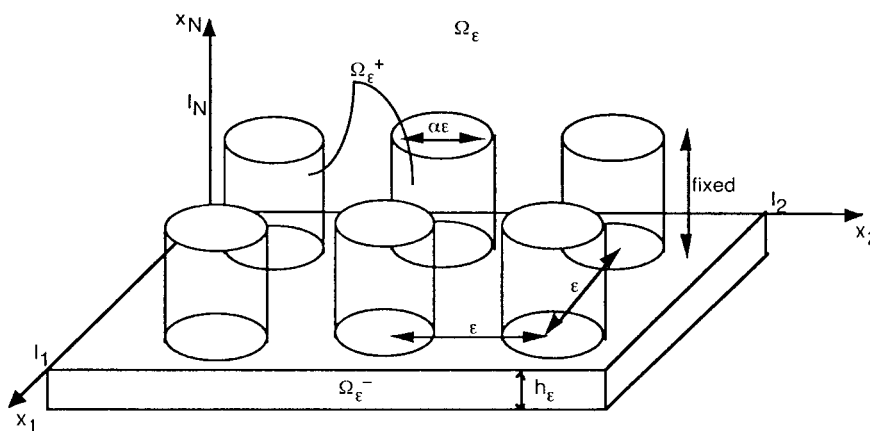


HOMOGENIZATION OF HIGHLY OSCILLATING BOUNDARIES AND REDUCTION OF DIMENSION FOR A MONOTONE PROBLEM

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Abstract. We investigate the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of a class of monotone nonlinear Neumann problems, with growth $p - 1$ ($p \in]1, +\infty[$), on a bounded multidomain $\Omega_\varepsilon \subset \mathbb{R}^N$ ($N \geq 2$). The multidomain Ω_ε is composed of two domains. The first one is a plate which becomes asymptotically flat, with thickness h_ε in the x_N direction, as $\varepsilon \rightarrow 0$. The second one is a “forest” of cylinders distributed with ε -periodicity in the first $N - 1$ directions on the upper side of the plate. Each cylinder has a small cross section of size ε and fixed height (for the case $N = 3$, see the figure). We identify the limit problem, under the assumption: $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^p}{h_\varepsilon} = 0$. After rescaling the equation, with respect to h_ε , on the plate, we prove that, in the limit domain corresponding to the “forest” of cylinders, the limit problem identifies with a diffusion operator with respect to x_N , coupled with an algebraic system. Moreover, the limit solution is independent of x_N in the rescaled plate and meets a Dirichlet transmission condition between the limit domain of the “forest” of cylinders and the upper boundary of the plate.



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1. MOTIVATION AND MAIN RESULT

Let $N \geq 2$, $l_1, \dots, l_{N-1}, l_N \in]0, +\infty[$, ω be an open smooth subset of \mathbb{R}^{N-1} such that $\omega \subset\subset]0, 1[^{N-1}$, and let $\{\varepsilon\}, \{h_\varepsilon\}_\varepsilon$ be two sequences of strictly positive numbers converging to 0. For every ε , let us consider the multidomain in \mathbb{R}^N (for the case $N = 3$, see the figure):

$$\Omega_\varepsilon = \Omega_\varepsilon^- \cup \Omega_\varepsilon^+,$$

where Ω_ε^- is a plate with constant cross section and small height h_ε :

$$\Omega_\varepsilon^- =]0, l_1[\times \dots \times]0, l_{N-1}[\times]-h_\varepsilon, 0[,$$

and Ω_ε^+ is a “forest” of cylinders with small cross section $\varepsilon\omega$, constant height l_N and distributed with ε -periodicity in the first $N - 1$ directions x_1, \dots, x_{N-1} on the upper side of Ω_ε^- :

$$\Omega_\varepsilon^+ = \bigcup_{k \in J_\varepsilon} (\varepsilon\omega + \varepsilon k) \times]0, l_N[,$$

where

$$J_\varepsilon = \{k \in \mathbb{N}^{N-1} : \varepsilon\omega + \varepsilon k \subset\subset]0, l_1[\times \dots \times]0, l_{N-1}[\}. \tag{1.1}$$

Let us observe that the volume of Ω_ε^+ does not vanish as $\varepsilon \rightarrow 0$, precisely:

$$\chi_{\Omega_\varepsilon^+} \rightharpoonup |\omega| \quad \text{in } L^\infty(\Omega^+) \text{ weak } *, \tag{1.2}$$

where $\chi_{\Omega_\varepsilon^+}$ denotes the characteristic function of Ω_ε^+ , $|\omega|$ denotes the $(N - 1)$ -dimensional Lebesgue measure of ω and Ω^+ is the “smallest” parallelepiped containing the interior of Ω_ε^+ for every ε :

$$\Omega^+ =]0, l_1[\times \dots \times]0, l_{N-1}[\times]0, l_N[.$$

This paper arises from the interest of studying the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the following Neumann problem:

$$\begin{cases} -\operatorname{div}(a(DU_\varepsilon)) + |U_\varepsilon|^{p-2}U_\varepsilon = F_\varepsilon & \text{in } \Omega_\varepsilon, \\ a(DU_\varepsilon) \cdot \nu = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.3}$$

where ν denotes the exterior unit normal to Ω_ε , $p \in]1, +\infty[$, $F_\varepsilon \in L^{\frac{p}{p-1}}(\Omega^+ \cup \Omega_\varepsilon^-)$ and

$$a = (a_1, \dots, a_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N \text{ is a monotone continuous function} \tag{1.4}$$

satisfying the usual growth conditions:

$$\exists \alpha \in]0, +\infty[: \alpha|\xi|^p \leq a(\xi)\xi \quad \forall \xi \in \mathbb{R}^N, \tag{1.5}$$

$$\exists \beta, \gamma \in]0, +\infty[: |a(\xi)| \leq \beta + \gamma|\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^N. \tag{1.6}$$

It is well known (see [18]) that problem (1.3) admits a unique weak solution $U_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$. To study the asymptotic behaviour of $\{U_\varepsilon\}_\varepsilon$, as $\varepsilon \rightarrow 0$, we introduce the classical transformation mapping Ω_ε^- onto the fixed domain

$$\Omega^- =]0, l_1[\times \dots \times]0, l_{N-1}[\times]-1, 0[$$

(compare, for instance [9, 14, 16] and [17]) and we set, for every ε ,

$$u_\varepsilon(x) = \begin{cases} U_\varepsilon(x) & x \text{ a.e. in } \Omega_\varepsilon^+, \\ U_\varepsilon(x', h_\varepsilon x_N) & x = (x', x_N) \text{ a.e. in } \Omega^-. \end{cases}$$

Obviously, u_ε is the unique solution of the following problem:

$$\begin{cases} \int_{\Omega_\varepsilon^+} a(Du_\varepsilon)Dv + |u_\varepsilon|^{p-2}u_\varepsilon v \, dx \\ + h_\varepsilon \int_{\Omega^-} a\left(D_{x'}u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N}\right) \left(D_{x'}v, \frac{1}{h_\varepsilon} \frac{\partial v}{\partial x_N}\right) + |u_\varepsilon|^{p-2}u_\varepsilon v \, dx \\ = \int_{\Omega_\varepsilon^+} f_\varepsilon v \, dx + h_\varepsilon \int_{\Omega^-} f_\varepsilon v \, dx \quad \forall v \in W^{1,p}(\Omega_\varepsilon^+ \cup \Omega^-), \\ u_\varepsilon \in W^{1,p}(\Omega_\varepsilon^+ \cup \Omega^-), \end{cases} \tag{1.7}$$

where $x = (x_1, \dots, x_{N-1}, x_N) = (x', x_N) \in \mathbb{R}^N$, $D_{x'} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{N-1}})$ and

$$f_\varepsilon(x) = \begin{cases} F_\varepsilon(x) & x \text{ a.e. in } \Omega^+, \\ F_\varepsilon(x', h_\varepsilon x_N) & x = (x', x_N) \text{ a.e. in } \Omega^-. \end{cases} \tag{1.8}$$

Then, we study the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of problem (1.7), under the following supplementary assumptions:

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^p}{h_\varepsilon} = 0 \tag{1.9}$$

$$\begin{cases} f_\varepsilon \rightarrow f \text{ strongly in } L^{\frac{p}{p-1}}(\Omega^+), \\ \exists c \in]0, +\infty[: \|f_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega^-)} \leq c, \quad \forall \varepsilon. \end{cases} \tag{1.10}$$

In terms of the sequence $\{F_\varepsilon\}_\varepsilon$, (1.10) means that $\{F_\varepsilon\}_\varepsilon$ converges strongly in $L^{\frac{p}{p-1}}(\Omega^+)$ and $\|F_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega^-)} \leq ch_\varepsilon^{\frac{p-1}{p}}$.

To describe the limit problem, we introduce the space

$$V^p(\Omega^+) = \left\{ v \in L^p(\Omega^+) : \frac{\partial v}{\partial x_N} \in L^p(\Omega^+) \right\}, \tag{1.11}$$

and we recall that functions of $V^p(\Omega^+)$ admit a trace on Σ :

$$\Sigma =]0, l_1[\times \dots \times]0, l_{N-1}[\times \{0\}.$$

In the sequel, \tilde{v} or $[v]^\sim$ denotes the zero-extension to Ω^+ of any (vector) function v defined on a subset of Ω^+ .

The main result of this paper is the following one:

Theorem 1.1. *Let u_ε be the unique solution of problem (1.7) under assumptions (1.4–1.6, 1.9, 1.10). Let $|\omega|$ be the constant given by (1.2) and let $V^p(\Omega^+)$ be the space defined in (1.11).*

Then, there exists $u \in V^p(\Omega^+)$ such that

$$\tilde{u}_\varepsilon \rightharpoonup |\omega|u, \quad \frac{\partial \tilde{u}_\varepsilon}{\partial x_N} \rightharpoonup |\omega| \frac{\partial u}{\partial x_N} \quad \text{weakly in } L^p(\Omega^+), \tag{1.12}$$

$$u_\varepsilon \rightharpoonup u|_\Sigma \quad \text{weakly in } L^p(\Omega^-), \tag{1.13}$$

$$\frac{\partial u_\varepsilon}{\partial x_N} \rightarrow 0 \quad \text{strongly in } L^p(\Omega^-), \tag{1.14}$$

as $\varepsilon \rightarrow 0$, where $u_{|\Sigma}$ denotes the function independent of x_N which is equal, on Σ , to the trace of u . Moreover, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $(d_1, \dots, d_{N-1}) \in (L^p(\Omega^+))^{N-1}$, depending possibly on the selected subsequence, such that

$$\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} \rightharpoonup d_i \quad \text{weakly in } L^p(\Omega^+) \quad \forall i \in \{1, \dots, N-1\}, \tag{1.15}$$

as $\varepsilon \rightarrow 0$, and (u, d_1, \dots, d_{N-1}) is a weak solution of the following problem:

$$\begin{cases} -\frac{\partial}{\partial x_N} a_N \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{N-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) + |u|^{p-2} u = f & \text{in } \Omega^+, \\ a_N \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{N-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) = 0 & \text{on the lower and upper boundary of } \Omega^+, \\ a_i \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{N-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) = 0 & \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \dots, N-1\}. \end{cases} \tag{1.16}$$

The function $u \in V^p(\Omega^+)$ satisfying problem (1.16) is unique. Furthermore, the energies converge in the sense that:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega_\varepsilon^+} a(Du_\varepsilon) Du_\varepsilon + |u_\varepsilon|^p \, dx + h_\varepsilon \int_{\Omega^-} a \left(D_{x'} u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) \left(D_{x'} u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) + |u_\varepsilon|^p \, dx \right) \\ = |\omega| \int_{\Omega^+} a_N \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) \frac{\partial u}{\partial x_N} + |u|^p \, dx = |\omega| \int_{\Omega^+} f u \, dx. \end{aligned} \tag{1.17}$$

If a is strictly monotone, problem (1.16) admits a unique solution $(u, d_1, \dots, d_{N-1}) \in V^p(\Omega^+) \times (L^p(\Omega^+))^{N-1}$ and, consequently, convergence (1.15) holds true for the whole sequence $\{u_\varepsilon\}_\varepsilon$.

We point out that the limit problem, in the limit domain corresponding to the “forest” of cylinders, identifies with a diffusion operator with respect to x_N coupled with an algebraic system for the limit fluxes. In particular, if $a(\xi) = |\xi|^{p-2} \xi$, with $p \in [2, +\infty[$, it results $d_1 = \dots = d_{N-1} = 0$ a.e. in Ω^+ .

As far as the asymptotic behaviour, as $\varepsilon \rightarrow 0$, of the solution U_ε of problem (1.3) is concerned, Theorem 1.1 leads immediately to the following result ($|\Omega_\varepsilon^-|$ and $|\Sigma|$ denote the N -dimensional Lebesgue measure of Ω_ε^- and the $(N-1)$ -dimensional Lebesgue measure of Σ , respectively):

Corollary 1.2. *Let U_ε be the unique solution of problem (1.3). Then, under the assumptions of Theorem 1.1 with f_ε defined by (1.8), it results*

$$\begin{aligned} \widetilde{U}_\varepsilon \rightharpoonup |\omega| u, \quad \frac{\partial \widetilde{U}_\varepsilon}{\partial x_N} \rightharpoonup |\omega| \frac{\partial u}{\partial x_N} \quad \text{weakly in } L^p(\Omega^+), \\ \frac{\partial \widetilde{U}_\varepsilon}{\partial x_i} \rightharpoonup d_i \quad \text{weakly in } L^p(\Omega^+) \quad \forall i \in \{1, \dots, N-1\}, \text{ up to a subsequence,} \\ \lim_\varepsilon \int_{\Omega_\varepsilon^-} |U_\varepsilon|^p \, dx = 0, \\ \lim_\varepsilon \frac{1}{|\Omega_\varepsilon^-|} \int_{\Omega_\varepsilon^-} U_\varepsilon \, dx = \frac{1}{|\Sigma|} \int_\Sigma u \, dx', \end{aligned} \tag{1.18}$$

as $\varepsilon \rightarrow 0$, and $(u, d_1, \dots, d_{N-1}) \in V^p(\Omega^+) \times (L^p(\Omega^+))^{N-1}$ is a weak solution of problem (1.16). Moreover, the energies converge in the sense that:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} a(DU_\varepsilon) DU_\varepsilon + |U_\varepsilon|^p \, dx = |\omega| \int_{\Omega^+} a_N \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) \frac{\partial u}{\partial x_N} + |u|^p \, dx.$$

Remark 1.3. We point out that we need assumption (1.9) only to derive convergence (1.13), and consequently (1.18). The remaining part of Theorem 1.1 and of Corollary 1.2 holds true without assumption (1.9). In particular, the limit problem in Ω^+ can be obtained independently of the rate of convergence to zero of h_ε with respect to ε . Assumption (1.9) is needed to describe the behaviour of the sequence $\{u_\varepsilon\}_\varepsilon$ in Ω^- .

Remark 1.4. We point out that, under assumption (1.9), the results of Theorem 1.1 and Corollary 1.2 do not change if we replace assumption (1.10) by the weaker one

$$\begin{cases} f_\varepsilon \rightarrow f \text{ strongly in } L^{\frac{p}{p-1}}(\Omega^+), \\ \exists c \in]0, +\infty[: \|h_\varepsilon^{\frac{p-1}{p}} f_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega^-)} \leq c, \quad \forall \varepsilon, \end{cases}$$

that is $\{F_\varepsilon\}_\varepsilon$ converges strongly in $L^{\frac{p}{p-1}}(\Omega^+)$ and $\|F_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega^-)} \leq c$.

Also in the case where the rate of convergence of h_ε is ε^p , under assumption (1.19), we obtain in Ω^+ the same limit problem as in Theorem 1.1.

If $h_\varepsilon = 1$, the asymptotic behaviour of problem (1.3) (or (1.7)) has been studied by Blanchard *et al.* in [5]. Let us also recall a few references in the case where $h_\varepsilon = 1$. For the linear problem, see [6] for homogeneous Neumann boundary condition, [12] for a non-homogeneous Neumann boundary condition, [13] for a transmission boundary condition, [20] for a spectral problem, and [3] and [19] for the construction of boundary layer correctors. Moreover, see [11] in the case of functionals with convex energies and [4] in the case of functionals with non-convex energies. Furthermore, see [15] for the Ginzburg–Landau equation with homogeneous Neumann boundary condition.

The asymptotic behaviour of the solution of problem (1.7) in Ω^+ is performed, with similar techniques than in [5], by making use of the method of oscillating test functions, introduced by Tartar in [22], combined with monotonicity arguments and density results (this is the object of Sect. 4). The originality of our paper consists in the study of the asymptotic behaviour of the solution of problem (1.7) in Ω^- and in the characterization of the limit in Ω^- in terms of the limit in Ω^+ (see (1.13)). The first task is to obtain *a priori* $L^p(\Omega^-)$ -norm estimate for the solution u_ε , independently of ε . These estimates are not obvious since directly from equation (1.7) it follows only that $\|h_\varepsilon^{\frac{1}{p}} u_\varepsilon\|_{L^p(\Omega^-)} \leq c$. Via trace arguments (Friedrichs inequality), in Proposition 3.3 we are able to prove that $\|u_\varepsilon\|_{L^p(\Omega^-)} \leq c$. Consequently, by passing to a subsequence, $\{u_\varepsilon\}_\varepsilon$ converges to a function w weakly in $L^p(\Omega^-)$ and moreover, since $\|\frac{\partial u_\varepsilon}{\partial x_N}\|_{L^p(\Omega^-)} \leq ch_\varepsilon^{\frac{p-1}{p}}$, it follows that w is independent of the last variable x_N . The second task is to identify w as the trace on Σ (the surface which separates Ω^- and Ω^+) of the solution u of the limit problem in Ω^+ (see (1.13)). To this aim, we have to pass to the limit in the product of weak convergences: $\widetilde{u}_\varepsilon = u_\varepsilon \chi_{\Omega_\varepsilon^+ \cap \Sigma}$ on Σ . In Proposition 3.4 we are able to solve this problem by making use of the two-scale convergence method introduced by Nguetseng in [21] and developed by Allaire in [1]. In Section 2 some preliminary results are recalled.

If the Neumann boundary condition in problem (1.3) is replaced by the homogeneous Dirichlet condition $U_\varepsilon = 0$ on $\partial\Omega_\varepsilon$, it becomes an easier task to prove that $\widetilde{u}_\varepsilon \rightharpoonup 0$ weakly in $W^{1,p}(\Omega^+)$, $u_\varepsilon \rightarrow 0$ strongly in $L^p(\Omega^-)$ (compare [3] and [6]). As far as this Dirichlet problem is concerned, the lower order term $|U_\varepsilon|^{p-2}U_\varepsilon$ may be removed in the whole analysis.

For the study of very rapidly oscillating boundaries we refer to [2, 4, 7] and [8]. For the junction of a plate with a beam we refer to [14, 16] and [17].

2. PRELIMINARY RESULTS

In this section, the main properties of the two-scale convergence method introduced by Nguetseng in [21] and developed by Allaire in [1] are recalled. Here, $Y = [0, 1]^N$ ($N \geq 1$), $1 < p < \infty$, $C_{\text{per}}^\infty(Y)$ denotes the space of infinitely differentiable functions in \mathbb{R}^N that are periodic of period Y and $W_{\text{per}}^{1,p}(Y)$ is the completion of $C_{\text{per}}^\infty(Y)$ for the norm of $W^{1,p}(Y)$.

Definition 2.1. Let A be an open subset of \mathbb{R}^N . A sequence $\{v_\varepsilon\}_\varepsilon \subset L^p(A)$ is said to “two-scale converge” to a limit $v \in L^p(A \times Y)$ if, for any function ψ in $D(A, C^\infty_{\text{per}}(Y))$, it results

$$\lim_{\varepsilon \rightarrow 0} \int_A v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{A \times Y} v(x, y) \psi(x, y) dx dy. \tag{2.1}$$

See [1] for the proof of the following result:

Proposition 2.2. *Let A be an open subset of \mathbb{R}^N .*

- i) Let $\{v_\varepsilon\}_\varepsilon \subset L^p(A)$ be a sequence converging to v strongly in $L^p(A)$. Then, $\{v_\varepsilon\}_\varepsilon$ two-scale converges to the same limit v .*
- ii) Let $\{v_\varepsilon\}_\varepsilon \subset L^p(A)$ be a sequence two-scale converging to $v \in L^p(A \times Y)$. Then, $\{v_\varepsilon\}$ converges to $\int_Y v(\cdot, y) dy$ weakly in $L^p(A)$.*
- iii) Let $\{v_\varepsilon\}_\varepsilon$ be a bounded sequence in $L^p(A)$. Then, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and a function $v \in L^p(A \times Y)$ such that $\{v_\varepsilon\}_\varepsilon$ two-scale converges to v .*
- iv) Let $\{v_\varepsilon\}_\varepsilon \subset W^{1,p}(A)$ be a sequence such that $\{v_\varepsilon\}_\varepsilon$ and $\{\varepsilon Dv_\varepsilon\}_\varepsilon$ are bounded in $L^p(A)$ and $(L^p(A))^N$, respectively. Then, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and a function $v \in L^p(A, W^{1,p}_{\text{per}}(Y))$ such that $\{v_\varepsilon\}_\varepsilon$ and $\{\varepsilon Dv_\varepsilon\}_\varepsilon$ two-scale converge to v and $D_y v$, respectively.*

Remark 2.3. Due to density properties, it is easily seen that if $\{v_\varepsilon\}_\varepsilon \subset L^p(A)$ two-scale converges to $v \in L^p(A \times Y)$, convergence (2.1) holds true also for any function ψ of the form $\psi(x, y) = \varphi_1(x)\varphi_2(y)$, where $\varphi_1 \in C_0(A)$ and $\varphi_2 \in L^\infty_{\text{per}}(Y)$.

This section concludes with the following well-known Friedrichs inequality which will be an helpful tool to obtain *a priori* estimates for the solution of problem (1.7).

Proposition 2.4. *Let A be an open bounded connected subset of \mathbb{R}^N with Lipschitz boundary and let $\Gamma \subset \partial A$ be such that the $(N - 1)$ -dimensional Hausdorff measure of Γ is positive. Then, there exists a constant c such that*

$$\int_A |v|^p dx \leq c \left(\int_\Gamma |v|^p d\sigma + \int_A |Dv|^p dx \right) \quad \forall v \in W^{1,p}(A). \tag{2.2}$$

3. A PRIORI NORM-ESTIMATES AND A CONVERGENCE RESULT

This section is devoted to prove convergences (1.12, 1.13) and (1.14). In Proposition 3.1, some *a priori* norm-estimates for u_ε , coming directly from equation (1.7), are given. The first difficulty is to obtain *a priori* $L^p(\Omega^-)$ -norm estimate for the solution u_ε , independently of ε . These estimates are obtained in Proposition 3.3 by making use of the Friedrichs inequality. The second difficulty is to identify the weak limit of $\{u_\varepsilon\}_\varepsilon$ in $L^p(\Omega^-)$ as the trace on Σ (the surface which separates Ω^- and Ω^+) of the weak limit of $\{\widetilde{u}_\varepsilon\}_\varepsilon$ in $V^p(\Omega^+)$. This result is proved in Proposition 3.4 by making use of the two-scale convergence method.

Proposition 3.1. *Let u_ε be the unique solution of problem (1.7) under assumptions (1.4–1.6) and (1.10). Then, there exists a constant c such that*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon^+)} \leq c, \tag{3.1}$$

$$\|a(Du_\varepsilon)\|_{\left(L^{\frac{p}{p-1}}(\Omega_\varepsilon^+)\right)^N} \leq c, \tag{3.2}$$

$$\left\| h_\varepsilon^{\frac{1}{p}} u_\varepsilon \right\|_{L^p(\Omega^-)} \leq c, \tag{3.3}$$

$$\left\| h_\varepsilon^{\frac{1}{p}} \left(D_{x'} u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) \right\|_{(L^p(\Omega^-))^N} \leq c, \tag{3.4}$$

$$\left\| h_\varepsilon^{\frac{p-1}{p}} a \left(D_{x'} u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) \right\|_{\left(L^{\frac{p}{p-1}}(\Omega^-)\right)^N} \leq c, \tag{3.5}$$

for every ε .

Proof. The *a priori* norm-estimates are obtained by choosing $v = u_\varepsilon$ as test function in (1.7) and by making use of (1.5, 1.6, 1.10) and the Young inequality. □

Corollary 3.2. *Let u_ε be the unique solution of problem (1.7) under assumptions (1.4–1.6) and (1.10). Then, there exists a constant c such that*

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^+ \cap \Sigma)} \leq c, \tag{3.6}$$

for every ε .

Proof. The estimate (3.6) follows from estimate (3.1), if one observes that there exists a constant c such that

$$\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^+ \cap \Sigma)}^p \leq c \left(\|u_\varepsilon\|_{L^p(\Omega_\varepsilon^+)}^p + \left\| \frac{\partial u_\varepsilon}{\partial x_N} \right\|_{L^p(\Omega_\varepsilon^+)}^p \right),$$

for every ε . □

Proposition 3.3. *Let u_ε be the unique solution of problem (1.7) under assumptions (1.4–1.6, 1.9) and (1.10). Then, there exists a constant c such that*

$$\|u_\varepsilon\|_{L^p(\Omega^-)} \leq c, \tag{3.7}$$

for every ε .

Proof. For the sake of clarity, first we prove (3.7) by assuming

$$\Omega^- = \bigcup_{k \in J_\varepsilon} (\varepsilon]0, 1^{[N-1+\varepsilon k)} \times]-1, 0[$$

for every ε , where J_ε is defined in (1.1). Then, we sketch the proof of (3.7) for the general case.

By making use of the change of variable $x' = \varepsilon y' + \varepsilon k$ with $y' \in]0, 1^{[N-1}$, it results

$$\int_{\Omega^-} |u_\varepsilon|^p dx = \sum_{k \in J_\varepsilon} \int_{(\varepsilon]0, 1^{[N-1+\varepsilon k)} \times]-1, 0[} |u_\varepsilon|^p d(x', x_N) = \varepsilon^{N-1} \sum_{k \in J_\varepsilon} \int_{]0, 1^{[N-1} \times]-1, 0[} |u_\varepsilon(\varepsilon y' + \varepsilon k, x_N)|^p d(y', x_N).$$

Then, the Friedrichs inequality (2.2) yields

$$\begin{aligned} \int_{\Omega^-} |u_\varepsilon|^p dx &\leq c\varepsilon^{N-1} \sum_{k \in J_\varepsilon} \left(\int_{\omega} |u_\varepsilon(\varepsilon y' + \varepsilon k, 0)|^p dy' + \varepsilon^p \int_{]0, 1[^{N-1} \times]-1, 0[} |(D_{x'} u_\varepsilon)(\varepsilon y' + \varepsilon k, x_N)|^p d(y', x_N) \right. \\ &\quad \left. + \int_{]0, 1[^{N-1} \times]-1, 0[} \left| \frac{\partial u_\varepsilon}{\partial x_N}(\varepsilon y' + \varepsilon k, x_N) \right|^p d(y', x_N) \right) \\ &= c \left(\int_{\Omega_\varepsilon^+ \cap \Sigma} |u_\varepsilon|^p dx' + \varepsilon^p \int_{\Omega^-} |D_{x'} u_\varepsilon|^p dx + \int_{\Omega^-} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^p dx \right) \end{aligned} \tag{3.8}$$

for every ε , where the constant c , given in (2.2), is independent of ε and $u_\varepsilon(\cdot, 0)$ denotes the trace of u_ε on $\Omega_\varepsilon^+ \cap \Sigma$. Consequently, by making use of estimates (3.4) and (3.6), it follows that

$$\int_{\Omega^-} |u_\varepsilon|^p dx \leq c \left(1 + \frac{\varepsilon^p}{h_\varepsilon} + h_\varepsilon^{p-1} \right) \tag{3.9}$$

for every ε , where c is a constant independent of ε . Finally, estimate (3.7) is obtained using assumption (1.9).

Now, let us consider the general case. Let $J'_\varepsilon = \{k \in \mathbb{N}^{N-1} : (\varepsilon]0, 1[^{N-1} + \varepsilon k) \cap (]0, l_1[\times \dots \times]0, l_{N-1}[) \neq \emptyset\}$, $C_\varepsilon = \bigcup_{k \in J'_\varepsilon} (\varepsilon]0, 1[^{N-1} + \varepsilon k) \times]-1, 0[$, $B \subset \mathbb{R}^{N-1}$ be a bounded open set such that $\bigcup_{k \in J'_\varepsilon} \varepsilon]0, 1[^{N-1} + \varepsilon k \subset C$ for every ε and $C = B \times]-1, 0[$. It is easy to prove the existence of a linear extension-operator $Q \in L(W^{1,p}(\Omega^-), W^{1,p}(C))$ such that (compare, for instance [6] and [10])

$$\|D_{x'} Qv\|_{(L^p(C))^{N-1}} \leq c \|D_{x'} v\|_{(L^p(\Omega^-))^{N-1}}, \quad \left\| \frac{\partial Qv}{\partial x_N} \right\|_{L^p(C)} \leq c \left\| \frac{\partial v}{\partial x_N} \right\|_{L^p(\Omega^-)} \quad \forall v \in W^{1,p}(\Omega^-), \tag{3.10}$$

where c is a constant independent of v . In particular, we have, on one hand,

$$\int_{\Omega^-} |u_\varepsilon|^p dx \leq \int_{C_\varepsilon} |Qu_\varepsilon|^p dx, \tag{3.11}$$

for every ε . On the other hand, by an argument similar to that used in the proof of (3.8), one can prove that

$$\int_{C_\varepsilon} |Qu_\varepsilon|^p dx \leq c \left(\int_{\Omega_\varepsilon^+ \cap \Sigma} |u_\varepsilon|^p dx' + \varepsilon^p \int_{C_\varepsilon} |D_{x'} Qu_\varepsilon|^p dx + \int_{C_\varepsilon} \left| \frac{\partial Qu_\varepsilon}{\partial x_N} \right|^p dx \right), \tag{3.12}$$

for every ε , where c is a constant dependent on N , but independent of ε . Indeed, the proof of (3.8) would rather lead to $\int_{\bigcup_{k \in J'_\varepsilon} (\varepsilon\omega + \varepsilon k)} |Qu_\varepsilon|^p dx'$ as first term in the right hand-side of (3.12). To obtain the smaller quantity $\int_{\Omega_\varepsilon^+ \cap \Sigma} |u_\varepsilon|^p dx'$, one has to use the definition of C_ε . This definition allows, upon joining up a fixed number (depending only on N) of cells of C_ε , to control the L^p -norm of Qu_ε on the cells of C_ε crossed by the lateral boundary of Ω^- , only in terms of the L^p -norm of $\varepsilon D_{x'} Qu_\varepsilon$ and $\frac{\partial Qu_\varepsilon}{\partial x_N}$ on the same cells, plus the L^p -norm of the trace of u_ε on a part of the upper surface of cells of C_ε completely included in Ω^- .

By combining (3.11) and (3.12) with (3.10), it follows that

$$\int_{\Omega^-} |u_\varepsilon|^p dx \leq c \left(\int_{\Omega_\varepsilon^+ \cap \Sigma} |u_\varepsilon|^p dx' + \varepsilon^p \int_{\Omega^-} |D_{x'} u_\varepsilon|^p dx + \int_{\Omega^-} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^p dx \right),$$

for every ε , where c is a constant independent of ε . Finally, by making use of (3.4, 3.6) and (1.9), we obtain (3.7). □

Proposition 3.4. *Let u_ε be the unique solution of problem (1.7) under assumptions (1.4–1.6) and (1.10), let $|\omega|$ be the constant given by (1.2) and let $V^p(\Omega^+)$ be the space defined in (1.11). Then (1.14) holds true. Moreover, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $u \in V^p(\Omega^+)$ such that (1.12) holds true. Furthermore, for this subsequence, under assumption (1.9), convergence (1.13) holds true.*

Remark 3.5. In the spirit of Remark 1.3, let us point out that convergences (1.12) and (1.14) hold true without assumption (1.9).

Proof of Proposition 3.4. Convergence (1.14) follows from (3.4). By virtue of (3.1), there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $u \in V^p(\Omega^+)$ such that (1.12) holds true (see [5, 6] and [11]). Moreover, under assumption (1.9), estimate (3.7) holds true. Consequently, in view of (1.14), there exist a subsequence of the previous one, still denoted by $\{\varepsilon\}$, and $w \in L^p(\Omega^-)$, independent of x_N , such that

$$u_\varepsilon \rightharpoonup w \text{ weakly in } L^p(\Omega^-). \tag{3.13}$$

In order to prove (1.13), we show that, up to a subsequence,

$$\{u_\varepsilon\}_\varepsilon \text{ two-scale converges to } w \text{ in } \Omega^-, \tag{3.14}$$

too. To this aim, first observe that (3.4) and (1.9) provide that $\lim_{\varepsilon \rightarrow 0} \|\varepsilon Du_\varepsilon\|_{(L^p(\Omega^-))^N} = 0$. Consequently, by making use of Proposition 2.2-i, it follows that

$$\{\varepsilon Du_\varepsilon\}_\varepsilon \text{ two-scale converges to } (0, \dots, 0) \text{ in } \Omega^-. \tag{3.15}$$

By combining (3.7) with (3.15), by virtue of Proposition 2.2-iv, there exist a subsequence of the previous selected one, still denoted by $\{\varepsilon\}$, and $g \in L^p(\Omega^-)$ (independent of $y \in [0, 1]^N$!) such that

$$\{u_\varepsilon\}_\varepsilon \text{ two-scale converges to } g \text{ in } \Omega^-. \tag{3.16}$$

Finally, since g is independent of y , (3.14) follows by comparing (3.13) with (3.16) and by making use of Proposition 2.2-ii.

Let us point out that the two-scale convergence allows us to pass to the limit in the product of weak convergences. In fact, by setting $\psi_\varepsilon(x) = \chi_{\Omega_\varepsilon^+ \cap \Sigma}(x')$ for $x = (x', x_N) \in \Omega^-$, where $\chi_{\Omega_\varepsilon^+ \cap \Sigma}$ denotes the characteristic function of $\Omega_\varepsilon^+ \cap \Sigma$ in Σ , convergence (3.14) provides that (see Def. 2.1 and Rem. 2.3)

$$\lim_\varepsilon \int_{\Omega^-} u_\varepsilon \psi_\varepsilon \varphi dx = |\omega| \int_{\Omega^-} w \varphi dx \quad \forall \varphi \in C_0(\Omega^-),$$

from which it follows that, since $\{u_\varepsilon \psi_\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega^-)$,

$$u_\varepsilon \psi_\varepsilon \rightharpoonup |\omega|w \text{ weakly in } L^p(\Omega^-). \tag{3.17}$$

Moreover, equation (1.14) provides that

$$\lim_\varepsilon \left\| \frac{\partial(u_\varepsilon \psi_\varepsilon)}{\partial x_N} \right\|_{L^p(\Omega^-)} = \lim_\varepsilon \left\| \psi_\varepsilon \frac{\partial u_\varepsilon}{\partial x_N} \right\|_{L^p(\Omega^-)} \leq \lim_\varepsilon \left\| \frac{\partial u_\varepsilon}{\partial x_N} \right\|_{L^p(\Omega^-)} = 0. \tag{3.18}$$

Then, by making use of (3.17) and (3.18), and by recalling that w is independent of x_N , it results that

$$\begin{aligned} \lim_\varepsilon \int_\Sigma \widetilde{u}_\varepsilon \varphi dx' &= \lim_\varepsilon \left(\int_{\Omega^-} \frac{\partial(u_\varepsilon \psi_\varepsilon)}{\partial x_N} \varphi dx + \int_{\Omega^-} u_\varepsilon \psi_\varepsilon \frac{\partial \varphi}{\partial x_N} dx \right) \\ &= |\omega| \int_{\Omega^-} w \frac{\partial \varphi}{\partial x_N} dx = |\omega| \int_\Sigma w \varphi dx' \quad \forall \varphi \in C_0^\infty(\Omega^+ \cup \Sigma \cup \Omega^-). \end{aligned} \tag{3.19}$$

On the other hand, from (1.12) it follows that

$$\begin{aligned} \lim_{\varepsilon} \int_{\Sigma} \widetilde{u}_{\varepsilon} \varphi dx' &= - \lim_{\varepsilon} \left(\int_{\Omega^+} \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_N} \varphi dx + \int_{\Omega^+} \widetilde{u}_{\varepsilon} \frac{\partial \varphi}{\partial x_N} dx \right) \\ &= -|\omega| \int_{\Omega^+} \frac{\partial u}{\partial x_N} \varphi dx - |\omega| \int_{\Omega^+} u \frac{\partial \varphi}{\partial x_N} dx = |\omega| \int_{\Sigma} u \varphi dx' \quad \forall \varphi \in C_0^{\infty}(\Omega^+ \cup \Sigma \cup \Omega^-). \end{aligned} \tag{3.20}$$

By comparing (3.19) with (3.20), we obtain

$$\int_{\Sigma} w \varphi dx' = \int_{\Sigma} u \varphi dx' \quad \forall \varphi \in C_0^{\infty}(\Sigma).$$

Then, w is the function independent of x_N which is equal, on Σ , to the trace of u . Finally, convergence (1.13) follows from (3.13). □

Remark 3.6. In order to obtain (3.7), and consequently (3.13), we may only assume $\left\{ \frac{\varepsilon^p}{h_{\varepsilon}} \right\}_{\varepsilon}$ bounded (see (3.9)), but in the proof of (3.15) we need $\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^p}{h_{\varepsilon}} = 0$.

4. PROOF OF THEOREM 1.1

We sketch the proofs of (1.12, 1.15, 1.16) and (1.17) which are similar (except for the last one) to the corresponding ones in Theorem 1.2 in [5].

By virtue of (3.1, 3.2) and Proposition 3.4, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $u \in V^p(\Omega^+)$, $d = (d_1, \dots, d_{N-1}) \in (L^p(\Omega^+))^{N-1}$, $z \in L^{\frac{p}{p-1}}(\Omega^+)$ and $\eta = (\eta_1, \dots, \eta_N) \in \left(L^{\frac{p}{p-1}}(\Omega^+) \right)^N$ satisfying (1.12, 1.15) and

$$|\widetilde{u}_{\varepsilon}|^{p-2} \widetilde{u}_{\varepsilon} \rightharpoonup z \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^+), \quad [a(Du_{\varepsilon})] \rightharpoonup \eta \text{ weakly in } \left(L^{\frac{p}{p-1}}(\Omega^+) \right)^N. \tag{4.1}$$

By arguing as in [5] (p. 1064), it results that

$$\eta_i = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \dots, N-1\}. \tag{4.2}$$

Now, let us prove the convergence of the energies. For any $v \in C^1(\overline{\Omega^+})$, let us pass to the limit in (1.7), as $\varepsilon \rightarrow 0$, with the test function:

$$w(x) = \begin{cases} v(x) & \text{if } x \in \Omega^+ \\ v(x', 0) & \text{if } x \in \Omega^- \end{cases} \in W^{1,p}(\Omega^+ \cup \Omega^-).$$

Then, by virtue of (1.2, 1.10, 3.3, 3.5, 4.1) and (4.2), one has that

$$\int_{\Omega^+} \eta_N \frac{\partial v}{\partial x_N} + z v dx = |\omega| \int_{\Omega^+} f v dx \quad \forall v \in C^1(\overline{\Omega^+}),$$

i.e., by density argument (compare Prop. 4.1 in [11]),

$$\int_{\Omega^+} \eta_N \frac{\partial v}{\partial x_N} + z v dx = |\omega| \int_{\Omega^+} f v dx \quad \forall v \in V(\Omega^+). \tag{4.3}$$

In particular, (4.3) holds true for $v = u$. Consequently, by choosing $v = u_\varepsilon$ as test function in (1.7), by virtue of (1.10, 1.12) and (3.3) one obtain the convergence of the energies:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega_\varepsilon^+} a(Du_\varepsilon)Du_\varepsilon + |u_\varepsilon|^p dx + h_\varepsilon \int_{\Omega^-} a \left(D_{x'}u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) \left(D_{x'}u_\varepsilon, \frac{1}{h_\varepsilon} \frac{\partial u_\varepsilon}{\partial x_N} \right) + |u_\varepsilon|^p dx \right) \\ = |\omega| \int_{\Omega^+} f u dx = \int_{\Omega^+} \eta_N \frac{\partial u}{\partial x_N} + z u dx. \end{aligned} \tag{4.4}$$

As in [5] (p. 1065) (1.2, 1.12, 1.15, 4.1, 4.2) and (4.4) allow us to obtain the following monotone relation:

$$\begin{aligned} \frac{1}{t} \int_{\Omega^+} \eta_N \left(\frac{\partial u}{\partial x_N} - \tau_N \right) - a(\tau) \left(\left(d, |\omega| \frac{\partial u}{\partial x_N} \right) - |\omega| \tau \right) dx \\ + \frac{1}{t} \int_{\Omega^+} (z - |\omega| |v|^{p-2} v) (u - v) dx \geq 0 \quad \forall \tau \in (L^p(\Omega^+))^n, \quad \forall v \in L^p(\Omega^+), \quad \forall t \in]0, +\infty[, \end{aligned}$$

which will enable us to derive (see again in [5], pp. 1065-1068)

$$z = |\omega| |u|^{p-2} u, \quad \eta_N = |\omega| a_N \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{N-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) \quad \text{a.e. in } \Omega^+, \tag{4.5}$$

$$a_i \left(\frac{d_1}{|\omega|}, \dots, \frac{d_{N-1}}{|\omega|}, \frac{\partial u}{\partial x_N} \right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \dots, N-1\}. \tag{4.6}$$

By combining (4.3) with (4.5) and (4.6), it results that (u, d_1, \dots, d_{N-1}) is a weak solution of problem (1.16). Since this problem admits a unique solution u (see [5], pp. 1068-1069), convergences (1.12) holds true for the whole sequence $\{u_\varepsilon\}_\varepsilon$. The convergence of the energies (1.17) follows from (4.4) and (4.5). If a is strictly monotone, the uniqueness of the solution $(u, d_1, \dots, d_{N-1}) \in V^p(\Omega^+) \times (L^p(\Omega^+))^{N-1}$ of problem (1.16) is proved in [5] (pp. 1068-1069). Consequently, in this case, convergence (1.15) holds true for the whole sequence $\{u_\varepsilon\}_\varepsilon$.

Convergences (1.13) and (1.14) are proved in Proposition 3.4, under assumption (1.9) for the first one. \square

Remark 4.1. If in (1.3) we replace $-\text{div}(a(DU_\varepsilon))$ with $-\text{div}(a(x, DU_\varepsilon))$ in Ω_ε^+ and $-\text{div}(a(x', \frac{x_N}{h_\varepsilon}, DU_\varepsilon))$ in Ω_ε^- , where $a(x, \xi)$ is a Carathéodory function satisfying usual monotonicity, coercivity and growth conditions, it is evident that Theorem 1.1 and Corollary 1.2 hold true with a depending also on x . Similarly, the results obtained in the present paper are still valid if the local nonlinearity $|U_\varepsilon|^{p-2} U_\varepsilon$ is replaced by $b(x, U_\varepsilon)$ in Ω_ε^+ and $b(x', \frac{x_N}{h_\varepsilon}, U_\varepsilon)$ in Ω_ε^- , where $b(x, s)$ is a Carathéodory function, strictly monotone with respect to s and with $p-1$ growth at infinity.

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