5.36) and exponential decreasing for $|y_2|$ tending to infinity. This problem is exactly analogous to (3.25–3.27) and solved in the same way using the separation of variables (5.37).

Once $\bar{v}$ is known, the function $v(y_1, y_2)$ may be obtained by integration with respect to y_2 (see (5.33)) but of course it is only defined up to an additive function of y_1.

Remark 5.6. As we pointed out, the solution of the problem for $\bar{v}$ (defined by (5.33) is uniquely defined. This is a genuine partial differential equation with respect to y_1 and y_2, analogous to the equation for v of Section 3 (see (3.25–3.27)). But in the present case the unknown is the derivative $D_2 v$ so that in this concern, the problem is analogous to that of Section 2.

6. Propagation properties for the previous problem

Let us first consider the limit problem (5.8, 5.9). The singularity along $y_2 = 0$ may be developed in terms of the sequences (1.16), namely

$$u \sim \delta^{(p-2)}(x_2)\ U(x_1) + ..., \quad p = 1, 2, 3 \tag{6.1}$$

(where $\delta^{(-1)}$ is obviously the step function Y). Substituting in (5.8) and (5.9) we have:

$$D_1^2 U(x_1) = F(x_1) \ \text{ on } (0, 1) \tag{6.2}$$

$$U(0) = U(\pi) = 0 \tag{6.3}$$

which determines the leading order of the singularity in (6.1). It should be noticed that, for $p = 2$ and 3, this is the exact solution; oppositely, for $p = 1$, the leading term of the right hand side of (6.1) is just the jump of the solution at $y_2 = 0$.

Coming back to the solution $\bar{v}$ of (5.34) (which is exponentially decreasing as $|y_2| \to +\infty$) we may define

$$\hat{\bar{v}}(y_2) = (-1)^{p-1} \int_{-\infty}^{+\infty} \frac{y_2^{p-1}}{(p-1)!} \bar{v}(y_1, y_2) \mathrm{d}y_2, \quad p = 1, 2, 3 \tag{6.4}$$

which makes sense as the action of the distribution $\bar{v}(y_1,.)$ on the test function y_2^{p-1} while y_1 being considered as a parameter. Taking the action of the different terms of (5.38) on that test functions, we obtain:

$$\left\langle D_1^2\bar{v}, (-1)^{p-1}\frac{y_2^{p-1}}{(p-1)!}\right\rangle = D_1^2\hat{\bar{v}}(y_1)$$
$$\left\langle D_2^6\bar{v}, (-1)^{p-1}\frac{y_2^{p-1}}{(p-1)!}\right\rangle = \left\langle D_2^{6-(p-1)}\bar{v}, 1\right\rangle = D_2^{6-p}\ \bar{v}|_{-\infty}^{+\infty} = 0$$
$$\left\langle F(y_1)\delta^{(p-1)}(y_2), (-1)^{p-1}\frac{y_2^{p-1}}{(p-1)!}\right\rangle = F(y_1)$$

so that

$$D_1^2\hat{\bar{v}}(y_1) = F(y_1) \tag{6.5}$$

and obviously

$$\hat{\bar{v}}(0) = \hat{\bar{v}}(\pi) = 0 \tag{6.6}$$

which is the global propagation property for $v(y_1, y_2)$. It is concerned with $\bar{v} = D_2 v$ and then uniquely defined. In particular, for $p = 1$, $\hat{\bar{v}}(y_1)$ is merely the integral of $\bar{v}$, in other words the jump of v between $-\infty$ and $+\infty$. For $p = 2$ and $p = 3$ they account for $\delta'(y_2)$ and $\delta''(y_2)$ -like singularities of $\bar{v}$, *i.e.* $\delta(x_2)$ and $\delta'(x_2)$ -like singularities of v, respectively, according to considerations analogous to those of Section 4.

7. Complements, open problems and comments

7.1. Layers along free boundaries

As in [11], Section 4, problems analogous to those of Sections 2, 3 and 5 may be handled using the same kind of methods in the case when the curve $x_2 = 0$ where the singular right hand side is applied is a boundary of Ω without "principal boundary conditions" prescribed in the space V. More precisely, we may consider the domain

$$\Omega = (0, \pi) \times (0, 1) \tag{7.1}$$

and the boundary conditions (2.6, 3.4) or (5.7) are only prescribed for $x_1 = 0$, $x_1 = \pi$ and $x_2 = 1$. Clearly the singular loading must be considered in the sense of (2.13, 2.14), and the analogous expressions in the other cases. This amounts to singular right hand sides in the natural boundary conditions on $x_2 = 0$.

The solution of these problems is analogous to that of Sections 2, 3 and 5, but obviously the half-axis $y_2 > 0$ replaces $\mathbb{R}_{y_2}$.

There is a variant of the problem of Section 2 in this context that deserves attention. If the bilinear form a of (2.2) us replaced by

$$a(u, v) = \int_\Omega \Delta u.\Delta v \ \mathrm{d}x \tag{7.2}$$

the problem in $H_0^4(\Omega)$ does not change. Oppositely in the case when there is a "free boundary" (as $x_2 = 0$ in (7.1), without principal boundary conditions in the space V), the limit problem for $\varepsilon = 0$ is not well posed as an elliptic problem because the natural boundary conditions do not satisfy the Shapiro–Lopatinskii condition. The structure of the $\mathcal{P}_\eta$ problem (2.21) is different involving other terms than the "sum of squares" of (2.22) for $v^\eta = w$. It appear that V_a is then a "very large space" going out of the distribution space. The problem for

$\varepsilon = 0$ is said to be "sensitive". Not very much is known on the asymptotic behaviour as ε tends to zero in this case. It is nevertheless likely that loadings as (2.13, 2.14) may be handled by the methods of the present paper.

7.2. A comment on the equivalence classes

Let us consider the case of Section 2 (Sect. 5 is analogous). The expression of a_0 in (2.22) of the limit energy form defines the square of a norm on

$$\bigcup_{\eta<1} H_0^4(B_\eta) \tag{7.3}$$

as the elements of (7.3) vanish for $\mid y_2 \mid$ sufficiently large. One may think that taking equivalence classes up to (2.25) is useless, and even leads to some loss of information. In fact this is not the case, as we are seeing that the associated topology on functions of the form (2.25) is "very poor" and they should be assimilated to the zero function in the completion process.

To be convinced of it, let us first consider the completion with the Dirichlet norm

$$\|u\|^2 = \int_{-\infty}^{+\infty} |\mathrm{d}u/\mathrm{d}y|^2 \mathrm{d}y \tag{7.4}$$

of the space

$$\bigcup_{\eta<1} H_0^1(-1/\eta, 1/\eta) \tag{7.5}$$

where, as usual, the functions are extended with value zero for $|\, y \,|> 1/\eta$.

Let us consider the continuous function u_L taking the value 1 (resp. 0) for $|\, y \,|< L$ (resp. $|\, y \,|> 2L$) which depends linearly on y for $L <|\, y \,|< 2L$. Its norm (7.4) tends to zero as L tends to ∞. Obviously u_L tends to the function equal to 1 in the distribution sense, but to the zero function in the completion of (7.5) for the norm (7.4).

It is easily seen that the above example may be adapted to the case when H_0^1 is replaced by H_0^4 in (7.5) with the norm

$$\|u\|^2 = \int_{-\infty}^{+\infty} \left(|\mathrm{d}u/\mathrm{d}y|^2 + |\mathrm{d}^2u/\mathrm{d}y^2|^2\right) \mathrm{d}y \tag{7.5}$$

instead of (7.4): we may, for instance, introduce small "curved regions" of length $O(1)$ in the neighbourhoods of $|y| = 1, 2$.

This shows that, in the convergence of the limit space $\mathcal{V}$, functions of the form (2.25) should be assimilated to zero. We preferred to avoid these functions by using equivalence classes in order to use in the sequal L^2-like "good topologies".

7.3. Right hand sides tending to singular functions

Instead of singular right hand sides as (2.11, 2.12), we may consider sequences of smooth functions depending on η with limits of the form (2.11, 2.12). For instance, if $\psi(y_1, y_2)$ is a function such that

$$\int_{-\infty}^{+\infty} \psi(y_1, y_2)\mathrm{d}y = \int_{-\infty}^{+\infty} y_2 y(y_1, y_2)\mathrm{d}y_2 = 0 \tag{7.6}$$

$$\int_{-\infty}^{+\infty} \frac{y_2^2}{2}\psi(y_1, y_2)\mathrm{d}y_\varepsilon = F(y_1) \tag{7.7}$$

we have

$$\frac{1}{\eta^3}\psi\left(x_1, \frac{x_2}{\eta}\right) \longrightarrow F(x_1)\delta''(x_2). \tag{7.8}$$

We may then take instead of (2.11):

$$f(x_1, x_1) = \frac{1}{\eta^3}\psi\left(x_1, \frac{x_2}{\eta}\right)\cdot \tag{7.9}$$

All the developments are the same as in Section 2, but (2.23) becomes

$$\langle \Phi, w\rangle = \int_{B_0} \psi(y_1, y_2) w(y_1, y_2) \mathrm{d}y. \tag{7.10}$$

7.4. **Open problems with interacting layers**

Let us consider for instance the problem of Section 5 with (instead of (5.10)):

$$f(x_1, x_2) = \delta'(x_1)\delta'(x_2) \tag{7.11}$$

i.e. (5.10) with $p = 1$ but the function F is in its turn singular. It is easily seen that f defined by (7.11) belongs to $V = H^{-4}$ but not to V_a.

Because of the symmetry of the problem by exchanging x_1 and x_2, it is clear that there will be two layers more or less analogous to that of Section 5 in the neighbourhoods of $x_1 = 0$ and $x_2 = 0$. But it is not possible to study each of these layers by the method of Section 5. Indeed, the right hand side of problem $\mathcal{P}_0$ should be

$$\langle \Phi, w\rangle = D_1 D_2 w(0, 0) \tag{7.12}$$

instead of (5.19), and the right hand side of (7.11) is not a continuous functional on $\mathcal{V}$ (see (5.22, 5.23)). Clearly, there is an interaction of the two layers near the origin.

7.5. **Indications on systems of equations**

In the case when the limit problem P_0 is given by a system of equations with different orders for the different unknowns (this is the case of shell theory) the singularities may have a very high intensity.

Let us consider a system of n equations with constant coefficients with n unknowns u_j in the variables x_1, x_2:

$$a_{ij}(\partial_1, \partial_2) u_j = f_i \quad i = 1, ..., n \tag{7.13}$$

with the indices (for equations as well as unknowns):

$$s_1 \geq s_2 \geq ... \geq s_n \geq 0. \tag{7.14}$$

This means that the a_{ij} are polynomials in ∂_1, ∂_2 of order $s_i + s_j$. For the sake of simplicity (this is not essential) we shall consider the case when a_{ij} is formed only by terms of the leading order $s_i + s_j$

Let

$$D(\partial_1, \partial_2) = \det(a_{ij}(\partial_1, \partial_2)), \tag{7.15}$$

which is a homogeneous polynomial in ∂_1, ∂_2 of degree 2N (called the total order of the system) where

$$N = \sum_{i=1}^{n} s_i. \tag{7.16}$$

Let $A_{ij}(\partial_1, \partial_2)$ be the cofactor of a_{ij}; it is a homogeneous polynomial of degree $2N - (s_i + s_j)$ in ∂_1, ∂_2.

In order to study the singularities of an unknown u_k, we may eliminate the other unknowns (note that this process ignores the boundary conditions, so that, in principle, it may only be used to study local properties). Multiplying (7.13) by A_{ik} (and adding in i, of course) we have

$$D(\partial_1, \partial_2)u_k = A_{ik}(\partial_1, \partial_2)f_i. \tag{7.17}$$

It is then apparent that u_k is a solution of an equation of order $2N$, but the right hand side contains derivatives of the f_i, so that it is much more singular than the f_i themselves.

Let us consider singularities along $x_2 = 0$. In the case when this curve is a characteristic of degree m (*i.e.* $D(\xi_1, \xi_2)$ vanishes m times at $\xi = (0,1)$). The left hand side of (7.17) takes the form:

$$D(\partial_1, \partial_2) = E(\partial_1, \partial_2)\partial_1^m \tag{7.18}$$

where $E(\xi_1, \xi_2)$ is microlocally elliptic at $\xi = (0,1)$, *i.e.* does not vanish at $(0,1)$. Then, the order of the singularity (in the sense of (1.15)) is $2N - m$ times lower than the order of the right hand side. When the order of transversal differentiation in the A_{ik} is higher than $2N - m$, u_k is more singular than the data f_i.

Let us give an example of this phenomenon. The system

$$\begin{cases} -\Delta u_1 + \partial_2 u_2 = 0 \\ -\partial_2 u_1 + u_2 = f_2 \end{cases} \tag{7.19}$$

is in the above framework with $s_1 = 1$, $s_2 = 0$. We have by elimination

$$-\partial_1^2 u_2 = -\Delta f_2 \tag{7.20}$$

so that $x_2 = 0$ is a double characteristic. We have $2M = m = 2$ and the right hand side of (7.20) contains transversal derivatives of order 2. The singularity of u_2 is two orders higher that the singularity of f_2. For instance, for f_2 having a step $Y(x_2)$ at $x_2 = 0$, u_2 is singular as $\delta'(x_2)$.

7.6. Case when P_0 is microlocally elliptic

All the developments of Section 2 hold true in the case when the limit problem is not elliptic, but only microlocally elliptic with respect to $\xi = (0,1)$ normal to the curve bearing the singularity. In fact, the only essential hypothesis is that the limit problem contains derivatives of the highest order transversal to the curve $x_2 = 0$. For instance, we may take

$$a(u, w) = \int_\Omega \partial_2^2 u \partial_2^2 w \, \mathrm{d}y \tag{7.21}$$

instead of (2.2.).

7.7. Comment on the applied singularities

We saw that the asymptotic behavior depends highly on the type of the singular right hand side that we took as an element of the hierarchy (1.15) (see also (1.16)). By the linearity of the problem, right hand sides which are finite sums of terms in (1.15) are also allowed. But this is not very general, as certain singular loadings cannot be separated out of non-singular parts. We may for instance think about the distributions associated with

$$Y(x_2)x_2^{\alpha},\ Y(x_2)x_x^{\alpha-1},\ Y(x_2)x_2^{\alpha-2}$$

with non-integer α (see [2] for instance) which have at $x_2 = 0$ singularities with intermediate intensity with respect to those of (1.15) but cannot be separated from their smooth parts for $x_2 > 0$. The corresponding problems are open.

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