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ON THE PANEITZ ENERGY ON STANDARD THREE SPHERE

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Abstract. We prove that the Paneitz energy on the standard three-sphere S^3 is bounded from below and extremal metrics must be conformally equivalent to the standard metric.

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1. Introduction

The study of the Q-curvature equations, a natural generalization of the Yamabe equation to higher order equation, began with the work of Paneitz [8], Branson [2] and Fefferman-Graham [5]. Several authors ([4, 6, 7, 10]) have studied this equation in dimensions higher than four due to the natural constraints from Sobolev inequalities. In [11], Xu and Yang first call attention to the problem in dimension three and started their preliminary study of the fourth order Paneitz equation in dimension three. The Paneitz operator on a three dimensional manifold M^3 is defined by

$$P_g = (-\Delta_g)^2 + \delta\left(\frac{5}{4}R_g g - 4Ric_g\right)d - \frac{1}{2}Q_g,$$

where the Q-curvature is given by

$$Q_g = -2|Ric_g|^2 + \frac{23}{32}R_g^2 - \frac{1}{4}\Delta R_g.$$

Under a conformal change of metrics $g_1 = \phi^{-4}g$ with $\phi > 0$, the Paneitz operator has the following property:

$$P_{g_1}(w) = \phi^7 P_g(\phi w), \qquad \forall w \in W^{2,2}(M^3). \tag{1.1}$$

Therefore, similar to the scalar curvature problems, the Q-curvature problems are related to the following fourth order nonlinear equation:

$$P(u) = -\frac{1}{2}Q_{g_1}u^{-7}. \tag{1.2}$$

It should be noted that the negative power $-7 = \frac{n+4}{n-4}$ only appears in the case of dimension $n = 3$. A similar situation arises for the conformally laplacian equation in dimension one [1].

Keywords and phrases. Paneitz operator, symmetrization, extremal metric.

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In [11], Xu and Yang studied equation (1.2) in three dimensional manifolds on which the Paneitz operator is positively. These do not include the standard three sphere, on which the Paneitz operator has a negatively eigenvalue. The difficulty comes from the verification that the Paneitz energy is bounded from below. In this note, we solve this problem and obtain the Liouvillve type theorem about the extremal metrics on the Paneitz energy on S^3.

For a given positive function $\phi(x) \in W^{2,2}(S^3)$, the Paneitz energy is defined by

$$I(\phi) = \frac{\int_{S^3} P\phi \cdot \phi}{\left(\int_{S^3} \phi^{-6}\right)^{-1/3}} = \frac{\int_{S^3} \left(|\Delta\phi|^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{15}{16}|\phi|^2\right)}{\left(\int_{S^3} \phi^{-6}\right)^{-1/3}}, \tag{1.3}$$

where $P = (-\Delta)^2 + \frac{1}{2}\Delta - \frac{15}{16}$ is the Paneitz operator with respect to the standard metric g_{S^3}, see, *e.g.* the paper of Xu and Yang [11].

We are going to prove the following.

Theorem 1.1.

$$\inf_{\phi>0, \phi\in W^{2,2}(S^3)} I(\phi) = -\frac{15}{16} \cdot (2\pi)^{4/3}$$

is attained by $u(x)$, where $u(x)$ is of the form that $u^{-4}g_{S^3}$ is the pullback of the standard metric via a conformal transformation.

Remark 1.1. The above theorem actually implies the following sharp inequality for the Paneitz operator: for any positive function $\phi \in W^{2,2}(S^3)$,

$$\int_{S^3} \left(|\Delta\phi|^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{15}{16}|\phi|^2\right) \geq -\frac{15}{16} \cdot (2\pi)^{4/3} \cdot \left(\int_{S^3} \phi^{-6}\right)^{-1/3},$$

and the equality holds if and only if $\phi(x)$ is of the form that $\phi^{-4}g_{S^3}$ is the pullback of the standard metric *via* a conformal transformation.

We shall sketch our arguments as follows. Let $\{w_k\}_{i=1}^{\infty} \in C^\infty(S^3)$ be a positive minimizing sequence with $\|w_k\|_{L^{-6}} = 1$. Based on Talenti's theorem of symmetrization for Laplace operator [9], we first obtain, in Section 2, a rotationally symmetric minimizing sequence from $\{w_k\}_{k=1}^{\infty}$; then in Section 3, we use the conformal invariant property of P to obtain a bounded minimizing sequence $\{h_k\}_{k=1}^{\infty}$, which eventually converges to a minimizer h_∞ in $W^{2,2}(S^3)$. A technical lemma is proved in the last section. Throughout the note, we denote N, S as the north and south poles of S^3, respectively. We may also use the common C to represent various constants.

2. Symmetrization

In this section, we prove

Proposition 2.1. *Let w be a positive smooth function on S^3 with $\|w\|_{L^{-6}} = 1$ and $\int_{S^3} Pw \cdot w < 0$. For any $\epsilon > 0$, there is a rotationally symmetric positive function $w^{\#} \in W^{2,2}(S^3)$ such that*

$$I(w^{\#}) \leq I(w) + \epsilon.$$

Proof. Without loss of generality, we may assume that $w(N) = \max_{S^3} w(x)$. For fixed ϵ, we choose small δ_1 such that for $\delta < \delta_1$,

$$\left|\int_{B_\delta(N)} Pw \cdot w\right| + \left|\int_{B_\delta(N)} w^{-6}\right| \leq \frac{\epsilon}{100}, \tag{2.4}$$

where and throughout this section, we denote $B_\delta(x)$ as the geodesic ball of radius δ with center at x.

Consider $w_1 = w \cdot \eta + (1-\eta) \cdot w(N)$, where

$$\eta = \begin{cases} 0, & dist(x, N) \leq \delta \\ 1, & dist(x, N) \geq 2\delta \end{cases}$$

and $|\nabla \eta| \leq 10/\delta$. From the definition of P we have

$$\begin{aligned} \Big|\int_{B_{2\delta}(N)} P w_1 \cdot w_1\Big| &\leq \left| \int_{B_{2\delta}(N)} \left(|\Delta w_1|^2 - \frac{1}{2}|\nabla w_1|^2 - \frac{15}{16}|w_1|^2 \right) \right| \\ &\quad + \left| \int_{\partial B_{2\delta}(N)} \left(\frac{\partial w_1}{\partial \nu} \cdot \Delta w_1 + \frac{\partial \Delta w_1}{\partial \nu} \cdot w_1 \right) \right| \\ &\leq C\delta^3 \cdot \left(\max_{B_{2\delta}} |\Delta w_1|^2 + \max_{B_{2\delta}} |\nabla w_1|^2 + C \right) \\ &\quad + C\delta^2 \cdot \left(\max_{\partial B_{2\delta}} |\partial \Delta w_1 / \partial \nu| + \max_{\partial B_{2\delta}} |\nabla w_1 \cdot \Delta w_1| \right). \end{aligned} \tag{2.5}$$

At the maximal point N of w, we have

$$|\nabla w(x)| = O(1)\delta, \quad w(x) - w(N) = O(1)\delta^2, \quad \forall x \in B_{2\delta}(N).$$

Thus for $x \in B_{2\delta}(N)$,

$$\Delta w_1 = w \cdot \Delta \eta - w(N) \cdot \Delta \eta + 2\nabla w \cdot \nabla \eta + \Delta w \cdot \eta \leq C.$$

On the boundary $\partial B_{2\delta}(N)$

$$\partial \Delta w_1 / \partial \nu \leq \frac{C}{\delta}\cdot$$

It follows from (2.5) that

$$\left| \int_{B_{2\delta}(N)} P w_1 \cdot w_1 \right| \leq C\delta. \tag{2.6}$$

We thus can choose $\delta_2 \leq \delta_1$ such that for $\delta < \delta_2$,

$$I(w_1) \leq I(w) + \frac{\epsilon}{100}\cdot \tag{2.7}$$

Therefore, without loss of generality, we can assume that for fixed $\bar{\delta} < \delta_2$

$$w(x) = w(N) \qquad \forall x \in B_{\bar{\delta}}(N). \tag{2.8}$$

Next, we reduce the problem onto $\mathbb{R}^3$ *via* the stereographic projection $\Phi : \ x \in S^3 \to y \in \mathbb{R}^3$, given by

$$x_i = \frac{2y_i}{1+|y|^2}, \quad \text{for } \ i = 1, 2, 3; \quad x_4 = \frac{|y|^2 - 1}{|y|^2 + 1}\cdot$$

Let $v(y)$ be the positive function such that

$$g_{S^3} = \sum_{i=1}^{4} \mathrm{d}x_i^2 = \left(\frac{2}{1+|y|^2} \right)^2 \mathrm{d}y^2 := v^{-4} \mathrm{d}y^2 := v^{-4} g_0.$$

From (2.6) and (1.1) we have for $\delta < \delta_2$,

$$\frac{\epsilon}{100} \geq \left| \int_{B_\delta(N)} Pw \cdot w \mathrm{d}v_g \right| = \left| \int_{B_\delta(N)} P_0(vw) \cdot vw \mathrm{d}v_{g_0} \right|$$
$$= \left| \int_{B^c_{R_\delta}(0)} (-\Delta_0)^2 (wv) \cdot (wv) \right| \mathrm{d}y, \tag{2.9}$$

where $B^c_{R_\delta}(0)$ is the exterior ball of radius R_δ centered at the origin in $\mathbb{R}^3$ and $\partial B_{R_\delta} := \Phi(\partial B_\delta(N))$. Integrating by parts, we have

$$\left| \int_{B_{R_\delta}(0)} (-\Delta)^2 (wv) \cdot (wv) \right| = \int_{B_{R_\delta}(0)} |\Delta(wv)|^2 - \int_{\partial B_{R_\delta}(0)} \frac{\partial(wv)}{\partial \nu} \cdot \Delta(wv)$$
$$+ \int_{\partial B_{R_\delta}(0)} \frac{\partial(\Delta(wv))}{\partial \nu} \cdot (wv). \tag{2.10}$$

Throughout the rest of this section, we fix $\bar{R}$ to be the radius of the ball $\Phi(\partial B_{\bar{\delta}}(N))$; And we always choose $\delta < \bar{\delta}$ (thus $R_\delta > \bar{R}$). For convenience, we denote $a := w(N)$. Since $w(x) = a$ in $B_{\bar{\delta}}(N)$, we know from (2.8) that $w(y) = a$ for $|y| \geq \bar{R}$. Thus, in $B^c_{\bar{R}} \subset \mathbb{R}^3$ one can check that

$$wv(y) = \frac{a}{\sqrt{2}} \left(1 + |y|^2\right)^{1/2}; \quad \frac{\partial(wv)}{\partial \nu}(y) = \frac{a}{\sqrt{2}} \left(1 + |y|^2\right)^{-1/2} |y|,$$

and

$$\Delta(wv)(y) = \frac{a}{\sqrt{2}} \left(3 + 2|y|^2\right) \cdot \left(1 + |y|^2\right)^{-\frac{3}{2}}; \quad \frac{\partial(\Delta(wv))}{\partial \nu}(y) = \frac{a}{\sqrt{2}} \left(5 + 2|y|^3\right) \cdot \left(1 + |y|^2\right)^{-\frac{5}{2}}. \tag{2.11}$$

Therefore on boundary $\partial B_R(0)$ for any $R > \bar{R}$,

$$wv(y) = \frac{aR}{\sqrt{2}} \left[1 + (1 + o(1))R^{-2}\right]; \tag{2.12}$$

$$\frac{\partial(wv)}{\partial \nu}(y) = \frac{a}{\sqrt{2}} \left[1 - (1 + o(1))R^{-2}\right]; \tag{2.13}$$

$$\Delta(wv)(y) = \frac{\sqrt{2}a}{R} \cdot \left(1 + o(1)R^{-2}\right); \tag{2.14}$$

$$\frac{\partial \Delta((wv))}{\partial \nu}(y) = -\frac{\sqrt{2}a}{R^2} \cdot \left(1 + o(1)R^{-2}\right), \tag{2.15}$$

where $o(1) \to 0$ as $R \to \infty$. It follows that

$$-\int_{\partial B_R(0)} \frac{\partial(wv)}{\partial \nu} \cdot \Delta(wv) + \int_{\partial B_R(0)} \frac{\partial(\Delta(wv))}{\partial \nu} \cdot (wv) = -6\omega_3 a^2 R + o(1)R^{-1}, \tag{2.16}$$

where $\omega_3 = 4\pi/3$ is the volume of the unit ball in $\mathbb{R}^3$.

We start the symmetrization procedure for $(wv)(y)$ in the ball $B_R(0)$ for any fixed $R > \bar{R}$. Let $h(y) := wv(y)$. Notice $h(R) = \max_{|y| \leq R} h(y)$. We consider $\tau(y) = h(R) - h(y)$ for $y \in B_R(0)$, and let $\tau^\#$ be the positive solution to

$$\begin{cases} \Delta \tau^\# = (\Delta \tau)^* & \text{in } B_R(0) \\ \tau^\#(R) = 0, \end{cases} \tag{2.17}$$

where $(\Delta\tau)^*$ is the non-increasing radially symmetric rearrangement of $|\Delta\tau|$. Let τ^* be the non-increasing symmetric rearrangement of τ in $B_R(0)$, then $\tau^*(y) = h(R) - h_*(y)$ for $y \in B_R(0)$, where $h_*(y)$ is the non-decreasing symmetric rearrangement of $h(y)$ in $B_R(0)$. Since $\tau(y) = 0$ on the boundary $\partial B_R(0)$, it follows from a theorem of Talenti [9] that

$$\tau^{\#}(y) \geq \tau^*(y) = h(R) - h_*(y) \qquad \forall y \in B_R(0).$$

Let

$$h^{\#} = h(R) - \tau^{\#}, \tag{2.18}$$

then

$$\begin{cases} \Delta h^{\#} = -(\Delta\tau)^* = -(|\Delta h|)^* & \text{in } \ B_R(0) \\ h^{\#} \leq h_* & \text{in } \ B_R(0). \end{cases} \tag{2.19}$$

Thus, for $r < R$,

$$\int_{B_r(0)} |\Delta(wv)^{\#}|^2 = \int_{B_r(0)} |(|\Delta h|)^*|^2 = \int_{B_r(0)} |\Delta h|^2 = \int_{B_r(0)} |\Delta(wv)|^2. \tag{2.20}$$

Lemma 2.1. *If we choose $R \geq 3\bar{R}$, and define $(wv)^{\#}$ in $B_R(0)$, then on the boundary $\partial B_{R-2\bar{R}}(0)$, we have the following equalities*

$$(wv)^{\#}\left(|y| = R - 2\bar{R}\right) = \frac{a\left(R - 2\bar{R}\right)}{\sqrt{2}}\left[1 + \frac{O(1)}{\left(R - 2\bar{R}\right)^2}\right]; \tag{2.21}$$

$$\frac{\partial (wv)^{\#}}{\partial \nu}\left(|y| = R - 2\bar{R}\right) = \frac{a}{\sqrt{2}} + \frac{O(1)}{\left(R - 2\bar{R}\right)^2}; \tag{2.22}$$

$$\Delta(wv)^{\#}\left(|y| = R - 2\bar{R}\right) = \frac{\sqrt{2}a}{R - 2\bar{R}} + \frac{O(1)}{\left(R - 2\bar{R}\right)^4}, \tag{2.23}$$

where $O(1)$ is a bounded term (bounded by a uniform constant independent of R). In addition, there is a sequence of radii $R_i \to \infty$ such that if we define $(wv)^{\#}$ in $B_{R_i}(0)$, then on the boundary $\partial B_{R_i - 2\bar{R}}(0)$

$$\frac{\partial \Delta((wv)^{\#})}{\partial \nu}\left(|y| = R_i - 2\bar{R}\right) = -\frac{\sqrt{2}a}{\left(R_i - 2\bar{R}\right)^2} + \frac{O(1)}{\left(R_i - 2\bar{R}\right)^4}, \tag{2.24}$$

where $O(1)$ is a bounded term (bounded by a uniform constant independent of R_i).

We relegate the proof of Lemma 2.1 to the last section. We now define $(wv)^{\#}$ as before in $B_{R_i}(0)$. From Lemma 2.1, one can check that

$$-\int_{\partial B_{R_i - 2\bar{R}}} \frac{\partial (wv)^{\#}}{\partial \nu} \cdot \Delta(wv)^{\#} + \int_{\partial B_{R_i - 2\bar{R}}} \frac{\partial (\Delta(wv)^{\#})}{\partial \nu} \cdot (wv)^{\#} = -6\omega_3\left(R_i - 2\bar{R}\right)a^2 + \frac{O(1)}{\left(R_i - 2\bar{R}\right)}. \tag{2.25}$$

For a small positive number $\gamma \ll 1$, we can choose a radially symmetric positive function $\tilde{w} \in W^{2,2}_{loc}(\mathbb{R}^3)$ such that

$$\tilde{w}(y) = \begin{cases} (wv)^{\#} & |y| \leq R_i - 2\bar{R} \\ \dfrac{a(1 + y^2)^{1/2}}{\sqrt{2}} & |y| \geq R_i - 2\bar{R} + \gamma, \end{cases}$$

and $\forall y \in B_{R_i - 2\bar{R} + \gamma}(0) \setminus B_{R_i - 2\bar{R}}(0)$,

$$\tilde{w}(y) \leq CR_i, \quad |\Delta\tilde{w}(y)|^2 \leq CR_i^{-2}, \quad (-\Delta)^2\tilde{w}(y) \leq CR_i^{-3}.$$

The existence of such $\tilde{w}$ is guaranteed by (2.12)–(2.15) and Lemma 2.1. We thus choose $\gamma < \gamma_1$ for some small γ_1, such that for sufficiently large R_i,

$$\int_{B_{R_i-2\bar{R}+\gamma}(0)\backslash B_{R_i-2\bar{R}}(0)} |\Delta\tilde{w}|^2 \mathrm{d}y + \int_{B_{R_i-2\bar{R}+\gamma}(0)\backslash B_{R_i-2\bar{R}}(0)} \tilde{w}(-\Delta)^2\tilde{w}\mathrm{d}y \le \frac{\epsilon}{100}. \tag{2.26}$$

Finally, we define $w^{\#} = \tilde{w}/v$ on S^3 and have: for large enough R_i,

$$\begin{aligned}\int_{S^3} Pw\cdot w\mathrm{d}v_g \pm \frac{\epsilon}{100} &= \int_{B_{R_i-2\bar{R}}} (-\Delta)^2(wv)\cdot(wv)\mathrm{d}v_{g_0}\ (by\ \ (2.9))\\ &= \int_{B_{R_i-2\bar{R}}} (-\Delta)^2(wv)^{\#}\cdot(wv)^{\#}\mathrm{d}v_{g_0} \pm \frac{\epsilon}{50} \qquad (by\ \ (2.10),(2.16),(2.20)\ and\ (2.25))\\ &= \int_{B_{R_i-2\bar{R}+\gamma}} (-\Delta)^2\tilde{w}\cdot\tilde{w}\mathrm{d}v_{g_0} \pm \frac{\epsilon}{25} \qquad (by\ \ (2.26))\\ &= \int_{S^3\backslash\Phi^{-1}(B_{R_i-2\bar{R}+\gamma})} Pw^{\#}\cdot w^{\#}\mathrm{d}v_g \pm \frac{\epsilon}{25}\\ &= \int_{S^3} Pw^{\#}\cdot w^{\#}\mathrm{d}v_g \pm \frac{\epsilon}{10},\end{aligned} \tag{2.27}$$

where we use the fact that $w^{\#}(y) = a$ in $\Phi^{-1}(B_{R_i-2\bar{R}+\gamma})$ in the last equality.

On the other hand, from (2.19) and the definition of $w^{\#}$, we have

$$\left(\int_{S^3}(w^{\#})^{-6}\mathrm{d}v_g\right)^{1/3} \ge \left(\int_{S^3} w^{-6}\mathrm{d}v_g\right)^{1/3} - \frac{\epsilon}{10}. \tag{2.28}$$

Notice that $\int_{S^3} Pw\cdot w < 0$, we thus obtain Proposition 2.1 from (2.27) and (2.28). □

3. Convergence

Existence of extremal functions

Let $\{w_k\}_{k=1}^{\infty}$ be a minimizing sequence of $\inf I(u)$ with the following properties:

$$w_k > 0, \quad \int_{S^3} w_k^{-6} = 1. \tag{3.29}$$

We shall consider two cases.

Case 1. Up to a subsequence of $\{w_k\}_{k=1}^{\infty}$,

$$||w_k||_{L^\infty} \le C < \infty. \tag{3.30}$$

As a consequence we have: $||w_k||_{L^2} \le C$. i Since $\int_{S^3} Pw_k\cdot w_k \le 0$, it follows from Bochner's formula that

$$\int_{S^3} Pw_k\cdot w_k = \int_{S^3}\left\{\left|\nabla^2 w_k\right|^2 + \frac{3}{2}|\nabla w_k|^2 - \frac{15}{16}w_k^2\right\},$$

thus $||w_k||_{W^{2,2}} \le C$. Therefore $w_k \to w_o$ weakly in $W^{2,2}(S^3)$. From Sobolev embedding theorem, we know that $w_o \in C^{0,\frac{1}{2}}(S^3)$, and $w_o \ge 0$. We claim that $w_o > 0$ on S^3. Otherwise, there is a point $x_0 \in S^3$ such that

$w_o(x_0) = 0$. This together with $w_o \in C^{0,\frac{1}{2}}(S^3)$ yields

$$\int_{S^3} w_o^{-6} = \infty.$$

On the other hand, from Fatou's lemma, we have

$$\int_{S^3} w_o^{-6} = \int_{S^3} \underline{\lim}_{k\to\infty} w_k^{-6} \leq \underline{\lim}_{k\to\infty} \int_{S^3} w_k^{-6} = 1.$$

in contradiction to the previous assertion. Thus $w_o > 0$. It follows that for any $\epsilon > 0$, as k becomes sufficiently large, $w_k^{-6} \leq w_o^{-6} + \epsilon$. From the dominated convergence theorem, we obtain:

$$\int_{S^3} w_o^{-6} = \int_{S^3} \underline{\lim}_{k\to\infty} w_k^{-6} = \underline{\lim}_{k\to\infty} \int_{S^3} w_k^{-6} = 1.$$

Also, by semi-continuity we have

$$\int_{S^3} P w_o \cdot w_o \leq \underline{\lim}_{k\to\infty} \int_{S^3} P w_k \cdot w_k,$$

these yield

$$I(w_o) \leq \inf I(u),$$

that is: w_o is a minimizer.

Case 2. $||w_k||_{L^\infty}$ is not bounded, that is

$$||w_k||_{L^\infty} \to \infty. \tag{3.31}$$

We will construct another minimizing sequence which is uniformly bounded. Due to Proposition 2.1, we can assume that $w_k(x)$ is rotationally symmetric and $w_k(x(y)) \cdot v(y)$ is non-decreasing in $|y|$ in any ball $B_R(0) \subset \mathbb{R}^3$ as $k \to \infty$.

Define

$$\lambda_k = w_k(S) \cdot w_k^{-1}(N); \quad x_{\lambda_k y} = \Phi^{-1}(\lambda_k y), \tag{3.32}$$

and

$$z_k(x) := \lambda_k^{-1/2} w_k(x_{\lambda_k y}) \cdot \left(\frac{1 + |\lambda_k y|^2}{1 + |y|^2}\right)^{1/2}, \tag{3.33}$$

where $x = \Phi^{-1}(y)$. It is easy to check that

$$I(z_k) = I(w_k), \quad \text{and} \quad \int_{S^3} z_k^{-6} = 1. \tag{3.34}$$

Therefore $\{z_k\}_{k=1}^{\infty}$ is a minimizing sequence. We need the following two lemmas.

Lemma 3.1. *Let*

$$\mathcal{L}_o = \left\{\varphi \in W^{2,2}(S^3) \setminus \{0\} \ : \ \varphi(x) \geq 0, \quad \textit{but} \ \ \varphi(x) \quad \textit{is} \quad \textit{not} \quad \textit{strictly} \quad \textit{positive}\right\},$$

and

$$\mathcal{L}_b = \left\{\varphi \in \mathcal{L}_o \ : \ \{x \in S^3 \ : \ \varphi(x) = 0\} \ \textit{has} \quad \textit{positive} \quad \textit{measure} \ \right\} \cdot$$

Then

$$\inf_{\varphi \in \mathcal{L}_o} \frac{\int_{S^3} P\varphi \cdot \varphi}{\int_{S^3} \varphi^2} \geq 0, \tag{3.35}$$

and there is a $\Lambda_1 > 0$*, such that*

$$\inf_{\varphi\in\mathcal{L}_b} \frac{\int_{S^3} P\varphi\cdot\varphi}{\int_{S^3}\varphi^2} \ge \Lambda_1. \tag{3.36}$$

Proof. We first prove (3.35) by contradiction. If (3.35) is not true, then there are a function $u(x) \in C^\infty(S^3)$ and a point $\bar{x}\in S^3$ satisfying $u(\bar{x}) = 0$, $u(x)\ge 0$, $\int_{S^3} u^2 = 1$, and

$$\int_{S^3} Pu\cdot u \le \frac{1}{2}\inf_{\varphi\in\mathcal{L}_o} I(\varphi) < 0.$$

Since $\bar{x}$ is the minimal point of $u(x)$, $|\nabla u(\bar{x})| = 0$. Using the stereographic projection with the north pole at $\bar{x}$ and integrating by parts, we obtain

$$\int_{S^3} Pu\cdot u = \int_{\mathbb{R}^3}\Delta^2(uv(y)) = \int_{\mathbb{R}^3}[\Delta(uv(y))]^2 \ge 0,$$

where $v(y) = \sqrt{(1+|y|^2)/2}$. Contradiction.

Notice that for any fixed $R > 0$,

$$\inf_{\phi\in C_o^\infty(B_R(0))}\frac{\int_{B_R(0)}|\Delta\phi|^2}{\int_{B_R(0)}|\phi|^2}\ge \Lambda(R) > 0,$$

we can obtain (3.36) using a similar argument. □

Lemma 3.2. *Let* $G(y) = G(|y|)\ge 0$ *be a positive radially symmetric function in* $\mathbb{R}^3 i\setminus\{0\}$*. If* $G(r)\in L^\infty_{loc}(\mathbb{R}^3)$ *and satisfies*

$$\begin{cases} \Delta^2 G = 0, & in\ \ \mathbb{R}^3\setminus\{0\} \\ \lim_{r\to\infty}\frac{G(r)}{r}\le C. \end{cases} \tag{3.37}$$

Then either $G(r) > 0$ *at* $r = 0$ *or* $G(r) = ar$ *for* $0\le r<\infty$*, where* a *is some positive constant.*

Proof. From the general solution to the equation, it follows that $G(r)$ is given by

$$G(r) = \frac{C_1}{r} + C_2 + C_3 r + C_4 r^2.$$

Since $G\in L^\infty_{loc}(\mathbb{R}^3)$, we find that $C_1 = 0$. From $\lim_{r\to\infty} G(r)/r\le C$, we find that $C_4 = 0$. If $G(0) = 0$, then $C_2 = 0$, thus $G(r) = C_3 r$, where C_3 must be positive since $G(r) > 0$ for $0 < r < \infty$. This proves the lemma. □

Return to the construction of a uniformly bounded minimizing sequence. For any k, we check that

$$z_k(N) = z_k(S) = [w_k(S)\cdot w_k(N)]^{1/2}. \tag{3.38}$$

If $||z_k||_{L^\infty}\le C$ up to a subsequence, we then can obtain a minimizer as in Case 1.

We are left to handle the case of $||z_k||_{L^\infty}\to\infty$. Assume that $z_k(\bar{x}_k) = \max_{S^3} z_k(x)$, and define

$$h_k(x) := \frac{z_k(x)}{z_k(\bar{x}_k)}.$$

Then $0 < h_k \le 1$ and $h_k(\bar{x}_k) = 1$. Moreover, $\{h_k\}_{k=1}^\infty$ is again a minimizing sequence of $\inf I(u)$. Therefore, up to a subsequence of k,

$$h_k\to h_\infty \ \text{ weakly in } W^{2,2}, \quad h_k\to h_\infty \ \text{ in } L^2,$$

for some $h_\infty \in W^{2,2}(S^3)$ with $h_\infty(N) = h_\infty(S)$, $h_\infty(\bar{x}) = 1$, where $\bar{x}$ is a limit point of $\{\bar{x}_k\}$. Also, it is obvious that h_∞ is rotationally symmetric. If $h_\infty > 0$, it is a minimizer. We need to rule out the possibility that h_∞ vanishes somewhere. We claim that $h_\infty(x) > 0$ for all $x \in S^3 \setminus \{N, S\}$. Suppose this is not so. If $1/C < \lambda_k < C$ for some positive constant, due to the monotonicity property of w_k, h_∞ may vanish in a small neighborhood of some point on S^3, which yields (due to Lem. 3.1) that $\int_{S^3} Ph_\infty \cdot h_\infty > 0$, contradiction! If $\lambda_k \to 0$ or $\lambda_k \to \infty$ up to a subsequence of k, then $h_\infty(N) = h_\infty(S) = 0$; it is not difficult to see (using Lem. 3.1) that h_∞ must satisfy (3.37). But this contradicts Lemma 3.2. We therefore complete the proof of the existence of a minimizer for $\inf I(u)$. □

Classification of extremal function

Let $u_o(x)$ be an extremal function for Paneitz energy with the maximal point at the north pole. Denote $v(y) = \sqrt{(1+|y|^2)/2}$. Using the stereographic projection, we know that $w(y) = u_o(x(y))v(y)$ is a positive solution to the following equation:

$$\begin{cases} \Delta^2 w = Ew^{-7} & \text{in } \mathbb{R}^3 \\ w(y) \to C|y| & \text{as } |y| \to \infty \end{cases} \tag{3.39}$$

for some positive constants E and C. It was proved by Choi and Xu [3] that

$$w(y) = C\sqrt{(1+\lambda|y-y_0|^2)}$$

for some positive constants C and λ, and any point $y_0 \in \mathbb{R}^3$. This yields that

$$\inf_{\phi>0,\phi\in W^{2,2}(S^3)} I(\phi) = -\frac{15}{16}\cdot(2\pi)^{4/3},$$

and $u_o^{-4}g_{S^3}$ is a pullback of the standard metric on S^3 *via* a conformal transformation. We therefore complete the proof of the theorem. □

4. Proof of Lemma 2.1

Define

$$t = (\Delta(wv))^*\left(|z| = R - 2\bar{R}\right). \tag{4.40}$$

For fixed $\bar{R}$, t is a function of R. We need to study the set $\{y \in \mathbb{R}^3 \ : \ |\Delta(wv)| > t\}\cdot$

Let

$$t_R = \frac{a}{\sqrt{2}}\left(3 + 2|R|^2\right)\cdot\left(1+|R|^2\right)^{-3/2},$$

ω_3 be the volume of the unit ball in $\mathbb{R}^3$, and $m_t = vol\{y \in B_{\bar{R}} \ : \ |\Delta(wv)| \le t\}\cdot$

We first claim that $t \ge t_R$. If not, $t < t_R$. This implies that (using (2.11))

$$\begin{aligned} mes\{y \in B_R \ : \ \Delta(wv))^* < t\} &= mes\{y \in B_R \ : \ |\Delta(wv)| < t\} \\ &= mes\{y \in B_{\bar{R}} \ : \ |\Delta(wv)| < t\} \\ &= m_{t_R} \\ &\le \bar{R}^3\omega_3. \end{aligned}$$

It follows from (4.40) that

$$mes\{y \in B_R \ : \ \Delta(wv))^* < t\} = vol(B_R) - vol(B_{R-2\bar{R}}) > \bar{R}^3\omega_3.$$

This is in contradiction with the previous assertion.

Since (2.11) holds for all $y \in B_R(0)\setminus B_{\bar{R}}(0)$, we see that for almost every $s \geq t$, the level set $\{y\ :\ |\Delta(wv)| = s\}$ consists of

$$\left\{y\ :\ \frac{a}{\sqrt{2}}(3+2|y|^2)\cdot(1+|y|^2)^{-3/2} = s\right\}$$

and some other level surface in a bounded (independent of t) subset of $B_{\bar{R}}(0)$. Therefore, we have

$$vol\{y \in \mathbb{R}^3\ :\ |\Delta(wv)| > s\} = r_s^3\omega_3 - m_s, \tag{4.41}$$

where r_s satisfies

$$\frac{a}{\sqrt{2}}\left(3+2|r_s|^2\right)\cdot\left(1+|r_s|^2\right)^{-3/2} = s, \tag{4.42}$$

thus

$$r_s = \frac{\sqrt{2}a}{s}\left(1+o_s(1)s^2\right), \tag{4.43}$$

where $o_s(1) \to 0$ as $s \to 0$.

If $(\Delta(wv))^*(|z|) = s$ for some $s \geq t_R$, we have

$$\begin{aligned} mes\{y \in B_R\ :\ \Delta(wv))^* > s\} &= mes\{y \in B_R\ :\ |\Delta(wv)| > s\} \\ &\quad - mes\{y \in B_{\bar{R}}\ :\ |\Delta(wv)| < s\}\cdot \end{aligned}$$

That is $|z|^3\cdot\omega_3 = r_s^3\omega_3 - m_s$. Using C_s to represent various uniformly bounded constants (bounded by a constant depending only on $\bar{R}$), we have (also using (4.43))

$$\begin{aligned} |z| &= (r_s^3 + C_s)^{1/3} \\ &= \frac{\sqrt{2}a}{s} + C_s s^2. \end{aligned}$$

Thus

$$s = \frac{\sqrt{2}a}{|z|} + \frac{C_s}{|z|^4}\cdot \tag{4.44}$$

Since $t \geq t_R$, it follows that for $|z| \leq R - 2\bar{R}$,

$$\Delta(wv)^{\#}(z) = (\Delta(wv))^*(z) = \frac{\sqrt{2}a}{|z|} + \frac{C_s}{|z|^4}\cdot \tag{4.45}$$

This yields (2.23).

If $(\Delta(wv))^*(|z|) = s$ for some $s \leq t_R$, we have

$$mes\{y \in B_R\ :\ \Delta(wv))^* > s\} = vol(B_R) - mes\{y \in B_{\bar{R}}\ :\ |\Delta(wv)| < s\}\cdot$$

That is $|z|^3 \cdot \omega_3 = R^3\omega_3 - m_s$. Since $R = r_{t_R}$, we have

$$\begin{aligned} |z| &= (R^3 + C_s)^{1/3} \\ &= \frac{\sqrt{2}a}{t_R} + C_s t_R^2. \end{aligned}$$

That is

$$t_R = \frac{\sqrt{2}a}{|z|} + \frac{C_s}{|z|^4}\cdot \tag{4.46}$$

We hereby have

$$s = \Delta(wv)^{\#}(z) = (\Delta(wv))^*(z) \begin{cases} = \dfrac{\sqrt{2}a}{|z|} + \dfrac{C_s}{|z|^4}, & \text{for } \ s \geq t_R \\ \leq \dfrac{\sqrt{2}a}{|z|} + \dfrac{C_s}{|z|^4}, & \text{for } \ s \leq t_R. \end{cases}$$

Notice that $s = t_R$ is equivalent to

$$|z| = R\left(1 + \frac{C_s}{R^3}\right)\cdot$$

We thus have

$$\Delta(wv)^{\#}(z) = (\Delta(wv))^*(z) \begin{cases} = \dfrac{\sqrt{2}a}{|z|} + \dfrac{C_s}{|z|^4}, & \text{for } \ |z| \leq R\left(1 + \dfrac{C_s}{R^3}\right) \\ \leq \dfrac{\sqrt{2}a}{|z|} + \dfrac{C_s}{|z|^4}, & \text{for } \ |z| \geq R\left(1 + \dfrac{C_s}{R^3}\right)\cdot \end{cases} \tag{4.47}$$

If we define

$$\varphi(r) := \frac{\mathrm{d}}{\mathrm{d}r}(wv)^{\#}(r) \quad \text{for } \ r = |z|,$$

from (4.47) we have

$$(r^2\varphi)' = \sqrt{2}ar + \frac{C_s}{r^2}, \qquad \text{for } \ r \in \left[2, R - 2\bar{R}\right).$$

Integrating the above from $r = 2$ to $r = |y| \leq R - 2\bar{R}$ yields $\varphi(|y|) = \frac{a}{\sqrt{2}} + \frac{C_s}{|y|^2}\cdot$ Thus

$$\frac{\partial}{\partial\nu}(wv)^{\#}(|y| = R - 2\bar{R}) = \frac{a}{\sqrt{2}} + \frac{C_R}{(R - 2\bar{R})^2}, \tag{4.48}$$

where C_R is a uniformly bounded term. This yields (2.22).

Similarly, using (4.47), we have

$$(r^2\varphi)' \begin{cases} = \sqrt{2}ar + \dfrac{C_s}{r^2} & \text{for } \ r \leq R\left(1 + \dfrac{C_s}{R^3}\right) \\ \leq \sqrt{2}ar + \dfrac{C_s}{r^2} & \text{for } \ r \geq R\left(1 + \dfrac{C_s}{R^3}\right)\cdot \end{cases}$$

Thus

$$\frac{\partial}{\partial\nu}(wv)^{\#}(|y| = r) = \frac{a}{\sqrt{2}} + \frac{O(1)}{r^2} \qquad \text{for } \quad r \in [2, R). \tag{4.49}$$

Since

$$(wv)^{\#}(R) = wv(R) = \frac{aR}{\sqrt{2}}\left[1 + (1 + o(1))R^{-2}/2\right],$$

we obtain (2.21) by integrating (4.49).

Finally, we prove (2.24). Again, we need to study the set $\{y \in \mathbb{R}^3 \ : \ |\Delta(wv)| > s\}$ for $s \geq t_R$. If $(\Delta(wv))^*(z) = s$, then $|z|^3\omega_3 = r_s^3\omega_3 - m_s$. We have

$$r_s^3 = |z|^3 + m_s\omega_3^{-1} \tag{4.50}$$

thus

$$r_s = |z| + \frac{C_s}{|z|^2}\cdot \tag{4.51}$$

From (4.50) we have

$$\frac{\mathrm{d}r_s^3}{\mathrm{d}|z|} = 3|z|^2 + \frac{1}{\omega_3} \cdot \frac{\mathrm{d}m_s}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}|z|}\cdot$$

Thus

$$\frac{\mathrm{d}r_s}{\mathrm{d}|z|} = \frac{|z|^2}{r_s^2} + \frac{1}{3r_s^2\omega_3} \cdot \frac{\mathrm{d}m_s}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}|z|}\cdot \tag{4.52}$$

Notice that

$$\int_0^\infty m_s'(s)\mathrm{d}s \le \bar{R}^3\omega_3 < \infty,$$

we know that there is a sequence $s_i \to 0$ as $i \to \infty$, such that

$$m_s'(s) \le \frac{1}{s^2} \quad \text{for} \quad s = s_i. \tag{4.53}$$

For these s_i, there are corresponding r_{s_i} and $|z_i|$, which satisfy

$$\frac{\mathrm{d}m_s}{\mathrm{d}s} < \frac{1}{s_i^2} \le Cr_{s_i}^2.$$

Notice that

$$\frac{\mathrm{d}s}{\mathrm{d}|z|} = \frac{\mathrm{d}s}{\mathrm{d}r_s} \cdot \frac{\mathrm{d}r_s}{\mathrm{d}|z|}\cdot$$

And from (4.42), we have

$$\frac{\mathrm{d}s}{\mathrm{d}r_s} = -\frac{\sqrt{2}a}{r_s^2}(1 + C_s r_s^{-2}).$$

Therefore

$$\frac{\mathrm{d}s}{\mathrm{d}|z|}(|z| = |z_i|) = -\frac{\sqrt{2}a}{r_{s_i}^2}\left(1 + C_s r_{s_i}^{-2}\right) \cdot \left(\frac{|z_i|^2}{r_{s_i}^2} + \frac{1}{3r_s^2\omega_3} \cdot \frac{\mathrm{d}m_s}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}|z|}\right)\cdot$$

Using (4.51) we obtain

$$\begin{aligned}\frac{\partial}{\partial\nu}(\Delta(wv))^{\#}(|z| = |z_i|) =& \frac{\partial}{\partial\nu}(\Delta(wv))^{*}(|z| = |z_i|)\\ =& \frac{\mathrm{d}s}{\mathrm{d}|z|}(|z| = |z_i|)\\ =& -\frac{\sqrt{2}a}{z_{s_i}^2} + \frac{C_s}{|z_i|^4}\cdot\end{aligned}$$

We therefore complete the proof of Lemma 2.1. □

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