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ASYMMETRIC HETEROCLINIC DOUBLE LAYERS

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Abstract. Let W be a non-negative function of class C^3 from $\mathbb{R}^2$ to $\mathbb{R}$, which vanishes exactly at two points $\mathbf{a}$ and $\mathbf{b}$. Let $S^1(\mathbf{a},\mathbf{b})$ be the set of functions of a real variable which tend to $\mathbf{a}$ at $-\infty$ and to $\mathbf{b}$ at $+\infty$ and whose one dimensional energy

$$E_1(v) = \int_{\mathbb{R}} \left[W(v) + |v'|^2/2\right] \,\mathrm{d}x$$

is finite. Assume that there exist two isolated minimizers z_+ and z_- of the energy E_1 over $S^1(\mathbf{a},\mathbf{b})$. Under a mild coercivity condition on the potential W and a generic spectral condition on the linearization of the one-dimensional Euler–Lagrange operator at z_+ and z_-, it is possible to prove that there exists a function u from $\mathbb{R}^2$ to itself which satisfies the equation

$$-\Delta u + \mathrm{D}W(u)^{\mathsf{T}} = 0,$$

and the boundary conditions

$$\lim_{x_2\to+\infty} u(x_1,x_2) = z_+(x_1 - m_+), \quad \lim_{x_2\to-\infty} u(x_1,x_2) = z_-(x_1 - m_-),$$
$$\lim_{x_1\to-\infty} u(x_1,x_2) = \mathbf{a}, \quad \lim_{x_1\to+\infty} u(x_1,x_2) = \mathbf{b}.$$

The above convergences are exponentially fast; the numbers m_+ and m_- are unknowns of the problem.

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1. Introduction and notations

I definitely learnt variational problems from J.-L. Lions; I cannot but recall the graduate course (attestation d'études approfondies) that he taught in 1969-1970: the subject matter was non linear boundary value problems, and the style of exposition made the contents look deceptively simple. Of course, they were not; I carefully kept the notes, written in a rigid cover exercise book, which I can locate in about 10 seconds in my office.

I learnt that a good way to catch a mathematical object is to patiently set up a functional trap, tailored to its size and to its behavior, and when everything is ready, you trail the animal, you set up the mechanism, and you catch it so fast that it does not even realize that it is caught before it is too late. Most probably, J.-L. Lions

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would not have compared mathematics to hunting: it was not his style. All differences in style being put apart, I learnt quite a few things from him, and I gratefully acknowledge my debt toward him.

In this article I describe how I caught a very elusive animal: a solution of an elliptic system of two equations in full two-dimensional space with given asymptotic behavior at infinity. This solution is a “minimizer” of a Landau functional of the theory of phase transition. The word minimizer is between quotes, because the Landau functional is infinite on any solution of interest; therefore, it has to be renormalized for the problem to make sense.

The renormalized functional and hence the corresponding elliptic system are translation invariant: the direct method of the calculus of variation does not work. The concentration-compactness method also fails because of the rigidity of the problem: left to itself the solution “wants” to behave in a definite way, approximating a one-dimensional behavior. The solution is also smooth: no fancy behavior is permissible once we know that the solution is bounded.

The animal is a fish which may well slip to infinity if we do not use very precise tools: asymptotics at infinity and detailed exploitation of the energy functional. In turn, the energy functional can be exploited if the behavior of the one-dimensional problem is understood in depth.

The problem stems from the Landau theory of second order phase transitions [5]. This theory introduces an energy functional of the form

$$\mathcal{E}(u) = \int \left[W(u) + |\mathrm{grad}\, u|^2 /2\right] \,\mathrm{d}x \tag{1}$$

where u is an order parameter, and W is a real valued potential which describes the physics of the system under consideration.

If u is scalar valued, much is known about the minimizers of the above energy, and about limits as ε tends to 0 of scaled problems whose energy is given by

$$\mathcal{E}^\varepsilon(u) = \int_\Omega \left[\varepsilon^{-n} W(u) + \varepsilon^{2-n} |\mathrm{grad}\, u|^2 /2\right] \,\mathrm{d}x. \tag{2}$$

[2, 3, 7–9] have given significant results on the case of a potential with two wells.

However, when u takes its values in $\mathbb{R}^2$, the minimization problem is not understood very well. When the spatial variable is one-dimensional, and the potential has two wells of equal depth at $\mathbf{a}$ and $\mathbf{b}$ the variational argument given in [9, 11] provides a solution of the Euler–Lagrange equations whose limits are $\mathbf{a}$ at $-\infty$ and $\mathbf{b}$ at $+\infty$.

The study of minimizers of the scaled Landau functional (2) under appropriate boundary conditions requires the study of full-space problems: the functional (1) can be retrieved by blowing up the coordinates according to the transformation $x \mapsto x/\varepsilon$.

Therefore, the study of the minimizers of the full-space problem for (1) is an elementary brick for the convergence of the minimizers of (2) in a bounded set.

Let me be more precise, defining notations and assumptions.

The potential W is a function of class C^3 from $\mathbb{R}^2$ to $\mathbb{R}$, and it has minima at $\mathbf{a}$ and $\mathbf{b}$; these minima are non-degenerate, *i.e.* $\mathrm{D}^2W(\mathbf{a})$ and $\mathrm{D}^2W(\mathbf{b})$ are strictly positive in the sense of quadratic forms. The potential W vanishes at $\mathbf{a}$ and $\mathbf{b}$, and is strictly positive elsewhere. We also assume that there exists $R > 0$ such that

$$|\xi| \geq R \Rightarrow \mathrm{D}W(\xi)\xi > 0. \tag{3}$$

Without loss of generality, we may assume that $\mathbf{a} = (-1, 0)$ and $\mathbf{b} = (1, 0)$.

The set $S(\mathbf{a},\mathbf{b})$ is the set of locally absolutely continuous functions v from $\mathbb{R}$ to $\mathbb{R}^2$ whose one-dimensional energy is finite

$$E_1(v)=\int_{\mathbb{R}}\left[W(v)+|v'|^2/2\right]\,\mathrm{d}x<+\infty$$

and which have the following behavior at infinity:

$$\lim_{x\to-\infty}v(x)=\mathbf{a},\quad \lim_{x\to\infty}v(x)=\mathbf{b}.$$

A function z is called a (minimal) heteroclinic connection if it minimizes E_1 over $S(\mathbf{a},\mathbf{b})$.

The minimal heteroclinic connections E_1 over $S(\mathbf{a},\mathbf{b})$ satisfy the Euler–Lagrange equation over $\mathbb{R}$

$$-z''+\mathrm{D}W(z)^{\mathsf{T}}=0. \tag{4}$$

Since W is of class C^3, $\mathrm{D}W$ is of class C^2 and therefore a solution of (4) is of class C^4. At infinity, any minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$ tends exponentially fast to its limits $\mathbf{a}$ and $\mathbf{b}$. Moreover, $z^{(j)}$ belongs to $\mathrm{L}^1(\mathbb{R})^2\cap\mathrm{L}^\infty(\mathbb{R})^2$ for $j=1,\dots,4$.

The set of minimizers of E_1 over $S(\mathbf{a},\mathbf{b})$ will be called $\mathcal{Z}$. Clearly, $\mathcal{Z}$ is translation invariant, and for each z in $\mathcal{Z}$, the operator A defined in $\mathrm{L}^2(\mathbb{R})^2$ by

$$D(A)=\mathrm{H}^2(\mathbb{R}),\quad Av=-v''+\left(\mathrm{D}^2W(z)v\right)^{\mathsf{T}} \tag{5}$$

is non negative; by the non degeneracy of $\mathrm{D}^2W(\mathbf{a})$ and $\mathrm{D}^2W(\mathbf{b})$, the essential spectrum of A is bounded away from 0; $\zeta=z'$ is an eigenvector of A relative to the eigenvalue 0. It is convenient to denote by $\mathcal{C}(z)$ the set of translates $z(\cdot-m),m\in\mathbb{R}$ of a minimizer z.

We assume that there are two distinct heteroclinic connections, *i.e.* minimizers z_+ and z_- of the one-dimensional energy which cannot be deduced by translation one from another. The operators A_+ and A_- are defined by (5), with z replaced respectively by z_+ and z_-.

The main non-degeneracy assumption is:

$$\text{the kernels of } A_+ \text{ and } A_- \text{ are one-dimensional,} \tag{6}$$

and it turns out to be generic, *i.e.* for any non negative potential W with at least two wells of equal depth, and at least two distinct minimal heteroclinic connections, there is an arbitrarily close potential $W+\delta W$ which has exactly two potential wells of equal depth and exactly two distinct minimal heteroclinic connections (see Th. 4.3 and Rem. 4.4).

The two-dimensional energy of any interesting function is infinite: let indeed z be a minimal heteroclinic connection and take $u(x_1,x_2)=z(z_1)$: $\mathcal{E}(u)$ is clearly infinite. This observation means that we have to renormalize the energy. For this purpose, we let e_1 be the minimum of the energy of one dimensional heteroclinic connections:

$$e_1=\min\{E_1(v):v\in S(\mathbf{a},\mathbf{b})\}\cdot$$

Then, the renormalized energy is

$$E_2(u)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left(\frac{1}{2}\left|\frac{\partial u}{\partial x_1}\right|^2+\frac{1}{2}\left|\frac{\partial u}{\partial x_2}\right|^2+W(u)\right)\,\mathrm{d}x_1-e_1\right)\,\mathrm{d}x_2. \tag{7}$$

We seek minimizers u of the renormalized energy belonging to the set S_2 defined by the following conditions:

$$\lim_{x_1\to-\infty} u(x_1,x_2)=\mathbf{a},\quad \lim_{x_1\to+\infty} u(x_1,x_2)=\mathbf{b},\tag{8}$$

and there exist m_+ and m_- such that

$$\lim_{x_2\to\pm\infty} u(x_1,x_2)=z_\pm(x_1-m_\pm).\tag{9}$$

A substantial difficulty immediately appears in the above statement: the translation parameters are unknowns of the problem.

A change of coordinates shows that we must be concerned only with m_+-m_-, and the essential step consists in proving that it is bounded.

When the potential is invariant by the reflexion which exchanges $\mathbf{a}$ and $\mathbf{b}$, the analogous study has been performed in [1], yielding a two-dimensional heteroclinic connection between two minimal one-dimensional heteroclinic connections. The symmetry assumption enabled the authors to control the translation parameter, since they considered only solutions which were equivariant by the reflexion.

The solution of the minimization problem will be a solution of the elliptic system

$$-\Delta u+\mathrm{D}W(u)^{\mathsf{T}}=0,$$

where it is important to transpose the derivative $\mathrm{D}W$ of W, which is a row vector, to obtain a column vector.

So how does the proof work? Though it is expressed in analytic language, the method is geometric in essence, and its argument can be given as follows: the cost of the phase transition is asymptotically proportional to the length of the layer, provided that the solution looks like a one dimensional connection in the transversal direction.

Given Dirichlet data, it is always possible to solve a minimization problem in the half-plane $x_2>0$, using the direct method of the calculus of variations (Sect. 7).

On the other hand Assumption (6) suffices to prove that if the data are close enough to z_+ or z_-, then for all $x_2>0$, $u(\cdot,x_2)$ is close to $z_\pm$ or rather to one of its translate; the translation parameter converges exponentially fast to its limit as x_2 tends to infinity, as is proved in Section 6. One might think that this result is a simple application of a fixed point argument; however, it is a rather tricky result, because the naïve formulation leads to loss of regularity. The first step consists in solving the linearized problem in a half plane (Sect. 5): an overly simplified explanation is that we can decompose the space into the sum of a stable, an unstable and a neutral linear manifold: functional trickery is needed to give a sense to this idea, since it is impossible to define the action of the semi-group in the unstable direction: it acts there more or less like the backward heat equation. The nonlinear step combines metric estimates and topological arguments.

That the solutions obtained by these two different methods coincide when the data are close enough to $\mathcal{C}(z_\pm)$ is proved thanks to a number of estimates given also in Section 7; exponential decay estimates on local norms must be obtained, so as to prove that the minimizer is indeed a solution of the fixed point problem, whose uniqueness is only local. In those proofs, estimates based on one-dimensional results play a prominent rôle.

The construction of the minimizer of the two-dimensional energy is done as follows: we start with a minimizing sequence (v_n) and we replace it by a smoother one, patched from minimizers in two half-planes $x_2\geq X_{n,+}$ and $x_2\leq X_{n,-}$ and in the strip in between; the strip is chosen so that on its upper and lower boundaries, the trace of v_n is close respectively to $\mathcal{C}(z_+)$ and $\mathcal{C}(z_-)$. For the new sequence of minimizers, (u_n), we choose a minimal strip $Y_{n,-}\leq x_2\leq Y_{n,+}$ on which $u_n(\cdot,x_2)$ is bounded away from $\mathcal{C}(z_+)\cup\mathcal{C}(z_-)$. The height of this strip is bounded independently of n, thanks to the local coercivity of the energy in one dimension. At the strip boundaries, $u_x(\cdot,Y_{n,\pm})$ is close to $z_\pm(\cdot-m_{n,\pm}$ and another energy argument shows that $|m_{n,+}-m_{n,-}|$ is bounded independently of n. After applying a translation, and replacing again v_n by the half-plane minimizers outside of the strip $Y_{n,-}\leq x_2\leq Y_{n,+}$, the passage to the limit is then straightforward.

The fundamental one-dimensional result is what I call the local coercivity of the one-dimensional transition energy: if $E_1(u)$ denotes the one-dimensional energy and e_1 its lower bound on the set on functions tending to $\mathbf{a}$ and $\mathbf{b}$ respectively at $-\infty$ and at $+\infty$, and if z is minimizer satisfying the spectral assumption, then in a translation invariant neighborhood of $\mathcal{C}(z)$, $E_1(u) - e_1$ is greater than or equal to a constant times the square of the H^1 distance from u to $\mathcal{C}(z)$.

This is a very technical result which is proved in many steps, starting from Section 2 where we study the properties of curves of minimizers through Section 3 dedicated to the compactness of sequences of minimizers, and the consequences of the spectral assumption in Section 4. These results are interesting in themselves, since they give some very precise informations on the properties of the energy functional in one dimension.

2. Projection on curves of minimizers

Define for each measurable subset I of $\mathbb{R}$

$$E_1(u, I) = \int_I \left(\frac{|u'|^2}{2} + W(u) \right) \, \mathrm{d}t. \tag{10}$$

If $I = \mathbb{R}$, we will simply write $E_1(u)$ instead of $E_1(u, \mathbb{R})$.

Let ψ be any function of class C^∞ which is constant for large values of its argument and equal to $\mathbf{a}$ in a neighborhood of $-\infty$ and to $\mathbf{b}$ in a neighborhood of $+\infty$. For $s \geq 0$, let $S^s(\mathbf{a}, \mathbf{b})$ be the set

$$S^s(\mathbf{a}, \mathbf{b}) = \mathrm{H}^s(\mathbb{R})^2 + \psi.$$

The definition of $S^s(\mathbf{a}, \mathbf{b})$ is clearly independent of the exact choice of the function ψ. For simplicity, we will write $S(\mathbf{a}, \mathbf{b})$ instead of $S^1(\mathbf{a}, \mathbf{b})$. As the difference of elements of $S(\mathbf{a}, \mathbf{b})$ belongs to $\mathrm{H}^s(\mathbb{R})^2$, the set $S^1(\mathbf{a}, \mathbf{b})$ is equipped with the topology defined by the distance in $\mathrm{H}^1(\mathbb{R})^2$.

Denote by $\| \, \|_s$ the $\mathrm{H}^s(\mathbb{R})$ norm of a function and by $(\; , \;)_s$ the scalar product in $\mathrm{H}^s(\mathbb{R})$. We do not use different notations for the norm and scalar product of vector valued H^s functions.

Let u belong to $S^s(\mathbf{a}, \mathbf{b})$ and let F be a subset of $S^s(\mathbf{a}, \mathbf{b})$; the distance from u to F in the s-norm is

$$d_s(u, F) = \inf\{\|u - v\|_s, v \in F\} \cdot$$

Let z be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$; we define

$$\mathcal{C}(z) = \{z(\cdot - m) : m \in \mathbb{R}\} \cdot$$

The curve $\mathcal{C}(z)$ can be seen as a curve in the affine functional space $S(\mathbf{a}, \mathbf{b})$, and by analogy to the finite dimensional case, if u is close enough to $\mathcal{C}$, we prove now that there exists a unique projection of u on $\mathcal{C}(z)$:

Lemma 2.1. *For all $s \in [0, 2]$ there exists $\beta(s) > 0$ such that if u belonging to $S^s(\mathbf{a}, \mathbf{b})$ satisfies $d_s(u, \mathcal{C}(z)) < \beta(s)$ there exists a unique real number $m_s(u)$ such that*

$$d_s(u, \mathcal{C}(z)) = \|u - z(\cdot - m_s(u))\|_s.$$

Moreover, if s is one of the integers 0, 1, 2, *m_s is a function of class C^{3-s} of u, and $\mathrm{D}m_s$ satisfies the estimate*

$$|\mathrm{D}m_s v| \leq \frac{\|v\|_s \|\zeta\|_s}{\|\zeta'\|_s \big(\beta(s) - d_s(u, \mathcal{C}(z))\big)} \cdot \tag{11}$$

Proof. It is clear that we have the following equivalent as $|m|$ tends to infinity:

$$\int_{\mathbb{R}} |z(t - m) - z(t)|^2 \, \mathrm{d}t \sim |\mathbf{b} - \mathbf{a}|^2 |m|.$$

Define a function y by

$$y(m, s) = \|u - z(\cdot - m)\|_s^2. \tag{12}$$

The inequality

$$\sqrt{y(m,s)} \geq \|z(\cdot - m) - z\|_0 - \|z - u\|_s$$

shows that the set of parameters m at which $y(\cdot, s)$ reaches its minimum is compact. Since $y(\cdot, s)$ is continuous, it is plain that it reaches its lower bound. There remains to prove that if the distance of u to $\mathcal{C}(z)$ is small enough, this minimum is unique, which we will prove with the help of an appropriate differential inequality.

The first two derivatives of $y(m, s)$ with respect to m are

$$\begin{aligned} \frac{\partial y(m,s)}{\partial m} &= 2(u - z(\cdot - m), \zeta(\cdot - m))_s, \\ \frac{\partial^2 y(m,s)}{\partial m^2} &= 2\|\zeta\|_s^2 + 2(u - z(\cdot - m), -\zeta'(\cdot - m))_s \end{aligned} \tag{13}$$

and these expressions make sense: we have seen that $z^{(4)}$ belongs to $\mathrm{L}^1(\mathbb{R})^2 \cap \mathrm{L}^\infty(\mathbb{R})^2$, so that ζ belongs to $\mathrm{H}^3(\mathbb{R})^2$; therefore ζ and ζ' belong to $\mathrm{H}^s(\mathbb{R})^2$ for $s \in [0, 2]$. Therefore, for all $a > 0$,

$$\frac{\partial^2 y(m,s)}{\partial m^2} \geq 2\|\zeta\|_s^2 - \frac{\|\zeta'\|_s^2}{a^2} - y(m,s)a^2.$$

We choose $a = \|\zeta'\|_s / \|\zeta\|_s$; the differential inequality becomes

$$\frac{\partial^2 y(m,s)}{\partial m^2} + a^2 y(m,s) \geq \|\zeta\|_s.$$

We integrate this differential inequality; assuming that the minimum is attained at $\bar{m}$, we have $\partial y(\bar{m}, s)/\partial m = 0$. Thus for $|m - \bar{m}| \leq \pi/2a$

$$y(m,s) \geq d_s(u, \mathcal{C}(z))^2 \cos((m - \bar{m})a) + \frac{\|\zeta\|_s^4}{\|\zeta'\|_s^2}\bigl(1 - \cos(a(m - \bar{m}))\bigr). \tag{14}$$

Therefore, if $d_s(u, \mathcal{C}(z)) < \|\zeta\|_s^2 / \|\zeta'\|_s$, the minimum of $y(\cdot, s)$ on $[\bar{m} - \pi/2a, \bar{m} + \pi/2a]$ is attained at $\bar{m}$. On the other hand

$$y(m,s) \geq \bigl(\|z(\cdot - m) - z(\cdot - \bar{m})\|_s - d_s(u, \mathcal{C}(z))\bigr)^2.$$

If $d_s(u, \mathcal{C}(z))$ is strictly inferior to the lower bound of the right hand side of the above equation over $\mathbb{R} \setminus [\bar{m} - \pi/2a, \bar{m} + \pi/2a]$, the minimum of $y(\cdot, s)$ is attained on $[\bar{m} - \pi/2a, \bar{m} + \pi/2a]$. If

$$d_s(u, \mathcal{C}(z)) < \min\left(\frac{1}{2}\inf\{\|z(\cdot + m) - z\|_s : |m| \geq \pi/2a\}, \frac{\|\zeta\|_s^2}{\|\zeta'\|_s}\right) = \beta(s)$$

there is a unique m_s at which $y(\cdot, s)$ attains its minimum over $\mathbb{R}$.

We observe that for $s = 0$, $s = 1$ or $s = 2$ and u satisfying $d_s(u, \mathcal{C}(z)) < \beta(s)$, m_s is the solution of the implicit equation

$$\bigl(u, \zeta(\cdot - m_s)\bigr)_s = (z, \zeta)_s. \tag{15}$$

As we have seen above, $z^{(j)}$ belongs to $\mathrm{L}^1(\mathbb{R})^2 \cap \mathrm{L}^\infty(\mathbb{R})^2$ for $j = 1, \dots, 4$; therefore, the following regularity results hold:

$$\begin{aligned} m \mapsto \big(u, \zeta(\cdot - m)\big)_0 &\text{ is of class } \mathrm{C}^3 \text{ if } u \in S(\mathbf{a}, \mathbf{b}),\\ m \mapsto \big(u', \zeta'(\cdot - m)\big)_0 &\text{ is of class } \mathrm{C}^2 \text{ if } u \in S^2(\mathbf{a}, \mathbf{b}),\\ m \mapsto \big(u'', \zeta''(\cdot - m)\big)_0 &\text{ is of class } \mathrm{C}^1 \text{ if } u \in S^3(\mathbf{a}, \mathbf{b}). \end{aligned}$$

This shows that the left hand side of (15) is of class C^{3-s}; therefore, for $d_s(u, \mathcal{C}(z))$, $u \mapsto m_s(u)$ is of class C^{3-s}. We have also an estimate on $\mathrm{D}m_s(u)$: if we differentiate (15) with respect to u, we find that

$$(v, \zeta(\cdot - m_s))_s - (u, \zeta'(\cdot - m_s))_s \mathrm{D}m_s v = 0.$$

But

$$(u, \zeta'(\cdot - m_s)) = (u - z(\cdot - m_s), \zeta'(\cdot - m_s)) - \|\zeta\|_s^2;$$

therefore,

$$|(u, \zeta'(\cdot - m_s))| \geq \|\zeta\|_s^2 - \|\zeta'\|_s d_s(u, \mathcal{C}(z))$$

never vanishes if $d_s(u, \mathcal{C}(z)) \leq \beta(s)$. This implies relation (11). □

If we already know that $\|u - z(\cdot - t)\|_s$ is small, we can estimate $t - m_s(u)$, thanks to the following result:

Lemma 2.2. *There exist positive numbers $\beta_0(s)$ and $\theta(s)$ such that if t satisfies*

$$\|u - z(\cdot - t)\|_s \leq \beta_0(s),$$

then it satisfies also

$$|t - m_s(u)| \leq \theta(s)\|u - z(\cdot - t)\|_s.$$

Proof. Without loss of generality, assume that $t = 0$. Let y be as in (12); the computation of (13) implies that

$$\frac{\partial^2 y(m, s)}{\partial m^2} \geq 2\|\zeta\|_s^2 - 2\|\zeta'\|_s\big(\|u - z\|_s + |m|\,\|\zeta\|_s\big) \cdot$$

If we assume that $\|u - z\|_s \leq \|\zeta\|_s^2/3\|\zeta'\|_s$, and $|m|\,\|\zeta\|_s/3\|\zeta'\|_s = \bar{m}$, then $\partial^2 y(m, s)/\partial m^2 \geq 2\|\zeta\|_s^2/3$. Therefore

$$\begin{aligned} 0 \leq m \leq \bar{m} \Rightarrow \frac{\partial y(m, s)}{\partial m} &\geq \frac{\partial y(0, s)}{\partial m} + \frac{2m\|\zeta\|_s^2}{3},\\ -\bar{m} \leq m \leq 0 \Rightarrow \frac{\partial y(m, s)}{\partial m} &\leq \frac{\partial y(0, s)}{\partial m} + \frac{2m\|\zeta\|_s^2}{3}\cdot \end{aligned}$$

If we assume that $\|u - z\|_s \leq \|\zeta\|_s^2/9\|\zeta'\|_s$, there exists a unique m in $[-\bar{m}, \bar{m}]$ such that $\partial y(m, s)/\partial m$ vanishes. Moreover, we have the estimate

$$|m| \leq 3\|u - z\|_s/\|\zeta\|_s. \tag{16}$$

Therefore, if $\beta_0(s) < \min\big(\beta(s), \|\zeta\|_s^2/9\|\zeta'\|_s\big)$ and $\theta(s) = 3/\|\zeta\|_s$, the statement of the lemma holds. □

We will need the following corollary of Lemma 2.2:

Corollary 2.3. *If $d_1(u, \mathcal{C}(z)) \le \beta_0(0)$, then $|m_0(u) - m_1(u)|$ is at most equal to $\theta(0)d_1(u, \mathcal{C}(z))$.*

Proof. Under the assumptions of the corollary, $\|u - z(\cdot - m_1(u)\|_0 \le \|u - z(\cdot - m_1(u))\|_1 = d_1(u, \mathcal{C}(z)) \le \beta_0(0)$; therefore, according to Lemma 2.2, $|m_1(u) - m_0(u)| \le \theta(0)d_0(u, \mathcal{C}(z))$, which is at most equal to $\theta(0)d_1(u, \mathcal{C}(z))$. □

3. Compactness of sequences of minimizers in dimension one

We show now that minimizing sequences for E_1 are in fact compact in $S(\mathbf{a}, \mathbf{b})$ up to translations. This technical result will be essential in the sequel.

Theorem 3.1. *Let $(u_n)_n$ be a minimizing sequence for E_1 over $S(\mathbf{a}, \mathbf{b})$. Then there exists an extracted subsequence still denoted by u_n and a sequence of numbers x_n such that the sequence $u_n(\cdot - x_n)$ converges to a minimizer of E_1 in the strong topology of $S(\mathbf{a}, \mathbf{b})$.*

Proof. We start by studying three auxiliary problems, for which we will need the constants α_0, α_1, δ_0 and ℓ, defined as follows. Since D^2W is continuous and non degenerate at $\mathbf{a}$ and $\mathbf{b}$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that

$$\min(|y - \mathbf{a}|, |y - \mathbf{b}|) \le \delta_0 \Longrightarrow \forall \xi \in \mathbb{R}^2, D^2W(y)\xi \otimes \xi \ge \alpha_0^2 |\xi|^2. \tag{17}$$

The positive number ℓ is defined by

$$\ell^2 = \sup\{|D^2W(y)\xi \otimes \xi| : \min(|y - \mathbf{a}|, |y - \mathbf{b}|) \le 1, |\xi| \le 1\} \tag{18}$$

and it is finite, thanks to the continuity of D^2W. Finally, as W vanishes only at $\mathbf{a}$ and $\mathbf{b}$, there exists $\alpha_1 > 0$ such that

$$\min(|y - \mathbf{a}|, |y - \mathbf{b}|) \le 1 \Longrightarrow W(y) \ge \alpha_1^2 \min\bigl(|y - \mathbf{a}|^2, |y - \mathbf{b}|^2\bigr)/2. \tag{19}$$

We start by studying two auxiliary minimization problems, where δ belongs to the interval $(0, 1)$:

$$\phi_1(\delta) = \inf\bigl\{E_1(v, \mathbb{R}^-) : v(-\infty) = \mathbf{a}, |v(0) - \mathbf{b}| \le \delta\bigr\}$$
$$\phi_2(\delta) = \inf\{E_1(v, [0, r]) : |v(0) - \mathbf{b}| \le \delta, (v(r))_1 = 0, r \ge 0\}\cdot$$

It is plain that each function $\phi_j(\delta)$ is finite.

Let u_n be a minimizing sequence for $\phi_1(\delta)$. We define x_n as the smallest number such that $|u_n(x_n) - \mathbf{b}| = \delta$. Let v_n be the restriction to $\mathbb{R}^-$ of the sequence $u_n(\cdot - x_n)$: it is also a minimizing sequence, and it satisfies the inequality

$$\forall x \le 0, \quad |v_n(x) - \mathbf{b}| \ge \delta.$$

Since v_n' is bounded in $\mathrm{L}^2(\mathbb{R}^-)$ and $v_n(0)$ is bounded, we extract a subsequence such that v_n' converges to v' in the weak L^2 topology and v_n converges to v uniformly on compact subsets of $\mathbb{R}^-$. In particular, we must have $|v(x) - \mathbf{b}| \ge \delta$ for all $x \le 0$. Thanks to Fatou's lemma and the properties of weak convergence, we must have

$$\liminf E_1(v_n) \ge E_1(v).$$

As x tends to $-\infty$, $W(v)$ tends to 0, since it is integrable and Lipschitz continuous; as v stays bounded away from $\mathbf{b}$, it must tend to $\mathbf{a}$. This shows that the limit of a minimizing sequence is indeed a minimizer. In particular, we have found a minimizer such that $|v(0) - \mathbf{b}| = \delta$.

Let us estimate the energy of such a minimizer from below: extend v to $\mathbb{R}^+$ as $v(x) = (v(0) - \mathbf{b})e^{-\ell x} + \mathbf{b}$. Then its energy can be decomposed as

$$E_1(v, \mathbb{R}) = E_1(v, \mathbb{R}^-) + E_1(v, \mathbb{R}^+) \geq e_1 + E_1(v, \mathbb{R}^+).$$

We can estimate $E_1(v, \mathbb{R}^+)$, with the help of the definition (18) of ℓ and of a Taylor expansion:

$$E_1(v, \mathbb{R}^+) \leq \int_0^\infty \ell^2 |v(0) - \mathbf{b}|^2 \mathrm{e}^{-2\ell x} \, \mathrm{d}x = \delta^2 \ell / 2.$$

We see now that

$$\phi_1(\delta) \geq e_1 - \delta^2 \ell / 2. \tag{20}$$

For the second minimization problem, let u_n be a minimizing sequence; assume that x'_n is the largest number for which $|u_n(x'_n) - \mathbf{b}| = \delta$ and that x''_n is the smallest number larger than x'_n for which $(u_n(x''_n))_1$ vanishes. If r_n is equal to $x''_n - x'_n$ and if v_n is the restriction of $u_n(\cdot - x'_n)$ to $[0, r_n]$, then v_n is also a minimizing sequence which satisfies for all $x \in [0, r_n]$

$$(v_n(x))_1 \geq 0, \quad |v_n(x) - \mathbf{b}| \geq \delta.$$

We have the estimate

$$\int_0^{r_n} \frac{|v'_n|^2}{2} \, \mathrm{d}x \geq \frac{|v_n(0) - v_n(r_n)|^2}{2r_n} \geq \frac{(1-\delta)^2}{2r_n} \cdot$$

As $\phi_2(\delta)$ is finite, this means that r_n is bounded from below. On the other hand

$$\omega = \inf\{W(y) : y_1 \geq 0, |y - \mathbf{b}| \geq \delta\}$$

is strictly positive, which implies that

$$\omega r_n \leq E_1(v_n, [0, r_n]).$$

Hence r_n is also bounded from above, and we may extract a convergent subsequence, still denoted by r_n, whose limit is $r > 0$. Extend v_n by the constant $v_n(r_n)$ for $x \geq r_n$; we can extract a subsequence such that the restriction of v_n to $[0, r]$ tends to a certain v, which realizes the desired minimum.

Let now $\hat{x}$ be the smallest number in $[0, r]$ for which $|v(x) - \mathbf{b}| = 1$. We use the definition (19) of α_1 to estimate $\phi_2(\delta)$ from below:

$$\begin{aligned}
\phi_2(\delta) \geq E_1(v, [0, \hat{x}]) &\geq \int_0^{\hat{x}} \frac{|v'(x)|^2 + \alpha_1^2 |v(x) - \mathbf{b}|^2}{2} \, \mathrm{d}x \\
&\geq \alpha_1 \int_0^{\hat{x}} |v'(x)| |v(x) - \mathbf{b}| \, \mathrm{d}x \\
&\geq \frac{\alpha_1}{2} \int_0^{\hat{x}} \frac{\mathrm{d}}{\mathrm{d}x} |v(x) - \mathbf{b}|^2 \, \mathrm{d}x = \frac{\alpha_1(1 - \delta^2)}{2} \cdot
\end{aligned}$$

Therefore, we have found the estimate

$$\phi_2(\delta) \geq \frac{\alpha_1(1 - \delta^2)}{2} \cdot \tag{21}$$

Let $(u_n)_n$ be a minimizing sequence for E_1; there exists an x_n such that $(u_n(x_n))_1$ vanishes. Then, $(u_n(\cdot - x_n))_n$ is also a minimizing sequence; therefore, without loss of generality, we may assume that $(u_n(0))_1$ vanishes for all n. It is plain that the derivatives u_n' are bounded in $\mathrm{L}^2(\mathbb{R})^2$; therefore, u_n and $W(u_n)$ are uniformly Hölder continuous, with exponent $1/2$. We infer from this fact and from assumption (3) that u_n is uniformly bounded in $\mathrm{L}^\infty(\mathbb{R})^2$. In particular, we may extract a subsequence still denoted by $(u_n)_n$ which converges uniformly on compact subsets of $\mathbb{R}$ to a certain Hölder continuous limit u, and whose first derivatives converges weakly in $\mathrm{L}^2(\mathbb{R})^2$ to u'. By Fatou's lemma and the properties of weak convergence, $E_1(u)$ is at most equal to e_1 and in particular, $W(u)$ is integrable, and u tends to $\mathbf{a}$ or $\mathbf{b}$ at $\pm\infty$. Let us prove that u tends to $\mathbf{a}$ at $-\infty$, arguing by contradiction. If u tended to $\mathbf{b}$ at $-\infty$, we could find for all $\delta \in (0,1)$ an $x_0 < 0$ such that $|u(x_0) - \mathbf{b}| \le \delta/2$. Then, for all large enough n, $|u_n(x_0 - \mathbf{b}| \le \delta$. But we can use inequalities (20) and (21) to estimate $E_1(u_n)$ from below: indeed,

$$E_1(u_n) \ge E_1(u_n, (-\infty, x_0]) + E_1(u_n, [x_0, 0]) \ge \phi_1(\delta) + \phi_2(\delta).$$

Choosing δ so small that

$$\phi_1(\delta) + \phi_2(\delta) \ge e_1 + \alpha_1/4,$$

bounds $E_1(u_n)$ away from e_1, which is a contradiction. Thus, we have proved that $u(x)$ tends to $\mathbf{a}$ at $-\infty$. A similar argument shows that $u(x)$ tends to $\mathbf{b}$ at $+\infty$.

We infer from the integrability of $W(u)$ over $\mathbb{R}$ that $z - \psi$ belongs to $\mathrm{L}^2(\mathbb{R})$; we already knew that u' is square integrable; therefore u belongs to $S(\mathbf{a}, \mathbf{b})$. As $E_1(u)$ is equal to e_1, u must be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$.

We will show now that the sequence u_n converges uniformly to u. Let $\eta < e_1$ be given, and let x_1 be such that

$$E_1(u, (-\infty, x_1]) = \eta/2. \tag{22}$$

General theorems of analysis imply that

$$\liminf_{n\to\infty} E_1(u_n, [x_1, +\infty)) \ge E_1(z, [x_1, +\infty)) = e_1 - \eta/2.$$

As the limit of $E_1(u_n)$ is precisely equal to e_1, there exists N such that for all $n \ge N$

$$E_1(u_n, (-\infty, x_1]) \le \eta.$$

Let x_2 be the largest number in $(-\infty, x_1]$ such that $|u(x_2) - \mathbf{a}| = \min(1, |u(x_1, \mathbf{a})|)$; then the energy of u_n over $(-\infty, x_2]$ can be estimated with the help of (19): with the same argument as for estimate (21),

$$\int_{-\infty}^{x_2} \left(\frac{|u_n'|^2}{2} + W(u_n) \right) \, \mathrm{d}x \ge \frac{\alpha_1 \min(1, |u_n(x_1) - \mathbf{a}|^2)}{2} \cdot$$

Therefore, if $\eta < \alpha_1/2$, we must have

$$|u_n(x_1) - \mathbf{a}| \le \sqrt{2\eta/\alpha_1}.$$

The same inequality holds for u:

$$|u(x_1) - \mathbf{a}| \le \sqrt{2\eta/\alpha_1}.$$

Given $\delta \in (0,2]$, we choose $\eta = \delta^2\alpha_1/8$ and the corresponding x_1 satisfying (22). For n large enough, we have over $(-\infty, x_1]$

$$|u_n(x) - u(x)| \leq 2\sqrt{2\delta^2\alpha_1/8\alpha_1} = \delta.$$

By uniform convergence over compact sets of u_n to u, for n large enough, $\sup\{|u_n(x) - u(x)| : x_1 \leq x \leq 0\}$ is at most equal to δ. This proves the uniform convergence over $\mathbb{R}^-$; the uniform convergence of u_n to u over $\mathbb{R}^+$ is proved identically.

We can prove now the strong convergence of $u_n - u$ to 0 in $\mathrm{H}^1(\mathbb{R})^2$. Define $r_n = u_n - u$. We may write the energy of u_n with the help of a Taylor formula with integral remainder:

$$E_1(u_n) = e_1 + \int_{\mathbb{R}} \left(u' \cdot r_n' + DW(u)r_n\right) \,\mathrm{d}x + \int \left(\frac{|r_n'|^2}{2} + \int_0^1 (1-s)\mathrm{D}^2W(u)r_n \otimes r_n \,\mathrm{d}s\right) \mathrm{d}x. \tag{23}$$

The first integral term in the right hand side of (23) cancels out by integration by parts, because u satisfies the Euler–Lagrange equation (4). We choose $x_1 > 0$ in such a way that for $|x| \geq x_1$, $\min(|u(x)-\mathbf{a}|, |u(x)-\mathbf{b}|) \leq \delta_0/2$, where δ_0 has been defined at (17). Then, for $|x| \geq x_1$ and n large enough, $\min(|u_n(x) - \mathbf{a}|, |u_n(x) - \mathbf{b}|) \leq \delta_0$. The integral

$$\int_{-x_1}^{x_1} \int_0^1 (1-s)\mathrm{D}^2W(u)r_n \otimes r_n \,\mathrm{d}s \,\mathrm{d}x$$

converges to 0 as n tends to infinity, thanks to the uniform convergence of r_n to 0. We see now that

$$\frac{1}{2}\int |r_n'|^2 \,\mathrm{d}x + \frac{\alpha_0^2}{2}\int_{|x|\geq x_1} |r_n|^2 \,\mathrm{d}x$$

converges to 0, and the theorem is proved. □

Let us prove now a corollary which relates the energy of $u \in S(\mathbf{a}, \mathbf{b})$ and the distance $d_1(u, \mathcal{Z})$:

Corollary 3.2. *For all $\beta > 0$ the following inequality holds:*

$$\inf\{E_1(u) : u \in S(\mathbf{a}, \mathbf{b}), d_1(u, \mathcal{Z}) \geq \beta\} > e_1. \tag{24}$$

Proof. Assume that the lower bound in the statement of Lemma 3.2 is equal to e_1 and let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Theorem 3.1 implies that there exists a sequence x_n and an element z of $\mathcal{Z}$ such that $u_n(\cdot - x_n) - z$ converges to 0 in the strong topology of $\mathrm{H}^1(\mathbb{R})^2$. Therefore, for n large enough, $d_1(u_n, \mathcal{Z})$ is at most equal to $\beta/2$, which contradicts our assumption. □

Another useful information is the following bound on the values taken by any minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$:

Lemma 3.3. *The minimizers of E_1 over $S(\mathbf{a}, \mathbf{b})$ take their values in the closed ball of radius R about 0.*

Proof. Let z be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$; define for x in $\mathbb{R}^2$

$$\tilde{z}(x) = \begin{cases} Rz(x)/|z(x)| & \text{if } |z(x)| > R, \\ z(x) & \text{if } |z(x)| \leq R. \end{cases}$$

Assumption (3) implies that for all $x \in \mathbb{R}^2$

$$W(\tilde{z}(x)) \leq W(z(x)),$$

and the inequality is strict on the set $\{x : |z(x)| > R\}$. As the mapping $z \mapsto \tilde{z}$ is a contraction, we have

$$|\tilde{z}'| \le |z'| \text{ a.e. on } \mathbb{R}.$$

Therefore, $E_1(\tilde{z}) \le E_1(z)$, and there is equality only if

$$\int_{|z|>R} \big(W(z) - W(\tilde{z})\big)\, \mathrm{d}x = 0.$$

This, together with the continuity of z and assumption (3) implies the conclusion of the lemma. $\square$

4. The spectral assumption

We define now the spectral assumption: let z be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$. The unbounded operator A in $\mathrm{L}^2(\mathbb{R})^2$ is defined by

$$D(A) = \mathrm{H}^2(\mathbb{R})^2, \quad Ay = -y'' + (\mathrm{D}^2 W(z) y)^{\mathsf{T}}. \tag{25}$$

We shall say that A is the linearization operator at z of the Euler–Lagrange operator at z. It is easy to see that A is self-adjoint since it is a symmetric bounded perturbation of $-\,\mathrm{d}^2/\,\mathrm{d}t^2$. Let M be the minimum of the lowest eigenvalues of $\mathrm{D}^2 W(\mathbf{a})$ and $\mathrm{D}^2 W(\mathbf{b})$; then, according to a theorem of Volpert *et al.* [12], the essential spectrum of A is $[M, +\infty)$.

Let us verify that A is a non negative operator: if v belongs to $\mathrm{H}^2(\mathbb{R})^2$,

$$0 \le \frac{\mathrm{d}^2}{\mathrm{d}t^2} E_1(z + tv)\Big|_{t=0} = \int_{\mathbb{R}} \big(|v'|^2 + \mathrm{D}^2 W(z) v \otimes v\big)\, \mathrm{d}x$$

so that by an integration by parts

$$\int_{\mathbb{R}} \big(-v'' + (\mathrm{D}^2 W(z) v)^{\mathsf{T}}\big) \cdot v\, \mathrm{d}x = (Av, v) \ge 0.$$

The function z is of class C^4; under the assumption of non degeneracy of $\mathrm{D}^2 W(\mathbf{a})$ and $\mathrm{D}^2 W(\mathbf{b})$, z and all its derivatives tend exponentially to their limits at $\pm\infty$. Therefore, $\zeta = z'$ is an eigenfunction for A, relative to the eigenvalue 0. The above considerations on the essential spectrum of A imply that the kernel of A is of finite dimension.

We say that the spectral assumption is satisfied for z if

$$\text{the kernel of } A \text{ is of dimension } 1. \tag{26}$$

We will show that this property is generic, *i.e.* it holds in an open dense set: first, if the spectral assumption is satisfied for a given potential, it holds also in a neighborhood of that potential, thanks to standard results on the perturbation of the isolated eigenvalues of unbounded operators. Moreover, for any W, and any minimizer of W there exists an arbitrary small perturbation δW of W such that for the new potential $W + \delta W$, z is still a minimizer with the same energy as before, and it satisfies the spectral assumption. Moreover, if z is an isolated minimizer, we can choose the perturbation δW so that it will vanish in the neighborhood of the union of the images $z(\mathbb{R})$ for $\hat{z} \in \mathcal{Z} \setminus \mathcal{C}(z)$.

The proof of this result depends on two lemmas which give detailed information on the minimizers of E_1 over $S(\mathbf{a}, \mathbf{b})$:

Lemma 4.1. *Let z be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$; then z is injective, it never takes the values $\mathbf{a}$ and $\mathbf{b}$ and z' never vanishes. Let $\hat{z}$ and $\bar{z}$ be minimizers of E_1; if $\hat{z}(\mathbb{R})$ intersects $\bar{z}(\mathbb{R})$ then $\hat{z}$ can be deduced from $\bar{z}$ by a translation.*

Proof. Let z be a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$ which takes the value $\mathbf{a}$ at some $\bar{x}\in\mathbb{R}$. Then if we define

$$\hat{z}(x)=\begin{cases}\mathbf{a} & \text{if } x\le x',\\ z(x) & \text{otherwise,}\end{cases}$$

we can see that $\hat{z}$ belongs also to $S(\mathbf{a},\mathbf{b})$ and that $E_1(\hat{z})$ is at most equal to $E_1(z)$. Thus $\hat{z}$ is a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$. Therefore, it satisfies the Euler–Lagrange equation (4). Moreover, we can see that

$$z(\bar{x})=\mathbf{a},\quad z'(\bar{x})=0. \tag{27}$$

The constant function $\mathbf{a}$ satisfies equation (4) with the initial conditions (27); therefore, by uniqueness of solutions of smooth differential equations, $\hat{z}$ must be equal to $\mathbf{a}$, which contradicts the assumption $z\in S(\mathbf{a},\mathbf{b})$.

Assume that z is a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$ which has a self-intersection: there exists x' and $x''>x'$ such that $z(x')=z(x'')$; define

$$\hat{z}(x)=\begin{cases}z(x) & \text{if } x\le x',\\ z(x-x''+x') & \text{if } x\ge x'.\end{cases}$$

It is clear that $\hat{z}$ belongs to $S(\mathbf{a},\mathbf{b})$ and that

$$E_1(z)-E_1(\hat{z})=\int_{x'}^{x''}\left(\frac{|z'|^2}{2}+W(z)\right)\,\mathrm{d}x$$

is strictly positive since $W(z)$ is strictly positive on (x',x''). Therefore $E_1(z)>E_1(\hat{z})\ge e_1$ which contradicts the assumption that z was a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$.

Let z be a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$; it satisfies the Euler–Lagrange equation (4); multiply this equation scalarly by z' in $\mathbb{R}^2$ and integrate with respect to x: we obtain the first integral

$$-\frac{|z'|^2}{2}+W(z)=\text{constant}.$$

By passing to the limit at infinity, we infer that the constant vanishes in the above equation. If there exists a number $\bar{x}\in\mathbb{R}$ where $z'(\bar{x})$ vanishes, then $W(z(\bar{x}))$ must also vanish; this means that $z(\bar{x})$ takes the value $\mathbf{a}$ or $\mathbf{b}$ and we have already seen that this is impossible.

If there exists x' and x'' such that $\hat{z}(x')$ is equal to $\bar{z}(x'')$ we observe that

$$E_1(\hat{z},(-\infty,x'])\le E_1(\bar{z},(-\infty,x'']) \text{ iff } E_1(\hat{z},[x',+\infty))\ge E_1(\bar{z},[x'',+\infty)).$$

Assume for instance that $E_1(\hat{z},(-\infty,x'])$ is at most equal to $E_1(\bar{z},(-\infty,x''])$. Define a function z by

$$z(x)=\begin{cases}\hat{z}(x) & \text{if } x\le x',\\ \bar{z}(x-x'+x'') & \text{if } x\ge x'.\end{cases}$$

Then z also belongs to $S(\mathbf{a},\mathbf{b})$ and is a minimizer of $E_1(u)$; thus it satisfies the Euler–Lagrange equation (4) which implies that

$$\hat{z}(x')=\bar{z}(x'')=z(x'),\quad \hat{z}'(x')=\bar{z}'(x'')=z'(x').$$

Therefore, by uniqueness of the solutions of smooth differential equation, $\hat{z}=\bar{z}(\cdot-x''+x')$. $\square$

We will say that a minimizer z of E_1 is isolated if

$$d_0(z, \mathcal{Z} \setminus \mathcal{C}(z)) > 0.$$

Lemma 4.2. *Let z be an isolated minimizer of E_1. For all $x \in \mathbb{R}$, there exists $\delta > 0$ such that the ball of center $z(x)$ and radius δ does not intersect the graph of any other minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$.*

Proof. Assume that there exists $\bar{x}$ and a sequence of minimizers z_n and of numbers x_n such that

$$\lim_{n\to\infty} z_n(x_n) = z(\bar{x}).$$

Thanks to Theorem 3.1, there exists a subsequence, still denoted b y z_n, and a minimizer z_∞ such that $z_n(\cdot - x_n)$ converges to z_∞ in the strong topology of $S(\mathbf{a}, \mathbf{b})$. But $z_\infty(0)$ is equal to $z(\bar{x})$ and Lemma 4.1 implies that z_∞ belongs to $\mathcal{C}(z)$, which precludes z from being isolated. □

We can prove now the genericity of the spectral assumption.

Theorem 4.3. *Let z be a minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$ and let A be the linearization at z of the Euler–Lagrange operator. There exists a non negative function $\delta W \geq 0$ of class C^p vanishing on $z(\mathbb{R})$ such that for all $s > 0$, z is a minimizer of*

$$E_1(u) + s\delta E_1(u) = E_1(u) + \int_{\mathbb{R}} \delta W(u)\ \mathrm{d}x$$

over $S(\mathbf{a}, \mathbf{b})$; moreover, if z is an isolated minimizer of E_1 over $S(\mathbf{a}, \mathbf{b})$, it is possible to choose the perturbation δW so that it will vanish on $\mathcal{Z} \setminus \mathcal{C}(z)$.

Proof. We treat first the case when ζ is parallel to a given direction, throughout the real line. Then, the conditions at infinity imply that ζ_2 vanishes over $\mathbb{R}$; in particular, $\partial_2 W(z_1(x), 0)$ vanishes, and therefore, for all $y_1 \in z_1(\mathbb{R})$, the cross derivative $\partial_1\partial_2 W(y_1, 0)$ vanishes. The positivity of the operator A can be expressed as follows: for all $w \in \mathrm{H}^1(\mathbb{R})$,

$$\int \left(|w'|^2 + \partial_1^2 W(z)w^2\right)\ \mathrm{d}x \geq 0, \text{ and } \int \left(|w'|^2 + \partial_2^2 W(z)w^2\right)\ \mathrm{d}x \geq 0. \tag{28}$$

Let f and g be two non negative functions of class C^∞; assume that f is equal to 1 on an interval $[-x_1, x_1]$ and that g is equal to 1 on an interval $[-x_2, x_2]$. We define

$$\delta W(y) = f(y_1)y_2^2 g(y_2).$$

If z is isolated among the minimal heteroclinic connections, we may always assume that δW vanishes on the images of the other heteroclinic connections, by choosing suitably small values for x_1 and x_2; we use of course Lemmas 4.1 and 4.2. Assume that v is an eigenvector of $A + s\delta A$ defined by

$$D(A + s\delta A) = \mathrm{H}^2(\mathbb{R})^2, \quad (A + s\delta A)v = -v'' + \left(\mathrm{D}^2 W(z)v + s\mathrm{D}^2\delta W(z)v\right)^{\mathsf{T}}.$$

The remark on the vanishing of the cross derivative of W along the image of z shows that if v is an eigenvector of $A + \delta A$, then

$$-v_1'' + \partial_1^2 W(z)v_1 = 0 \text{ and } -v_2'' + \partial_2^2 W(z)v_2 + sf(z_1)v_2 = 0. \tag{29}$$

We multiply the v_2 equation by v_2, we integrate over $\mathbb{R}$ and we find that

$$\int \left(|v_2'|^2 + \partial_2^2 W(z)v_2^2 + sf(z)v_2^2\right)\ \mathrm{d}x = 0;$$

we infer from the second inequality in (28) that v_2 vanishes on the support of f; therefore, according to the classical theory of linear ordinary differential equations, v_2 vanishes everywhere. Consider now the first component of v: it solves the first ordinary differential equation in (29) and its tends at $\pm\infty$ to ± 1. The Wronskian of v_1 and ζ_1 is constant; its value is found by letting x tend to infinity: it is thus clear that it vanishes, and that v_1 is proportional to ζ_1. Therefore, in the case where ζ is parallel to a constant direction, we have proved that an arbitrary small modification of W satisfies the conditions of the theorem.

Assume now that ζ does not have a constant direction. We can thus find an interval of $\mathbb{R}$ such that ζ does not have a constant direction on any sub-interval of that interval. Without loss of generality, we may translate the space variable in such a way that this interval is of the form $(-x_1, x_1)$, with x_1 some strictly positive number.

Let $\nu(x)$ be a unit vector of class C^4 defined in $(-x_1, x_1)$ and such that the vector product $\nu(x) \wedge \zeta(x)$ stays bounded away from 0 on that interval. Define a transformation

$$\Phi(x, \lambda) = z(x) + \lambda\nu(x).$$

The implicit function theorem implies that Φ is a diffeomorphism of class C^3 in a neighborhood of $0 \in \mathbb{R}^2$; we choose $\bar{x} > 0$ and $\bar{\lambda} > 0$ small enough for Φ to be a diffeomorphism from $R = (-\bar{x}, \bar{x}) \times (-\bar{\lambda}, \bar{\lambda})$ to its image. If z is an isolated minimizer, we also choose R so small that $\Phi(R)$ has an empty intersection with the union of the images $z(\mathbb{R})$, for $z \in \mathcal{Z} \setminus \mathcal{C}(z)$; in particular, it does not contain $\mathbf{a}$ or $\mathbf{b}$. This last condition can be satisfied thanks to Lemma 4.2. Denote by Ψ the inverse diffeomorphism of Φ; it is defined on $\Phi(R)$.

Let now f and g be two non negative functions of class C^∞ with support respectively in $(-\bar{x}, \bar{x})$ and $(-\bar{\lambda}, \bar{\lambda})$; assume that f is equal to 1 over $(-\bar{x}/2, \bar{x}/2)$ and that g is equal to 1 over $(-\bar{\lambda}/2, \bar{\lambda}/2)$. We let

$$Z(x, \lambda) = f(x)\lambda^2 g(\lambda)$$

and

$$\delta W(y) = \begin{cases} Z(\Psi(y)) & \text{if } y \in \Phi(R) \\ 0 & \text{otherwise.} \end{cases}$$

The potential δW is of class C^4, and it vanishes on $z(\mathbb{R})$; it also vanishes on the image of the other minimizers if z is isolated. Therefore the energy of z for the new potential $W + s\delta W$ is equal to e_1. The new linearized operator is $A + s\delta A$ defined by

$$D(A + s\delta A) = \mathrm{H}^2(\mathbb{R})^2, \quad A + s\delta A v = -v'' + \left[(\mathrm{D}^2 W(z) + s\mathrm{D}^2\delta W(z))v\right]^\mathsf{T}.$$

We do not have to consider the essential spectrum of $A + s\delta A$: by construction, δW vanishes in a neighborhood of $\mathbf{a}$ and of $\mathbf{b}$; therefore, $A + s\delta A$ is a relatively compact perturbation of A and it has the same essential spectrum as A which is $[M, +\infty)$, $M > 0$.

We have to calculate the second derivative $\mathrm{D}^2\delta W(z)$; if y belongs to $\Phi(R)$ and $\hat{y}$ to $\mathbb{R}^2$,

$$\mathrm{D}^2\delta W(y)\hat{y}^{\otimes 2} = \mathrm{D}^2 Z \circ \Psi(y)(\mathrm{D}\Psi(y)\hat{y})^{\otimes 2} + \mathrm{D}Z \circ \Psi(y)\mathrm{D}^2\Psi(y)\hat{y}^{\otimes 2}.$$

When λ vanishes, $\mathrm{D}Z(x, 0)$ vanishes and $\mathrm{D}^2 Z(x, 0)(\hat{x}, \hat{\lambda})^{\otimes 2}$ is equal to $2f(x)\hat{\lambda}^2$. On the other hand,

$$\left(\mathrm{D}\Psi(z(x))\hat{y}\right)_2 = \frac{\zeta \wedge \hat{y}}{\zeta \wedge \nu} \cdot$$

Therefore, we find that if v is an eigenvector of $A + s\delta A$ relative to the eigenvalue 0,

$$\int_{\mathbb{R}} \left(|v'|^2 + \mathrm{D}^2 W(z)v \otimes v + 2s(\zeta \wedge v)^2(\zeta \wedge \nu)^{-2} f\right) \mathrm{d}x = 0.$$

The positivity properties of A imply that

$$\int (z' \wedge v)^2 (z' \wedge \nu)^{-2} f \, \mathrm{d}x \leq 0$$

if $s > 0$; all the eigenfunctions of A are of class C^3; therefore $z' \times v$ vanishes on the interval $(-\bar{x}, \bar{x})$; this shows that there exists a scalar function μ such that $v' = \mu\zeta$ on the support of f. As ζ never vanishes (see Lem. 4.1), μ is of class C^2; we substitute $\mu\zeta$ in the equation for v, and we find that

$$\mu''\zeta + 2\mu'\zeta' = 0. \tag{30}$$

If v is not proportional to ζ, μ' cannot vanish: indeed, if there existed x_0 in the support of f such that $\mu'(x_0)$ vanishes, we could write the relations

$$v(x_0) = \mu(x_0)\zeta(x_0) \text{ and } v'(x_0) = \mu(x_0)\zeta'(x_0).$$

The classical theory of ordinary differential equations implies immediately that $v = \mu(x_0)\zeta$, which is a contradiction. Without loss of generality, we may assume that μ' is strictly positive over the interior of the support of f. Then, we divide (30) by $\sqrt{\mu'}$ and we find the following equality

$$\sqrt{\mu'}\zeta = \text{constant}$$

but this cannot be true, since ζ is never parallel to any direction on any subinterval of $[-x_1, x_1]$. □

Remark 4.4. Given W with $N \geq 2$ wells of equal depth at $\mathbf{a}_1, \ldots, \mathbf{a}_N$, and at least n distinct minimal heteroclinic connections from $\mathbf{a}_1$ to $\mathbf{a}_2$, it is quite clear that an arbitrarily small addition to W in the neighborhood of $\mathbf{a}_j$ for $j \geq 3$ makes W into a potential with only two deeper wells. It is an obvious consequence of Theorem 4.3 that we can apply an arbitrarily small modification of W, so that it will have exactly $k \leq n$ distinct minimal heteroclinic connections from $\mathbf{a}_1$ to $\mathbf{a}_2$.

If the spectral assumption is satisfied for a minimizer z of E_1 then z is isolated; more precisely we can estimate $E_1(u) - e_1$ from below in terms of $d_1(u, \mathcal{C}(z))$. We will denote by $\nu > 0$ the square root of the lower bound of the spectrum of A without 0:

$$\nu = \sqrt{\inf \operatorname{spec}(A) \setminus \{0\}} \cdot \tag{31}$$

We need the modulus of continuity of $\mathrm{D}^2 W$; since we shall need later other moduli of continuity, we define them together here. The assumption that W is of class C^3 implies that there exist two continuous increasing functions ϖ_1 and ϖ_2 from $\mathbb{R}^+$ to itself and a continuous function ϖ_3 from $\{(r, R) : 0 \leq r \leq 2R\}$ to $\mathbb{R}^+$, increasing with respect to its two arguments, which have the following property:

$$\text{for all } \xi_1 \text{ and } \xi_2 \text{ in } \mathbb{R}^2 \text{ satisfying } \max(|\xi_1|, |\xi_2|) \leq R,$$

$$|\mathrm{D}W(\xi_1) - \mathrm{D}W(\xi_2)| \leq \varpi_1(R)|\xi_1 - \xi_2|; \tag{32}$$

$$\left|\mathrm{D}^2 W(\xi_1) - \mathrm{D}^2 W(\xi_2)\right| \leq \varpi_2(R)|\xi_1 - \xi_2|; \tag{33}$$

$$\left|\mathrm{D}^3 W(\xi_1) - \mathrm{D}^3 W(\xi_2)\right| \leq \varpi_3(|\xi_1 - \xi_2|, R). \tag{34}$$

Moreover, the function ϖ_3 vanishes at $(0, 0)$.

We relate now the spectral assumption to the coercivity of the energy, starting with a local result.

Lemma 4.5. *Assume that z is a minimizer of E_1 which satisfies the spectral assumption. Then, z is isolated in $\mathcal{Z}$ and there exist two strictly positive numbers α_2 and β_2 such that if $d_1(u, \mathcal{C}(z)) \leq \beta_2$, then*

$$E_1(u) - e_1 \geq \alpha_2 d_1(u, \mathcal{C}(z))^2.$$

Proof. The number β_2 will be less than or equal to $\min(\beta(1), \beta_0(1))$ defined at Lemma 2.2; therefore, there exists a unique m_1 such that $d_1(u, \mathcal{C}(z)) = \|u - z(\cdot - m_1)\|_1$. Without loss of generality, we may assume this m_1 vanishes. There exists also an unique m_0 such that

$$(u - z(\cdot - m_0), \zeta(\cdot - m_0)) = 0.$$

The following estimate is a consequence of (16):

$$|m_0| \leq \frac{3\beta_2\|\zeta\|_{\mathrm{L}^1}}{\|\zeta\|_0^2} \cdot$$

Define $r = u(\cdot + m_0) - z$; then r satisfies the estimates

$$\|r\|_{\mathrm{L}^\infty} \leq \|u - z\|_{\mathrm{L}^\infty} + |m_0|\|\zeta\|_{\mathrm{L}^\infty} \leq \beta_2\left(1 + \frac{3\|\zeta\|_{\mathrm{L}^1}\|\zeta\|_{\mathrm{L}^\infty}}{\|\zeta\|_0^2}\right) = \gamma_2\beta_2.$$

We expand $E_1(u)$ as in (23), and we obtain

$$E_1(u) = e_1 + \int\left(\frac{|r'|^2}{2} + \int_0^1 (1-s)D^2W(z+sr)r\otimes r\ \mathrm{d}s\right)\ \mathrm{d}x. \tag{35}$$

The first way to estimate the integral term from below is as follows: we observe that

$$|\big(D^2W(z+sr) - D^2W(z)\big)r\otimes r| \leq \varpi_2(\|z\|_{\mathrm{L}^\infty} + \gamma_2\beta_2)\gamma_2\beta_2|r|^2.$$

According to the definition (31), we have the inequality

$$E_1(u) \geq e_1 + \frac{\nu^2 - \gamma_2\beta_2\varpi_2(\|z\|_{\mathrm{L}^\infty} + \gamma_2\beta_2)}{2}\int |r|^2\ \mathrm{d}x. \tag{36}$$

We choose β_2 small enough to have the inequality

$$\gamma_2\beta_2\varpi_2(\|z\|_{\mathrm{L}^\infty} + \gamma_2\beta_2) \leq \frac{\nu^2}{2} \cdot$$

On the other hand, if we denote by γ_3 the norm of $D^2W(0)$, *i.e.*

$$\gamma_3 = \sup\{|D^2W(0)\xi\otimes\xi| : |\xi| \leq 1\}, \tag{37}$$

we have

$$|D^2W(z+rs)r\otimes r| \leq (\gamma_3 + \gamma_2\beta_2\varpi_2(\|z\|_{\mathrm{L}^\infty} + \gamma_2\beta_2)|r^2| = \gamma_1|r^2|.$$

Then, we have the other inequality for E_1:

$$E_1(u) \geq e_1 + \frac{1}{2}\int |r'|^2\ \mathrm{d}x - \frac{\gamma_1}{2}\int |r|^2\ \mathrm{d}x. \tag{38}$$

We can find a convex combination of (36) and (38) such that the coefficient of the term in r and the coefficient of the term in r' are both strictly positive, yielding the inequality

$$E_1(u) \geq e_1 + \alpha_2\|u - z(\cdot - m)\|_1^2,$$

and the conclusion is readily obtained since $\|u - z(\cdot - m)\|_1 \geq d_1(u, \mathcal{C}(z))$. $\square$

The following corollary gives a more global result.

Corollary 4.6. *Assume that the spectral assumption for z holds. For all $\beta \in (0, d_1(z, \mathcal{Z} \setminus \mathcal{C}(z)))$, there exists α such that*

$$d_1(u, \mathcal{Z} \setminus \mathcal{C}(z)) \geq \beta \Longrightarrow E_1(u) \geq e_1 + \alpha \min(1, d_1(u, \mathcal{C}(z))^2). \tag{39}$$

Proof. The proof is an immediate corollary of Corollary 3.2 and Lemma 4.5. □

The inequality opposite to (39) is an easy result, proved at the following lemma:

Lemma 4.7. *Let γ_3 be as in* (37). *The following inequality holds:*

$$E_1(u) \leq e_1 + \frac{\max(1, \gamma_3 + d_1(v, \mathcal{Z})\varpi_2(R + d_1(v, \mathcal{Z})))d_1(v, \mathcal{Z})^2}{2} \cdot \tag{40}$$

Proof. We let $d_1(u, \mathcal{Z}) = \|u - z\|_1$ and $r = u - z$. We use the expression (35) of $E_1(u)$, and we observe that

$$\left| \int_0^1 (1-s) D^2 W(z + sr) r \otimes r \, \mathrm{d}s \right| \leq \frac{|r|^2}{2} \big(\gamma_3 + \varpi_2(R + \|r\|_{\mathrm{L}^\infty})\big).$$

The conclusion is then immediate. □

5. A linear elliptic problem

We assume in this section that the spectral assumption (26) holds. We need some analytic information on A and the semi-group generated by $\sqrt{A}$ in different functional spaces. Define for $s \geq 0$ the space

$$V_s = \{v \in \mathrm{H}^s(\mathbb{R})^2 : (v, \zeta) = 0\} \cdot$$

The operator A_0 is an unbounded operator in V_0 defined by

$$D(A_0) = D(A) \cap V_0, \quad A_0 y = Ay.$$

It is clear that A_0 is self adjoint and that $A_0 - \nu^2$ is non negative, where ν has been defined by (31).

Lemma 5.1. *Assume that the spectral assumption* (26) *is satisfied. For all $s \in [0, 1]$, the expressions $\|u\|_s$ and $\|A_0^{s/2} u\|_0$ define equivalent norms on V_s.*

Proof. The proof of this result is by interpolation. We use one of the simplest cases of interpolation theory, namely Theorem 15.1 of [6]. The content of this result is the following: let X and Y be Hilbert spaces such that X is continuously and densely embedded in Y; let Λ be the self-adjoint positive operator in Y defined by

$$D(\Lambda^2) = \big\{u \in X : \sup\big(|(u, v)|_X / |v|_Y\big) < \infty\big\}, \quad (\Lambda^2 u, v)_Y = (u, v)_X \quad \forall v \in X.$$

The interpolation space $[X, Y]_\theta$ is defined as the domain of $\Lambda^{1-\theta}$ and it is equipped with the norm $\|\Lambda^{1-\theta}\|_Y$. Moreover, if X is equipped with two equivalent Hilbertian norms, corresponding to operators Λ_1 and Λ_2, the interpolation spaces can be identified, and their norms are equivalent. We apply this to $X = \mathrm{H}^2(\mathbb{R})^2$, $Y = \mathrm{L}^2(\mathbb{R})^2$ and to the following operators: Λ_1 is $\mathbf{1} + A$ and the operator Λ_2 is defined as follows:

$$D(\Lambda_2) = \mathrm{H}^2(\mathbb{R})^2, \quad \Lambda_2 v = -v'' + v.$$

It is well known that the domain of $\Lambda_2^{1-\theta}$ is the space $\mathrm{H}^{2(1-\theta)}(\mathbb{R})^2$. Therefore, we have proved that for all $s \in [0,2]$, there exists $k(s)$ such that for all $v \in \mathrm{H}^s(\mathbb{R})^2$

$$k(s)^{-1}\|v\|_s \leq \|(\mathbf{1}+A)^{s/2}v\|_0 \leq k(s)\|v\|_s. \tag{41}$$

To conclude the proof of the lemma, we use the spectral theorem. Recall that a resolution of the identity in a Hilbert space H is defined by the data of orthogonal projections $P(\lambda)$ parameterized by $\lambda \in \mathbb{R}$; it is assumed that $\lambda \mapsto \operatorname{Im} P(\lambda)$ is increasing; this is equivalent to $P(\lambda)P(\lambda') = P(\lambda)$ whenever $\lambda' \geq \lambda$. It is also assumed that P is continuous on the right and that

$$\bigcap_{\lambda\in\mathbb{R}} \operatorname{Im} P(\lambda) = \{0\}, \quad \overline{\bigcup_{\lambda\in\mathbb{R}} \operatorname{Im} P(\lambda)} = H.$$

With these definitions we can see that the function $\lambda \mapsto (P(\lambda)u,u)$ is increasing and right continuous for all $u \in H$ and $\mathrm{d}(P(\lambda)u,u)$ is a non-negative Stieltjes measure. The function $(P(\lambda)u,v)$ is the difference of two increasing function so that $\mathrm{d}(P(\lambda)u,v)$ is also a Stieltjes measure. Define

$$D(L) = \left\{u \in H : \int \lambda^2 \,\mathrm{d}\big(P(\lambda)u,u\big) < +\infty\right\}; \tag{42}$$

if u belongs to $D(L)$, Lu is defined by the condition

$$\forall v \in H, \quad (Lu,v) = \int \lambda \,\mathrm{d}\big(P(\lambda)u,v\big). \tag{43}$$

The spectral theorem (see for instance [4], VI Paragraph 5 or [10], VII for a full description of the theory) asserts that for all self-adjoint operator L in H there exists a resolution of the identity for which the domain of L and L are given by (42) and (43). If ϕ is a continuous function defined on the spectrum of L, the operator $\phi(L)$ is defined by

$$D(\phi(L)) = \left\{u \in H : \int \phi(\lambda)^2 \;\mathrm{d}(P(\lambda)u,u) < +\infty\right\},$$
$$\big(\phi(A)u,v\big) = \int \phi(\lambda)\,\mathrm{d}\big(P(\lambda)u,v\big).$$

In particular, if ϕ is bounded on the spectrum of L then $\phi(L)$ is bounded and

$$\|\phi(L)\|_{\mathcal{L}(H)} \leq \sup\{|\phi(\lambda)| : \lambda \in \operatorname{spec}(L)\}\cdot$$

The most important property of these operator functions are that $\phi_1(A)\phi_2(A)$ is identical to $(\phi_1\phi_2)(A)$ and $\phi_1(\phi_2(L))$ is identical to $(\phi_1 \circ \phi_2)(L)$.

In our special case, the projection $P(0)$ is equal to the projection on the kernel of A, $\mathbb{R}\zeta$. Let I be the injection $V_0 \to \mathrm{L}^2(\mathbb{R})^2$ and let I^* be its adjoint, *i.e.* the projection $\mathrm{L}^2(\mathbb{R})^2 \to V_0$. The resolution of the identity associated to A_0 is $I^*P(\lambda)I = P_0(\lambda)$ and its support is included in $[\nu^2,\infty)$. If v belongs to V_s, we can write

$$A_0^{s/2}v = \int \lambda^{s/2}\,\mathrm{d}P_0(\lambda)v$$

and thanks to the inequalities

$$\lambda \geq \nu^2 \Rightarrow \frac{\nu^s}{(1+\nu)^s}(1+\lambda)^{s/2} \leq \lambda^{s/2} \leq (1+\lambda)^{s/2}$$

we can see that

$$\frac{\nu^s}{(1+\nu)^s}\|(1+A)^{s/2}v\|_0 \le \|A_0^{s/2}v\|_0 \le \|(1+A)^{s/2}v\|_0.$$

This relation together with (41) enables us to conclude. □

Henceforth, V_s is equipped with the norm $\|A^s v\|_0 = \|v\|_{V_s}$. The number $\tau(s)$ is such that for all $v \in V_s$

$$\tau(s)^{-1}\|v\|_s \le \|v\|_{V_s} \le \tau(s)\|v\|_s. \tag{44}$$

It should be noted that for all $s \ge 0$, $\tau(s) \ge 1 = \tau(0)$

Lemma 5.2. *Assume that the spectral assumption* (26) *is satisfied. Then the following assertions hold:*

(1) *The operator $\sqrt{A_0}$ is an unbounded self-adjoint operator in V_0 with domain V_1; the inverse of $\sqrt{A_0}$ is bounded and of norm at most equal to ν^{-1} as an operator from V_0 to itself.*

(2) *The operator $\sqrt{A_0}$ generates a holomorphic semi-group of contractions in V_0, which satisfies the estimate*

$$\left\|\exp\bigl(-t\sqrt{A_0}\bigr)\right\|_{\mathcal{L}(V_0)} \le \exp(-\nu t). \tag{45}$$

(3) *For all $s \in [0,2]$, $A_0^{-1/2}$ is a bounded operator from V_s into itself.*

(4) *The operator $\exp\bigl(-t\sqrt{A_0}\bigr)$ maps V_s into itself and satisfies the estimate*

$$\left\|\exp\bigl(-t\sqrt{A_0}\bigr)\right\|_{\mathcal{L}(V_s)} \le \exp(-\nu t). \tag{46}$$

For all $v \in V_s$ the mapping $t \mapsto \exp\bigl(-t\sqrt{A_0}\bigr)v$ is continuous at 0.

Proof. The proof is an exercise in spectral representations, and left to the reader. □

We will use these functional results to solve a linear problem in a half-plane: given $g \in V_s$ and a bounded $f \in \mathrm{C}^0(\mathbb{R}^+, V_s)$, we would like to find a solution of

$$-\Delta y + (\mathrm{D}^2 W(z)y)^{\mathsf{T}} + f = 0, \quad y(\cdot,0) = g \tag{47}$$

which is a continuous bounded function of x_2 with values in V_s.

Lemma 5.3. *For all $g \in V_s$ and all bounded $f \in \mathrm{C}^0(\mathbb{R}^+, V_s)$, there exists a unique bounded solution $y \in \mathrm{C}^0(\mathbb{R}^+, V_s)$ of* (47)*; it is given explicitly by*

$$\begin{aligned} y(\cdot,x_2) = {} & \exp\bigl(-x_2\sqrt{A_0}\bigr)\left(g + \bigl(2\sqrt{A_0}\bigr)^{-1}\int_0^\infty \exp\bigl(-t\sqrt{A_0}\bigr)f(\cdot,t)\,\mathrm{d}t\right)\\ & - \bigl(2\sqrt{A_0}\bigr)^{-1}\int_0^{x_2} \exp\bigl(-(x_2-t)\sqrt{A_0}\bigr)f(t)\,\mathrm{d}t\\ & - \bigl(2\sqrt{A_0}\bigr)^{-1}\int_{x_2}^\infty \exp\bigl((x_2-t)\sqrt{A_0}\bigr)f(t)\,\mathrm{d}t. \end{aligned}$$

Proof. Define

$$(B_1 g)(\cdot,x_2) = \exp\bigl(-x_2\sqrt{A_0}\bigr)g \tag{48}$$

and

$$(B_2 f)(\cdot, x_2) = \big(2\sqrt{A_0}\big)^{-1} \Big(\exp\big(-x_2\sqrt{A_0}\big) \int_0^\infty \exp\big(-t\sqrt{A_0}\big) f(\cdot, t)\ \mathrm{d}t$$
$$- \int_0^{x_2} \exp\big(-(x_2 - t)\sqrt{A_0}\big) f(t)\ \mathrm{d}t \tag{49}$$
$$- \int_{x_2}^\infty \exp\big((x_2 - t)\sqrt{A_0}\big) f(t)\ \mathrm{d}t \Big).$$

The following estimate is plain:

$$\|(B_1 g)(\cdot, x_2)\|_{V_s} \le \exp(-\nu x_2)\|g\|_{V_s}.$$

If we let

$$\|f\|_{\mathrm{L}^\infty(V_s)} = \sup\{\|f(\cdot, x_2)\|_{V_s} : x_2 \in \mathbb{R}^+\},$$

we have also

$$\|(B_2 f)(\cdot, x_2)\|_{V_s} \le \nu^{-2}\|f\|_{\mathrm{L}^\infty(V_s)}.$$

Therefore $(B_1 g + B_2 f)(\cdot, x_2)$ is bounded in V_s uniformly with respect to $x_2 \ge 0$. The continuity of $B_1 g$ with respect to x_2 with values in V_s is a consequence of the semi-group property. The continuity of the term

$$\left(2\sqrt{A_0}\right)^{-1} \exp\left(-x_2\sqrt{A_0}\right) \int_0^\infty \exp\left(-t\sqrt{A_0}\right) f(t)\ \mathrm{d}t$$

is clear. The continuity of the other terms is more delicate: if x_2 is fixed, the set $f([0, x_2]) = \{f(t) : 0 \le t \le x_2\}$ is compact in V_s and therefore as h decreases to 0, the convergence of $\big[\exp\big(-(x_2-t-h)\sqrt{A_0}\big) - \exp\big(-(x_2-t)\sqrt{A_0}\big)\big]v$ toward 0 in V_s is uniform with respect to t in $[0, x_2]$; therefore

$$\int_0^{x_2} \left[\exp\left(-(x_2 - t - h)\sqrt{A_0}\right) - \exp\left(-(x_2 - t)\sqrt{A_0}\right)\right] f(t)\ \mathrm{d}t$$

tends to 0 as h decreases to 0; the term

$$\int_{x_2}^{x_2+h} \exp\left(-(x_2 + h - t)\sqrt{A_0}\right) f(t)\ \mathrm{d}t$$

can be treated immediately. If h increases to 0 there is an analogous treatment of the two different terms. For the term integrated on a semi-infinite interval, we have to argue differently: given $x_2 > 0$ and $\delta > 0$, we find $x_2' > x_2$ such that for all $h \in [0, 1]$

$$\left\| \int_{x_2'}^\infty \left(\exp\left((x_2 + h - t)\sqrt{A_0}\right) - \exp\left((x_2 - t)\sqrt{A_0}\right) \right) f(t)\ \mathrm{d}t \right\|_{V_s} \le \frac{\delta}{2}\cdot$$

It is possible to find such an x_2' because of the exponential estimate on the semi-group, which implies that

$$\int_{x_2'}^\infty \left\| \exp\left((x_2 - t)\sqrt{A_0}\right) f(t) \right\|_s\ \mathrm{d}t \le \|f\|_{\mathrm{L}^\infty(V_s)} \exp(-\nu(x_2' - x_2))$$

with a similar estimate for the other term. Fix this x_2', and now the compactness argument shows that it is possible to choose h so small that the remaining terms are less than $\delta/2$ in the norm of V_s. An adequately modified argument holds in the case $h \leq 0$.

We prove that $B_1 g + B_2 f$ is a solution of (47) in the most direct fashion: we differentiate twice $(B_1 g)(\cdot, x_2)$ with respect to x_2; since $\exp\bigl(-t\sqrt{A_0}\bigr)$ is a holomorphic semi-group, it is clear that for all $x_2 > 0$

$$-\frac{\partial^2 B^1 g}{\partial x_2^2} + A_0 B^1 g = 0. \tag{50}$$

Similarly, we differentiate $(B_2 f)(\cdot, x_2)$ twice; using arguments similar to those of the proof of continuity, we find that

$$\begin{aligned}
&\frac{d^2}{\mathrm{d}x_2^2}\left(2\sqrt{A_0}\right)^{-1}\left(\int_0^{x_2} \exp\left(-(x_2 - t)\sqrt{A_0}\right) f(t)\ \mathrm{d}t\right.\\
&\qquad\left. + \int_0^{x_2} \exp\left((x_2 - t)\sqrt{A_0}\right) f(t)\ \mathrm{d}t\right)\\
&= -f(x_2) + \frac{1}{2}\int_0^{x_2} \sqrt{A_0}\exp\left(-(x_2 - t)\sqrt{A_0}\right) f(t)\ \mathrm{d}t\\
&\qquad + \frac{1}{2}\int_{x_2}^{\infty} \sqrt{A_0}\exp\left((x_2 - t)\sqrt{A_0}\right) f(t)\ \mathrm{d}t.
\end{aligned} \tag{51}$$

Therefore, $B_2 f$ satisfies the relation

$$-\frac{\partial^2 B_2 f}{\partial x_2^2} + A_0 B_2 f = f. \tag{52}$$

It should be observed that the integral terms on the right hand side of (51) make sense in H^{s-1}: therefore (52) is satisfied in the sense of distributions with values in V_{s-1}. The boundary condition are obviously satisfied. If we put together (50) and (52) we can see that $B_1 g + B_2 f$ solves (47).

There remains to prove the uniqueness of bounded solutions of (47). It suffices to prove that when g and f vanish, the only bounded solution of (47) vanishes. Let us assume therefore that u is continuous with values in V_0, that $u(\cdot, x_2)$ is bounded in V_0 independently of $x_2 \geq 0$ and that u satisfies in the sense of distributions the equation

$$-\frac{\partial^2 u}{\partial x_2^2} + Au = 0, \quad u(\cdot, 0) = 0. \tag{53}$$

Define $v = A_0^{-1} u$; then v satisfies the same relations as u, and moreover, since $\partial^2 v/\partial x_2^2$ is equal to $A_0 v = u$, v is of class C^2 with values in V_0. In particular, the boundary value $(\partial v/\partial x_2)(\cdot, 0)$ is well defined and belongs to V_0. Assume that y_0 belongs to V_2; two integrations by parts show that

$$\int_0^\infty \left(\frac{\partial^2 v}{\partial x_2^2}, \exp\left(-t\sqrt{A_0}\right) y_0\right)\ \mathrm{d}t = \int_0^\infty \left(v, A_0 \exp\left(-t\sqrt{A_0}\right) y_0\right)\ \mathrm{d}t - \left(\frac{\partial v}{\partial x_2}(\cdot, 0), y_0\right).$$

If we use (53), we can see that $(\partial v/\partial x_2)(\cdot, 0)$ is orthogonal to V_2; as V_2 is dense in V_0, this proves that $(\partial v/\partial x_2)(\cdot, 0)$ vanishes. The function v has a partial Laplace transform with respect to x_2 denoted by $V(x_1, p)$ and defined by the formula

$$V(x_1, p) = \int_0^\infty \exp(-p x_2) v(x_1, x_2)\ \mathrm{d}x_2.$$

For $\Re p > 0$, $p \mapsto V(\cdot, p)$ is holomorphic with values in V_s, and it satisfies the relation

$$-p^2 V + A_0 V = 0.$$

But for $0 < \Re p < \nu$, $A_0 - p^2$ has an inverse in V_0; therefore V vanishes identically on the strip $0 < \Re p < \nu$. Thus, we can see that V vanishes identically and the uniqueness is proved. □

We prove another lemma which gives more precise estimates in the space

$$\mathcal{V}_s = \{u \in \mathrm{C}^0(\mathbb{R}^+; V_s) : \sup \exp(\gamma x_2) \|u(\cdot, x_2)\|_s < +\infty\},$$

where γ is chosen in the interval $(0, \nu)$. Observe that here we take the $\| \, \|_s$ norm on V_s and not the $\| \, \|_{V_s}$ norm.

Lemma 5.4. *For all $s \in [0, 2]$, B_1 maps continuously V_s into $\mathcal{V}_s$ and B_2 maps continuously $\mathcal{V}_s$ into itself. More precisely, there exists $K_1(s)$ such that for all $g \in V_s$*

$$\|B_1 g\|_{\mathcal{V}_s} \leq K_1(s) \|g\|_{V_s}, \tag{54}$$

and there exists $K_2(s)$ such that for all f in $\mathcal{V}_s$

$$\|B_2 f\|_{\mathcal{V}_s} \leq K_2(s) \|f\|_{\mathcal{V}_s}. \tag{55}$$

Proof. We use the definition (48) of B_1, and we find immediately that

$$\sup_{x_2 \geq 0} \exp(-\gamma x_2) \|(B_1 g)(\cdot, x_2)\|_{V_s} \leq \|g\|_{V_s},$$

and with the help of the definition (44) of $\tau(s)$, we infer that

$$K_1(s) = \tau(s)^2$$

answers the question.

We estimate term by term $B_2 f$ as defined by (49): observe first that

$$\begin{aligned} \left\| \int_0^\infty \exp\left(-t\sqrt{A_0}\right) f(\cdot, t) \, \mathrm{d}t \right\|_{V_s} &\leq \int_0^\infty \exp(-t\nu) \tau(s) \|f\|_{\mathcal{V}_s} \exp(-\gamma t) \, \mathrm{d}t \\ &\leq \frac{\tau(s) \|f\|_{\mathcal{V}_s}}{\gamma + \nu} \cdot \end{aligned}$$

Therefore

$$\left\| (2 \left(\sqrt{A_0}\right)^{-1} \int_0^\infty \exp\left(-t\sqrt{A_0}\right) f(\cdot, t) \, \mathrm{d}t \right\|_s \exp(\gamma x_2) \leq \frac{\tau(s)^2 \|f\|_{\mathcal{V}_s} \exp\big((\gamma - \nu) x_2\big)}{2\nu(\gamma + \nu)} \cdot$$

The second term in the expression of $B_2 f$ is estimated similarly:

$$\left\| \int_0^{x_2} \exp\left(-(x_2 - t)\sqrt{A_0}\right) f(\cdot, t) \, \mathrm{d}t \right\|_{V_s} \leq \frac{\big[\exp\big(-\gamma x_2\big) - \exp\big(-\nu x_2\big)\big] \tau(s) \|f\|_{\mathcal{V}_s}}{\nu - \gamma},$$

so that

$$\left\| (2\left(\sqrt{A_0}\right)^{-1} \int_0^{x_2} \exp\left(-(x_2-t)\sqrt{A_0}\right) f(\cdot,t)\ \mathrm{d}t \right\|_s \exp\bigl(\gamma x_2\bigr)$$
$$\leq \frac{\tau(s)^2\bigl[1-\exp\bigl((\gamma-\nu)x_2\bigr)\bigr]\|f\|_{\mathcal{V}_s}}{\nu-\gamma}.$$

The computation is analogous for the third term in the expression of $B_2 f$:

$$\left\| \int_{x_2}^{\infty} \exp\left((x_2-t)\sqrt{A_0}\right) f(\cdot,t)\ \mathrm{d}t \right\|_{V_s} \leq \frac{\tau(s)\exp\bigl(-\gamma x_2)\bigr)\|f\|_{\mathcal{V}_s}}{\nu+\gamma},$$

so that

$$\left\| \left(2\sqrt{A_0}\right)^{-1} \int_{x_2}^{\infty} \exp\left(-(x_2-t)\sqrt{A_0}\right) f(\cdot,t)\ \mathrm{d}t \right\|_s \exp\bigl(\gamma x_2\bigr) \leq \frac{\tau(s)^2\|f\|_{\mathcal{V}_s}}{2\nu(\nu+\gamma)}\,.$$

Write $\alpha = \exp\bigl(-(\nu-\gamma)x_2\bigr)$; from the elementary calculation

$$\frac{\alpha}{\gamma+\nu} + \frac{1-\alpha}{\nu-\gamma} + \frac{1}{\gamma+\nu} \leq \frac{2\nu}{\nu^2-\gamma^2},$$

we infer that

$$\|B_2 f\|_{\mathcal{V}_s} \leq \frac{\tau(s)^2}{\nu^2-\gamma^2}\,\|f\|_{\mathcal{V}_s}\,.$$

Thus

$$K_2(s) = \frac{\tau(s)^2}{\nu^2-\gamma^2} \tag{56}$$

satisfies the announced properties and our proof is complete. □

Remark 5.5. Without loss of generality, we may assume that

$$K_1(0) \leq K_1(1). \tag{57}$$

6. An elliptic problem in a half-plane

In this section we show that if z is a minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$ satisfying the spectral condition and if $d_1(g,\mathcal{C}(z))$ is small enough we can construct a solution of

$$-\Delta u + \mathrm{D}W(u)^{\mathsf{T}} = 0 \text{ in } \mathbb{R}\times(0,\infty), \tag{58}$$
$$u(\cdot,0) = g \tag{59}$$

such that $u(\cdot,x_2)$ converges exponentially as x_2 tends to infinity to some $z(\cdot-m)$.

The idea is to use the implicit function theorem and a monotonicity argument. For this purpose, we define the functional space

$$\mathcal{W} = \left\{\lambda \in \mathrm{C}^0(\mathbb{R}^+) : \quad \sup_{x_2\geq 0} \exp(\gamma x_2)|\lambda(x_2)| < \infty\right\}$$

and the operator B_3 on $\mathcal{W}$

$$(B_3\lambda)(x_2) = \int_{x_2}^{\infty}\int_s^{\infty} \lambda(\sigma)\,\mathrm{d}\sigma\ \mathrm{d}s.$$

The reader will check that for $K_3 = \gamma^{-2}$ and for all $\lambda \in \mathcal{W}$

$$\|B_3\lambda\|_{\mathcal{W}} \le K_3\|\lambda\|_{\mathcal{W}}.$$

We introduce new unknowns $v \in \mathcal{V}_1$, $\lambda \in \mathcal{W}$ and $m \in \mathbb{R}$; we write

$$u(\cdot + m, \cdot) = z + v + \lambda\zeta.$$

We will assume that v is orthogonal to ζ. Equation (58) can be rewritten

$$-\frac{\partial^2 v}{\partial x_2^2} - \frac{\mathrm{d}^2\lambda}{\mathrm{d}x_2^2}\zeta + Av + Q(v + \lambda\zeta) = 0, \tag{60}$$

provided that we define for all $y \in \mathrm{H}^1(\mathbb{R})^2$

$$Q(y) = \left(\mathrm{D}W(z + y) - \mathrm{D}W(z) - \mathrm{D}^2W(z)y\right)^{\mathsf{T}}.$$

The boundary data for (60) becomes

$$v(\cdot, 0) + \lambda(0)\zeta = g(\cdot + m) - z. \tag{61}$$

We define a function F from $\mathcal{V}_s \times \mathcal{W} \times \mathrm{H}^1(\mathbb{R})^2 \times \mathbb{R}$ to $\mathcal{V}_s \times \mathcal{W} \times \mathbb{R}$ by

$$F(X) = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \end{pmatrix}, \quad X = \begin{pmatrix} v \\ \lambda \\ g \\ m \end{pmatrix},$$

and

$$\begin{aligned}
F_1(X) &= v - B_1(\mathbf{1} - P(0))\big(g(\cdot + m) - z\big) + B_2(\mathbf{1} - P(0))Q(v + \lambda\zeta),\\
F_2(X) &= \lambda - B_3(Q(v + \lambda\zeta), \zeta)/\|\zeta\|_0^2,\\
F_3(X) &= \lambda(0) - (g(\cdot + m) - z, \zeta)/\|\zeta\|_0^2.
\end{aligned}$$

Lemma 5.3 implies that it is equivalent to solve $F(\cdot, \cdot, g, m) = 0$ and to find a solution of a solution of (60) with boundary data (61).

It is impossible to apply directly the implicit function theorem to F because F is not of class C^1 from $\mathcal{V}_1 \times \mathcal{W} \times \mathrm{H}^1(\mathbb{R}^2) \times \mathbb{R}$ to $\mathcal{V}_1 \times \mathcal{W} \times \mathbb{R}$: when we differentiate with respect to m, one degree of differentiability is lost in the F_1 equation. However this defect cannot be taken care of by considering that F takes its values in $\mathcal{V}_0 \times \mathcal{W} \times \mathbb{R}$, since in that case, the derivative of F with respect to (v, λ, m) stops being invertible at $(0, 0, 0)$.

In order to get around this difficulty, we proceed as follows: we first solve the first two equations for v and λ in terms of m and g; since we need uniform estimates locally in m and g, we do not use the general implicit function theorem and we write the conditions under which we can use the strict contraction theorem. Denoting the solution obtained by $v(m)$, $\lambda(m)$, we solve for m by finding a zero of $F(v(m), \lambda(m), g, m)$: the lower regularity with respect to m is enough to differentiate $F(v(m), \lambda(m), g, m)$ with respect to m, and for m and $\|g\|_1$ small enough, we find a solution.

We estimate now the norm of $\mathrm{D}Q(y)\hat{y}$ in $\mathrm{H}^1(\mathbb{R})^2$ with the help of the moduli of continuity defined at (33) and (34):

Lemma 6.1. *There exists an increasing continuous function ϖ from $\mathbb{R}^+$ to itself and vanishing at 0 such that for all y and $\hat{y}$ in $\mathrm{H}^1(\mathbb{R})^2$*

$$\|\mathrm{D}Q(y)\hat{y}\|_0 \le \varpi(\|y\|_1)\|\hat{y}\|_0 \tag{62}$$

$$\|\mathrm{D}Q(y)\hat{y}\|_1 \le \varpi(\|y\|_1)\|\hat{y}\|_1. \tag{63}$$

Proof. The proof is immediate; recall the definition (33) of ϖ_2 and (34) of ϖ_3 and the inequality $\|y\|_{\mathrm{L}^\infty} \le \|y\|_{\mathrm{H}^1}$ for all function $y \in \mathrm{H}^1(\mathbb{R})^2$; we may take

$$\varpi(r) = \varpi_2(R+r) + r\sup\{|D^3W(y)\xi^{\otimes 3}| : |y| \le R+r, |\xi| \le 1\} + \varpi_3(R+r)\|\zeta\|_{\mathrm{L}^\infty}.$$

□

Equip the space $\mathcal{V}_1 \times \mathcal{W}$ with the norm

$$\|(v,\lambda)\|_{\mathcal{V}_1\times\mathcal{W}} = \max(\|v\|_{\mathcal{V}_1}, \|\lambda\|_{\mathcal{W}})$$

and define the mapping

$$G(v,\lambda;g,m) = \begin{pmatrix} B_1(\mathbf{1}-P(0))(g(\cdot+m)-z) - B_2(\mathbf{1}-P(0))Q(v+\lambda\zeta) \\ B_3\big(Q(v+\lambda\zeta),\zeta\big)/\|\zeta\|_0^2 \end{pmatrix}.$$

Next lemma proves that for (m,g) close enough to $(0,z)$, it is possible to find a fixed point of $G(.,.;g,m)$ in a small enough ball about 0 in $\mathcal{V}_1 \times \mathcal{W}$; it is also possible to estimate the derivatives $\partial v/\partial m$ and $\partial\lambda/\partial m$ respectively in $\mathcal{V}_0$ and $\mathcal{W}$.

Lemma 6.2. *There exist $\rho > 0$ and $\rho_1 > 0$ such that if*

$$\max(\|g-z\|_1, |m|) \le \rho_1, \tag{64}$$

G maps the ball of radius ρ about 0 of $\mathcal{V}_1 \times \mathcal{W}$ into itself and is a strict contraction on that ball. Moreover, the unique fixed point $(v(m),\lambda(m))$ of G in the ball satisfies the estimate

$$\max(\|v(m)\|_{\mathcal{V}_1}, \|\lambda(m)\|_{\mathcal{W}} \le 2\tau(1)K_1(1)\big(\|g-z\|_1 + |m|\|\zeta\|_1\big). \tag{65}$$

The derivatives $\partial v/\partial m$ and $\partial\lambda/\partial m$ are well defined as respective elements of $\mathcal{V}_0$ and $\mathcal{W}$ and they satisfy the estimates

$$\max\left(\left\|\frac{\partial v}{\partial m}(m)\right\|_{\mathcal{V}_0}, \left\|\frac{\partial\lambda}{\partial m}(m)\right\|_{\mathcal{W}}\right) \le \frac{K_1(0)(\|g'-\zeta\|_0 + |m|\|\zeta'\|_0)}{1-\varpi(\kappa\rho)(K_2(0)+K_3)}. \tag{66}$$

Proof. Thanks to Lemma 6.1, Q is locally Lipschitz continuous, and for all y_1 and y_2 in $\mathrm{H}^1(\mathbb{R})^2$,

$$\|Q(y_2) - Q(y_1)\|_1 \le \varpi(\max(\|y_1\|_1, \|y_2\|_1)\|y_2 - y_1\|_1.$$

Since $Q(0)$ vanishes, we have in particular $\|Q(y)\|_1 \le \varpi(\|y\|_1)\|y\|_1$. Assume $\max(\|v\|_{\mathcal{V}_1}, \|\lambda\|_{\mathcal{W}}) \le \rho$ and define $\kappa = 1 + \|\zeta\|_1$. Then, we may estimate the first component of G, under assumption (64):

$$\|B_1(\mathbf{1}-P(0))(g(\cdot+m)-z)\|_{\mathcal{V}_1} \le K_1(1)\tau(1)\kappa\rho_1,$$

$$\|B_2(\mathbf{1}-P(0))Q(v+\lambda\zeta)\|_{\mathcal{V}_1} \le K_2(1)\varpi(\kappa\rho)\kappa\rho,$$

and similarly the second component of G:

$$\|B_3\big(Q(v+\lambda\zeta),\zeta\big)/\|\zeta\|_0^2\|_{\mathcal{W}} \leq K_3\varpi(\kappa\rho)\kappa\rho.$$

Therefore, if

$$\kappa\rho_1 K_1(1)\tau(1) + K_2(1)\varpi(\kappa\rho)\kappa\rho \leq \rho \text{ and } K_3\varpi(\kappa\rho)\kappa\rho/\|\zeta\|_0 \leq \rho,$$

the ball of radius ρ about 0 in $\mathcal{V}_1 \times \mathcal{W}$ is invariant by G. The Lipschitz constant of G is estimated as follows:

$$\begin{aligned}
&\|B_2(\mathbf{1}-P(0))(Q(v_1+\lambda_1\zeta)-Q(v_2+\lambda_2\zeta))\|_{\mathcal{V}_1} \leq K_2(1)\varpi(\kappa\rho)(\|v_2-v_1\|_{\mathcal{V}_1}+\|\lambda_2-\lambda_1\|_{\mathcal{W}}),\\
&\|B_3\big(Q(v_1+\lambda_1\zeta)-Q(v_2+\lambda_2\zeta),\zeta\big)/\|\zeta\|_0^2\big)\|_{\mathcal{W}} \leq K_3\varpi(\kappa\rho)(\|v_2-v_1\|_{\mathcal{V}_1}+\|\lambda_2-\lambda_1\|_{\mathcal{W}})/\|\zeta\|_0.
\end{aligned}$$

The mapping G will be a contraction of ratio $1/2$ in $\mathcal{V}_1 \times \mathcal{W}$ if

$$\max(K_2(1), K_3/\|\zeta\|_0)\max(1,\|\zeta\|_1)\varpi(\kappa\rho) \leq 1/2. \tag{67}$$

At this point, we require therefore (67) and

$$K_1(1)\tau(1)\kappa\rho_1 \leq \rho/2. \tag{68}$$

Conditions (67) and (68) imply that there exists a unique fixed point $(v(m),\lambda(m))$ of $G(\cdot,\cdot,;g,m)$ in the ball of radius ρ about 0. This fixed point satisfies the estimate (65), according to a classical argument. Consider now the system in $(\hat{v},\hat{\lambda})$:

$$\begin{aligned}
&\hat{v} = B_1\big(\mathbf{1}-P(0)\big)g'(\cdot+m) - B_2(\mathbf{1}-P(0))\mathrm{D}Q(v+\lambda\zeta)(\hat{v}+\hat{\lambda}\zeta)\\
&\hat{\lambda} = B_3\big(\mathrm{D}Q(v+\lambda\zeta)(\hat{v}+\hat{\lambda}\zeta),\zeta\big)/\|\zeta\|_0^2.
\end{aligned} \tag{69}$$

If (v,λ) belongs to the ball of radius ρ about 0 in $\mathcal{V}_1 \times \mathcal{W}$, the following estimate holds:

$$\|\hat{v}\|_{\mathcal{V}_0} + \|\zeta\|_0\|\hat{\lambda}\|_{\mathcal{W}} \leq K_1(0)\|g'(\cdot+m)-\zeta\|_0 + \varpi(\kappa\rho)(K_2(0)+K_3),$$

where we have used the identity $(\mathbf{1}-P(0))\zeta = 0$. We infer from condition (67) that

$$\varpi(\kappa\rho)(K_2(0)+K_3) \leq \frac{K_2(0)}{2K_1(0)} + \frac{\|\zeta\|_0}{2\|\zeta\|_1}$$

which is strictly less than 1 thanks to assumption (57) and the obvious inequality $\|\zeta\|_0 < \|\zeta\|_1$. This shows that there exists a unique solution of system (69). Its solution can be classically identified to the partial derivatives of v and λ with respect to m. Estimates (66) are deduced by a straightforward calculation. $\square$

We are able now to solve $F(X) = 0$ when the data g is close enough to z.

Theorem 6.3. *There exist $\rho_1' \in (0,\rho_1]$, $\rho_2 \in (0,\rho_1']$ and K_4 such that if $\|g-z\|_1 \leq \rho_2$, there exists a unique solution (v,λ,m) of $F(v,\lambda,g,m) = 0$ such that $\|v\|_{\mathcal{V}_1} \leq \rho$, $\|\lambda\|_{\mathcal{W}} \leq \rho$ and $|m| \leq \rho_1'$. Moreover, this solution satisfies the estimate*

$$\max\big(\|v\|_{\mathcal{V}_1}, \|\lambda\|_{\mathcal{W}}, |m|\big) \leq K_4\|g-z\|_1. \tag{70}$$

Proof. Let us calculate the derivative of $F_3(v(m),\lambda(m),m,g)$ with respect to m:

$$\frac{\partial F_3}{\partial m}(v(m),\lambda(m),g,m)=\frac{\partial\lambda}{\partial m}\Big|_0-\frac{(g'(\cdot+m),\zeta)}{\|\zeta\|_0^2}\cdot$$

We have the estimate

$$\left|\frac{\partial F_3}{\partial m}(v(m),\lambda(m),g,m)+1\right|\le\frac{K_1(0)\big(\|g'-\zeta\|_0+|m|\|\zeta'\|_0\big)}{1-\varpi(\kappa\rho)(K_2(0)+K_3)}+\frac{\|g'-\zeta\|_0+|m|\|\zeta'\|_0}{\zeta_0}\cdot$$

We choose $\rho_1'\in(0,\rho_1]$ such that

$$\left(\frac{K_1(0)}{1-\varpi(\kappa\rho)(K_2(0)+K_3)}+\frac{1}{\|\zeta\|_0}\right)(1+\|\zeta'\|_0)\rho_1'\le\frac12\cdot$$

Then, $F_3(v(m),\lambda(m),g,m)$ satisfies the inequalities

$$\begin{aligned}0\le m\le\rho_1'&\Longrightarrow F_3(v(m),\lambda(m),g,m)\ge F_3(v(0),\lambda(0),g,0)+\frac m2,\\-\rho_1'\le m\le 0&\Longrightarrow F_3(v(m),\lambda(m),g,m)\le F_3(v(0),\lambda(0),g,0)+\frac m2\end{aligned}$$

which imply that if $|F_3(v(0),\lambda(0),g,0)|\le\rho_1'/2$, the equation $F_3(v(m),\lambda(m),g,m)=0$ possesses a unique solution in the interval $|m|\le\rho_1'$. We use now (65):

$$|F_3(v(0),\lambda(0),g,0)|\le 2\tau(1)K_1(1)\|g-z\|_1+\frac{\|g-z\|_0}{\|\zeta\|_0}\cdot$$

We choose $\rho_2\in(0,\rho_1']$ such that

$$2\tau(1)K_1(1)\rho_2+\frac{\rho_2}{\|\zeta\|_0}\le\frac{\rho_1'}{2}\cdot$$

Choose $\rho_2\in(0,\rho_1]$ so small that

$$\frac{\rho_2}{2K_1(1)\|\zeta\|_0}+\frac{\varpi(\kappa\rho_2)K_1(0)\|g\|_1}{\kappa\big(2\varpi(\kappa\rho)-\varpi(\kappa\rho_2)\big)}\le\frac12\cdot$$

If $\|g-z\|_1\le\rho_2$, it is clear now that the conclusion of the lemma is verified; estimate (70) is now a straightforward consequence of (65) and the obvious inequality $|m|\le 2|F_3(v(0),\lambda(0),g,0)|$. ☐

7. A minimization problem in a half-plane

We can also solve (58) by minimizing an appropriately renormalized energy on the half-plane $x_2\ge 0$, with boundary data g. The existence of a minimizer is straightforward; the main result of this section is that this minimizer coincides with the solution we just found by a fixed point argument, provided that g is close enough to z.

Let I be an interval of $\mathbb{R}$; define the class $S_2(I)$ by

$$\begin{aligned}S_2(I)=\Big\{u\in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}\times I)^2:\ &\frac{\partial u}{\partial x_2}\in \mathrm{L}^2(\mathbb{R}\times I)^2;\\&\text{for almost every } x_2\in I,\ u(\cdot,x_2)\in S(\mathbf{a},\mathbf{b});\\&E_1(u(\cdot,x_2))-e_1 \text{ belongs to }\ \mathrm{L}^1(I)\Big\}\cdot\end{aligned}\tag{71}$$

For $u \in S_2(I)$ we define a renormalized energy

$$E_2(u,I) = \int_I \left[E_1(u(\cdot,x_2)) - e_1 + \int_{\mathbb{R}} \frac{1}{2} \left| \frac{\partial u}{\partial x_2} \right|^2 \, \mathrm{d}x_1 \right] \mathrm{d}x_2.$$

Given $z \in \mathcal{Z}$ and $g \in \mathrm{H}^1(\mathbb{R})^2$ we consider the minimization problem

$$\begin{aligned}&\text{Minimize } E_2(u,\mathbb{R}^+) \text{ over } S_2(\mathbb{R}^+) \text{ under the boundary condition}\\ &u(\cdot,0) = g.\end{aligned} \tag{72}$$

Lemma 7.1. *For all $g \in \mathrm{H}^1(\mathbb{R})^2$, problem (72) possesses a solution.*

Proof. Observe first that for all $u \in S_2(\mathbb{R}+)$, $E_1(u(\cdot,x_2)) - e_1$ is nonnegative; therefore, for all $u \in S_2(\mathbb{R}^+)$, $E_2(u,\mathbb{R}^+)$ is nonnegative. We exhibit an element of $S_2(\mathbb{R}^+)$ satisfying the boundary condition for which $E_2(u,\mathbb{R}^+)$ is bounded: let z be an element of $\mathcal{Z}$ such that

$$d_1(g,\mathcal{Z}) = \|g - z\|_1.$$

We define the test function

$$u(x_1,x_2) = g(x_1) + (1-x_2)^+ \big(z(x_1) - g(x_1)\big).$$

With the help of inequality (40), we see immediately that

$$E_2(u,\mathbb{R}^+) \le \chi(d_1(g,\mathcal{Z}))^2 d_1(g,\mathcal{Z}),$$

where the function χ is defined as

$$\chi(r) = \left(\frac{1}{2} + \frac{\max(1, \gamma_3 + \varpi_2(r+R)}{2} \right)^{1/2}. \tag{73}$$

Define

$$e_2(g,\mathbb{R}^+) = \inf\{ E_2(u,\mathbb{R}^+) : u \in S_2(\mathbb{R}^+), u(\cdot,0) = g \} \cdot \tag{74}$$

We have just proved that

$$e_2(g,\mathbb{R}^+) \le \chi(d_1(g,z))^2 d_1(g,\mathcal{Z})^2. \tag{75}$$

Let $(u_n)_n$ be a minimizing sequence; without loss of generality, we may assume that for all $n \ge 0$

$$E_2(u_n,\mathbb{R}^+) \le e_2(g,\mathbb{R}^+) + 1 = \eta.$$

The following estimates hold:

$$\int_{\mathbb{R}\times\mathbb{R}^+} \left| \frac{\partial u_n}{\partial x_2} \right| \mathrm{d}x_1 \, \mathrm{d}x_2 \le 2\eta; \quad \forall L > 0, \quad \int_{\mathbb{R}\times(0,L)} \left(\left| \frac{\partial u_n}{\partial x_2} \right|^2 + 2W(u) \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \le 2(\eta + Le_1),$$

$$\forall L > 0, \quad \int_{\mathbb{R}\times(0,L)} |u_n(x_1,x_2) - g(x_1)|^2 \, \mathrm{d}x_1 \le L^2 \eta.$$

Thus, we may extract a subsequence, still denoted by u_n which converges to a certain function u, in the following sense:

$$\forall L > 0, \quad (u_n - z)\big|_{\mathbb{R}\times(0,L)} \text{ converges weakly in } \mathrm{H}^1(\mathbb{R}\times(0,L)) \text{ to } (u_n - z)\big|_{\mathbb{R}\times(0,L)},$$
$$\partial u_n/\partial x_2 \text{ converges weakly in } \mathrm{L}^2(\mathbb{R}\times\mathbb{R}^+)^2 \text{ to } \partial u/\partial x_2,$$
$$u_n \text{ converges to } u \text{ almost everywhere in } \mathbb{R}\times\mathbb{R}^+.$$

In particular, for almost every x_2, $u(\cdot, x_2) - z$ belongs to $\mathrm{H}^1(\mathbb{R})^2$, and therefore $u(\cdot, x_2)$ belongs to $S(\mathbf{a}, \mathbf{b})$. Therefore, $E_1(u(\cdot, x_2)) - e_1$ is non-negative for almost every x_2. A classical passage to the limit gives

$$e_2(g, \mathbb{R}^+) \geq \liminf_{n\to\infty} E_2(u_n, [0, L]) \geq \int_{\mathbb{R}\times(0,L)} \left|\frac{\partial u}{\partial x_2}\right|^2 \mathrm{d}x_1\ \mathrm{d}x_2 + \int_0^L \big(E_1(u(\cdot, x_2)) - e_1\big)\ \mathrm{d}x_2.$$

As L is arbitrary, u belongs to $S_2(\mathbb{R}^+)$ and $E_2(u, I) \leq e_2(g, \mathbb{R}^+)$; moreover, it is clear that the boundary condition is satisfied, and the lemma is proved. □

In order to show that this u coincides with the solution found in Section 6, we must obtain some regularity results on u. We start by proving that the minimizers are bounded; it could be thought that the maximum principle applied to $|u|^2$ gives the answer; however, this argument is valid only if we have obtained the Euler–Lagrange equation; but this is a very delicate step if we know only that u is locally in H^1 and $W(u)$ is locally in L^1. This explains the strategy used here.

Lemma 7.2. *Let R be as in* (3)*; then if u solves* (72) *it satisfies*

$$\|u\|_{\mathrm{L}^\infty} \leq \max(R, \|g\|_{\mathrm{L}^\infty}). \tag{76}$$

Proof. Let

$$R' = \max(R, \|g\|_{\mathrm{L}^\infty})$$

and define a new function $\tilde{u}$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq R', \\ R'u(x)/|u(x)| & \text{otherwise.} \end{cases}$$

It is immediate that $\tilde{u}$ belongs to $S_2(\mathbb{R}^+)$ and that almost everywhere on $\mathbb{R}\times\mathbb{R}^+$

$$|\mathrm{D}\tilde{u}(x)| \leq |\mathrm{D}u(x)|, \quad W(\tilde{u}(x)) \leq W(u(x)).$$

Therefore, $E_2(\tilde{u}, \mathbb{R}^+) \leq E_2(u, \mathbb{R}^+)$. It is also clear that $\tilde{u}$ satisfies the boundary condition. Therefore $\tilde{u}$ is also a minimizer and we must have

$$\int_{|u|>R} \big[W(u) - W(\tilde{u})\big]\ \mathrm{d}x = 0.$$

But assumption (3) implies that the integrand in the above expression is strictly positive; therefore, the set $\{|u| > R'\}$ is negligible and (76) is proved. □

The Euler–Lagrange equation is obtained now in a straightforward fashion: let ϕ belong to $\mathrm{C}_0^\infty(\mathbb{R}\times(0,\infty))$ and assume that its support is included in $[-L, L) \times [0, L]$. We have for all $t > 0$

$$\int_0^L \int_{-L}^L \left[\mathrm{D}u\cdot\mathrm{D}\phi + \frac{t}{2}|\mathrm{D}\phi|^2 + \frac{W(u + t\phi) - W(u)}{t}\right]\ \mathrm{d}x_1\ \mathrm{d}x_2 \geq 0.$$

As u is bounded we have uniformly on $[-L, L] \times [0, L]$

$$\lim_{t\to 0} \frac{W(u+t\phi) - W(u)}{t} = \mathrm{D}W(u)\phi.$$

Now it is clear that by classical arguments, u satisfies the Euler–Lagrange equation (58). It is also clear that it satisfies the boundary condition (59).

The local regularity is easy to obtain: as u is bounded, $\mathrm{D}W(u)$ is also bounded, and by classical interior estimates, for all $q \in (1, \infty)$, u is locally in the Sobolev space $W^{2,q}$. As W is at least of class C^3, this means that locally $\mathrm{D}W(u)$ is in $W^{2,q}$ and therefore, u is locally in $W^{4,q}$. By interior Schauder estimates, u is locally in $\mathrm{C}^{3,\alpha}$ for all $\alpha \in (0, 1)$.

Let us obtain now some uniform estimates on strips of fixed height; for this purpose, we first prove an auxiliary estimate:

Lemma 7.3. *For all $a > 0$ and for all $\varepsilon \in (0, a/2)$ there exists $K > 0$ such that for all y satisfying*

$$y \in \mathrm{L}^2(\mathbb{R} \times (0, a)), \quad \Delta y \in \mathrm{L}^2(\mathbb{R} \times (0, a)) \tag{77}$$

the following estimate holds:

$$\max\left(\|y\|_{\mathrm{H}^2(\mathbb{R}\times(\varepsilon,a-\varepsilon))}, \max_{x_2\in[\varepsilon,a-\varepsilon]} \|y(\cdot, x_2)\|_{\mathrm{H}^{3/2}(\mathbb{R})} \right) \leq K\big(\|y\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))} + \|\Delta y\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))}\big). \tag{78}$$

Under assumption (77), $x_2 \mapsto y(\cdot, x_2)$ is continuous from $(0, a)$ to $\mathrm{H}^1(\mathbb{R})$.

For all $a > 0$ and all $\varepsilon \in (0, a)$, there exists $K > 0$ such that for all y satisfying (77) and

$$y(\cdot, 0) \in \mathrm{H}^1(\mathbb{R}) \tag{79}$$

the following estimate holds:

$$\begin{aligned} &\max\left(\|y\|_{\mathrm{H}^{3/2}(\mathbb{R}\times(0,a-\varepsilon))}, \max_{x_2\in[0,a-\varepsilon]} \|y(\cdot, x_2)\|_1 \right) \\ &\qquad \leq K\big(\|y\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))} + \|\Delta y\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))} + \|y(\cdot, 0)\|_{\mathrm{H}^1(\mathbb{R})}\big). \end{aligned} \tag{80}$$

Under assumptions (77) and (79), $x_2 \mapsto y(\cdot, x_2)$ is continuous from $[0, a)$ to $\mathrm{H}^1(\mathbb{R})$.

Proof. This result is almost in [6], but I did not find it; so I sketch the proof, which is obtained by performing a partial Fourier transform in x_1:

$$\hat{y}(\xi_1, x_2) = \int \mathrm{e}^{-ix_1\xi_1} u(x_1, x_2)\, \mathrm{d}x_1.$$

Our assumptions imply that $\hat{y}$ satisfies the ordinary differential equation

$$\frac{\partial^2 \hat{y}}{\partial x_2^2} = \big(1 + |\xi_1|^2\big)\hat{y} + \hat{h}(\xi_1, x_2). \tag{81}$$

The function $h = y - \Delta y$ belongs to $L^2(\mathbb{R} \times (0, a))$; define $\sigma(\xi_1) = \sqrt{1 + \xi_1^2}$; the general solution of (81) is

$$\hat{y}(\xi_1, x_2) = A(\xi_1)\mathrm{e}^{-\sigma(\xi_1)x_2} + \hat{B}(\xi_1)\mathrm{e}^{\sigma(\xi_1)(x_2-a)} - \hat{y}_1(\xi_1, x_2) - \hat{y}_2(\xi_1, x_2),$$

with $\hat{y}_1$ and $\hat{y}_2$ given by

$$\hat{y}_1(\xi_1, x_2) = \int_{x_2}^{a} \frac{\mathrm{e}^{-(t-x_2)\sigma(\xi_1)\hat{h}(\xi_1,t)}}{2\sigma(\xi_1)}\ \mathrm{d}t, \quad \hat{y}_2(\xi_1, x_2) = \int_{0}^{x_2} \frac{\mathrm{e}^{(t-x_2)\sigma(\xi_1)\hat{h}(\xi_1,t)}}{2\sigma(\xi_1)}\ \mathrm{d}t.$$

We use a Cauchy–Schwartz inequality with the weight $\mathrm{e}^{-t\sigma}/\sigma$ to estimate $\hat{y}_1$, assuming that h is extended by 0 for negative values of its argument:

$$\int_0^a |\hat{y}_1(\xi_1, x_2)|^2\ \mathrm{d}x_2 = \int_0^a \left| \int_0^a \frac{\mathrm{e}^{-t\sigma} 1_{\mathbb{R}^+}(t) \hat{h}(\xi_1, t - x_2)}{2\sigma}\ \mathrm{d}t \right|^2\ \mathrm{d}x_2 \le \left(\int_0^a \frac{\mathrm{e}^{-t\sigma}}{2\sigma}\ \mathrm{d}t \right)^2 \int_0^a |\hat{h}(\xi_1, s)|^2\ \mathrm{d}s.$$

This inequality enables us to see that

$$\int_0^a \int_{\mathbb{R}} |\hat{y}_1(\xi_1, x_2)|^2 \xi_1^4\ \mathrm{d}\xi_1 \le \int_0^a \int_{\mathbb{R}} \frac{\xi_1^4}{8\sigma^4} |h(\xi_1, x_2)|^2\ \mathrm{d}\xi_1\ \mathrm{d}x_2 \le \frac{\|\hat{h}\|^2_{\mathrm{L}^2(\mathbb{R}\times(0,a))}}{8}\cdot$$

The derivative $\partial\hat{y}_1/\partial x_2$ is treated in an analogous fashion, and we use equation (81) to estimate $\partial^2\hat{y}_1/\partial x_2^2$. Thus, we proved

$$\|y_1\|_{\mathrm{H}^2(\mathbb{R}\times(0,a))} \le C\|y - \Delta y\|_{\mathrm{L}^2} \text{ and analogously } \|y_2\|_{\mathrm{H}^2(\mathbb{R}\times(0,a))} \le C\|y - \Delta y\|_{\mathrm{L}^2}.$$

The uniform estimate on the norm of $y_1(\cdot, x_2)$ in $\mathrm{H}^{3/2}(\mathbb{R})$ is obtained as follows: by Cauchy–Schwarz inequality, we have

$$|\hat{y}(\xi_1, x_2)| \le \frac{1}{2\sigma} \left(\int_{x_2}^a \mathrm{e}^{-2(t-x_2)\sigma}\ \mathrm{d}t \right)^{1/2} \left(\int_{x_2}^a \left|\hat{h}(\xi_1, t)\right|^2\ \mathrm{d}t \right)^{1/2} \le \frac{1}{\sqrt{8\sigma^3}} \left(\int_{x_2}^a \left|\hat{h}(\xi_1, t)\right|^2\ \mathrm{d}t \right)^{1/2}.$$

Then, it is clear that y_1 belongs to $\mathrm{L}^\infty(0, a; \mathrm{H}^{3/2}(\mathbb{R}))$. The continuity is proved by a repeated application of Lebesgue's theorem. The same results also hold for y_2. If y belongs to $\mathrm{L}^2(\mathbb{R} \times (0, a))$, the function $\hat{p}(\xi_1, x_2) = \hat{A}(\xi_1)\mathrm{e}^{-x_2\sigma} + \hat{B}(\xi_1)\mathrm{e}^{(x_2-a)\sigma}$ belongs to $\mathrm{L}^2(\mathbb{R} \times (0, a))$. We differentiate $\hat{p}$ to x_2, we multiply it by σ and using linear combinations, we can check that $\sigma\hat{A}\mathrm{e}^{-x_2\sigma}$ and $\sigma\hat{B}\mathrm{e}^{(x_2-a)\sigma}$ belong to the Fourier transform of the space $\mathrm{H}^1(\mathbb{R} \times (0, a))$. Therefore, $\hat{A}\mathrm{e}^{-x_2\sigma}$ and $\hat{B}\mathrm{e}^{(x_2-a)\sigma}\hat{B}$ belong to $\mathrm{L}^2(\mathbb{R} \times (0, a))$, from which we deduce immediately that

$$\int \left(|\hat{A}(\xi_1)|^2 + |\hat{B}(\xi_1)^2|\right) \frac{1 - \mathrm{e}^{-2a\sigma}}{2\sigma}\ \mathrm{d}\xi_1 < \infty. \tag{82}$$

Relation (82) also implies

$$\int_\varepsilon^{a-\varepsilon} \int \left(|\hat{p}(\xi_1, x_2)|\sigma^4 + |\partial_2\hat{p}(\xi_1, x_2)|\sigma^2 + |\partial_2^2\hat{p}(\xi_1, x_2)|^2 \right)\ \mathrm{d}\xi_1\ \mathrm{d}x_2 < \infty.$$

It can be seen easily that u has a trace on $\{x_2 = 0\}$ which belongs to $\mathrm{H}^{-1/2}(\mathbb{R})$; its Fourier transform is given by

$$\hat{y}(\xi_1, 0) = \hat{A}(\xi_1) + \hat{B}(\xi_1)\mathrm{e}^{-a\sigma} - \hat{y}_1(\xi_1, 0). \tag{83}$$

Finally, if $y(\cdot, 0)$ belongs to $\mathrm{H}^1(\mathbb{R})$, relation (83) implies that A belongs to $\mathrm{H}^1(\mathbb{R})$ and it is then clear that y belongs to $C^0([0, a - \varepsilon); \mathrm{H}^1(\mathbb{R}))$. $\square$

We are able to give now some important regularity properties of a solution of (72).

Lemma 7.4. *Let $d_1(g, \mathcal{Z}) = \delta$. All solutions of* (72) *are continuous from $\mathbb{R}^+$ to $S(\mathbf{a}, \mathbf{b})$. Moreover, there exists a continuous function $M(\delta)$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that for all solution u of* (72)*, the following estimates hold:*

$$\forall x_2 \in [0,1], \quad d_1(u(\cdot, x_2), \mathcal{Z}) \le M(\delta)\delta, \tag{84}$$

$$\forall x_2, \bar{x}_2 \in [1, +\infty), \quad \|\partial_1 u(\cdot, \bar{x}_2) - \partial_1 u(\cdot, x_2)\|_1 \le M(\delta)\,|\bar{x}_2 - x_2|\,, \tag{85}$$

$$\forall x_2, \bar{x}_2 \in [1, +\infty), \quad \|\partial_2 u(\cdot, \bar{x}_2) - \partial_2 u(\cdot, x_2)\|_1 \le \delta M(\delta)\,|\bar{x}_2 - x_2|\,. \tag{86}$$

Proof. Let $z \in \mathcal{Z}$ be such that $d_1(g, \mathcal{Z}) = \|g - z\|_1$. We estimate the L^2 norm of $u - z$ over the strip $[0,2] \times \mathbb{R}$: from the inequality

$$|u(x_1, t) - u(0, t)|^2 \le t \int_0^t |\partial_2 u(x_1, s)|^2 \, \mathrm{d}s \tag{87}$$

we infer

$$\int_0^2 \int |u(x_1, x_2) - g(x_2)|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \le 2 \int_0^2 \int |\partial_2 u|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \le 4\delta^2 \chi(\delta)^2.$$

Thanks to the triangle inequality,

$$\|u - z\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))} \le \big(\sqrt{2} + 2\chi(\delta)\big)\delta.$$

We subtract from (58) the identity

$$-\Delta z + \mathrm{D}W(z)^{\mathsf{T}} = 0,$$

and we obtain the equation

$$-\Delta(u - z) + \mathrm{D}W(u)^{\mathsf{T}} - \mathrm{D}W(z)^{\mathsf{T}} = 0.$$

We use the modulus of continuity ϖ_1 defined by (32):

$$\|DW(u) - DW(z)\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))} \le \varpi(R + \delta)\|u - z\|_{\mathrm{L}^2(\mathbb{R}\times(0,a))};$$

as $g - z$ belongs to $\mathrm{H}^1(\mathbb{R})^2$, we may apply estimate (80), *i.e.*

$$\max_{0 \le x_2 \le 1} \|u(\cdot, x_2) - z\|_1 \le K\delta\big(1 + \varpi(R + \delta)\big)\big(\sqrt{2} + 2\chi(\delta)\big). \tag{88}$$

We differentiate now the Euler–Lagrange equation (58) with respect to x_1 and x_2:

$$-\Delta \partial_j u + \big(D^2 W(u)\partial_j u\big)^{\mathsf{T}} = 0.$$

We have already the following estimates

$$\int_{\mathbb{R}\times\mathbb{R}^+} |\partial_2 u|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \le 2\chi(\delta)^2\delta^2 \text{ and } \int_{\mathbb{R}\times(L-1/2, L+3/2)} |\partial_1 u|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \le 2\chi_1(\delta)^2\delta^2 + 4e_1.$$

This is a situation in which we may apply (78), and we obtain for all $L \geq 1/2$

$$\|\partial_1 u\|_{H^2(\mathbb{R}\times(L,L+1))} \leq K\big(1+\gamma_3+\varpi_2(R+\delta)\big)\sqrt{2\chi(\delta)^2\delta^2+1},$$
$$\|\partial_2 u\|_{H^2(\mathbb{R}\times(L,L+1))} \leq K\big(1+\gamma_3+\varpi_2(R+\delta)\big)\sqrt{2}\,\chi(\delta)\delta.$$

The conclusion of the lemma is now clear. □

An important corollary is the following:

Corollary 7.5. *Let $\delta = d_1(g,\mathcal{Z})$. For all solution u of* (72) *the function $x_2 \mapsto d_1(u(\cdot,x_2),\mathcal{Z})$ is Lipschitz continuous over* $[1,\infty)$*, with Lipschitz constant $\delta M(\delta)$.*

Proof. Let x_2 and $\bar{x}_2$ be given in $[1,\infty)$; there exist z and $\bar{z}$ in $\mathcal{Z}$ such that

$$d_1(u(\cdot,x_2),\mathcal{Z}) = \|u(\cdot,x_2)-z\|_1, \quad d_1(u(\cdot,\bar{x}_2),\mathcal{Z}) = \|u(\cdot,\bar{x}_2)-\bar{z}\|_1.$$

In the right hand side of the identity $d_1(u(\cdot,\bar{x}_2),\mathcal{Z}) - d_1(u(\cdot,x_2),\mathcal{Z}) = \|u(\cdot,\bar{x}_2)-\bar{z}\|_1 - \|u(\cdot,x_2)-z\|_1$, we substitute $\|u(\cdot,\bar{x}_2)-\bar{z}\|_1$ by $\|u(\cdot,\bar{x}_2)-z\|_1$ which is larger, we apply the triangle inequality, obtaining thus $d_1(u(\cdot,\bar{x}_2),\mathcal{Z}) - d_1(u(\cdot,x_2),\mathcal{Z}) \leq \|u(\cdot,\bar{x}_2)-(\cdot,x_2)\|_1$; we obtain the conclusion of the lemma by exchanging the rôles of x_2 and $\bar{x}_2$. □

We can prove now that if z is an isolated minimizer of E_1 and if g is small enough, then $u(\cdot,x_2)$ remains bounded away from $\mathcal{Z}\setminus\mathcal{C}(z)$ for all $x_2 > 0$:

Theorem 7.6. *Assume that z is an isolated minimizer of E_1 over $S(\mathbf{a},\mathbf{b})$. There exists $\delta_1 > 0$ such that for all g satisfying $d_1(g,\mathcal{C}(z)) \leq \delta_1$, any minimizer of* (72) *is in fact the unique solution of* (58) *and* (59) *obtained at Theorem 6.3.*

Proof. Take $\beta = 3\min(1, d_1(\mathcal{C}(z),\mathcal{Z}\setminus\mathcal{C}(z)))/4$; we know from Corollary 4.6 that there exists $\alpha > 0$ such that if $d_1(v,\mathcal{Z}\setminus\mathcal{C}(z)) \geq d_1(\mathcal{C}(z),\mathcal{Z}\setminus\mathcal{C}(z))/4$, then

$$E_1(v) - e_1 \geq \alpha\min\big(1, d_1(v,\mathcal{C}(z))^2\big).$$

Define

$$h(x_2) = d_1(u(\cdot,x_2),\mathcal{C}(z)).$$

The first assumption we make on δ_1 is

$$M(\delta_1)\delta_1 < \beta, \tag{89}$$

and therefore, relation (84) implies $h(x_2) \leq \beta$ for $0 \leq x_2 \leq 1$.

Let us prove that for δ_1 small enough, $h(x_2)$ is at most equal to β for all x_2. Define indeed

$$\bar{x}_2 = \inf\{x_2 \geq 1 : h(x_2) = \beta\}.$$

By continuity of h, $h(\bar{x}_2) = \beta$. We infer from the inequality $E_2(u,\mathbb{R}^+) \leq \delta^2\chi(\delta)^2$ and from the definition of α that

$$\alpha\int_0^{\bar{x}_2} h(x_2)^2\,\mathrm{d}x_2 \leq \delta^2\chi(\delta)^2.$$

Let $\tilde{h}$ be the maximum of $h(x_2)$ over $[0, \bar{x}_2]$; as h is Lipschitz continuous with Lipschitz constant $\delta M(\delta)$, we have the estimate

$$\int_0^{\bar{x}_2} h(x_2)^2 \, \mathrm{d}x_2 \geq \tilde{h}^3/(3\delta M(\delta)),$$

and therefore

$$\tilde{h} \leq \delta \sqrt[3]{3M(\delta)\chi(\delta)^2}.$$

If we choose δ_1 so small that

$$\delta_1 \sqrt[3]{3M(\delta_1)\chi(\delta_1)^2} < \beta, \tag{90}$$

we obtain a contradiction, and thus $h(x_2) < \beta$ for all $x_2 \geq 0$.

We minimize now $E_2(v, [x_2, \infty))$ over $S_2([x_2, \infty))$ with boundary data $u(\cdot, x_2)$. It is clear that

$$e_2(u(\cdot, x_2), [x_2, \infty)) \leq \chi(h(x_2))^2 h(x_2)^2$$

and that the restriction of u to $\mathbb{R} \times [x_2, \infty)$ provides a solution of this minimization problem. Therefore, we must have

$$\alpha \int_{x_2}^{\infty} h(t)^2 \, \mathrm{d}t \leq \chi(\beta)^2 h(x_2)^2. \tag{91}$$

We choose

$$3\gamma \in (0, \min(\nu, \alpha/\chi(\beta)^2)). \tag{92}$$

If we let

$$H(x_2) = \int_{x_2}^{\infty} h(t)^2 \, \mathrm{d}t,$$

we observe that (91) implies $H'(x_2) + 3\gamma H(x_2) \leq 0$, a differential inequality which is immediately integrated:

$$H(x_2) \leq H(0) \exp(-3\gamma x_2).$$

But

$$H(0) = \int_0^{\infty} d_1(u(\cdot, x_2), \mathcal{C}(z))^2 \, \mathrm{d}x_2 \leq e_2(g, \mathbb{R}^+)/\alpha \leq \delta^2/3\gamma.$$

Therefore, we have found the estimate for all $t \geq 0$:

$$H(t) \leq \delta^2 \exp(-3\gamma t)/\gamma. \tag{93}$$

We use again the fact that h is Lipschitz continuous to infer from (93) that h decreases exponentially fast to 0. Indeed, if $\tilde{h}(x_2)$ is the maximum of h over the interval $[x_2, \infty)$, we must have

$$\tilde{h}(x_2)^3 \leq 3\delta M(\delta) H(x_2). \tag{94}$$

Define

$$C_0 = \sqrt[3]{3M(\beta)/\gamma}.$$

We infer from (94):

$$\forall x_2 \geq 0, \quad h(x_2) \leq C_0 \delta \mathrm{e}^{-\gamma x_2}. \tag{95}$$

We will obtain now an exponential bound for $\|\partial_2 u(\cdot, x_2)\|_1$: for $x_2 \geq 1$, we have the estimates

$$\|\partial_2 u\|_{\mathrm{L}^2(\mathbb{R}\times(x_2-1,x_2+1))} \leq \delta \mathrm{e}^{-\gamma x_2}\sqrt{2}\chi(\beta)e^{\gamma/2}C_0,$$

and

$$\|\partial_2 \Delta u\|_{\mathrm{L}^2(\mathbb{R}\times(x_2-1,x_2+1))} \leq (\gamma_3 + \varpi_2(R+\beta))\|\partial_2 u\|_{\mathrm{L}^2(\mathbb{R}\times(x_2-1,x_2+1))}.$$

According to (78), there exists a constant K such that for all $x_2 \geq 1$:

$$\|\partial_2 u(\cdot, x_2)\|_1 \leq K(1+\gamma_3+\varpi_2(R+\beta))\|\partial_2 u\|_{\mathrm{L}^2(\mathbb{R}\times(x_2-1,x_2+1))},$$

and therefore, if

$$C_1 = K(1+\gamma_3+\varpi_2(R+\beta))\chi(\beta)\mathrm{e}^{\gamma}\sqrt{2}C_0,$$

we have

$$\forall x_2 \geq 1, \quad \|\partial_2 u(\cdot, x_2)\|_1 \leq C_1 \delta \mathrm{e}^{-\gamma x_2}. \tag{96}$$

Assume that

$$\delta_1 \max(1, C_1) \leq \beta(0)/2, \tag{97}$$

then we may define for all $x_2 \geq 1$ the function

$$\mu(x_2) = m_0(u(\cdot, x_2));$$

this function is continuously differentiable and we infer from (11) the inequality

$$\forall x_2 \geq 1, |\mu'(x_2)| \leq C_2' \delta \mathrm{e}^{-\gamma x_2}$$

with

$$C_2' = 2C_1\|\zeta\|_0/(\|\zeta'\|_0\beta(0)).$$

This implies in particular that $\mu(x_2)$ tends to a limit $\mu(\infty)$ as x_2 tends to infinity.

We have to estimate $\mu(x_2) - \mu(0)$ for $0 \leq x_2 \leq 1$: if we assume

$$2M(\delta_1)\delta_1 \leq \beta(0), \tag{98}$$

$\mu(x_2)$ is well defined on $[0,1]$, and thanks to (87),

$$\|u(\cdot, x_2) - z(\cdot - \mu(0))\|_0 \leq \|g - z(\cdot - \mu(0))\|_0 + \delta\chi(\delta)\sqrt{2} \leq \delta(1+\chi(\delta)\sqrt{2}).$$

Therefore, if we assume

$$\delta_1(1+\chi(\delta_1)\sqrt{2}) \le \beta_0(0). \tag{99}$$

Lemma 2.2 implies $|\mu(x_2)-\mu(0)| \le \delta\theta(0)(1+\chi(\delta)\sqrt{2})$, and therefore $|\mu(x_2)-\mu(1)| \le 2\delta\theta(0)(1+\chi(\delta)\sqrt{2})$. If we define

$$C_2 = 2\theta(0)(1+\chi(\beta)\sqrt{2}) + C_2'/\gamma$$

we obtain the inequality

$$\forall x_2 \ge 0, \quad |\mu(x_2)-\mu(\infty)| \le C_2\delta \mathrm{e}^{-\gamma x_2}.$$

We define now

$$\lambda(x_2) = \big(u(\cdot+\mu(\infty)-z,\zeta)/\|\zeta\|_0^2, \quad v(\cdot,x_2) = (\mathbf{1}-P(0))(u(\cdot+\mu(\infty),x_2)-z),$$

which we will estimate respectively in $\mathcal{W}$ and $\mathcal{V}_1$, and this depends on the following estimates on $\|u(\cdot+\mu(\infty),x_2)-z\|_1$. Indeed, $\|u(\cdot+\mu(\infty),x_2)-z\|_1 = \|u(\cdot,x_2)-z(\cdot-\mu(\infty)\|_1$ and thanks to the triangle inequality,

$$\begin{aligned}\|u(\cdot,x_2)-z(\cdot-\mu(\infty)\|_1 \le \|u(\cdot,x_2)-z(\cdot-m_1(u(\cdot,x_2)))\|_1 + \|z(\cdot-m_1(u(\cdot,x_2)))-z(\cdot-\mu(x_2)\|_1 \\ + \|z(\cdot-\mu(x_2))-z(\cdot-\mu(\infty))\|_1.\end{aligned}$$

If we assume that

$$\delta_1 C_0 \le \beta_0(0), \tag{100}$$

we may apply Corollary 2.3, obtaining thus the estimate

$$\|u(\cdot+\mu(\infty))-z\|_1 \le C_3\delta\mathrm{e}^{-\gamma x_2}$$

with

$$C_3 = C_0 + \|\zeta\|_1(\theta(0)C_0 + C_2).$$

If the following conditions are satisfied:

$$\delta_1\|\mathbf{1}-P(0)\|_{\mathcal{L}(\mathrm{H}^1,\mathrm{H}^1)}C_3 \le \rho, \quad \delta_1 C_3/\|\zeta\|_0 \le \rho, \quad C_2\delta_1 \le \rho_1', \quad \delta_1 \le \rho_2, \tag{101}$$

we are in the conditions of application of Theorem 6.3 and Theorem 7.6 is proved. Observe that δ_1 must satisfy only a finite number of inequalities, namely (89, 90, 97–100) and (101). □

We will need an corollary on the continuity with respect to boundary data:

Corollary 7.7. *Given a sequence of elements g_n of $S^1(\mathbf{a},\mathbf{b})$ satisfying $\|g_n-z\|_1 \le \delta_1$ and such that g_n-g tends to 0 in the weak topology of H^1, the corresponding sequence of solutions u_n of the minimization problem in $\mathbb{R}\times\mathbb{R}^+$ defined at Theorem 7.6 converges to the solution u of the same problem with data g, in the following sense: u_n converges to u almost everywhere and weakly in $\mathrm{H}^1(\mathbb{R}\times(0,L))$ for all $L<\infty$.*

Proof. It is clear that we can extract a subsequence which converges almost everywhere and weakly in $\mathrm{H}^1(\mathbb{R}\times(0,L))^2$ for all $L<\infty$; we still denote this sequence by (u_n); it is quite clear that all the inequalities satisfied by u_n pass to the limit, namely (84–86, 95), and the estimates on v_n, λ_n and m_n:

$$\|v_n\|_{\mathcal{V}_1} \le \delta_1 C_3\|\mathbf{1}-P(0)\|_{\mathcal{L}(\mathrm{H}^1,\mathrm{H}^1)}, \quad \|\lambda_n\|_{\mathcal{W}} \le \delta_1 C_3/\|\zeta\|_0 \text{ and } |m_n| \le C_2\delta_1.$$

It is quite easy to see that $\liminf E_2(u_n, \mathbb{R}^+) \geq \liminf E_2(u, \mathbb{R}^+)$. As u satisfies the conditions of Theorem 6.3, it solves the elliptic problem in the half-plane, with boundary data g and by uniqueness, all the sequence converges to u. □

8. Construction of a heteroclinic connection

We assume here that there are exactly two minimizers of E_1 over $S(\mathbf{a}, \mathbf{b})$, up to translation; these minimizers are denoted by z_+ and z_-.

We denote by the common energy of z_+ and z_-

$$e_1 = E_1(z_+) = E_1(z_-). \tag{102}$$

We denote by $\zeta_\pm$ the derivative of $z_\pm$. We also assume that z_+ and z_- satisfy the spectral assumption. Finally, δ_1 is chosen so as to satisfy the conditions of Theorem 7.6 relatively to z_+ and z_-. Without loss of generality, we may assume that $\delta_1 < d_1(\mathcal{C}(z_+), \mathcal{C}(z_-))$.

Theorem 8.1. *There exists a solution of*

$$-\Delta u + \mathrm{D}W(u)^\mathsf{T} = 0$$

in the plane $\mathbb{R}^2$ with the following behavior at infinity:

$$\lim_{x_1 \to -\infty} u(x_1, x_2) = \mathbf{a}, \quad \lim_{x_1 \to \infty} u(x_1, x_2) = \mathbf{b},$$

uniformly in x_2 and there exists m_+ and m_- such that

$$\lim_{x_2 \to -\infty} \|u(\cdot, x_2) - z_-(\cdot - m_-)\|_1 = 0, \quad \lim_{x_2 \to \infty} \|u(\cdot, x_2) - z_+(\cdot - m_+)\|_1 = 0,$$

the convergence being exponentially fast.

Proof. It is clear that the renormalized energy of any element of $S_2(\mathbb{R})$ is non-negative; moreover, the element $u(x_1, x_2) = z_-(x_1) + (x_2^+ - (x_2 - 1)^+)(z_+(x_1) - z_-(x_1))$ has finite renormalized energy. Define

$$\bar{S}_2 = \left\{ u \in S_2 : \lim_{x \to \pm\infty} d_0(u(\cdot, x_2), \mathcal{C}(z_\pm)) = 0 \right\},$$

and let

$$e_2 = \inf\{E_2(u, \mathbb{R}) : u \in \bar{S}_2\} \cdot$$

Observe that the definition of $\bar{S}_2$ makes sense, since $u(x_1, x_2) - z(x_1)$ has $\mathrm{H}^{1/2}$ traces on every line $x_2 = \text{constant}$.

Let $(u_n)_n$ be a minimizing sequence; we shall construct a smoother minimizing sequence. By definition of $S_2(\mathbb{R})$, we know that for almost every $x_2 \in \mathbb{R}$, $u_n(\cdot, x_2) - z$ belongs to $\mathrm{H}^1(\mathbb{R})^2$; we know also that there exist sequences $x_{2,k}$ and $x'_{2,k}$ tending to infinity such that $d_1(u(\cdot, x_{2,k})\mathcal{C}(z_+))$ and $d_1(u(\cdot, -y_{2,k}), \mathcal{C}(z_-)$ tend to 0 as k tends to infinity.

Therefore, we may choose $X_{n,+}$ and $X_{n,-}$ such that

$$d_1(u_n(\cdot, X_{n,\pm})), \mathcal{C}(z_\pm)) \leq \delta_1/2,$$

and we replace u_n by a smoother function defined as follows: for $\pm x_2 \geq \pm X_{n,\pm}$, v_n is the solution of the half-plane minimization problem with boundary data $u_n(\cdot, x_{n,\pm})$; in the strip $\mathbb{R} \times (X_{n,-}, X_{n,+})$, v_n is a minimizer of $E_2(u,(X_{n,-},X_{n,+}))$ with boundary data $u_n(\cdot,x_{n,\pm})$. We have not treated this problem, but it is almost classical, and certainly easier to solve than the problem in the half-plane. Details are left to the reader. In particular, the renormalized energy of v_n is at most equal to the renormalized energy of u_n and v_n is also a minimizing sequence. The function $v_n(\cdot,x_2)$ is continuous from $\mathbb{R}$ to $H^1(\mathbb{R})^2$ thanks to Lemma 7.4, which applies also to the problem in the strip, after appropriate modifications.

Without loss of generality, we may always assume that

$$E_2(v_n,\mathbb{R}) \le e_2 + 1.$$

Define now

$$\begin{aligned} Y_{n,+} &= \inf\{x_2 : d_1(v_n(\cdot,x_2),\mathcal{C}(z_+)) \le \delta_1\},\\ Y_{n,-} &= \sup\{x_2 \le Y_{n,+} : d_1(v_n(\cdot,x_2),\mathcal{C}(z_-)) \le \delta_1\}\cdot \end{aligned}$$

Thanks to our assumptions relative to δ_1, we know that $Y_{n,-} < Y_{n,+}$. Since, on the interval $[Y_{n,-},Y_{n,+}]$, $d_1(u_n(\cdot,x_2),\mathcal{Z}) \ge \delta_1$, Lemma 3.2 implies the existence of an $\eta > 0$ such that for all n:

$$\forall x_2 \in [Y_{n,-},Y_{n,+}], \quad E_1(u(\cdot,x_2)) \ge e_1 + \eta.$$

Therefore,

$$\eta(Y_{n,+} - Y_{n,-}) \le 2(e_2+1).$$

This inequality implies that $Y_{n,+} - Y_{n,-}$ is bounded independently of n; without loss of generality, we may translate in the x_2 variable and assume

$$0 \le Y_{n,+} = -Y_{n,-} = Y_n \le (e_2+1)/\eta.$$

On the other hand, we have the inequality

$$\int |u_n(x_1,Y_n) - u_n(x_1,-Y_n)|^2 \, \mathrm{d}x_1 \le 2Y_n \int_{-Y_n}^{Y_n} \int |\partial_2 u_n|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \le 4Y_n(e_2+1), \tag{103}$$

and we may also estimate from below the first expression in (103): define $d_1(u(\cdot,\pm Y_n),\mathcal{C}(z_{pm}) = \|u(\cdot,\pm Y_n) - z_\pm(\cdot - m_{n,\pm})\|_1$; then

$$\begin{aligned} \int |u_n(x_1,Y_n) - u_n(x_1,-Y_n)|^2 \, \mathrm{d}x_1 \ge \frac{1}{2} \int &|z_+(x_1 - m_{n,+}) - z_-(x_1 - m_{n,-})|^2 \, \mathrm{d}x_1 \\ &- 2\int \left(|u_n(\cdot,Y_n) - z_+(\cdot - m_{n,+})|^2 + |u_n(\cdot,-Y_n) - z_+(\cdot - m_{n,-})|^2\right) \mathrm{d}x_1, \end{aligned}$$

so that

$$\frac{1}{2}\int |z_+(x_1 - m_{n,+}) - z_-(x_1 - m_{n,-})|^2 \, \mathrm{d}x_1 - 4\delta_1^2 \le 4Y_n(e_2+1). \tag{104}$$

But we have the equivalent for $|m| \gg 1$:

$$\int |z_+(x_1) - z_-(x_1 - m)|^2 \, \mathrm{d}x_1 \sim |\mathbf{a} - \mathbf{b}|^2 |m|. \tag{105}$$

Relations (104) and (105) imply that $|m_{n,+} - m_{n,-}|$ is bounded independently of n. Without loss of generality, we may assume that $m_{n,+} = -m_{n,-} = m_n$ which is bounded independently of n.

We perform a last modification of the sequence of minimizers: we replace u_n by the half-space minimizers for $|x_2| \geq Y_n$, with boundary data $u_n(\cdot, \pm Y_n)$. Define $g_{n,+} = u_n(\cdot - m_n, Y_n)$ and $g_{n,-} = u_n(\cdot + m_n, -Y_n)$. We are in the conditions of application of Corollary 7.7, and thanks to Corollary 7.7, we can pass safely to the limit in both half planes after extraction as n tends to infinity. The passage to the limit in the strip is easy. It is then clear that the limiting u is a minimizer of $E_2(\cdot, \mathbb{R})$ over $\bar{S}_2$, and that $d_1(u(\cdot, x_2), \mathcal{C}(z_\pm))$ tends to 0 exponentially fast as $\pm x_2$ tends to infinity. □

We establish now some interesting identities:

Lemma 8.2. *Let u be a solution as constructed at Theorem 7.6. The following identities hold for all x_2:*

$$\int \frac{\partial u(x_1, x_2)}{\partial x_1} \cdot \frac{\partial u(x_1, x_2)}{\partial x_2} \, \mathrm{d}x_1 = 0, \tag{106}$$

$$E_1(u(\cdot, x_2)) = \frac{1}{2} \int \left| \frac{\partial u(x_1, x_2)}{\partial x_2} \right|^2 \, \mathrm{d}x_1. \tag{107}$$

Proof. For the first identity, we define

$$\tilde{u}(x_1, x_2) = u(x_1 - t\phi(x_2), x_2)$$

and we write that for all $t > 0$ and all ϕ with compact support in $(0, \infty)$ we have

$$E_2(\tilde{u}, \mathbb{R}^+) \geq E_2(u, \mathbb{R}^+).$$

By differentiating the inequality with respect to t at $t = 0$ we find that

$$\int \phi'(x_2) \int \frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2} \, \mathrm{d}x_1 \, \mathrm{d}x_2 = 0$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}x_2} \int \frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_2} \, \mathrm{d}x_1 = 0.$$

But at infinity, this quantity vanishes thanks to estimate (96); we have proved (107).

For any smooth ϕ with compact support in $(0, \infty)$ we define
the flow $X(x_2, t)$ of the differential equation

$$\frac{\partial X(x_2, t)}{\partial x_2} = \phi(X(x_2, t)), \quad X(x_2, 0) = x_2.$$

This time, we define $\tilde{u}$ by

$$\tilde{u}(x_1, x_2) = u(x_1, X(x_2, t)).$$

The same type of argument as above enables us to conclude (107). □

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References

[1] S. Alama, L. Bronsard and C. Gui, Stationary layered solutions in $\mathbb{R}^2$ for an Allen–Cahn system with multiple well potential. *Calc. Var. Partial Diff. Eqs.* **5** (1997) 359-390.

[2] I. Fonseca and L. Tartar, The gradient theory of phase transitions for systems with two potential wells. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989) 89-102.

[3] K. Ishige, The gradient theory of the phase transitions in Cahn–Hilliard fluids with Dirichlet boundary conditions. *SIAM J. Math. Anal.* **27** (1996) 620-637.

[4] T. Kato, *Perturbation theory for linear operators*. Springer-Verlag, Berlin (1995). Reprint of the 1980 edition.

[5] L.D. Landau and E.M. Lifchitz, *Physique statistique*. Ellipses (1994).

[6] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications,* Vol. 1. Dunod, Paris (1968). Travaux et Recherches Mathématiques, No. 17.

[7] L. Modica, The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* **98** (1987) 123-142.

[8] L. Modica, Gradient theory of phase transitions with boundary contact energy. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987) 487-512.

[9] N.C. Owen, J. Rubinstein and P. Sternberg, Minimizers and gradient flows for singularly perturbed bi-stable potentials with a Dirichlet condition. *Proc. Roy. Soc. London Ser. A* **429** (1990) 505-532.

[10] M. Reed and B. Simon, *Methods of modern mathematical physics. I.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, Second Edition (1980). Functional analysis.

[11] P. Sternberg, Vector-valued local minimizers of nonconvex variational problems. *Rocky Mountain J. Math.* **21** (1991) 799-807. Current directions in nonlinear partial differential equations. Provo, UT (1987).

[12] A.I. Volpert, V.A. Volpert and V.A. Volpert, *Traveling wave solutions of parabolic systems.* American Mathematical Society, Providence, RI, 1994. Translated from the Russian manuscript by James F. Heyda.