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ON THE LOWER SEMICONTINUITY OF SUPREMAL FUNCTIONALS

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Abstract. In this paper we study the lower semicontinuity problem for a supremal functional of the form $F(u,\Omega)=\operatorname*{ess\,sup}_{x\in\Omega} f(x,u(x),Du(x))$ with respect to the strong convergence in $L^\infty(\Omega)$, furnishing a comparison with the analogous theory developed by Serrin for integrals. A sort of Mazur's lemma for gradients of uniformly converging sequences is proved.

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Introduction

Let Ω be an open set in $\mathbb{R}^n$ and let $f=f(x,s,\xi):\Omega\times\mathbb{R}^m\times\mathbb{R}^{mn}\to\mathbb{R}\cup\{\infty\}$ be a $\mathcal{L}^n(\Omega)\otimes\mathcal{B}^m\otimes\mathcal{B}^{mn}$ measurable function, where $\mathcal{L}^n(\Omega)$ denotes the Lebesgue measurable subsets of Ω and $\mathcal{B}^m$ denotes the Borel subsets of $\mathbb{R}^m$. If $f(x,\cdot,\cdot)$ is lower semicontinuous for almost every fixed x in Ω and if $u\in W^{1,\infty}(\Omega)^m$, then the composition $f(x,u(x),Du(x))$ is a measurable map, so that we can consider the functional

$$F(u,\Omega)=\operatorname*{ess\,sup}_{x\in\Omega} f(x,u(x),Du(x)).$$

F is called supremal, while f is referred to as the supremand generating F. The dependence of F on Ω sometimes is dropped.

In order to apply the direct methods of the Calculus of Variations, the study of lower semicontinuity properties of $F(\cdot,\Omega)$ with respect to a given convergence τ on $W^{1,\infty}(\Omega)^m$ is of interest, *i.e.*, we want to find out conditions on f sufficient in order to have that $u_h,u\in W^{1,\infty}(\Omega)^m$, $u_h\to u$ in τ, implies

$$\liminf_{h\to\infty} F(u_h,\Omega)\geq F(u,\Omega). \tag{1}$$

To understand the real nature of the problem, let us define for every $t\in\mathbb{R}$ and every $(x,s)\in\Omega\times\mathbb{R}^m$ the sublevel sets

$$E_t(x,s):=\{\xi\in\mathbb{R}^{mn}: f(x,s,\xi)\leq t\}\cdot$$

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Then we can note that (1) holds if and only if for every $u_h, u \in W^{1,\infty}(\Omega)^m$, with $u_h \to u$ in τ, and for every $t > \liminf_{h\to\infty} F(u_h, \Omega)$ we have that

$$Du_h(x) \in E_t(x, u_h(x)), \quad \text{for a.e.} \quad x \in \Omega,$$

and for infinitely many $h \in \mathbb{N}$, implies

$$Du(x) \in E_t(x, u(x)), \quad \text{for a.e.} \quad x \in \Omega.$$

This formulation clearly suggest the need to assume the convexity of the sets $E_t(x, s)$. This property is, by definition, the *level convexity* of $f(x, s, \cdot)$. In fact (see Barron and Liu [8]) level convexity turns out to be necessary to lower semicontinuity only in the scalar case ($m = 1$), while in the vectorial case more general conditions should be considered (see [7]).

As Barron *et al.* showed in [7], the lower semicontinuity problem for a supremal can always be reduced to a lower semicontinuity problem for an integral functional. With this strategy they proved the following theorem:

Theorem 0.1 (*Barron et al.* [7])**.** *Let us consider a $\mathcal{L}^n(\Omega) \otimes \mathcal{B}^m \otimes \mathcal{B}^{mn}$ measurable function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to [0, \infty]$, such that, for a.e. $x \in \Omega$, $f(x, \cdot, \cdot)$ is lower semicontinuous, and, for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, $f(x, s, \cdot)$ is level convex. Then the lower semicontinuity inequality (1) holds on every sequence such that*

$$\begin{cases} u_h,\, u \in W^{1,\infty}(\Omega)^m, \\ u_h \rightharpoonup u \quad w^* - W^{1,\infty}(\Omega)^{mn}. \end{cases} \tag{2}$$

Here we want to study the lower semicontinuity problem with respect to the strong convergence in L^∞, *i.e.*, on sequences such that

$$\begin{cases} u_h,\, u \in W^{1,\infty}(\Omega)^m, \\ u_h \to u \quad \text{in} \quad L^\infty(\Omega)^m. \end{cases} \tag{3}$$

The analogous problem for integral functionals was originated by the famous counterexample to lower semicontinuity by Aronszajn [20] and from the classical paper by Serrin [21], producing since then a great deal of works (see for example Ambrosio [2], Dal Maso [11], De Giorgi *et al.* [13], Fonseca and Leoni [15] and [16], Gori and Marcellini [18], Gori *et al.* [17], and their bibliographies). Let us recall with some details the terms of the problem. A reasonable set of hypotheses to put on an integrand $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to [0, \infty)$ in order to have the lower semicontinuity of the functional

$$G(u, \Omega) = \int_\Omega g(x, u(x), Du(x))\mathrm{d}x,$$

with respect to the convergence

$$\begin{cases} u_h,\, u \in W^{1,1}(\Omega)^m, \\ u_h \to u \quad \text{in} \quad L^1(\Omega)^m, \end{cases} \tag{4}$$

is, at least when $m = 1$,

$$\begin{cases} g(x, s, \xi) \quad \text{continuous}, \\ g(x, s, \cdot) \quad \text{convex}. \end{cases} \tag{5}$$

Aronszajn gave in [20] a g satisfying (5) for which the lower semicontinuity of G with respect to the convergence in (4) does not hold. Then one has to add some additional hypotheses to (5) in order to prove lower semicontinuity. Serrin's theorem and its extensions and variations cited above provide such analysis, mainly in the case $m = 1$.

In the present paper we prove that, as a consequence of our main result (Th. 1.4 below), in the case of supremals an example analogous to Aronszajn's one cannot be done. Indeed Corollary 1.5 states the continuity of the supremand f and level convexity in the gradient variable are sufficient to lower semicontinuity with respect to (3).

We come back to the presentation of our work. In the study of the lower semicontinuity properties of F with respect to convergence (3), the "integral reduction" approach by Barron *et al.* cited above, does not seem to be adequate. Indeed, the theory of lower semicontinuity for integrals with respect to this kind of convergence requires the integrand to be regular enough in the lower order variables and to be finite (see [21] and the other references we have quoted): but usually these assumptions are not satisfied by the integrands generated from generic supremals.

To overcome such difficulties we give in this paper a direct approach to the study of lower semicontinuity with respect to convergence (3), based on an appropriate weak version of Mazur's lemma for gradients of uniformly converging sequences, valid only for scalar valued functions (see Lem. 1.7).

In the scalar case $m = 1$, Theorem 1.4 is the main result of this approach. Corollary 1.5 is a less general, but more elegant version of this theorem. Furthermore we prove that surprisingly level convexity is not needed when we are dealing with C^1 sequences (see Th. 1.1). These results are stated and proved in Section 2.

In Section 3 we show two counterexamples to lower semicontinuity. Example 2.1 proves that Theorems 1.1 and 1.4 are false in the vectorial case. Example 2.2 shows that, when giving up the weak convergence of the gradients, lower semicontinuity cannot be expected to hold true with only measurability assumptions in the position variable x. These two examples reveal to be meaningful even in the context of the lower semicontinuity theory of convex integrals, as explained in Remark 2.3.

1. Lower semicontinuity theorems

As said above, level convexity in the ξ variable when $m = 1$ is necessary to lower semicontinuity. Then it is quite surprising to discover that in the scalar case we still have lower semicontinuity with respect to (3) on C^1 sequences u_h, even if level convexity is dropped.

Theorem 1.1. *Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ be a lower semicontinuous function. Then the lower semicontinuity inequality (1) holds with respect to the convergence (3), provided that the u_h are of class $C^1(\Omega)$.*

The key step in the proof of Theorem 1.1 is the following convergence lemma. Note that we can prove it only in the scalar case.

Lemma 1.2. *Let us consider u, $u_h : \Omega \to \mathbb{R}$, with u_h converging uniformly to u and u_h differentiable in Ω. Then for every point x_0 of differentiability of u there exists $x_h \to x_0$ such that $Du_h(x_h) \to Du(x_0)$.*

Proof. By subtracting the affine function $w(x) := u(x_0) + \langle Du(x_0), x - x_0\rangle$ to the functions u and u_h, we can reduce ourselves to consider the case $u(x_0) = 0, Du(x_0) = 0$. By a simple diagonal argument, we achieve the thesis if we prove that, for every $\varepsilon > 0$, it is possible to find x_h such that $|x_h - x_0| < \varepsilon$ and $|Du_h(x_h)| < 6\varepsilon$, for h large enough.

Let us fix $\varepsilon > 0$. By the differentiability of u at x_0 there exists $0 < \delta < \varepsilon$ such that $\overline{B}_\delta(x_0) \subset \Omega$ and, for every $x \in \overline{B}_\delta(x_0)$, $|u(x)| < \varepsilon|x - x_0|$. Let us define $v : \overline{B}_\delta(x_0) \to \mathbb{R}$ as

$$v(x) := u(x) + \frac{2\varepsilon}{\sqrt{2}-1}\sqrt{\delta^2 + |x - x_0|^2}.$$

It results that, for every $x \in \partial\overline{B}_\delta(x_0)$,

$$v(x) \geq \left(\frac{2\sqrt{2}\varepsilon}{\sqrt{2}-1} - \varepsilon\right)\delta > \frac{2\varepsilon}{\sqrt{2}-1}\delta = v(x_0).$$

Then v attains its minimum in the interior of $\overline{B}_\delta(x_0)$.

The uniform convergence of u_h to u implies the existence of $h(\varepsilon)$ such that, for every $h \geq h(\varepsilon)$, the functions

$$v_h(x) := u_h(x) + \frac{2\varepsilon}{\sqrt{2}-1}\sqrt{\delta^2 + |x - x_0|^2},$$

attain their minima in the interior of $\overline{B}_\delta(x_0)$ too. For this reason, if x_h is one of the minimum point of v_h, then $|x_h - x_0| < \delta < \varepsilon$, and

$$0 = Dv_h(x_h) = Du_h(x_h) + \frac{2\varepsilon}{\sqrt{2}-1} \cdot \frac{x_h - x_0}{\sqrt{\delta^2 + |x_h - x_0|^2}}.$$

Thus we conclude the proof since

$$|Du_h(x_h)| \leq \frac{2\varepsilon}{\sqrt{2}-1} \cdot \frac{|x_h - x_0|}{\sqrt{\delta^2 + |x_h - x_0|^2}} \leq \frac{2\varepsilon}{\sqrt{2}-1} \leq 6\varepsilon.$$

We thank the referee for suggesting us this proof, that is simpler than the one we were first able to provide. □

Proof of Theorem 1.1. Let us take $t > \liminf_{h\to\infty} F(u_h)$. By the continuity of u_h and Du_h, the function $f(x, u_h(x), Du_h(x))$ of the x variable is lower semicontinuous, and hence the essential supremum is a pointwise supremum:

$$F(u_h) = \sup_{x\in\Omega} f(x, u_h(x), Du_h(x)). \tag{6}$$

Then, for every $x \in \Omega$,

$$f(x, u_h(x), Du_h(x)) \leq t. \tag{7}$$

Applying Lemma 1.2, the thesis follows, since by Rademacher's theorem $Du(x_0)$ exists for a.e. $x_0 \in \Omega$, and by the uniform convergence of u_h to u, we have always $u_h(x_h) \to u(x_0)$. Hence, by the lower semicontinuity of f, it results that, for a.e. $x_0 \in \Omega$,

$$f(x_0, u(x_0), Du(x_0)) \leq \liminf_{h\to\infty} f(x_h, u_h(x_h), Du_h(x_h)) \leq t,$$

as desired. □

By the usual strategy of Direct Methods, as a consequence of Theorem 1.1, an existence result of minima for non level convex supremands can be established.

Corollary 1.3. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a lower semicontinuous function such that there exists $\theta : [0, \infty) \to [0, \infty)$, with $\theta(r) \to \infty$ when $r \to \infty$ and*

$$f(x, s, \xi) \geq \theta(|\xi|),$$

for every $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Then for every $u_0 \in C^1(\Omega)$ there exists at least an $\bar{u} \in u_0 + W_0^{1,\infty}(\Omega)$ such that

$$F(\bar{u}) \leq F(v), \quad \forall v \in u_0 + C_0^1(\Omega).$$

To prove a general lower semicontinuity theorem we have necessarily to assume the level convexity of f in the gradient variable. Moreover (maybe only for technical reasons), we need some kind of continuity of f in the lower order variables (x, s). The precise form in which we have to do this, stated in hypothesis (8) of

Theorem 1.4 below, could appear a little tricky: for this reason we give an immediate corollary to our theorem, that loses something in generality but gains a lot in simplicity. We note however that there is a concrete reason to consider (8) rather than some simplified version of it, since it allows us to assume only lower semicontinuity in the gradient variable, a property that is more natural than continuity when combined with level convexity.

Theorem 1.4. *Let us consider a lower semicontinuous function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, such that, for every $(x,s) \in \Omega \times \mathbb{R}$, $f(x,s,\cdot)$ is level convex. Moreover, we ask that, for every $K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a modulus of continuity ω_K such that*

$$|f(x,s,\xi) - f(\bar{x},\bar{s},\xi)| \leq \omega_K\left(|x-\bar{x}| + |s-\bar{s}|\right), \tag{8}$$

whenever (x,s,ξ), $(\bar{x},\bar{s},\xi) \in K$. Then the lower semicontinuity inequality (1) holds with respect to the convergence (3).

As a consequence, an "Aronszajn's counterexample" cannot exist in the theory of supremals.

Corollary 1.5. *Let us consider a continuous $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, such that for every $(x,s) \in \Omega \times \mathbb{R}$, $f(x,s,\cdot)$ is level convex. Then the lower semicontinuity inequality (1) holds with respect to the convergence (3).*

In proving Theorem 1.4 we shall need two lemmas:

Lemma 1.6. *Let (X,μ) be a probability space and $V : X \to \mathbb{R}^N$ a μ-summable function. Then for every μ measurable $Y \subset X$ with $\mu(Y) = 1$ it is*

$$\int_X V \mathrm{d}\mu \in \overline{co}\left\{V(y) : y \in Y\right\} \cdot \tag{9}$$

Combining Lemmas 1.2, 1.6 and Carathéodory's theorem, we find a sort of Mazur's lemma for gradients.

Lemma 1.7. *Let u_h, $u \in W^{1,\infty}(\Omega)$ and $u_h \to u$ in $L^\infty(\Omega)$. Let Ω_0 be a subset of full measure in Ω and suppose that $Du(x)$ and $Du_h(x)$ exist for every $x \in \Omega_0$. Then for every $\rho_h \searrow 0^+$ and for every $x_0 \in \Omega_0$, there exist $x_h \to x_0$, $(\lambda_h^i)_{i=1}^{n+1} \subset [0,1]$ with $\sum_{i=1}^{n+1} \lambda_h^i = 1$, and $(y_h^i)_{i=1}^{n+1} \subset B_{\rho_h}(x_h) \cap \Omega_0$, such that*

$$\lim_{h\to\infty} \sum_{i=1}^{n+1} \lambda_h^i Du_h(y_h^i) = Du(x_0). \tag{10}$$

Remark 1.8. This lemma is false if we pretend to have x_0 in the place of x_h. Equivalently, we cannot prescribe the rate of convergence of the x_h to x_0. An example is provided considering $\Omega = (0,1)$, $x_0 = 1/2$ and defining

$$u_h(x) = \begin{cases} -\gamma_h, & x \in \left(0, \dfrac{1}{2} - \gamma_h\right), \\[2ex] x - \frac{1}{2}, & x \in \left(\dfrac{1}{2} - \gamma_h, \dfrac{1}{2} + \gamma_h\right), \\[2ex] \gamma_h, & x \in \left(\dfrac{1}{2} + \gamma_h, 1\right), \end{cases}$$

where $\gamma_h \searrow 0^+$ with $\gamma_h \leq 1/2$. Clearly u_h converges uniformly to $u(x) \equiv 0$. Then if we put $\Omega \backslash \Omega_0 = \{\frac{1}{2} - \gamma_h, \frac{1}{2} + \gamma_h : h \in \mathbb{N}\}$, and consider any $\rho_h < \gamma_h$ we find that

$$\begin{cases} 0 \leq \mu_h \leq 1, \quad y_h^1, y_h^2 \in \Omega_0 \cap B_{\rho_h}\left(\dfrac{1}{2}\right), \\[2ex] \text{with} \quad \mu_h y_h^1 + (1-\mu_h) y_h^2 = \dfrac{1}{2}, \end{cases}$$

$$\Longrightarrow \mu_h u_h'(y_h^1) + (1-\mu_h)u_h'(y_h^2) = 1, \quad \forall h \in \mathbb{N},$$

while $u'(\frac{1}{2}) = 0$.

Proof of Lemma 1.7. Let us fix $x_0 \in \Omega_0$, and take ρ_0 such that $B_{2\rho_0}(x_0) \subset\subset \Omega$. Possibly discarding a finite number of h we can suppose that $B_{\rho_h}(x) \subset B_{2\rho_0}(x_0)$ whenever $x \in B_{\rho_0}(x_0)$. Let us define

$$w_h(x) = \int_{B_{\rho_h}(x)} u_h(y)\alpha_{\rho_h}(x-y)\mathrm{d}y, \quad x \in B_{\rho_0}(x_0),$$

where $\alpha \in C_c^\infty(B_1(0))$, $\alpha \geq 0$, $\int \alpha(x)\mathrm{d}x = 1$, and $\alpha_\rho(x) := \rho^{-n}\alpha(x/\rho)$. It can be proved that $w_h \in C^\infty(B_{\rho_0}(x_0))$ and that $w_h \to u$ uniformly in $B_{\rho_0}(x_0)$. By Lemma 1.2, we can find $x_h \to x_0$, $x_h \in B_{\rho_0}(x_0)$, such that $Dw_h(x_h) \to Du(x_0)$. Clearly

$$Dw_h(x_h) = \int_{B_{\rho_h}(x_h)} Du_h(y)\alpha_{\rho_h}(x_h - y)\mathrm{d}y,$$

and hence, since $|\Omega\backslash\Omega_0| = 0$, by Lemma 1.6 we have, for every $h \in \mathbb{N}$,

$$Dw_h(x_h) \in \overline{co}\left\{Du_h(y) : y \in B_{\rho_h}(x_h) \cap \Omega_0\right\}\cdot$$

Then by Carathéodory's theorem there exists a sequence $\xi_{h,j} \to Dw_h(x_h)$ of the form

$$\xi_{h,j} = \sum_{i=1}^{n+1} \lambda_{h,j}^i Du_h(y_{h,j}^i),$$

with $\lambda_{h,j}^i \in [0,1]$, $\sum_{i=1}^{n+1}\lambda_{h,j}^i = 1$ and, this is the key point of this argument, $y_{h,j}^i \in B_{\rho_h}(x_h) \cap \Omega_0$. Then extracting a suitable $j(h) \to \infty$ we conclude the proof. □

Proof of Theorem 1.4. We fix $t > \liminf_{h\to\infty} F(u_h)$, and we want to prove that for a.e. $x \in \Omega$ it results $f(x, u(x), Du(x)) \leq t$. By Rademacher's theorem and the definition of essential supremum, the set

$$\Omega_0 = \{x \in \Omega : \exists\, Du(x),\, Du_h(x), \quad f(x, u_h(x), Du_h(x)) \leq t\},$$

is of full measure in Ω. Let us fix $x_0 \in \Omega_0$. We only need to show that $f(x_0, u(x_0), Du(x_0)) \leq t$, and hence, since $|\Omega\backslash\Omega_0| = 0$, the thesis will follow.

We start fixing $\rho_0 > 0$ such that $B_{2\rho_0}(x_0) \subset\subset \Omega$. In correspondence of the compact set

$$K_h = \overline{B_{2\rho_0}(x_0)} \times \overline{B_{||u_h||_{L^\infty(B_{2\rho_0}(x_0))}}(0)} \times \overline{B_{||Du_h||_{L^\infty(B_{2\rho_0}(x_0))}}(0)}$$

we find, according to hypothesis (8), a modulus of continuity $\omega_{K_h} = \omega_h$ such that

$$|f(x,s,\xi) - f(\bar{x},\bar{s},\xi)| \leq \omega_h\left(|x-\bar{x}| + |s-\bar{s}|\right) \tag{11}$$

whenever (x,s,ξ), $(\bar{x},\bar{s},\xi) \in K_h$. In particular it must be $\omega_h(\sigma) \to 0^+$ whenever $\sigma \to 0^+$, and hence we can choose a sequence $\sigma_h \to 0^+$, such that

$$\omega_h(\sigma_h) \leq \frac{1}{h}\cdot$$

Then there exists $\rho_h \to 0^+$ such that

$$\left(1 + ||Du_h||_{L^\infty(B_{2\rho_0}(x_0))}\right) \rho_h \leq \sigma_h, \quad \rho_h \leq \rho_0.$$

We apply Lemma 1.7 to u_h, u, Ω_0 and ρ_h. Then by the global lower semicontinuity of f, for (10), we have

$$f(x_0, u(x_0), Du(x_0)) \leq \liminf_{h\to\infty} f\left(x_h, u_h(x_h), \sum_{i=1}^{n+1} \lambda_h^i Du_h(y_h^i)\right),$$

where $y_h^i \in B_{\rho_h}(x_h) \cap \Omega_0$, $\lambda_h^i \in [0,1]$ and $\sum_{i=1}^{n+1} \lambda_h^i = 1$. By our choice of the constants it results

$$\begin{cases} (x_h, u_h(x_h), Du_h(y_h^i)), \quad (y_h^i, u_h(y_h^i), Du_h(y_h^i)) \in K_h, \\ \left|y_h^i - x_h\right| + \left|u_h(y_h^i) - u_h(x_h)\right| \leq \left(1 + ||Du_h||_{L^\infty(B_{2\rho_0}(x_0))}\right) \rho_h \leq \sigma_h, \end{cases}$$

so that the level convexity of f in the gradient variable and (11) imply

$$\begin{aligned} f(x_0, u(x_0), Du(x_0)) &\leq \liminf_{h\to\infty} \max_{1\leq i\leq n+1} f(x_h, u_h(x_h), Du_h(y_h^i)) \\ &\leq \liminf_{h\to\infty} \left\{\omega_h(\sigma_h) + \max_{1\leq i\leq n+1} \left(f(y_h^i, u_h(y_h^i), Du_h(y_h^i))\right)\right\} \leq t, \end{aligned}$$

since we have selected the y_h^i in Ω_0 and σ_h in a way that $\omega_h(\sigma_h) \leq 1/h$. □

2. Counterexamples to lower semicontinuity

In this section we show two counterexamples related to the previous results. This two examples are significant for the lower semicontinuity theory of integral functionals too: see Remark 2.3 below.

The first one shows that Theorems 1.1 and 1.4 do not hold in the vectorial case $m > 1$. In particular, for what concerns the comparison with Theorem 1.1, it should be noted that the supremand is even convex. Note also that the function f we show is of class C^∞.

Example 2.1. Let us define

$$f(s_1, s_2, \xi_1, \xi_2) = (\xi_1 s_2 - \xi_2 s_1 - 1)^2,$$

where $(s_1, s_2, \xi_1, \xi_2) = (s, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, and consider the sequence

$$u_h \in C^\infty(0,1)^2, \quad u_h(x) = \left(\frac{1}{2\pi h}\sin(2\pi h^2 x), \frac{1}{2\pi h}\cos(2\pi h^2 x)\right).$$

An easy computation gives $u_h \to 0$ in $L^\infty(0,1)^2$ while

$$F(0) = 1, \quad F(u_h) = 0, \quad \forall h \in \mathbb{N}.$$

Note that $f(u_h(x), Du_h(x)) = 0$ for *every* $x \in (0,1)$.

An important remark about Theorem 0.1 is that it holds true even with respect to the more general convergence given by

$$\begin{cases} u_h,\, u \in W^{1,1}_{\text{loc}}(\Omega)^m, \\ u_h \rightharpoonup u \quad w - W^{1,1}_{\text{loc}}(\Omega)^m. \end{cases} \tag{12}$$

In the following Example 2.2 we show a nonnegative function $f = f(x, \xi)$ upper semicontinuous in the x variable, convex in the ξ variable, for which lower semicontinuity inequality (1) does not hold on a sequence of C^1 functions converging uniformly and with L^1-bounded derivatives. This means that if in Theorem 0.1 we want to consider weaker convergences than (12), such as the weak convergence in BV and in particular the convergence in (3), we have to strengthen the measurability assumption on $f(\cdot, s, \xi)$. Moreover, this example shows that in Theorem 1.1 the lower semicontinuity of f in the x variable cannot be dropped.

Example 2.2. We shall construct a closed set $K \subset [0,1]$ of positive measure and a sequence of $C^\infty([0,1])$ functions u_h uniformly converging to 0, with derivatives bounded in the L^1 norm, such that

$$0 < c < u_h'(x) < \mathrm{d}, \quad \forall x \in K, \forall h \in \mathbb{N}. \tag{13}$$

Then, if χ_K is the characteristic function of the set K and $g(\xi) = (\xi - (\mathrm{d} - c)/2)^2$, defining $f(x, \xi) = \chi_K(x) g(\xi)$, we have the desired example.

Let us fix $t > 3$. For all $0 < a < b < 1$ with $b - a > t^{-h}$ we define

$$T_h([a,b]) = \left[a, \frac{b-a}{2} - \frac{1}{2t^h}\right] \cup \left[\frac{b-a}{2} + \frac{1}{2t^h}, b\right].$$

Then we put

$$\mathcal{E}_1 = T_1([0,1]) = \left\{I_i^1\right\}_{i=1}^2, \quad \mathcal{E}_h = \left\{T_h\left(I_i^{h-1}\right)\right\}_{i=1}^{2^{h-1}} = \left\{I_i^h\right\}_{i=1}^{2^h},$$

where the intervals I_i^h are enumerated in such a way that $\sup I_i^h < \inf I_{i+1}^h$. Let us define

$$K_h = \bigcup_{i=1}^{2^h} I_i^h, \quad K = \bigcap_{h=1}^{\infty} K_h.$$

K is a closed set, and since

$$\operatorname{diam}\left(I_i^h\right) = \frac{1}{2^h}\left\{1 - \frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\},$$

we have $\operatorname{meas}(K) = (t-3)/(t-2)$ (K is simply one of the standard variants of the Cantor's Set). Now we define the sequence u_h in the following way. On every interval I_i^h, u_h is the affine function that takes the value 0 in the left extreme of I_i^h and the value 2^{-h-1} in the right extreme. Then we extend u_h on all $[0,1]$ in a C^∞ way, under the constraint that, for every i, on the interval between I_i^h and I_{i+1}^h, the total variation of u_h is controlled by $3 \times 2^{-h-1}$. It results $|u_h| \leq 3 \times 2^{-h-1}$ everywhere, so that the sequence u_h converges uniformly to 0. The total variation can be estimated looking carefully at the construction:

$$\int_0^1 |u_h'(x)|\,\mathrm{d}x \leq \left(2^h + (2^h - 1)\right)\frac{3}{2^{h+1}},$$

so that the derivatives are bounded in L^1. Finally, on K_h it is

$$u_h'(x) = \frac{1}{2^{h+1}} \times \frac{1}{\operatorname{diam}\left(I_i^h\right)} = \frac{1}{2 \times \left\{1 - \frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\}}.$$

In particular, since we have

$$\lim_{h\to\infty}\left\{1-\frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\}=\frac{t-3}{t-2}\in(0,1),$$

we can choose c and d in such a way that (13) holds, and then conclude the construction.

Remark 2.3. We remark that the construction in Example 2.1 furnishes an analogous counterexample to the one given by Eisen in [14], and in fact refines it in the sense that our sequence converges uniformly to 0, while Eisen's one does it only in the strong norm topology of L^1. Note also that we gain something in simplicity. Example 2.2 says also that the De Giorgi–Ioffe Lower Semicontinuity theorem (see [12, 19]) does not hold with respect to weak BV convergence, at least with measurability in the position variable x. This fact is classically shown with a counterexample by Carbone and Sbordone [10] (see also [9]). However we did not see how to adapt their argument to supremals, and so we have constructed a direct counterexample.

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References

[1] E. Acerbi, G. Buttazzo and F. Prinari, The class of functionals which can be represented by a supremum. *J. Convex Anal.* **9** (to appear).
[2] L. Ambrosio, New lower semicontinuity results for integral functionals. *Rend. Accad. Naz. Sci. XL* **11** (1987) 1-42.
[3] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. *Ark. Mat.* **6** (1965) 33-53.
[4] G. Aronsson, Extension of functions satisfying Lipschitz conditions. *Ark. Mat.* **6** (1967) 551-561.
[5] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ II. *Ark. Mat.* **6** (1969) 409-431.
[6] G. Aronsson, Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$ III. *Ark. Mat.* **7** (1969) 509-512.
[7] E.N. Barron, R.R. Jensen and C.Y. Wang, Lower semicontinuity of L^∞ functionals. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001) 495-517.
[8] E.N. Barron and W. Liu, Calculus of variations in L^∞. *Appl. Math. Optim.* **35** (1997) 237-263.
[9] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Longman, Harlow, *Pitman Res. Notes Math. Ser.* **207** (1989).
[10] L. Carbone and C. Sbordone, Some properties of Γ-limits of integral functionals. *Ann. Mat. Pura Appl.* **122** (1979) 1-60.
[11] G. Dal Maso, Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals. *Manuscripta Math.* **30** (1980) 387-416.
[12] E. De Giorgi, *Teoremi di semicontinuità nel calcolo delle variazioni.* Istituto Nazionale di Alta Matematica, Roma (1968).
[13] E. De Giorgi, G. Buttazzo and G. Dal Maso, On the lower semicontinuity of certain integral functionals. *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur., Rend.* **74** (1983) 274-282.
[14] G. Eisen, A counterexample for some lower semicontinuity results. *Math. Z.* **162** (1978) 241-243.
[15] I. Fonseca and G. Leoni, Some remarks on lower semicontinuity. *Indiana Univ. Math. J.* **49** (2000) 617-635.
[16] I. Fonseca and G. Leoni, On lower semicontinuity and relaxation. *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001) 519-565.
[17] M. Gori, F. Maggi and P. Marcellini, On some sharp lower semicontinuity condition in L^1. *Differential Integral Equations* (to appear).
[18] M. Gori and P. Marcellini, An extension of the Serrin's lower semicontinuity theorem. *J. Convex Anal.* **9** (2002) 1-28.
[19] A.D. Ioffe, On lower semicontinuity of integral functionals. *SIAM J. Control Optim.* **15** (1977) 521-538.
[20] C.Y. Pauc, *La méthode métrique en calcul des variations.* Hermann, Paris (1941).
[21] J. Serrin, On the definition and properties of certain variational integrals. *Trans. Amer. Math. Soc.* **101** (1961) 139-167.