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ON THE INSTANTANEOUS SPREADING FOR THE NAVIER–STOKES SYSTEM IN THE WHOLE SPACE*

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Abstract. We consider the spatial behavior of the velocity field $u(x,t)$ of a fluid filling the whole space $\mathbb{R}^n$ $(n \geq 2)$ for arbitrarily small values of the time variable. We improve previous results on the spatial spreading by deducing the necessary conditions $\int u_h(x,t)u_k(x,t)\,\mathrm{d}x = c(t)\delta_{h,k}$ under more general assumptions on the localization of u. We also give some new examples of solutions which have a stronger spatial localization than in the generic case.

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Introduction

We consider the Navier–Stokes system for an incompressible viscous fluid in the absence of external forces

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) = \Delta u - \nabla p \\ u(x,0) = a(x) \\ \operatorname{div}(u) = 0. \end{cases} \tag{NS}$$

Here $u : \mathbb{R}^n \times [0,\infty[\to \mathbb{R}^n$ $(n \geq 2)$ denotes the velocity field and $p(x,t)$ is the pressure.

A very natural question is to know whether localization with respect to the space variables is preserved by the Navier–Stokes evolution.

It is now well known that mild localization properties are conserved for small time (see *e.g.* [4,5,7,8]). In order to give an elementary justification of this fact, let us write (NS) in the usual equivalent integral formulation, which is obtained after applying the Leray–Hopf projector $\mathbb{P}$ (which is the orthogonal projector onto the field of soleinoidal vectors):

$$u(t) = \mathrm{e}^{t\Delta}a - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s)\,\mathrm{d}s, \qquad \operatorname{div}(a) = 0. \tag{IE}$$

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If we consider the space L^∞_γ of all measurable functions f such that

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^\gamma|f(x)|<\infty \qquad (0\le\gamma\le n+1),$$

then a straightforward application of the standard fixed point argument yields the existence and unicity of a solution of (IE) in $C([0,T],\mathrm{L}^\infty_\gamma)$, for some $T>0$ which may depend on a, but not on γ. Here the continuity for $t=0$ should be defined in the distributional sense, as it is usually done in non-separable spaces.

Miyakawa [7] and He–Xin [4] first achieved the construction of (local and global) solutions to the Navier–Stokes equations with a decay $O(|x|^{-(n+1)})$, uniformly in time.

The restriction $\gamma\le n+1$ is very natural if we look at the kernel $F(x,t)$ of the matrix-operator $\mathrm{e}^{t\Delta}\mathbb{P}\nabla$. Indeed, we easily see that $|F(x,t)|\le C|x|^{-(n+1)}$ and this decay-rate is known to be optimal. We will recall hereafter some other useful properties of F.

This obviously does not mean that the bound $|u(x,t)|\le C(1+|x|)^{-(n+1)}$ for solutions generated by well-localized initial conditions will be optimal. To give a simple example, it is well known that there exists a classical two-dimensional solution u of (NS), with radial vorticity, which has a rapid spatial decay at infinity. This solution turns out to solve also the linear heat system with the same initial condition.

However, generic solutions to the Navier–Stokes equations do not have a fast decay at infinity, even if the initial data are smooth and compactly supported. This is usually seen by means of the Fourier transform.

For instance, the argument that follows goes through if we suppose $u(t)\in\mathrm{L}^2(\mathbb{R}^n)$ and $p(t)\in\mathrm{L}^1(\mathbb{R}^n)$ for some t. We just start with the classical relation $-\Delta p=\sum_{h,k}\partial_h\partial_k(u_hu_k)$ which is obtained by applying the divergence operator to (NS). On the Fourier transform side this reads

$$-\widehat{p}(\xi,t)=\sum_{h,k=1}^n\xi_h\xi_k|\xi|^{-2}\widehat{u_hu_k}(\xi,t),\qquad t\in[0,T]. \tag{1}$$

Since the left hand side is continuous at 0, the right hand side should also be continuous for $\xi=0$ which implies $\widehat{u_hu_k}(0,t)=-\widehat{p}(0)\delta_{hk}$. This is equivalent to

$$\int u_h(x,t)u_k(x,t)\,dx=-\delta_{hk}\int p(x,t)\,\mathrm{d}x \qquad (\delta_{hk}=1 \text{ if } h=k,\ \delta_{hk}=0 \text{ if } h\ne k). \tag{2}$$

Moreover, one shows that, if $u\in C(]t-\delta,t+\delta[,\mathrm{L}^1(\mathbb{R}^n,(1+|x|)\mathrm{d}x))$ $(\delta>0)$, the condition on the pressure is satisfied. Thus, a decay at infinity of the form $|x|^{-(n+1)}\log(|x|)^{-1-\epsilon}$ $(\epsilon>0)$ is forbidden for generic solutions (see [1,3]).

A slightly deeper argument allowed the first author to prove that a decay $|x|^{-(n+1)}\log(|x|)^{-1/2-\epsilon}$ is also forbidden. Indeed, he showed that the condition $a\in\mathrm{L}^2(\mathbb{R}^n,(1+|x|)^{n+2}\mathrm{d}x)$, in general, is not conserved during the evolution.

Our main goal will be to get rid of these logarithmic factors. Another important issue of this paper consists in proving that the pressure is localized, at least in a weak sense, under a mild localization assumption on the velocity. Then the orthogonality relations $\int u_h(x,t)u_k(x,t)\,\mathrm{d}x=c(t)\delta_{hk}$ will follow in a straightforward manner.

We now state our main result. Let us introduce the space E of all functions $f\in\mathrm{L}^1(\mathbb{R}^n)$ such that

$$\lim_{R\to\infty}R\int_{|x|\ge R}|f(x)|\,\mathrm{d}x=0. \tag{3}$$

We norm this space by

$$\int_{|x|\le 1} |f(x)|\,\mathrm{d}x + \sup_{R\ge 1} R \int_{|x|\ge R} |f(x)|\,\mathrm{d}x. \tag{4}$$

Observe that $f(x) = (1+|x|)^{-(n+1)}$ satisfies $||f||_E < \infty$, but this function does not belong to E. On the other hand, an integrable function which is $o(|x|^{-(n+1)})$ at infinity does belong to E. Thus we can say that that E is the $\mathrm{L}^1(\mathbb{R}^n)$ version of a localization of the form $o(|x|^{-(n+1)})$ at infinity.

Then we have:

Theorem 0.1. *Let $u(x,t)$ be a solution of the Navier–Stokes system in $[0,T]$ $(0 < T \le \infty)$ such that $a = u(0) \in \mathrm{L}^2(\mathbb{R}^n) \cap E$ and satisfying the following properties*

$$u \in C([0,T], \mathrm{L}^2(\mathbb{R}^n)) \tag{5}$$

$$u(\cdot,t) \in \mathrm{L}^\infty([0,T], E) \tag{6}$$

$$|u(\cdot,t)|^2 \in \mathrm{L}^\infty([0,T], E). \tag{7}$$

If we note by $a_1, \ldots, a_n$ the components of the initial data, then we have

$$\int a_h(x) a_k(x)\,\mathrm{d}x = c\delta_{hk}, \qquad h,k = 1,\ldots,n \tag{8}$$

for some constant c.

It should be observed that under some mild assumptions on the localization of the data (such as, for example, $|a(x)| \le C(1+|x|)^{-(n+1+\epsilon)/2}$) then (5) and (7) are always satisfied, at least in some small interval $[0,T]$. Hence this theorem states that *generic solutions do not belong to* $\mathrm{L}^\infty([0,T], E)$. Since E contains both $\mathrm{L}^1(\mathbb{R}^n, (1+|x|)\mathrm{d}x)$ and $\mathrm{L}^2(\mathbb{R}^n, (1+|x|)^{n+2}\mathrm{d}x)$, Theorem 0.1 improves all previous results on the instantaneous spreading. As we will see later on, even if we drop the assumption (5) then the solution must satisfy the orthogonality relations (8) at least for almost all $t \in [0,T]$.

Before going further, let us mention that the importance of (2) in the context of the spatial localization was first noticed by Dobrokhotov and Shafarevich [2]. They deduced these necessary conditions as a consequence of some remarkable integral identities. We will briefly recall such identities in the following section, since they do not seem to be much known. We point out, however, that their approach leads to very stringent assumptions on the decay of p and u.

In Section 2 we prove Theorem 0.1. Then we state a generalization of this result to the case of solutions with a stronger spatial localization. Many other necessary non-linear conditions on the initial data can be derived in this setting.

Sections 3 and 4 are fully borrowed from the first author's doctoral dissertation. There we will discuss the converse problem of finding sufficient conditions in order to obtain a space-decay for u faster than expected. Under a localization assumption for the data, we show that if (8) is satisfied, and *if this condition remains true* in a small interval of time, then the corresponding solution will decay faster than $|x|^{-(n+1)}$, uniformly on this interval. Let us however stress that (a) the orthogonality relations do not persist under the Navier–Stokes evolution and (b) we were not able to characterize the subclass of initial data for which (8) is conserved during the time evolution. In the last part of this paper we provide several new examples of such data. The interesting point is that the corresponding solutions are non-trivial in the following sense: they solve the Navier–Stokes system without being solutions of the heat equation.

1. Integral identities

We start with recalling the integral identities of Dobrokhotov and Shafarevich (see [2] for a proof). These identities read as follows ($i, j = 1, 2, 3$):

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\partial\Omega}\langle n,u\rangle x_i x_j\,\mathrm{d}S \quad &+ \quad \int_{\partial\Omega}\langle n,u\rangle(x_i u_j + x_j u_i)\,\mathrm{d}S + \int_{\partial\Omega}\langle n,a_{ij}\rangle\,\mathrm{d}S = \int_{\partial\Omega}\langle n,\nabla\rangle(x_i u_j + x_j u_i)\mathrm{d}S \\ &- \quad 2\int_{\partial\Omega}\langle n,b_{ij}\rangle\,\mathrm{d}S + 2\left[\int_\Omega u_i u_j\,\mathrm{d}x + \delta_{ij}\int_\Omega p\,\mathrm{d}x\right]\cdot \end{aligned} \tag{9}$$

Here $u(x,t)$ and $p(x,t)$ is any solution of $\partial_t u + u\cdot\nabla u = \Delta u - \nabla p$ which is two times differentiable in $(x,t) \in \bar{\Omega}\times(t-\delta,t+\delta)$, $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain, n is the exterior normal to $\partial\Omega$, $\mathrm{d}S$ the surface element on $\partial\Omega$ and the vectors a_{ij} and b_{ij} are defined by:

$$\begin{array}{llll} (a_{ij})_i = px_j, & (a_{ij})_j = px_i & (a_{ij})_k = 0, & i,j,k = 1,2,3 \\ (b_{ij})_i = u_j, & (b_{ij})_j = u_i, & (b_{ij})_k = 0, & i\neq j,\ j\neq k,\ k\neq i \\ (a_{ii})_j = 2\delta_{ij}px_j, & (b_{ii})_j = 2\delta_{ij}u_j. & & \end{array}$$

As an immediate consequence, we see that if $p(t)\in \mathrm{L}^1(\mathbb{R}^3)$ in some interval $t\in]0,T[$ and if u and p verify the following decay properties

$$\begin{array}{ll} \partial_t u(x,t) = o(|x|^{-4}) & \nabla u(x,t) = (|x|^{-3}) \\ p(x,t) = o(|x|^{-3}) & u(x,t) = o(|x|^{-2}), \end{array} \qquad 0<t<T, \tag{10}$$

when $x\to\infty$, then (2) holds for $t\in]0,T[$.

1.1. Estimates for the kernel of $\mathrm{e}^{t\Delta}\mathbb{P}\nabla$

Here we recall some known estimates for the kernel $F(x,t) = (F_{jhk}(x,t))$ of $\mathrm{e}^{t\Delta}\mathbb{P}\nabla$ $(j,h,k = 1,\dots,n)$. The components of its Fourier transform are given by

$$\widehat{F}_{jhk}(\xi,t) = i\xi_h\left(\delta_{jk} - \xi_j\xi_k|\xi|^{-2}\right)\exp\left(-t|\xi|^2\right)\cdot \tag{11}$$

Hence, $F(x,t)$ is smooth, $F(x,t) = t^{-1/2}F(xt^{-1/2},1)$ and $|F(x,1)| \le |x|^{-(n+1)}$.

This estimate for the kernel is optimal, as it is trivially seen from the singularity of $\widehat{F}(\xi,t)$ at $\xi = 0$. Another useful property, which directly follows from the scaling law is $||F(\cdot,t)||_1 \le Ct^{-1/2}$.

For later use, the derivatives of F can be estimated by

$$|\partial_x^\beta F(x,t)| \le C|x|^{-(n+1+|\beta|)}, \qquad |\beta|\ge 0. \tag{12}$$

2. Proof of the main result

Before entering into more details, let us sketch the proof of Theorem 0.1. The first step consists in checking that the localization of the velocity implies a similar property for the gradient of the pressure. Next we will show that the pressure itself is localized. Finally the localization of both the pressure and the velocity easily implies (8).

In order to establish Theorem 0.1 we place the Banach space E on a ladder of functional spaces E_α.

Since $\int_{|x|\ge R}|f(x)|\,\mathrm{d}x \to 0$ if $R\to\infty$ whenever $f\in \mathrm{L}^1(\mathbb{R}^n)$, we may want to know at which speed this convergence holds. For $\alpha > 0$, we introduce the space E_α of all integrable functions which satisfy

$$\lim_{R\to\infty} R^\alpha \int_{|x|\ge R}|f(x)|\,\mathrm{d}x = 0. \tag{13}$$

Hence E_α is a Banach space for the norm

$$\int_{|x|\le 1} |f(x)|\,\mathrm{d}x + \sup_{R\ge 1} R^\alpha \int_{|x|\ge R} |f(x)|\,\mathrm{d}x \tag{14}$$

and it is easy to show that continuous and compactly supported functions form a dense sub-space in E_α. Accordingly with the introduction, we call $E = E_1$. Let us state without proof some simple but useful properties of these spaces.

Lemma 2.1. *If $\alpha > 0$ and f is a locally integrable function such that*

$$\lim_{R\to+\infty} R^\alpha \int_{R\le|x|\le 2R} |f(x)|\,\mathrm{d}x = 0, \tag{15}$$

then $f \in E_\alpha$. Moreover, $||f||_{E_\alpha}$ and

$$\int_{|x|\le 1} |f(x)|\,\mathrm{d}x + \sup_{R\ge 1} R^\alpha \int_{R\le|x|\le 2R} |f(x)|\,\mathrm{d}x \tag{16}$$

are two equivalent norms.

Lemma 2.2. *For $\alpha > 0$, E_α is a Banach algebra with respect to the convolution product.*

Lemma 2.1 paves the road to a natural definition of the spaces E_α when $\alpha = 0$: E_0 is the space of locally integrable functions satisfying

$$\lim_{R\to\infty} \int_{R\le|x|\le 2R} |f(x)|\,\mathrm{d}x = 0 \tag{17}$$

and we norm it by

$$\int_{|x|\le 1} |f(x)|\mathrm{d}x + \sup_{R\ge 1} \int_{R\le|x|\le 2R} |f(x)|\,\mathrm{d}x < \infty. \tag{18}$$

The next important lemma shows that if one is assuming a localization condition on the gradient ∇f of a function f defined on $\mathbb{R}^n$, $(n \ge 2)$, then this function f is itself a localized function, up to an additive constant. This is obviously false in one dimension. A first instance of this general fact is provided by the space BV consists of the functions f whose gradient is a bounded Borel measure. If $f \in BV$, then there exists a constant c and a function g in $L^{n/(n-1)}(\mathbb{R}^n)$ such that $f = c + g$. Our aim is to extend this localization property to the context of the E_α spaces.

Lemma 2.3. *Let $n \ge 2$ and $f \in BV(\mathbb{R}^n)$ such that $\nabla f \in E_\alpha$, with $\alpha \ge 1$. Then we can find a constant c and $g \in E_{\alpha-1}$ such that $f = c + g$.*

Proof. To establish this property, we consider the unit sphere S^{n-1} in $\mathbb{R}^n$ and we renormalize its area element $\mathrm{d}\sigma$ in order to have $\int_{S^{n-1}} \mathrm{d}\sigma = 1$. Our assumption reads

$$\int_{\lambda\ge R} \int_{S^{n-1}} |\nabla f(\lambda\nu)|\lambda^{n-1}\,\mathrm{d}\lambda\,\mathrm{d}\sigma(\nu) \le \epsilon_R R^{-\alpha}$$

with $\epsilon_R \le C$ and $\epsilon_R \to 0$ if $R \to \infty$. We start by studying the total variation of $I(\lambda) \equiv \int_{S^{n-1}} f(\lambda\nu)\,\mathrm{d}\sigma(\nu)$. We have $I'(\lambda) = \int_{S^{n-1}} \nu \cdot \nabla f(\lambda\nu)\,\mathrm{d}\sigma\,(\nu)$ and so $|I'(\lambda)| \le \int_{S^{n-1}} |\nabla f(\lambda\nu)|\,\mathrm{d}\sigma(\nu)$. This implies $\int_{\lambda\ge R} |I'(\lambda)|\lambda^{n-1}\,\mathrm{d}\lambda \le \epsilon_R R^{-\alpha}$. Then

$$R^{n-1} \int_{\lambda\ge R} |I'(\lambda)|\,\mathrm{d}\lambda \le \epsilon_R R^{-\alpha}.$$

Hence the limit $c = \lim_{\lambda\to\infty} I(\lambda)$ does exist and

$$|I(\lambda) - c| \leq \epsilon_\lambda \lambda^{-\alpha-n+1}.$$

We next apply Hölder and Poincaré's inequalities on S^{n-1} and we get

$$\int_{S^{n-1}} |f(\lambda\nu) - I(\lambda)| \,\mathrm{d}\sigma(\nu) \leq C_n \lambda \int_{S^{n-1}} |\nabla f(\lambda\nu)| \,\mathrm{d}\sigma(\nu).$$

This finally gives

$$\begin{aligned}\int_R^{2R}\int_{S^{n-1}} |f(\lambda\nu) - c|\lambda^{n-1} \,\mathrm{d}\lambda\,\mathrm{d}\sigma(\nu) &\leq \int_R^{2R}\int_{S^{n-1}} |f(\lambda\nu) - I(\lambda)|\lambda^{n-1} \,\mathrm{d}\lambda\,\mathrm{d}\sigma(\nu)\\ &\quad + \int_R^{2R}\int_{S^{n-1}} |I(\lambda) - c|\lambda^{n-1} \,\mathrm{d}\lambda\,\mathrm{d}\sigma(\nu)\\ &\leq 2R\int_R^{2R}\int_{S^{n-1}} |\nabla f(\lambda\nu)|\lambda^{n-1} \,\mathrm{d}\lambda\,\mathrm{d}\sigma(\nu) + \epsilon_R R^{-\alpha+1} \leq \tilde{\epsilon}_R R^{-\alpha+1}\end{aligned}$$

where $\tilde{\epsilon}_R \to 0$ if $R \to \infty$. Lemma 2.3 is thus proved. □

The following proposition will be an essential step in deducing the orthogonality relations.

Proposition 2.4. *Let $g_{h,k}$ $(h, k = 1, \ldots, n)$ be a family of integrable functions and f a tempered distribution. We assume that*

$$\lim_{R\to\infty} \langle\, f, \varphi(\cdot/R)\,\rangle = 0 \tag{19}$$

for all smooth functions φ supported in $1 \leq |x| \leq 2$. If

$$\Delta f = \sum_{h,k=1}^{n} \partial_h\partial_k g_{h,k} \tag{20}$$

then we have, for some constant c,

$$\int g_{h,k}(x)\,\mathrm{d}x = c\delta_{hk} \qquad (h, k = 1, \ldots, n). \tag{21}$$

Proof. Taking the Fourier transform in (20) we get $\widehat{f}(\xi) = \sum_{h,k} \xi_h\xi_k|\xi|^{-2}\widehat{g}_{h,k}(\xi) + \widehat{P}(\xi)$, where $\widehat{P}$ is a sum of derivatives of Dirac masses supported by the origin (more precisely P is a harmonic polynomial). Since f tends to 0 at infinity in a weak sense, we must have $P \equiv 0$. Indeed, one first observes that $I(R) = \langle\, \widehat{P}, R^n\widehat{\varphi}(R\cdot)\,\rangle$ is a polynomial in R of the form $I(R) = C(\beta)\partial^\beta\widehat{\varphi}(0)R^{n+|\beta|}$, with $|\beta| \geq 0$. The coefficients $C(\beta)$ arise from the expansion $\widehat{P} = \sum_\beta C(\beta)\partial^\beta\delta_0$. Next we observe that $\widehat{g}_{h,k}(\xi)$ are bounded on $\mathbb{R}^n$ which implies that $J(R) = \int \sum_{h,k} \xi_h\xi_k|\xi|^{-2}\widehat{g}_{h,k}(\xi)R^n\widehat{\varphi}(R\xi)\,\mathrm{d}\xi$ is uniformly bounded in R. Finally (20) tells us that $I(R) + J(R)$ tends to 0 when $R \to \infty$. Therefore the polynomial $I(R)$ is bounded which implies $C(\beta) = 0$.

This discussion yields

$$\widehat{f}(\xi) = \sum_{h,k=1}^{n} \xi_h\xi_k|\xi|^{-2}\widehat{g}_{h,k}(\xi).$$

We now use again the fact that the functions $\widehat{g}_{h,k}(\xi)$ are bounded on $\mathbb{R}^n$ which implies that the family $\widehat{f}(\xi/R)$ is bounded in $\mathrm{L}^\infty(\mathbb{R}^n)$. The continuity at 0 of $\widehat{g}_{h,k}(\xi)$ is now used and $\widehat{f}(\xi/R)$ converges for $R \to \infty$ to

$\widehat{S}(\xi) \equiv \sum_{h,k} \xi_h \xi_k |\xi|^{-2} \widehat{g}_{h,k}(0)$ in the weak-$*$ topology $\sigma(\mathrm{L}^\infty, \mathrm{L}^1)$. This implies that the inverse Fourier transform f_R of $\widehat{f}(R\xi)$ converges to a tempered distribution S in the distributional sense. But if $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ and 0 does not belong to the support of φ, (19) yields

$$\langle\, f_R, \varphi \,\rangle = \langle\, f, \varphi(\cdot/R) \,\rangle \longrightarrow 0, \quad R \to \infty.$$

Hence S is supported at the origin. But $S = \lim_{R\to\infty} R^{-n} f(x/R)$ is also a homogeneous distribution of degree $-n$. Thus, $S = c\delta_0$ (Dirac mass at 0) for some constant c. This gives (21). □

Remark 2.5. We could now directly apply Proposition 2.4 to deduce (2) under a much weaker assumption on the pressure than $p \in \mathrm{L}^1(\mathbb{R}^n)$ (as we made in Sect. 2). Indeed let us just suppose that, at some instant t we have $u(t) \in \mathrm{L}^2(\mathbb{R}^n)$ and $p(t) \in \mathcal{S}'(\mathbb{R}^n)$ satisfying the mild decay condition (19). Then $\int u_h(x,t) u_k(x,t)\,\mathrm{d}x = c\delta_{hk}$.

We are now ready to prove Theorem 0.1. We write (NS) in the integral form

$$u(x,t) = \mathrm{e}^{t\Delta} a - \int_0^t \mathrm{e}^{(t-s)\Delta} \partial_h (u_h u)\,\mathrm{d}s - \int_0^t \mathrm{e}^{(t-s)\Delta} \nabla p(s)\,\mathrm{d}s. \tag{22}$$

It is easily seen that $\int_0^t \mathrm{e}^{(t-s)\Delta} \partial_h (u_h u)\,\mathrm{d}s$ belongs to $\mathrm{L}^\infty([0,T], E)$. Indeed $u_h u \in \mathrm{L}^\infty([0,T], E)$ by (7). Moreover, $\mathrm{e}^{t\Delta}\partial_h$ is a convolution operator with kernel $ct^{-(n+1)/2} G_h(\frac{x-y}{\sqrt{t}})$ and $G_h \in E$. Then we observe that the norm of $h^{-n}\phi(h^{-1}\cdot)$ in E is bounded by the norm of ϕ in E, at least when $0 < h \le 1$, and we just apply Lemma 2.2.

Let us come back to (22). By our hypotheses, all terms but the last belong to $\mathrm{L}^\infty([0,T], E)$. We thus have

$$\int_0^t \mathrm{e}^{(t-s)\Delta} \nabla p(s)\,\mathrm{d}s = \nabla \tilde{p}(t) \in \mathrm{L}^\infty([0,T], E),$$

where we wrote $\tilde{p}(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} p(s)\,\mathrm{d}s$. Let $\tilde{u}_{h,k}(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} u_h(\cdot,s) u_k(\cdot,s)\,\mathrm{d}s$. If we apply the divergence operator to (22), we get

$$-\Delta \tilde{p} = \sum_{h,k=1}^n \partial_h \partial_k \tilde{u}_{h,k}.$$

By Lemma 2.3, $\tilde{p}(t)$, up to a constant, belongs to $\mathrm{L}^\infty([0,T], E_0)$. Then Proposition 2.4 applies and we obtain $\int \tilde{u}_{h,k}(x,t)\,\mathrm{d}x = c(t)\delta_{hk}$. Then we just differentiate in t and deduce

$$\int u_h(x,t) u_k(x,t)\,\mathrm{d}x = c(t)\delta_{hk},$$

for almost every $t \in [0,T]$. The assumption (5) allows us to conclude. This completes the proof of Theorem 0.1.

If a stronger spatial localization is imposed to u, many other necessary algebraic conditions on the higher-order moments of $u_h u_k(x,t)$ must be satisfied. In order to state these conditions, let us consider the higher moments

$$\lambda_{\alpha hk}(t) = \int x^\alpha u_h(x,t) u_k(x,t)\,\mathrm{d}x, \qquad |\alpha| = 0, \ldots, m, \;\; m \in \mathbb{N}. \tag{23}$$

We first state a generalization of Proposition 2.4 in the following remark:

Remark 2.6. Let $m = 0, 1, \ldots$ be a fixed integer and $u(x,t)$ and $p(x,t)$ a solution of the Navier–Stokes equations such that, at some time t, $u(t) \in \mathrm{L}^2(\mathbb{R}^n, (1+|x|)^m \mathrm{d}x)$ and $p(t) \in E_m$. Then the moments $\lambda_{\alpha hk}(t)$

must satisfy the following identity:

$$\sum_{h,k=1}^{n} \sum_{|\alpha|=\ell} \frac{\lambda_{\alpha hk}(t)}{\alpha!} \xi^{\alpha} \xi_h \xi_k = (\xi_1^2 + \cdots + \xi_n^2) P_l(\xi) \tag{24}$$

where P_l is a homogeneous polynomial of degree l for $\ell = 0, \ldots, m$.

Here again this statement would become obvious if the pressure is also localized, *i.e.* if $p(t) \in \mathrm{L}^1(\mathbb{R}^n, (1+|x|)^m \mathrm{d}x)$. Indeed, this would give $\widehat{p}(\xi) \in \mathrm{C}^m(\mathbb{R}^n)$ and the conclusion would follow from the Taylor formula.

The case $m = 0$ follows from Proposition 2.4. In the general case, the conclusion of Remark 2.6 can be obtained by induction on m, following the same arguments of Proposition 2.4. We leave the details to the reader.

As a consequence of this observation, we easily obtain the following result, which is the natural generalization of Theorem 0.1.

Corollary 2.7. *Let $m \geq 0$ an integer and $a = u(0) \in \mathrm{L}^2(\mathbb{R}^n, (1+|x|)^m \mathrm{d}x) \cap E_{m+1}$ a soleinoidal field.* Let $T > 0$ and $u(x,t)$ be a solution of the Navier–Stokes equations such that

$$u \in C([0,T], \mathrm{L}^2(\mathbb{R}^n, (1+|x|)^m \mathrm{d}x)) \tag{25}$$

$$u(\cdot,t) \quad \text{and} \quad |u(\cdot,t)|^2 \in \mathrm{L}^{\infty}([0,T], E_{m+1}). \tag{26}$$

Then (24) holds for all $t \in [0,T]$ and $\ell = 0, 1 \ldots, m$.

In the following section we will show that whenever the initial condition is localized, these necessary conditions turn out to be also sufficient. They imply a good localization of the solution, at least for small time.

3. Sufficient conditions

As we have already mentioned in the introduction, in order to obtain bounded solutions $u(x,t)$ which decay at a low rate $|x|^{-\gamma}$ $(0 \leq \gamma \leq n+1)$ at the beginning of the evolution, we can simply suppose $|a(x)| \leq C(1+|x|)^{-\gamma}$. Moreover, the unicity of such solutions is granted in the natural functional space associated to this decay property.

This remark leads to studying the case $\gamma \geq n+1$, *i.e.* the spatial localization of u, when $u(0)$ is well-localized and the necessary conditions (8) (or, more generally, Eq. (24)) are satisfied at $t = 0$.

A first difficulty comes from the fact that, for generic initial data satisfying (8), this cancellation property instantaneously breaks down during the evolution. In three dimensions a simple example (which should be compared to the specific examples of Sect. 4) is given by

$$\begin{cases} a_1(x) = \left[x_3\left(1-2x_2^2\right) - x_2\left(1-2x_3^2\right)\right] \exp\left(-|x|^2/2\right) \\ a_2(x) = \left[x_1\left(1-2x_3^2\right) - x_3\left(1-2x_1^2\right)\right] \exp\left(-|x|^2/2\right) \\ a_3(x) = \left[x_2\left(1-2x_1^2\right) - x_1\left(1-2x_2^2\right)\right] \exp\left(-|x|^2/2\right). \end{cases}$$

Indeed, let us fix a small $t_0 > 0$. A trivial computation shows that $\int \exp(-2t_0|\xi|^2)\widehat{a}_1(\xi)\widehat{a}_2(\xi)\,\mathrm{d}x \neq 0$. Hence the cancellation $\int a_1(x)a_2(x)\,\mathrm{d}x = 0$ breaks down for the solution of the heat system. By choosing $\alpha > 0$ small enough, we thus see that the solution u of the Navier–Stokes equations, starting from $\alpha a(x)$, cannot verify (8) for $t = t_0$.

Therefore it does not suffice to impose this condition to the initial data in order to ensure an over-critical spatial decay for u. We will come back to this problem in the following section.

In this section we rather consider the class of initial data such that (8) *remains true* during the evolution. Before studying the spatial properties of the corresponding solutions, let us first show, by means of a classical two-dimensional example, that this class is non-empty.

We start with considering the vorticity $\omega(x,t) = \nabla \wedge u = \partial_1 u_2(x,t) - \partial_2 u_1(x,t)$. Here $x = (x_1, x_2)$. Then we choose $\omega_0 = \omega(0)$ such that $\widehat{\omega}_0$ is radial, smooth, compactly supported in $\mathbb{R}^2$ and such that 0 does not belong to the support of $\widehat{\omega}_0$. Then we pose $\omega(x,t) = \mathrm{e}^{t\Delta}\omega_0(x)$, in a such way that the vorticity solves the heat equation $\partial_t \omega = \Delta\omega$. Since $\omega(x,t)$ is radial, the Biot–Savart law yields $u \cdot \nabla\omega \equiv 0$. Hence, $\partial_t\omega - \Delta\omega + u \cdot \nabla\omega = 0$ which is nothing but the formulation of the Navier–Stokes equation in the velocity-vorticity formulation. Moreover, again by the Biot–Savart law, the solution u is given by $\widehat{u}(\xi,t) = (-i\xi_2, i\xi_1)|\xi|^{-2}\exp(-t|\xi|^2)\widehat{\omega}_0(\xi)$. Hence $u(t)$ is in the Schwartz class for all $t \geq 0$. This solution u satisfies (24) for all $t \geq 0$ and all $\ell = 0, 1, 2, \ldots$

In the following theorem we show that the converse of Corollary 2.7 is true:

Theorem 3.1. *Let L^∞_γ be defined as in the introduction and, for easing the notations, let us write $F = \mathrm{L}^\infty_{(n+1)}$. Let $m \geq 0$ be an integer and $a = u(0)$ be a divergence-free vector such that $|a(x)| \leq C(1+|x|)^{-\gamma}$, with $n+1+m < \gamma \leq n+2+m$. For a sufficiently small $T > 0$, we know that there exists a unique solution $u(x,t) \in C([0,T], F)$ of the Navier–Stokes equations with initial condition a. We then obtain $|u(x,t)| \leq C(1+|x|)^{-(n+1)}$, uniformly in $[0,T]$. Then, if (24) holds for all $t \in [0,T]$ and $\ell = 0, \ldots, m$, the decay of this solution is improved into*

$$|u(x,t)| \leq C(1+|x|)^{-\gamma}, \qquad x \in \mathbb{R}^n, \quad t \in [0,T]. \tag{27}$$

Proof. We consider the integral formulation (IE) of the Navier–Stokes equations, which we write in the following way

$$u(t) = \mathrm{e}^{t\Delta}a - B(u,u)(t),$$

where B is the bilinear operator given by

$$B(u,v)(t) = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes v)(s)\,\mathrm{d}s.$$

We already know that $|u(x,t)| \leq C(1+|x|)^{-(n+1)}$, uniformly in $[0,T]$. But since $|a(x)| \leq C(1+|x|)^{-\gamma}$, the linear evolution $\mathrm{e}^{t\Delta}a(x) = \int g_t(x-y)a(y)\,\mathrm{d}y$ satisfies $|\mathrm{e}^{t\Delta}a(x)| \leq C_T(1+|x|)^{-\gamma}$, uniformly in $[0,T]$. We now show that $|B(u,u)(x,t)| \leq C(1+|x|)^{-(n+2+m)}$. This is an immediate consequence of the following lemma:

Lemma 3.2. *Let $K \geq 0$ an integer and $T > 0$. Let also $u(x,t)$ be a function satisfying $|u(x,t)| \leq C(1+|x|)^{-(n+1+K)}$ and (24), for $\ell = 0, \ldots, K$. Then*

$$|B(u,u)(x,t)| \leq C_{K,T}(1+|x|)^{-(n+2+K)}.$$

Proof. We sketch the proof in the case $K = 0$ while the general case is treated in the first author's doctoral dissertation.

We consider the j-component of the bilinear term $B(u,u)$ which we shall write in the following way (accordingly with the notations fixed in the introduction): $B(u,u)_j = \int_0^t F_{jhk}(\cdot, t-s) * (u_h u_k)(s)\,\mathrm{d}s$. In order to simplify the notation, here and in what follows, we shall omit the summations on the repeated indexes h and k $(h, k = 1, \ldots, n)$.

Let $r_{h,k}(x,t)$ be such that

$$u_h u_k(x,t) = r_{h,k}(x,t) + \left(\int u_h u_k(y,t)\,\mathrm{d}y\right) g(x),$$

where g is the Gaussian, normalized by $\int g(y)\,\mathrm{d}y = 1$. By (24), with $\ell = 0$, we have $\int r_{h,k}(x,t)\,\mathrm{d}x = 0$, for all $t \in [0,T]$. We first consider the term

$$I(x,t) \equiv \int_0^t \int F_{jhk}(x-y, t-s) * r_{h,k}(s)\,\mathrm{d}y\,\mathrm{d}s.$$

By $\int r_{h,k} = 0$ and an integration by parts, using (12) with $|\beta| = 1$, we easily obtain $|I(x,t)| \le C(1+|x|)^{-(n+2)}$.

Consider next the second term which contains the expression $F_{jhk} * (\int u_h u_k) g$. It easily seen that this term vanishes. Indeed, $\int u_h u_k(x,t) = c(t)\delta_{hk}$ and $(\delta_{hk} F_{jhk})\widehat{\ }(\xi,t) = i \sum_{h=1}^n (\xi_j - \xi_j \xi_h \xi_h / |\xi|^2) \mathrm{e}^{-t|\xi|^2} \equiv 0$. Since the proof of the general case can be found in the first author's doctoral dissertation, we will limit the discussion to some heuristical comments. Indeed we write $v_{h,k} = u_h(x,t) u_k(x,t)$ and $w_j(\xi,t) = i \int_0^t \exp[-(t-s)\xi^2] \xi_h (\delta_{jk} - \xi_j \xi_k |\xi|^{-2}) \widehat{v}_{h,k}(\xi,s) \mathrm{d}s$. The assumption implies that $\widehat{v}_{h,k}$ is killing the singularity of the symbol of the Leray–Hopf projector. The improved smoothness on the Fourier transform side reflects into the improved decay in the space variables. This is the meaning of our lemma. □

The proof of Theorem 3.1 is now immediate: we simply iterate Lemma 3.2 for $K = 0, \ldots, m$. □

4. Well-localized flows

4.1. Symmetric solutions

We give here some examples of initial data such that the orthogonality relations $\int a_h(x) a_k(x)\,\mathrm{d}x = c\delta_{h,k}$ hold and remain true during the evolution. Moreover, we will choose a with a strong spatial localization in a such way that Theorem 3.1 implies the existence of a unique local solution with a faster than $|x|^{-(n+1)}$ decay.

We have already given in the previous section a two-dimensional example of such data. We point out, however, that this example is a quite trivial one since the corresponding solution satisfies $\mathbb{P}\nabla \cdot (u \otimes u) \equiv 0$ and thus u solves also the heat equation. In the case $n = 3$, even such trivial solutions do not seem to be known to exist.

We should look for some more stringent conditions than (8) which are conserved during the Navier–Stokes evolution. The following construction is borrowed from [1]. We choose the initial data in the class of the so called *symmetric vector fields*. We recall that $a(x) = (a_1(x), \ldots, a_n(x))$ is said to be symmetric, if the two following properties are satisfied:

1. $a_1(x) = a_2(\sigma x) = \ldots a_n(\sigma^{n-1} x)$, where $x = (x_1, \ldots, x_n)$ and σ is the permutation $\sigma(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1})$;
2. $a_1(x_1, \ldots, x_n)$ is odd with respect to x_1 and even with respect to x_j, $j = 2, \ldots, n$.

Starting from well-localized and symmetric data, we obtain solutions which decay at least with the over-critical rate $|x|^{-(n+3)}$. This was announced in [1], but the proof was only sketched. As an immediate consequence of Theorem 3.1, we can now give a complete proof.

We just need to state two more remarks. The first one is obvious:

Remark 4.1. Let a a symmetric vector field. If $a \in \mathrm{L}^2(\mathbb{R}^n)$, then $\int a_h(x) a_k(x)\,\mathrm{d}x = c\delta_{hk}$. Moreover, if $a \in \mathrm{L}^2(\mathbb{R}^n, (1+|x|)\mathrm{d}x)$ then $\int x_j a_h(x) a_k(x)\,\mathrm{d}x = 0$, for all $j, h, k = 1, \ldots, n$.

In particular, (24) holds for $\ell = 0, 1$.

The second remark simply states that the symmetry properties are conserved by the Navier–Stokes evolution. This is true under general assumptions on the initial data, but since in the last two sections only bounded solutions are studied, we just treat this particular case.

Remark 4.2. Let $a \in \mathrm{L}^\infty(\mathbb{R}^n)$ a divergence-free symmetric vector field. Then the corresponding solution $u \in C([0,T], \mathrm{L}^\infty(\mathbb{R}^n))$ (T small enough) is symmetric for all t ($0 \le t \le T$).

Indeed, u is obtained by means of the usual fixed point argument: $u_0 = \mathrm{e}^{t\Delta} a$, $u_{n+1} = \mathrm{e}^{t\Delta} a - B(u_n, u_n)$, $n = 0, 1, \ldots$ Hence, we just need to show that, for all n, $u_n(t)$ is symmetric. First note that the convolution with a radial function does not destroy the symmetries. Thus $\mathrm{e}^{t\Delta} a$ is symmetric.

Next we need to show that if $v(x,t)$ and $w(x,t)$ are symmetric for all t, then $B(v,w)(x,t)$ is also symmetric. This can be easily done by showing that $\widehat{B}(v,w)(\xi,t)$ is symmetric in the ξ variable. Indeed, we first observe that the vectors $\theta(\xi,t) \equiv \sum_{h=1}^n \xi_h \widehat{u_h v}(\xi,t)$ and $\phi(\xi,t) \equiv \xi|\xi|^{-2} \sum_{h,k=1}^n \xi_h \xi_k \widehat{u_h v_k}(\xi,t)$ are symmetric for all t. Then $\widehat{B}(v,w)(\xi,t) = i \int_0^t \mathrm{e}^{-(t-s)|\xi|^2} (\theta(\xi,s) - \phi(\xi,s))\,\mathrm{d}s$ is obviously symmetric.

We just proved the following:

Corollary 4.3. *Let a be a divergence-free symmetric vector field, such that $|a(x)| \le C(1+|x|)^{-\gamma}$, with $0 \le \gamma \le n+3$. Then there exists $T > 0$ and a solution u of the Navier–Stokes equations in $\mathbb{R}^n$ such that $u(0) = a$ and $|u(x,t)| \le C'(1+|x|)^{-\gamma}$.*

In order to build solutions which decay faster than $|x|^{-(n+3)}$, some supplementary symmetries on the data are probably necessary.

4.2. Some explicit examples

We now give some examples of initial data which satisfy the assumptions of Corollary 4.3, in order to show that the two symmetry properties are consistent with the divergence-free condition. Let us start with the case $n = 3$. We take $\varphi \in \mathcal{S}(\mathbb{R})$ (the Schwartz class) and we define

$$a(x_1, x_2, x_3) = \begin{pmatrix} x_1(x_3^2 - x_2^2) \\ x_2(x_1^2 - x_3^2) \\ x_3(x_2^2 - x_1^2) \end{pmatrix} \varphi(|x|^2) \qquad (n = 3).$$

This example generalizes in a obvious manner to the case $n \ge 3$. Indeed, we can take the vector field $a = (a_1, \dots, a_n)$ such that

$$a_h(x_1, \dots, x_n) = x_h \left(x_{h-1}^2 - x_{h+1}^2\right) \varphi\left(|x|^2\right), \qquad h = 1, \dots, n \qquad (n \ge 3). \tag{28}$$

Here we noted $x_0 = x_n$ and $x_{n+1} = x_1$. Both the symmetry properties and the divergence-free condition hold, as trivially checked.

This example cannot be adapted to the two-dimensional case. Somewhat surprisingly, as we will see hereafter, the examples for $n = 2$ turn out to be more difficult.

We now present a quite general method that can be applied to construct many other examples of symmetric initial data. This allows us to circumvent the difficulties arising from the condition $\mathrm{div}(a) = 0$. We recall that, for a n-dimensional vector field $a(x)$, its vorticity is given by the $n \times n$ antisymmetric matrix $\Omega = \nabla a - (\nabla a)^*$. In the two-dimensional case, Ω is usually identified to the scalar $\omega = \partial_1 a_2 - \partial_2 a_1$. If $n = 3$, Ω can be identified to the vector

$$\omega = \nabla \wedge a = \begin{pmatrix} \partial_2 a_3 - \partial_3 a_2 \\ \partial_3 a_1 - \partial_1 a_3 \\ \partial_1 a_2 - \partial_2 a_1 \end{pmatrix}.$$

By the Biot–Savart law, we have

$$a_h = (-\Delta)^{-1} \sum_{k=1}^{n} \partial_k \Omega_{hk}, \qquad h = 1, \dots, n. \tag{29}$$

Then a is a symmetric field if and only if

$$\Omega_{hk}(x) \text{ is } \textit{odd} \text{ with respect to } x_h \text{ and } x_k \tag{30}$$

$$\Omega_{hk}(x) \text{ is } \textit{even} \text{ with respect to } x_j, \quad j = 1, \dots, n \;\text{ and }\; j \ne h, k \tag{31}$$

$$\Omega_{hk}(x) = \Omega_{h+1,k+1}(x_n, x_1, \dots, x_{n-1}), \qquad h, k = 1, \dots, n. \tag{32}$$

Since $(-\Delta)^{-1}$ does not affect the symmetries, in order to construct a soleinoidal symmetric vector field $a(x)$, we can just define $a_h(x) = \sum_{k=1}^n \partial_k \tilde{\Omega}_{hk}(x)$, where $\tilde{\Omega}$ is any antisymmetric matrix which satisfies (30, 31) and (32).

Let us come back to the case $n \ge 3$. By (32) $\tilde{\Omega}$ is completely determined by the choice of its elements $\tilde{\Omega}_{1,2}$, $\tilde{\Omega}_{1,3}, \dots$ and $\tilde{\Omega}_{1,[n/2]}$. Here $[\cdot]$ is the integer part. These $[n/2]$ functions are just supposed to verify (30) and (31).

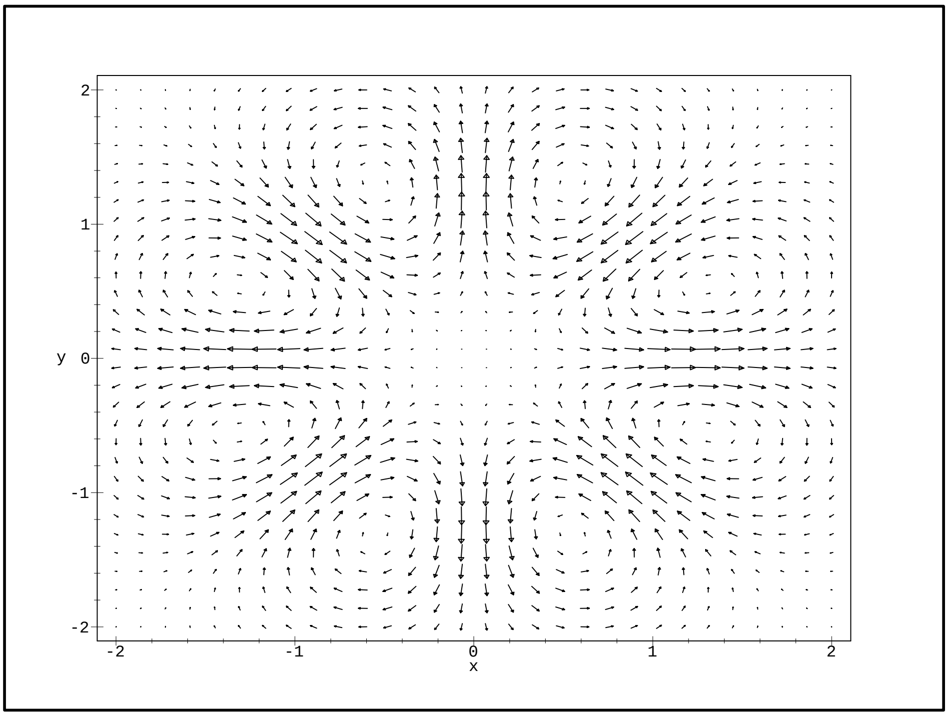

FIGURE 1. The field $a(x_1, x_2)$ corresponding to the choice $\theta(\rho) = \exp(-\rho)$.

Note that the examples (28) that we gave before for $n \geq 3$ are obtained by simply choosing $\tilde{\Omega}_{1,2}(x) = x_1 x_2 \theta(|x|^2)$, with $\theta \in \mathcal{S}(\mathbb{R})$ and $\tilde{\Omega}_{1,3} = \ldots = \tilde{\Omega}_{1,[n/2]} = 0$. Here φ and θ are relied by $\varphi = -2\theta'$.

In the case $n = 2$, $\tilde{\Omega}(x_1, x_2)$ is completely determined by $\tilde{\Omega}_{1,2}(x_1, x_2)$. This function must be odd with respect to x_1 and x_2 and such that $\tilde{\Omega}_{1,2}(x_1, x_2) = -\tilde{\Omega}_{1,2}(x_2, x_1)$. The simplest example is $\tilde{\Omega}_{1,2}(x_1, x_2) = x_1 x_2 (x_1^2 - x_2^2)\theta(|x|^2)$, with $\theta \in \mathcal{S}(\mathbb{R})$. Then $a_h = \sum_{k=1}^{2} \partial_k \tilde{\Omega}_{hk}$ yields

$$a(x_1, x_2) = \begin{pmatrix} (x_1^3 - 3x_1 x_2^2)\theta(|x|^2) + 2(x_1^3 x_2^2 - x_1 x_2^4)\theta'(|x|^2) \\ (x_2^3 - 3x_1^2 x_2)\theta(|x|^2) + 2(x_1^2 x_2^3 - x_1^4 x_2)\theta'(|x|^2) \end{pmatrix} \tag{33}$$

(see Fig. 1). Note that $\mathbb{P}\nabla \cdot (a \otimes a)$ does not vanish identically. Hence the corresponding solution $u(x,t)$ of the Navier-Stokes equations is not a trivial one (*i.e.* it is not a solution of the heat equation).

5. CONCLUSIONS

We showed that most of the localized initial data lead to solutions of the Navier–Stokes equations which instantaneously spread-out. In other terms the generic solutions of the Navier–Stokes equations turn out to have a poor spatial localization. The spreading effect holds, in particular, whenever the initial data have non-orthogonal components with respect to the $\mathrm{L}^2(\mathbb{R}^n)$ inner product.

However, we constructed some exceptional solutions on the whole space $\mathbb{R}^n$ $(n \geq 2)$ which decay at infinity with a faster decay than in the generic case. Some special symmetries of the data guarantee this unexpected spatial behavior. Another point of interest of such "symmetric solutions" is that the non-linear term of the Navier–Stokes equations $\mathbb{P}\nabla\cdot(u\otimes u)$ does not vanish identically. Hence these exceptional solutions are non-trivial in the following sense: they do not solve the linear heat system starting with the same initial data.

In this paper we dealt only with local strong solutions. In an independent paper we will treat the case of global weak (or strong) solutions. Starting from a symmetric data allows then to obtain a decay of the energy $||u(t)||_2^2$ *faster than expected*, when $t \to \infty$.

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