

ABSTRACT VARIATIONAL PROBLEMS WITH VOLUME CONSTRAINTS

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Abstract. Existence results for a class of one-dimensional abstract variational problems with volume constraints are established. The main assumptions on their energy are additivity, translation invariance and solvability of a transition problem. These general results yield existence results for nonconvex problems. A counterexample shows that a naive extension to higher dimensional situations in general fails.

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1. INTRODUCTION

Recently, constrained variational problems of the type

$$\left\{ \begin{array}{l} \text{Minimize } E(u) := \int_{\Omega} f(u(x), \nabla u(x)) \, dx, \\ u \in W^{1,p}(\Omega, \mathbb{R}), \quad \Omega \subset \mathbb{R}^n, \\ |\{u = 0\}| = \alpha, \quad |\{u = 1\}| = \beta \end{array} \right. \quad (1)$$

have been studied under different assumptions on the energy density f [1, 3, 5]. This was initially suggested by Gurtin in 1992, see also [2]. The aim of the present article is to find general conditions on the energy $E(u)$ which entail the existence of solutions to volume constrained problems of type (1) in the one-dimensional case, and which generalize the results of Morini and Rieger [3] to energy functionals which are not necessarily of integral form.

In Section 2 two existence results for a very general class of admissible energies are presented. Their proofs turn out to be surprisingly simple. In Section 3 these results are applied to minimization problems of integral type which are nonconvex in u' . In Section 4 a higher dimensional example illustrates how non-smoothness of the boundary may lead to non-existence of solutions.

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2. ABSTRACT VARIATIONAL PROBLEMS

We consider the following one-dimensional variational problem on an open interval $I = (a, b)$:

$$\begin{cases} \text{Minimize } E(u, I), \text{ for } u \in W^{1,p}(I, [0, 1]), \\ |\{u = 0\}| = \alpha, |\{u = 1\}| = \beta, \end{cases} \quad (2)$$

where $\alpha + \beta < b - a = |I|$ and $1 < p \leq +\infty$.

The aim of this section is to prove two existence results under certain assumptions on the energy E . We collect the main assumptions in the following definition:

Definition 2.1 (Admissible energies). Let $I = (a, b)$ be an open interval in \mathbb{R} , let $1 < p \leq +\infty$, and let \mathcal{I} denote the set of all open intervals in I . Consider a function

$$E: W^{1,p}(I, [0, 1]) \times \mathcal{I} \rightarrow \mathbb{R},$$

bounded from below, with $E(u, T) = E(v, T)$ for $T \in \mathcal{I}$ whenever $u|_T = v|_T$. (Thus we sometimes write $E(u, T)$ even if u is only defined on T .)

Then E is called an *admissible energy* if it satisfies the following three conditions:

- (i) let $T \in \mathcal{I}$ be an arbitrary open interval $(x_0, x_1) \subset I$. Consider the problem

$$\begin{cases} \text{Minimize } E(u, T), \text{ for } u \in W^{1,p}(T, [0, 1]), \\ \text{such that there exist } \xi_0, \xi_1 \in [x_0, x_1] \\ \text{with } u(\xi_0) = 0, \text{ and } u(\xi_1) = 1. \end{cases} \quad (3)$$

This problem admits a solution (u, ξ_0, ξ_1) ;

- (ii) (Additivity) for $x_0, x_1, x_2 \in I$, $x_0 < x_1 < x_2$ and $u \in W^{1,p}(I, [0, 1])$ we have:

$$E(u, (x_0, x_1)) + E(u, (x_1, x_2)) = E(u, (x_0, x_2));$$

- (iii) (Translation invariance) for all $x_0, x_1 \in \bar{I}$, $u \in W^{1,p}((x_0, x_1), [0, 1])$ and $\tau \in \mathbb{R}$ such that $x_0 + \tau, x_1 + \tau \in \bar{I}$ we have

$$E(u, (x_0, x_1)) = E(u(\cdot - \tau), (x_0 + \tau, x_1 + \tau)).$$

It turns out to be useful to extend the notion of admissible energies to functions which are only piecewise in $W^{1,p}$:

Definition 2.2. Let E be an admissible energy on $W^{1,p}(I, [0, 1]) \times \mathcal{I}$. Let $(x_i)_{i=1, \dots, n} \in I$ with $x_{i+1} > x_i$, and $u|_{(x_i, x_{i+1})} \in W^{1,p}((x_i, x_{i+1}), [0, 1])$, then define $E(u) := \sum_{i=1}^n E(u, (x_i, x_{i+1}))$.

A special type of admissible energies is given in the following definition:

Definition 2.3. We call an energy E *symmetric* if for all $(x_0, x_1) \in \mathcal{I}$ and all $u \in W^{1,p}((x_0, x_1), [0, 1])$ we have

$$E(u(x_0 + \cdot), (0, x_1 - x_0)) = E(u(x_1 - \cdot), (0, x_1 - x_0)).$$

The properties (ii) and (iii) of Definition 2.1 can be easily verified for a given energy E . However, condition (i) is a little involved. The following remark gives an easy characterization of (i) for symmetric energies:

Remark 2.4. If E is symmetric, and satisfies (ii) and (iii), then E is an admissible energy if and only if for every $T := (x_0, x_1) \subset I$ the Dirichlet boundary value problem

$$\begin{cases} \text{Minimize } E(u, T), \text{ for } u \in W^{1,p}(T, [0, 1]), \\ u(x_0) = 0, \text{ and } u(x_1) = 1, \end{cases} \quad (4)$$

admits a solution.

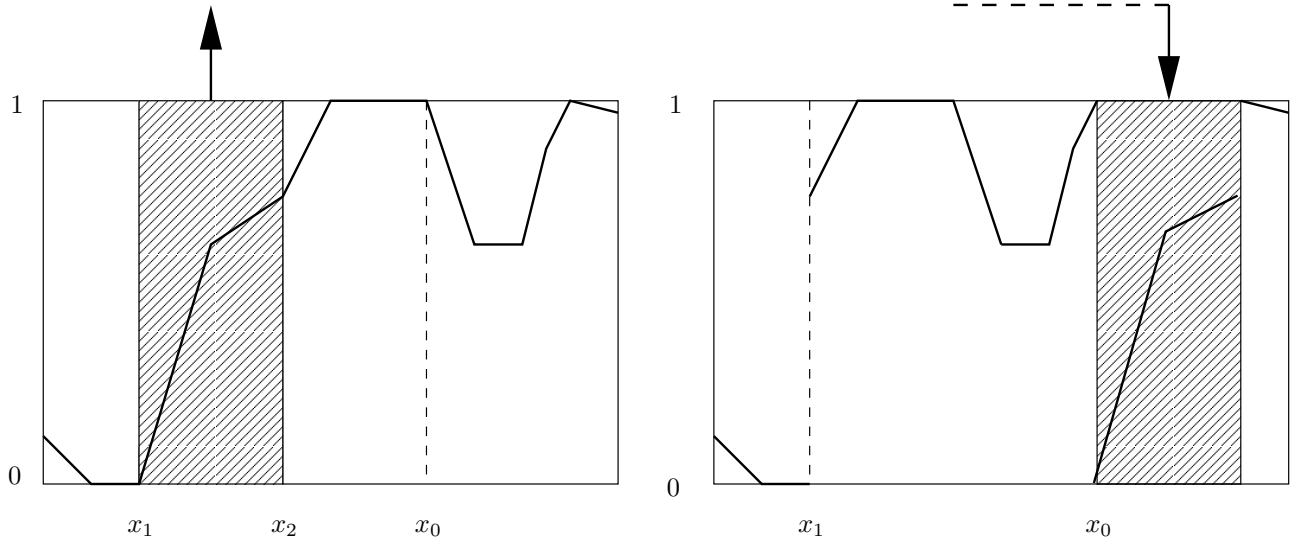


FIGURE 1. The shift operator S cuts out the piece of the function between x_1 and x_2 (left picture), glues the remaining parts of the function together and inserts the cut out piece at x_0 (right).

Proof. Any solution of (4) obviously solves (3). Hence the condition is sufficient. Necessity can be proved using the same argument as in Lemma 2.7, see below. \square

Before we state existence results for problem (2) we give some useful lemmata:

Lemma 2.5 (Energy conserving shifts). *Let E be an admissible energy. Let $x_0, x_1, x_2 \in I = (a, b)$ with $x_2 > x_1$ and $x_2 - x_1 \leq b - x_0$. We define an operation $S((x_1, x_2), x_0)$ on the functions which are piecewise $W^{1,p}$ on I in the following way (compare Fig. 1): If $x_1 \geq x_0$ we define*

$$S((x_1, x_2), x_0)u(x) := \begin{cases} u(x), & x < x_0, \\ u(x + x_1 - x_0), & x_0 \leq x < x_0 + x_2 - x_1, \\ u(x - (x_2 - x_1)), & x_0 + x_2 - x_1 \leq x < x_2, \\ u(x), & x \geq x_2, \end{cases}$$

else we define

$$S((x_1, x_2), x_0)u(x) := \begin{cases} u(x), & x < x_1, \\ u(x + x_2 - x_1), & x_1 \leq x < x_0, \\ u(x + x_1 - x_0), & x_0 \leq x < x_0 + x_2 - x_1, \\ u(x), & x \geq x_0 + x_2 - x_1. \end{cases}$$

The so defined operator S is energy conserving, i.e. $E(S((x_1, x_2), x_0)u, I) = E(u, I)$.

The shift operator $S((x_1, x_2), x_0)$ can be described as “cutting out” the values of a function on the interval (x_1, x_2) and inserting them at the position x_0 .

Proof. The proof follows immediately from (ii) and (iii) of the definition of admissible energies. \square

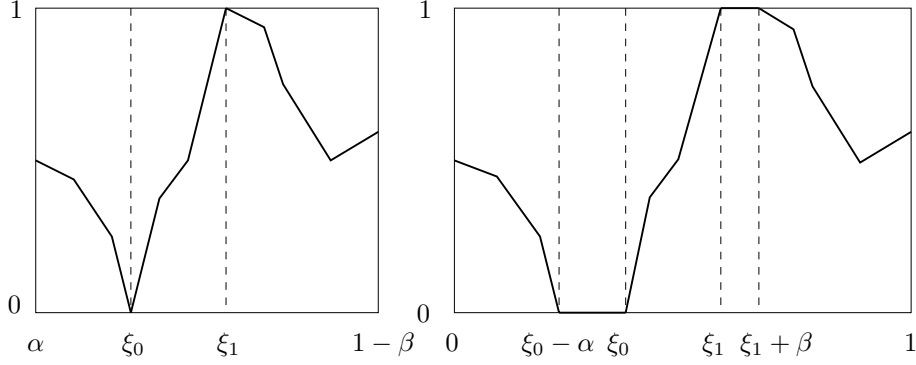


FIGURE 2. Left: the function \tilde{u} solves problem (3) on the interval $(\alpha, 1-\beta)$. Right: construction of the function u .

The operation defined in this lemma is the basic step in the constructions of the following results.

Lemma 2.6 (Existence for the relaxed problem). *Let E be an admissible energy for the open interval I , continuous with respect to the strong $W^{1,p}$ -topology on all open sets in I . Then for $\alpha, \beta > 0$ with $\alpha + \beta < |I|$ the relaxed problem*

$$\begin{cases} \text{Minimize } E(u, I), \text{ where } u \in W^{1,p}(I, [0, 1]), \\ |\{u = 0\}| \geq \alpha, |\{u = 1\}| \geq \beta, \end{cases} \quad (5)$$

admits a solution u .

Moreover we can construct a solution with the following properties:

$$\begin{aligned} \{x \in I, u(x) = 0\} &= A_0 \cup R_0, \\ \{x \in I, u(x) = 1\} &= A_1 \cup R_1, \end{aligned} \quad (6)$$

where A_0, A_1 are intervals closed in I and R_0, R_1 are countable sets.

Proof. For simplicity take $I = (0, 1)$. We will first prove that the following function u (compare Fig. 2) is a minimizer to the relaxed problem. Let $(\tilde{u}, \xi_0, \xi_1)$ be a minimizer of (3) on $(\alpha, 1-\beta)$. Without loss of generality assume $\xi_0 < \xi_1$. Define

$$u(x) := \begin{cases} \tilde{u}(x + \alpha), & x < \xi_0 - \alpha, \\ 0, & \xi_0 - \alpha \leq x < \xi_0, \\ \tilde{u}(x), & \xi_0 \leq x \leq \xi_1, \\ 1, & \xi_1 < x \leq \xi_1 + \beta, \\ \tilde{u}(x - \beta), & x > \xi_1 + \beta. \end{cases}$$

Take an arbitrary function $w \in W^{1,p}(I, [0, 1])$ with $E(w, I) \leq E(u, I)$, satisfying $|\{w = 0\}| \geq \alpha$ and $|\{w = 1\}| \geq \beta$. The following construction will exclude the case $E(w, I) < E(u, I)$. Moreover it will show that u can be chosen such that the additional properties as stated in the lemma are satisfied.

We denote $T := \{x \in I, w(x) \in (0, 1)\}$. We refer to T as the “transition set”.

T is an open set, hence there exists a countable set of disjoint open intervals T_j with $T = \cup_j T_j$. Consider one of these intervals $T_j = (l, r)$. Since w is continuous, there are only three possibilities for $w(l)$:

- (i) $w(l) = 0$;
- (ii) $w(l) = 1$;
- (iii) $w(l) \in (0, 1)$ and $l = 0$, *i.e.* the set T_j lies on the left boundary of $(0, 1)$.

Case (iii) can only occur once, call the corresponding set $T^L = (0, \hat{a})$. (In the case that $w(0) \in \{0, 1\}$, you may set $T^L = \emptyset$ and $\hat{a} = 0$.) Using the same reasoning for $w(r)$ we define a set $T^R = (\hat{b}, 1)$ where $w(r) \in (0, 1)$ and

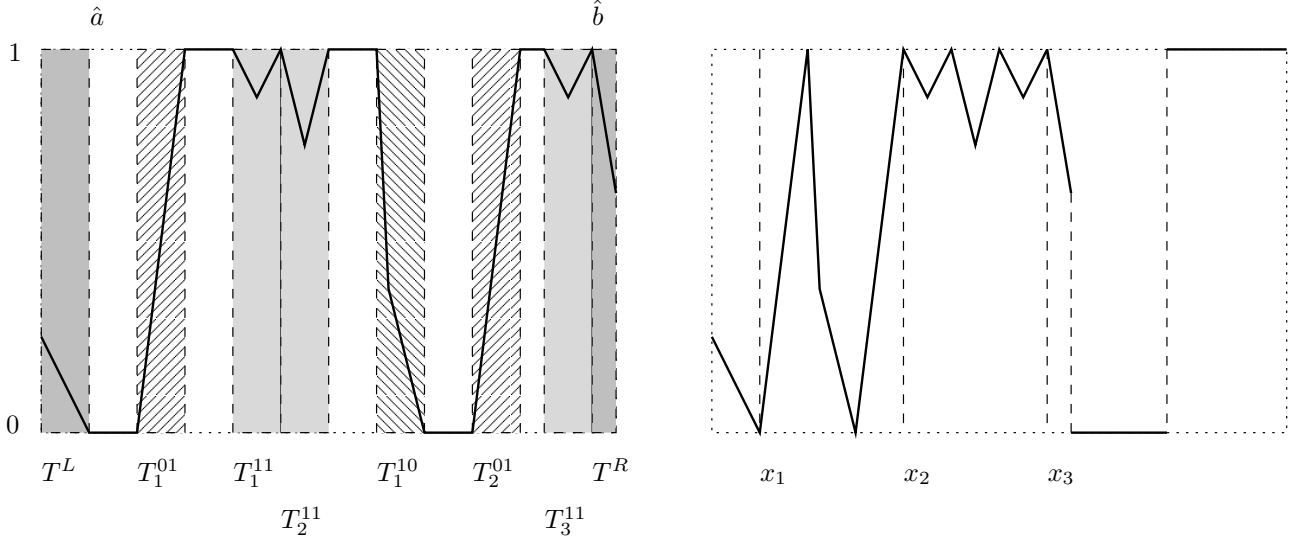


FIGURE 3. Left: illustration of the decomposition of the transition set T into open intervals T^L (left boundary), T^R (right boundary), T_j^{00} and T_j^{11} (possibly infinitely many), T_j^{01} and T_j^{10} (finitely many). Right: the construction in the proof of Lemma 2.6 collects iteratively the sets T^L , then T_j^{00} , then T_j^{01} and T_j^{10} , then T_j^{11} and finally T^R using energy preserving shifts. The construction shows that an optimal minimizer of the relaxed problem (5) is given by u (see text).

get the following decomposition of T :

$$T = \left(\bigcup_i T_i^{00} \right) \cup \left(\bigcup_{i=1}^{i^{01}} T_i^{01} \right) \cup \left(\bigcup_{i=1}^{i^{10}} T_i^{10} \right) \cup \left(\bigcup_i T_i^{11} \right) \cup T^L \cup T^R,$$

such that for $T_i^{yz} = (l, r)$ we have $u(l) = y$, $u(r) = z$.

Since $w \in W^{1,p}(I, [0, 1])$, we see that i^{01} and i^{10} are finite. (Otherwise $\int |w'|^p = +\infty$.) By continuity of w , there are now three possibilities: either $i^{01} = i^{10} + 1$, $i^{01} = i^{10} - 1$, or $i^{01} = i^{10}$. We consider here the first case and assume without loss of generality that $w(\hat{a}) = 0$ and $w(\hat{b}) = 1$. The other cases can be handled in a similar way. We define a function \hat{u} with $E(\hat{u}, I) = E(w, I)$ by an iterative energy conserving construction using Lemma 2.5 in several steps (see Fig. 3).

Step 1: First define $T_0^{00} := T^L$ and $v_0 := w$. For $T_j^{00} = (l_j, r_j)$ and $s := \sum_{k=0}^{j-1} (r_k - l_k)$ define iteratively $v_j := S(T_j^{00}, s)v_{j-1}$ for $j \geq 1$. The limit function $\lim_j v_j$ denote by \hat{u}_1 . (Since $\sum |T_j^{00}|$ is bounded, it is easy to see that v_j converges in $W^{1,p}$.) By continuity of E we have $E(\hat{u}_1, I) = \lim_j E(v_j, I) = E(w, I)$, and for the limit point $x_1 := \sum_k (r_k - l_k)$ we have $\lim_{x \rightarrow x_1} \hat{u}_1(x) = 0$, since $w \in W^{1,p}$. The result of step 1 is a function \hat{u}_1 with transition set $\hat{T} := \{\hat{u}_1(x) \in (0, 1)\}$ represented by the disjoint sum

$$\hat{T} = \hat{T}_0 \cup \left(\bigcup_{i=1}^{i^{01}} \hat{T}_i^{01} \right) \cup \left(\bigcup_{i=1}^{i^{10}} \hat{T}_i^{10} \right) \cup \left(\bigcup_i \hat{T}_i^{11} \right) \cup \hat{T}^R,$$

such that $\hat{T}_0 = (0, x_1) \setminus R$, R countable set, and for all i and $y, z \in \{0, 1\}$ and for $\hat{T}_i^{yz} = (l, r)$ we have again $\hat{u}_1(l) = y$, $\hat{u}_1(r) = z$ and $\hat{T}^R = (\hat{b}, 1)$.

Step 2: Now continue in the same way by shifting the values of \hat{u}_1 on \hat{T}_i^{01} and \hat{T}_i^{10} for all (finitely many) i in the order $\hat{T}_1^{01}, \hat{T}_1^{10}, \hat{T}_2^{01}, \hat{T}_2^{10} \dots$. Call the result \hat{u}_2 and define $x_2 := x_1 + \cup_{i=1}^{i^{01}} |\hat{T}_i^{01}| + \cup_{i=1}^{i^{10}} |\hat{T}_i^{10}|$. We have now for $\check{T} := \{\hat{u}_2(x) \in (0, 1)\}$ the representation

$$\check{T} = \check{T}_0 \cup \left(\bigcup_{i=1} \check{T}_i^{11} \right) \cup \check{T}^R,$$

where $\check{T}_0 = (0, x_2) \setminus R$, R countable set, and \check{T}_i^{11} and \check{T}^R satisfy the same conditions as \hat{T}_i^{11} and \hat{T}^R above. Since we assumed $i^{01} = i^{10} + 1$, we get $\lim_{x \rightarrow x_2} \hat{u}_2(x) = 1$.

Step 3: Finally shift the values of \hat{u}_2 on \check{T}_i^{11} following the idea of Step 1. The resulting function \hat{u}_3 has only finitely many non-trivial intervals on which $\hat{u}_3 \in \{0, 1\}$. By applying appropriate shifts, we can assume that \hat{u}_3 has only two such intervals. Call the closure of these intervals $\tilde{A}_0 = [a_0, b_0]$ and $\tilde{A}_1 = [a_1, b_1]$.

Step 4: Now define $\delta := |\tilde{A}_0| - \alpha$ and $\varepsilon := |\tilde{A}_1| - \beta$. By construction we have $\delta, \varepsilon \geq 0$. If $\delta > 0$ then choose a $x_{i_0} \in [0, 1 - \beta + \alpha]$ such that $\hat{u}_3(x_{i_0}) = 0$ and define $\hat{u}_4 := S((a_0, a_0 + \delta), x_{i_0})$. Applying the same idea for the case $\varepsilon > 0$ we obtain a function \hat{u}_5 , and we immediately see that the triple $(\hat{u}_5|_{(0, 1 - (\beta + \alpha))}, x_1, x_2)$ solves (3). Hence $E(\hat{u}_5, (0, 1 - (\beta + \alpha))) = E(\hat{u}, (\alpha, 1 - \beta))$, and $E(\hat{u}_5, I) = E(u, I)$. Finally shift \tilde{A}_0 and \tilde{A}_1 in order to get a continuous function with only two non-trivial intervals on which the function is zero resp. one. Call the resulting function \hat{u} and these intervals A_0 and A_1 . Taking everything together we deduce that $E(w, I) = E(\hat{u}, I) = E(u, I)$. Thus u and \hat{u} are both minimizers of (5), and by construction \hat{u} satisfies the additional properties (6). \square

Lemma 2.7 (Boundary conditions). *Let E be a symmetric admissible energy on $I = (a, b)$ and there exists a solution to the relaxed problem (5). Then there exists a solution u to (5) with $u(a) = 0$ and $u(b) = 1$.*

Proof. For simplicity take $I = (0, 1)$. Let w be a solution to (5) as given by Lemma 2.6. First, we show that we can construct a solution \tilde{w} to (5) with $\tilde{w}(0) = 0$ or $\tilde{w}(0) = 1$.

If $w(0) = 0$ or $w(0) = 1$ we are done. Hence let us assume that $L := \min\{x \in I, w(x) \in \{0, 1\}\}$ is positive. In the following we assume that $w(L) = 0$ and construct a function \tilde{w} with $\tilde{w}(0) = 0$. In the case where $w(L) = 1$ the same construction would give us $\tilde{w}(0) = 1$.

We distinguish two cases (see Fig. 4):

Case 1: $E(w, (0, L/2)) \geq E(w, (L/2, L))$.

In this case, we define

$$\tilde{w}(x) := \begin{cases} w(L - x), & x < L/2, \\ w(x), & x \geq L/2. \end{cases}$$

By the symmetry of E , we have $E(\tilde{w}, I) \leq E(w, I)$, moreover $\tilde{w}(0) = w(L) = 0$.

Case 2: $E(w, (0, L/2)) \leq E(w, (L/2, L))$.

In this case, the construction of \tilde{w} is nearly as easy: we choose $x_0 > L$ such that $w(x_0) = w(L/2)$. (By the intermediate value theorem such an x_0 exists.) Then we define

$$\tilde{w}(x) := \begin{cases} w(x + L), & x \leq x_0 - L, \\ w(x_0 - x - L/2), & x_0 - L < x \leq x_0 - L/2, \\ w(x - x_0 + L/2), & x_0 - L/2 < x \leq x_0, \\ w(x), & x > x_0. \end{cases}$$

Again by the symmetry of E we have $E(\tilde{w}, I) \leq E(w, I)$. Moreover $\tilde{w}(0) = w(L) = 0$.

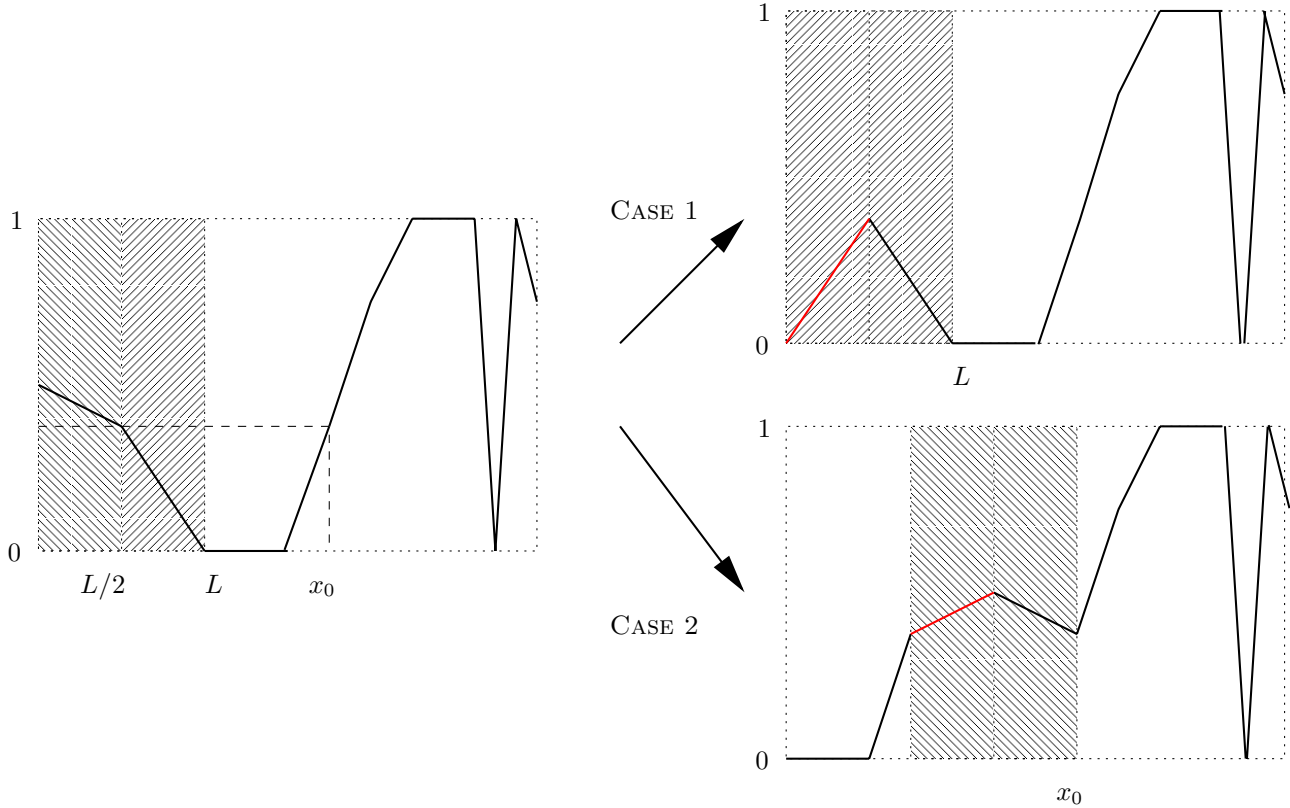


FIGURE 4. Starting from a solution w of the relaxed problem, we construct another solution \tilde{w} with $\tilde{w}(0) = 0$. To this aim, we compare the energy of w on the shaded regions (left). Depending on which is larger, we perform one of the constructions illustrated on the right.

Applying the same construction at $x = 1$, we get a function v with $v(0), v(1) \in \{0, 1\}$. If $v(0) = 0$ and $v(1) = 1$ we just set $u := v$. If $v(0) = 1$ and $v(1) = 0$ we set $u := v(1 - \cdot)$. For the remaining cases we use the following construction (performed without loss of generality for the case $v(0) = v(1) = 1$).

Let $L_1 := \min\{x \in I, v(x) = 0\}$ and $L_2 := \max\{x < L_1, v(x) = 1\}$. Define $\tilde{v} := S((0, L_2), 1 - L_2)v$, then \tilde{v} is continuous since $v(0) = v(L_2) = v(1) = 1$ and $E(\tilde{v}, I) = E(v, I)$. By a final application of the construction above (see Fig. 4) we can now get a function u with $u(0) = 0$, $u(1) = 1$ and $E(u, I) = E(w, I)$. \square

Using the lemmata proved above we can prove the following existence result:

Theorem 2.8 (Existence for small γ). *Let $I = (a, b)$ be an open interval, $1 < p \leq +\infty$, and $\alpha, \beta > 0$ with $\alpha + \beta < |I|$. Let E be a symmetric admissible energy, continuous with respect to the $W^{1,p}$ -topology on all open sets in I . Then there exists a constant $\gamma_0 > 0$, such that the abstract variational problem (2) admits a solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.*

Proof. First, we see that by (ii) and (iii) the solvability of the problem depends only on γ , but not on α and β . Hence we can speak of a solution for a specific transition width γ instead of α and β . An illustration of our construction is given in Figure 5.

Now we consider the relaxed problem (5) for $|\{u = 0\}| \geq \alpha_0$ and $|\{u = 1\}| \geq \beta_0$, where we assume without loss of generality that $I = (0, 1)$. By Lemma 2.6 and Lemma 2.7 there exists a solution u to this problem with $(0, \alpha_0) \subset \{u = 0\}$, $(1 - \beta_0, 1) \subset \{u = 1\}$ and $T := u^{-1}((0, 1))$ satisfying $\bar{T} \subset [\alpha_0, 1 - \beta_0]$.

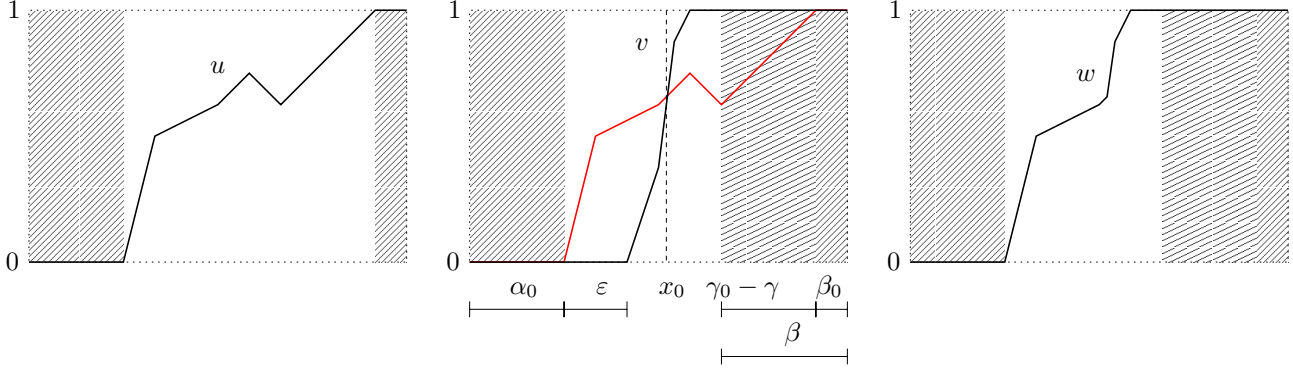


FIGURE 5. Left: a solution u for the relaxed problem (5) also solves the problem (2) for certain α_0 and β_0 . Middle: the relaxed solution v for $\alpha := \alpha_0$ and $\beta = \beta_0 + (\gamma_0 - \gamma) > \beta_0$ intersects with u at some point x_0 . Right: a solution w of the relaxed problem (5) satisfying the first constraint of (2) can be constructed by combining u and v . An analogous construction for the second constraint yields a solution for (2).

Now define $\gamma_0 := |T|$. Since u is continuous we have $\gamma_0 > 0$. Moreover, u is a solution of the non-relaxed problem (2) for the transition width γ_0 . Now take $\gamma \in (0, \gamma_0)$, and let v be a solution of (5) with the constraints $|\{v = 0\}| = \alpha_0 =: \alpha$ and $|\{v = 1\}| = \beta_0 + (\gamma_0 - \gamma) =: \beta$. Our goal is to modify v to a function w such that $|\{x \in (\alpha, 1 - \beta) : w(x) = 0\}| = 0$ and $|\{x \in (\alpha, 1 - \beta) : w(x) = 1\}| = 0$. We prove the first statement, the latter one can be asserted in a similar way. By Lemma 2.6 and Lemma 2.7 we can assume that $v|_{(\alpha, \alpha + \varepsilon]} = 0$, $v|_{[1 - \beta, 1]} = 1$ and $v|_{(\alpha + \varepsilon, 1 - \beta)} > 0$ a.e. for some $\varepsilon \geq 0$. If $\varepsilon = 0$, the function v already satisfies $|\{x \in (\alpha, 1 - \beta), v(x) = 0\}| = 0$, hence we have only to consider the case where $\varepsilon > 0$. Since $u(\alpha + \varepsilon) \geq v(\alpha + \varepsilon) = 0$ and $u(1 - \beta) \leq v(1 - \beta) = 1$, by the intermediate value theorem for continuous functions there exists a point $x_0 \in [\alpha + \varepsilon, 1 - \beta]$ such that $u(x_0) = v(x_0)$. Now define the function

$$w(x) := \begin{cases} u(x), & x < x_0, \\ v(x), & x \geq x_0. \end{cases}$$

Since u is a minimizer of (5) with transition width γ_0 we have in particular $E(u, (\alpha, 1 - \beta_0)) \leq E(w, (\alpha, 1 - \beta_0))$ and $E(u, (\alpha, x_0)) + E(u, (x_0, 1 - \beta_0)) \leq E(v, (\alpha, x_0)) + E(u, (x_0, 1 - \beta_0))$. Since $u|_{(\alpha, x_0)} = w|_{(\alpha, x_0)}$, we conclude that $E(w, (\alpha, x_0)) = E(u, (\alpha, x_0)) \leq E(v, (\alpha, x_0))$. Hence $E(w, (\alpha, 1 - \beta_0)) \leq E(v, (\alpha, 1 - \beta_0))$ and thus w is a solution of (5) with $|\{x \in (\alpha, 1 - \beta) : w(x) = 0\}| = 0$. \square

The continuity condition of Lemma 2.6 and Theorem 2.8 is only used if a minimizer u of the relaxed problem may have infinitely many transition layers $T_j = (l, r)$ with $u(l) = u(r) \in \{0, 1\}$. The following corollary catches a situation where this can be excluded *a priori*:

Corollary 2.9. *Let $I = (a, b)$ be an open interval, $1 < p \leq +\infty$, and $\alpha, \beta > 0$ with $\alpha + \beta < |I|$. Let E be a symmetric admissible energy such that for all $T \in \mathcal{I}$*

$$\begin{aligned} E(0, T) &= \min\{E(u, T), u|_{\partial T} = 0\}, \\ E(1, T) &= \min\{E(u, T), u|_{\partial T} = 1\}. \end{aligned}$$

Then there exists a constant $\gamma_0 > 0$, such that the abstract variational problem (2) admits a solution whenever $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

Proof. We follow the proof of Lemma 2.6 and Theorem 2.8. The transition layers of the form T_j^{00} and T_j^{11} can be omitted, by the following argument: if e.g. $u|_{(x_0, x_1)} \in (0, 1)$ and $u(x_0) = u(x_1) = 0$, we have

$E(u, (x_0, x_1)) \geq E(0, (x_0, x_1))$ and we can replace u by 0 on (x_0, x_1) . Hence the proofs work without using the continuity of E . \square

The following theorem gives existence for arbitrarily large transition layers, but only for a special class of energies:

Theorem 2.10 (Existence for special energies). *Let $I = (a, b)$ be an open interval, $1 < p \leq +\infty$, and let $\alpha, \beta > 0$ be such that $\alpha + \beta < |I|$. Let E be an admissible energy continuous with respect to the strong $W^{1,p}$ -topology on all open set in I . Suppose that there exists a constant function $\lambda \in (0, 1)$ such that*

$$E(\lambda, T) = \min_{\mu \in [0,1]} E(\mu, T) \quad (7)$$

for all nontrivial $T \in \mathcal{I}$. Then there exists a solution to the abstract variational problem (2).

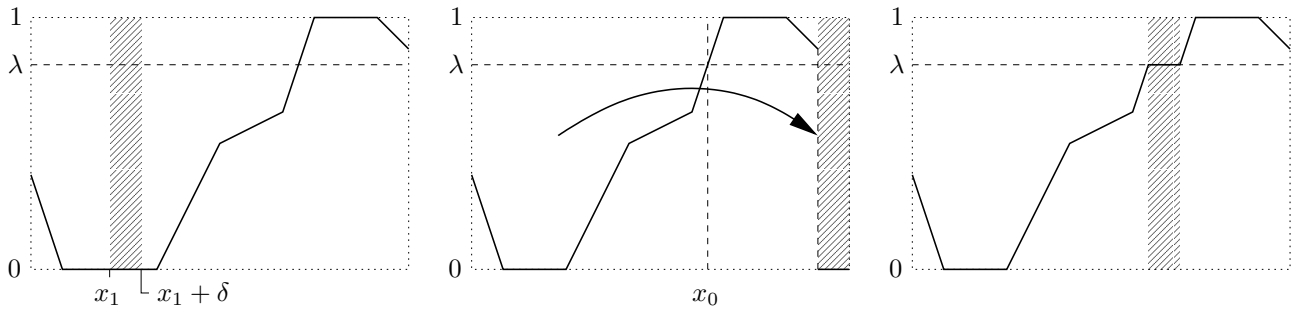


FIGURE 6. Left: assume the solution of the relaxed problem is zero on the interval $[x_1, x_1 + \delta]$. Middle: shift this interval to the right. Right: insert a constant piece with the optimal value λ such that the resulting function is continuous.

Proof. The following construction is illustrated in Figure 6. Let v be a solution of the relaxed problem (5) for I , α and β as given above satisfying for some $\delta, \varepsilon \geq 0$ the constraints $|\{v = 0\}| = \alpha + \delta$, $|\{v = 1\}| = \alpha + \varepsilon$. By Lemma 2.6 such a v exists and we can assume that there exists an $x_1 \in I$ such that $v|_{[x_1, x_1 + \delta]} = 0$. Let $\tilde{v} := S((x_1, x_1 + \delta), 1 - \delta)v$. Let x_0 be such that $\tilde{v}(x_0) = \lambda$. Then define

$$u(x) := \begin{cases} \tilde{v}(x), & x < x_0, \\ \lambda, & x_0 \leq x \leq x_0 + \delta, \\ \tilde{v}(x - \delta), & x > x_0 + \delta. \end{cases}$$

By applying Lemma 2.5 and Condition (7) we have $E(u, I) \leq E(v, I)$, moreover $|\{u = 0\}| = |\{v = 0\}| - \delta = \alpha$. In the same way we can take care of the second constraint, and the so defined u solves the original problem (2). \square

From Theorem 2.8 we have immediately the following corollary which was proved in Theorem 2.1 (H2) of [3] using an ODE method:

Corollary 2.11. *Let θ be a continuous function with $\operatorname{argmin} \theta \subset [0, 1]$ and $\min \theta = 0$, let I be an open interval, and $\alpha, \beta > 0$ with $\alpha + \beta < |I|$. Then there exists a constant $\gamma_0 > 0$ such that the problem*

$$\begin{cases} \text{Minimize } E(u) := \int_I \frac{1}{2} |u'(x)|^2 + \theta(u(x)) \, dx, \\ |\{u = 0\}| = \alpha, \quad |\{u = 1\}| = \beta, \end{cases}$$

admits a minimizer in $W^{1,2}(I, \mathbb{R})$, provided that $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

Proof. For $T \in \mathcal{I}$ take

$$E(u, T) := \int_T \frac{1}{2} |u'(x)|^2 + \theta(u(x)) \, dx.$$

Let u be a minimizer for $E(\cdot, \cdot)$ obtained by Theorem 2.8. The only condition that has to be checked is that no function $v \in W^{1,2}(I, \mathbb{R}) \setminus W^{1,2}(I, [0, 1])$ has a lower energy than u , but this can be excluded by a simple construction, cutting out the set I_0 where $v(x) \notin [0, 1]$ and inserting an interval of length $|I_0|$ where we define the function as a constant v_0 such that $\theta(v_0) = 0$. This gives a new function w with values in $[0, 1]$ and using the assumption $\operatorname{argmin} \theta \subset [0, 1]$ one easily checks that

$$\begin{aligned} E(v) &= \int_I \frac{1}{2} |v'(x)|^2 + \theta(v(x)) \, dx \\ &= \int_{I \setminus I_0} \frac{1}{2} |v'(x)|^2 + \theta(v(x)) \, dx + \int_{I_0} \frac{1}{2} |v'(x)|^2 + \theta(v(x)) \, dx \\ &> \int_{I \setminus I_0} \frac{1}{2} |v'(x)|^2 + \theta(v(x)) \, dx = \int_I \frac{1}{2} |w'(x)|^2 + \theta(w(x)) \, dx \\ &= E(w). \end{aligned}$$

This gives a contradiction to our assumption, and hence such a v cannot exist. □

Similarly, the following result is an immediate consequence of Theorem 2.10 (compare Th. 2.1 (H1) in [3]):

Corollary 2.12. *Let $1 < p < +\infty$, and let $f \in \mathcal{C}^1([0, 1] \times \mathbb{R}, \mathbb{R})$ satisfy $f(a, b) \geq C|b|^p$ for some constant $C > 0$. Furthermore assume that f is convex in the second variable, and that there exists a constant $\lambda \in (0, 1)$ such that $f(\lambda, 0) = 0$. Then the problem*

$$\begin{cases} \text{Minimize } E(u) := \int_I f(u(x), u'(x)) \, dx, \\ |\{u = 0\}| = \alpha, |\{u = 1\}| = \beta, \end{cases}$$

admits a minimizer in $W^{1,p}(I)$.

Admissible energies are not necessarily of (Lebesgue) integral form. This is illustrated by the following example (suggested to me by Massimiliano Morini):

Remark 2.13. Let $F: [0, 1] \rightarrow \mathbb{R}$ be a continuous function, then

$$E(u, (x_0, x_1)) := F(u(x_0)) - F(u(x_1))$$

is an admissible energy on $W^{1,2}$ which is continuous with respect to the strong $W^{1,2}$ -norm.

We can even have symmetric admissible energies which are not of integral form:

Remark 2.14. Let $F: [0, 1] \rightarrow \mathbb{R}$ be a continuous function and define $v(x) := u'(x)$ whenever $u'(x)$ exists, and $v(x) := 0$ elsewhere, then

$$E(u, (x_0, x_1)) := F(v(x_0)) - F(v(x_1))$$

is a symmetric admissible energy on $W^{1,p}$ for $p > 2$.

In the next section we will apply Theorem 2.8 to nonconvex variational problems with volume constraints.

3. APPLICATIONS TO NONCONVEX PROBLEMS

The main result of this section is:

Theorem 3.1 (Existence for nonconvex problems). *Let $1 < p < +\infty$, and let $f \in C^1([0, 1] \times \mathbb{R}, \mathbb{R})$ be a nonnegative function with $f(a, b) \geq C|b|^p$ for a constant $C > 0$. Moreover, assume that*

- (a) $f(0, 0) = 0$;
- (b) $\operatorname{sgn} \partial_a f(a, b_1) = \operatorname{sgn} \partial_a f(a, b_2)$ for all $a \in [0, 1]$ and all $b_1, b_2 \in \mathbb{R}$;
- (c) $f(a, b) = f(a, -b)$ for all $a \in [0, 1]$ and all $b \in \mathbb{R}$.

Then there exists $\gamma_0 > 0$ such that the abstract variational problem (2) with the energy

$$E(u, T) := \int_T f(u(x), u'(x)) \, dx$$

admits a $W^{1,p}$ -solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

The main purpose of this theorem is to illustrate a concrete application of the abstract theory, hence we are not attempting to give the most general result possible.

Proof. To apply Theorem 2.8 we have to prove that E is symmetric, continuous and satisfies the conditions (i–iii). E is obviously continuous and symmetric and satisfies (ii) and (iii), hence it remains to assert (i). Here we apply Remark 2.4, so we only have to check the existence of a solution to the transition problem. For this purpose we consider the convexified transition problem (4), where we replace f by f^{**} (the convexification of f with respect to the second variable). Due to the convexity of f^{**} and the coercivity condition that we have assumed, standard results from the calculus of variation guarantee the existence of a solution $u \in W^{1,p}(I)$. If $f(u(x), u'(x)) = f^{**}(u(x), u'(x))$ for a.e. $x \in I$, then the function u also solves the transition problem for f . If not, let $M \subset I$ denote the set of points x such that $f(u(x), u'(x)) \neq f^{**}(u(x), u'(x))$. M is open and hence a union of disjoint open intervals. We deduce from the coercivity bound $f(a, b) \geq C|b|^p$ that for all $y \in \mathbb{R}$ there exists a minimal $|y_1|$ with $y_1/y \geq 1$ such that $f(\cdot, y_1) = f^{**}(\cdot, y_1)$ and a maximal $|y_0|$ with $y_0/y \leq 1$ such that $f(\cdot, y_0) = f^{**}(\cdot, y_0)$. By this and Condition (b) we can decompose M (up to a set of measure zero) into open intervals $M_i = (m_i, n_i)$ such that on each M_i the function $\operatorname{sgn} \partial_u f(u, u')$ is constant and $u'(x) \in (y_0, y_1)$ for all $x \in M_i$ such that

$$f^{**}(u(x), u'(x)) = \frac{y_1 - u'(x)}{y_1 - y_0} f(u(x), y_0) + \frac{u'(x) - y_0}{y_1 - y_0} f(u(x), y_1)$$

and $f(\cdot, u'(y_i)) = f^{**}(\cdot, u'(y_i))$ for $i = 0, 1$.

Consider now the case that $\partial_u f(u(x), u'(x)) < 0$ on M_i . Then we define the affine interpolation

$$v_i(x) := \begin{cases} u(x) & \text{for } x \notin M_i, \\ u(m_i) + y_0(x - m_i) & \text{for } m_i < x \leq \xi, \\ u(n_i) - y_1(x - n_i) & \text{for } \xi < x < n_i, \end{cases}$$

where

$$\xi := \frac{u(n_i) - u(m_i) - (n_i - m_i)y_1}{y_0 - y_1}.$$

Now $\int_{M_i} f^{**}(v_i'(x), v_i(x)) \, dx \leq \int_{M_i} f^{**}(u'(x), u(x)) \, dx$, and for every $x \in M_i$ we have by construction that $f^{**}(v_i(x), v_i'(x)) = f(v_i(x), v_i'(x))$. An analogous argument applies to the cases where $\partial_u f(u(x), u'(x)) \geq 0$. The constraint $v_i \in [0, 1]$ is ensured by the Condition (a).

Define $v|_{\bar{M}_i} := v_i$ to get a solution to the nonconvex problem (4). Thus Condition (i) is valid. It remains to apply Theorem 2.8 to get the existence result. \square

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $1 < p < +\infty$, let $\theta \in C^1([0, 1], \mathbb{R}_+)$ and $\phi \in C^1(\mathbb{R}, \mathbb{R}_+)$. Assume that ϕ is symmetric and $\phi(b) \geq C|b|^p$ for a constant $C > 0$, and that $\phi(0) = 0$. Then there exists a $\gamma_0 > 0$ such that*

$$\begin{cases} \text{Minimize } E(u) := \int_I \phi(u') + \theta(u) \, dx, \\ |\{u = 0\}| = \alpha, \quad |\{u = 1\}| = \beta, \end{cases}$$

admits a solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

Without a coercivity condition, existence for the transition problem fails in general. A typical example is given by $f(u, u') := \exp(-|u'|^2)$. However, we have the following result, which can be derived from the proof of Theorem 3.1:

Corollary 3.3. *Let $1 < p \leq +\infty$, and let $f \in C^1([0, 1] \times \mathbb{R}, \mathbb{R})$ be a nonnegative function, symmetric in the second variable. Moreover assume that*

- (a) $f(0, 0) = 0$;
- (b) $\text{sgn } \partial_u f(a, b_1) = \text{sgn } \partial_u f(a, b_2)$ for all $a \in [0, 1]$ and all $b_1, b_2 \in \mathbb{R}$;
- (c) the transition problem (4) for f^{**} , the convexification of f with respect to the second variable, admits a $W^{1,p}$ -solution;
- (d) for all $y \in \mathbb{R}$ there exists b' with $b'/y \geq 1$ such that $\phi(b') = \phi^{**}(b')$.

Then there exists a $\gamma_0 > 0$ such that the abstract variational problem (2) with the energy $E(u, I_0) := \int_{I_0} f(u(x), u'(x)) \, dx$ admits a $W^{1,p}$ -solution if $\gamma := |I| - (\alpha + \beta) < \gamma_0$.

As a trivial example of an energy density which satisfies the conditions of Corollary 3.3, but violates the coercivity condition of Theorem 3.1, take $f(a, b) := \sin^2 b$.

4. HIGHER DIMENSIONAL PROBLEMS

It would seem natural to extend the one-dimensional existence results above, and in particular Theorem 2.10, to higher dimensional problems. However, without specific assumptions, *e.g.* on the smoothness of the boundary of the domain $\Omega \subset \mathbb{R}^n$, an extension of Theorem 2.10 to the higher dimensional case fails. We demonstrate this with the following counterexample:

Example 4.1. Let $p > 2$. There exist a bounded domain $\Omega \subset \mathbb{R}^2$ and a smooth, positive, convex function ψ with $C_1|Y|^p \leq \psi(Y) \leq C_2|Y|^p$ for some $C_1, C_2 > 0$ such that there exists **no** $\gamma_0 > 0$ with the property that the volume constraint problem

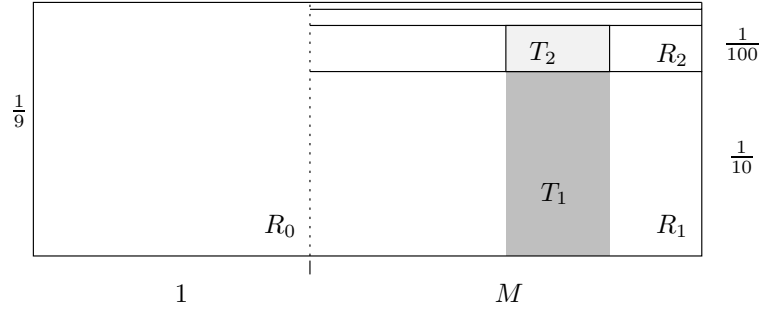
$$\begin{cases} \text{Minimize } E(u) := \int_{\Omega} \psi(|\nabla u(x)|) + |u(x)| \, dx, \\ |\{u = 0\}| = \alpha, \quad |\{u = 1\}| = \beta \end{cases} \quad (8)$$

admits a $W^{1,p}$ -solution for all α, β with $\gamma := |\Omega| - (\alpha + \beta) \leq \gamma_0$.

Outline of the proof: first we define ψ . Let $\psi|_{(-1,1)} := \frac{1}{2}|\cdot|^2$, $\psi|_{\mathbb{R} \setminus (-2,2)} := |\cdot|^p$ and interpolate smoothly, such that ψ is convex. We consider now an auxiliary problem: We want to find sequences α_j, β_j with $\alpha_j + \beta_j < |\Omega|$ and $\gamma_j := |\Omega| - (\alpha_j + \beta_j) \rightarrow 0$, such that the volume constraint problems

$$\begin{cases} \text{Minimize } E(u) := \int_{\Omega} \frac{1}{2}|\nabla u(x)|^2 + |u(x)| \, dx, \\ |\{u = 0\}| = \alpha_j, \quad |\{u = 1\}| = \beta_j \end{cases} \quad (9)$$

do not admit a $W^{1,p}$ -solution.

FIGURE 7. The set Ω and the transition layers T_j .

We use Proposition 2.6 of [3] which asserts that the one-dimensional volume constrained problem (2) with $E(u, I) := \int_I \frac{1}{2}|u'|^2 + |u| dx$ does not admit a solution for $\gamma > \sqrt{2}$. Define $\Omega := R_0 \cup R \subset \mathbb{R}^2$ (see Fig. 7), where

$$R_0 := (-1, 0) \times (0, 1/9), \quad R := \bigcup_{i=1}^{\infty} R_i, \quad R_i := [0, M) \times (10^{1-i}, 10^{-i}),$$

and $M > 0$ will be chosen later. As sequences α_j and β_j we set

$$\alpha_j := \frac{1+M}{9} - \frac{1}{4}10^{-j}M - 2 \times 10^{-j}, \quad \beta_j := \frac{1}{4}10^{-j}M,$$

hence $\gamma_j = 2 \times 10^{-j}$. We denote the $W^{1,p}$ -solution of the relaxed problem

$$\begin{cases} \text{Minimize } E(u) := \int_{\Omega} \frac{1}{2}|\nabla u(x)|^2 + |u(x)| dx, \\ |\{u = 0\}| \geq \alpha_j, \quad |\{u = 1\}| \geq \beta_j \end{cases}$$

by u_j .

We claim that the transition layer $T_j := \{x \in \Omega : u_j(x) \in (0, 1)\}$ of the minimizer u_j has the form $T_j = \{(x_1, x_2) \in R_j : 3M/4 - \sqrt{2} < x_1 < 3M/4\}$. Since $\gamma_j > 10^{-j}\sqrt{2} = |T_j|$, we conclude that u_j cannot be a solution of problem (9). Since u_j can be approximated by functions satisfying the volume constraints with energy arbitrarily close to the energy of u_j , this proves the non-solvability of the auxiliary problem (9).

Since $W^{1,p}(\Omega) \subset C(\Omega)$, the transition layer is open. The main idea is now to observe that the energy of a transition layer T can be estimated from below by $Cl(T)$, where $l(T)$ is the ‘‘length’’ of the transition layer and $C > 0$ is a constant. More precisely we define for every line g in Ω

$$l(g) := |\{x \in g : n_x \cap \{u = 0\} \neq \emptyset, n_x \cap \{u = 1\} \neq \emptyset\}|,$$

where n_x is the line through x orthogonal to g .

We denote the energy of the solution of the one-dimensional transition problem on $(0, t)$ by $\phi(t)$. By Proposition 2.6 in [3] we know that $\phi(t) \geq \phi(\sqrt{2})$. Hence we conclude that

$$E(T_j) := \int_{T_j} \frac{1}{2}|\nabla u|^2 + |u| dx \geq \int_0^{l(g)} \phi(\sqrt{2}) = l(g)\phi(\sqrt{2}).$$

Now define

$$l(T_j) := \sup_g l(g).$$

If T_j does not connect the upper and lower boundary in some R_i , $i \leq j$, then $l(T_j) \geq C_0 10^{-j}M$, and hence the energy is bounded from below by $E_1 := C_0\phi(\sqrt{2}) 10^{-j}M$. If T_j connects the upper and lower boundary in some R_i , $i < j$, then its energy can be estimated from below by $E_2 := 10^{-i}\phi(\sqrt{2})$, this can be shown by means of an argument similar to the one given below.

If T_j is as in our claim and $u_j(x_1, x_2) = v(x_1 - 3M/4 - 2)$ on T_j , where v is the solution of the one dimensional transition problem on $(0, \sqrt{2})$, then the energy on the transition layer is $10^{-j}\phi(\sqrt{2})$. This is smaller than E_1 and E_2 for M sufficiently large.

Now we use the estimate

$$\int_{T_j} \frac{1}{2} |\nabla u_j|^2 + |u_j| \, dx \geq \int_l^h \int_0^M \frac{1}{2} |\partial_x u_j|^2 + |u_j| \, dx \, dy$$

with $l := \sum_{i=1}^{j-1} 10^{-i}$, $h := \sum_{i=1}^j 10^{-i}$. We can estimate further where we denote $x_1(y) := \inf\{x \in (0, M) : (x, y) \in T_j\}$, $x_2(y) := \sup\{x \in (0, M) : (x, y) \in T_j\}$ and get

$$\int_l^h \int_0^M \frac{1}{2} |\partial_x u_j|^2 + |u_j| \, dx \, dy \geq \int_l^h \int_{x_1(y)}^{x_2(y)} \phi(x_2(y) - x_1(y)) \, dx \, dy.$$

Now, since ϕ is convex (see [3]), applying Jensen's inequality we deduce that a straight transition layer

$$\begin{aligned} T_j &= (3M/4 - \sqrt{2}, 3M/4) \times (l, h) \\ &= \{(x_1, x_2) \in R_j : 3M/4 - \sqrt{2} < x_1 < 3M/4\} \end{aligned}$$

is optimal. Since $|T_j| = 10^{-j}\sqrt{2} < \gamma_j$, we have non-existence for the auxiliary problem (9).

Now let w_j be a solution for the relaxed version of (8) for α_j, β_j , then since $\psi(w_j) \geq \frac{1}{2}|w|^2$ and $\psi(u_j) = \frac{1}{2}|u_j|^2$, we have $E(w_j) \geq \int \frac{1}{2} |\nabla w_j|^2 + |w_j| \geq \int \frac{1}{2} |\nabla u_j|^2 + |u_j|$. And since $|\nabla u_j| \leq 1$ (see [3]), we get $E(w_j) \geq E(u_j)$, where equality holds only if $u_j = w_j$. Hence we have proved non-existence for the original problem. \square

In the last example we made crucial use of the non-smoothness of the boundary of Ω . In fact it seems natural to propose the following conjecture:

Conjecture 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain. Suppose that $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is positive, continuous, strictly convex in the second variable and that there exists a constant $C > 0$ such that $f(a, b) \geq C|b|^p$ for all $a \in [0, 1]$. Then there exists a $\gamma_0 > 0$ with the property that the volume constraint problem*

$$\begin{cases} \text{Minimize } E(u) := \int_{\Omega} f(u(x), |\nabla u(x)|) \, dx, \\ |\{u = 0\}| = \alpha, |\{u = 1\}| = \beta \end{cases}$$

admits a $W^{1,p}$ -solution for all α, β with $\gamma := |\Omega| - (\alpha + \beta) \leq \gamma_0$.

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