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A RELAXATION RESULT FOR AUTONOMOUS INTEGRAL FUNCTIONALS WITH DISCONTINUOUS NON-COERCIVE INTEGRAND

Carlo Mariconda[1] and Giulia Treu[1]

Abstract. Let $L:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ be a Borelian function and consider the following problems

$$\inf\left\{F(y)=\int_a^b L(y(t),y'(t))\,\mathrm{d}t:\ y\in AC([a,b],\mathbb{R}^N), y(a)=A,\ y(b)=B\right\} \qquad (P)$$

$$\inf\left\{F^{**}(y)=\int_a^b L^{**}(y(t),y'(t))\,\mathrm{d}t:\ y\in AC([a,b],\mathbb{R}^N), y(a)=A,\ y(b)=B\right\}\cdot \qquad (P^{**})$$

We give a sufficient condition, weaker then superlinearity, under which $\inf F=\inf F^{**}$ if L is just continuous in x. We then extend a result of Cellina on the Lipschitz regularity of the minima of (P) when L is not superlinear.

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1. Introduction

We consider the relationships between the problems

$$\inf\left\{F(y)=\int_a^b L(y(t),y'(t))\,\mathrm{d}t:\ y\in AC\left([a,b],\mathbb{R}^N\right), y(a)=A,\ y(b)=B\right\} \qquad (P)$$

$$\inf\left\{F^{**}(y)=\int_a^b L^{**}(y(t),y'(t))\,\mathrm{d}t:\ y\in AC\left([a,b],\mathbb{R}^N\right), y(a)=A,\ y(b)=B\right\}\cdot \qquad (P^{**})$$

It is well known that $\inf F=\inf F^{**}$ if L is super-linear and continuous. Recently Cellina in [5] proved that the same conclusion holds true assuming, instead of superlinearity, a weaker growth condition that we will call (GA). Roughly, a convex function $L(x,\xi)$ satisfies (GA) if the intersection of the supporting hyperplane to its epigraph at $(\xi,L(x,\xi))$ with the ordinate axis tends to $-\infty$ as $|\xi|$ tends to $+\infty$, uniformly with respect to x in compact sets. This condition implies, but is not equivalent to, a sort of conical growth: we say that L

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[1] Dipartimento di Matematica pura e applicata, Università di Padova, 7 via Belzoni, 35131 Padova, Italy;
e-mail: `maricond@math.unipd.it; treu@math.unipd.it`

satisfies (CGA) if for every ξ_0 there exist $\varepsilon, R > 0$ such that, for every $|\xi| \geq R$,

$$L(x,\xi) \geq L(x,\xi_0) + p(x,\xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + \text{ const.} \tag{CGA}$$

whenever x belongs to a prescribed compact set and $p(x,\xi_0)$ belongs to the subdifferential of $\xi \mapsto L(x,\xi)$ in ξ_0. We weaken here the continuity assumption of L in both variables and we prove that, if $L(x,\xi)$ is just continuous in x and satisfies (CGA), then $\inf F = \inf F^{**}$.

The proof of the result is based on Theorem 3.2, a uniform approximation of the bipolar of a (discontinuous) function $L(\xi)$ satisfying (CGA) in terms of the convex hull of the graph of L; this kind of result is classical when L is supposed to be lower semi-continuous and superlinear in ξ [7].

In the last part of the paper we are concerned with an application to the Lipschitz regularity of the minima of (P). It is well known that, if $L(x,\xi)$ is superlinear and convex in ξ, then every minimizer of (P) is Lipschitz. The same result was obtained recently by dropping some of the assumptions: no continuity and no convexity but superlinearity is assumed in [6], continuity, no convexity and assumption (GA) instead of superlinearity is assumed in [5], no continuity and no convexity but the requirement that every section $\lambda \mapsto L(x,\lambda u)$ ($\lambda \geq 0$, $|u| = 1$) satisfies (GA) in [8], extending [6].

As a consequence of our relaxation result we prove that the minima of (P) are Lipschitz if $L(x,\xi)$ is just continuous in x and satisfies (GA), thus extending the main result in [5].

We point out that there are several results concerning the representation of the lower semi-continuous envelope of integral functionals; we just mention [2, 4] for some recent results and references. Here we are interested in comparing the values of the infima of problems (P) and (P^{**}) instead of establishing a representation formula.

2. Notation and preliminary results

In this paper $|\cdot|$ is the Euclidean norm and "$\cdot$" the scalar product in $\mathbb{R}^N$. For a function $L(x,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ we denote by $L^{**}(x,\xi)$ (resp. $\partial L^{**}(x,\xi)$) the bipolar (resp. the subdifferential of the bipolar) of $\xi \mapsto L(x,\xi)$. Finally, $AC([a,b],\mathbb{R}^N)$ is the space of absolutely continuous functions on $[a,b]$ with values in $\mathbb{R}^N$.

Here $L : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is just a Borelian function. We assume moreover that $L^{**}(x,\xi) \neq -\infty$ for every x and ξ; this is the case, for instance, if L is bounded below by an affine function of ξ.

The following growth condition will be assumed in the main result.

Conical growth assumption (CGA)**.** For every compact subset C of $\mathbb{R}^N$ and $R_0 \geq 0$ there exist $\varepsilon > 0$, $R > 0$ and $c \in \mathbb{R}$ such that

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R \qquad L^{**}(x,\xi) \geq L^{**}(x,\xi_0) + p(x,\xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + c$$

for every $x \in C$, $|\xi_0| \leq R_0$ and $p(x,\xi_0)$ in $\partial L^{**}(x,\xi_0)$.

The following growth assumption was introduced by Cellina in [5] in the case where L is continuous.

Growth assumption (GA)**.**
We say that L satisfies (GA) if there exist $p(x,\xi)$ in $\partial L^{**}(x,\xi)$ such that

$$\lim_{|\xi| \to +\infty} p(x,\xi) \cdot \xi - L^{**}(x,\xi) = +\infty \tag{2.1}$$

uniformly for x in a compact set.

Remark 2.1.

i) We point out that, in [5], the definition of (GA) is slightly different: it is formulated in an equivalent way in terms of the polar of L in $(x, p(x,\xi))$; moreover the uniformity with respect to the first variable is not required since it is a consequence of the continuity of L. We use it here since we drop the continuity assumption.

ii) Assumption (GA) is fulfilled if, for instance, $L(x,\xi)$ is superlinear with respect to ξ; the proof can be easily done following the lines of [5].

We refer to [8] for a survey on the properties of the functions that satisfy (GA).

Theorem 2.2. *[8, Cor 4.4] Assume that L is bounded on compact sets and satisfies the Growth Assumption (GA). Then L satisfies (CGA).*

3. Relaxation

It is well known that if $L : \mathbb{R}^N \to \mathbb{R}$ is a function whose bipolar is finite, then, for every $\varepsilon > 0$ and ξ in $\mathbb{R}^N$, there exists $\xi_1, ..., \xi_m$ $(m \leq N+1)$ in $\mathbb{R}^N$ and coefficients of a convex combination $\alpha_1, ..., \alpha_m$ such that $\sum_i \alpha_i L(\xi_i) \leq L^{**}(\xi) + \varepsilon$ and $\sum_i \alpha_i \xi_i = \xi$. We prove in the next Theorem 3.2 that if L satisfies (CGA) then, allowing $m \leq 2N+2$, the points ξ_i may be bounded uniformly with respect to ξ in compact sets. For this purpose we first quote, in a more general setting, a powerful consequence of (CGA) that was established in [5] in the continuous case. For every (x,ξ) we set

$$\overline{L}(x,\xi) = \liminf_{\eta \to \xi} L(x,\eta),$$

i.e. $\overline{L}(x,\xi)$ denotes the lower semi-continuous envelope of the map $\eta \mapsto L(x,\eta)$. The proof of the following result is based on the fact that if $f : \mathbb{R}^N \to \mathbb{R}$ is convex and satisfies (CGA) then the intersection of its epigraph with any supporting hyperplane is bounded. This condition is referred in [5] as the *Bounded Intersection Property.*

Theorem 3.1. *Assume that L satisfies (CGA) and let $p(x,\xi) \in \partial L^{**}(x,\xi)$. Then given $R_0 > 0$ and a compact subset C of R^N there exists $R > 0$ (depending only on R_0 and C) such that for every $x \in C$; for every ξ, with $|\xi| \leq R_0$, there exist at most $\nu \leq N+1$ points ξ_i, with $|\xi_i| \leq R$, and coefficients of a convex combination α_i, such that*

$$\begin{pmatrix} \xi \\ L^{**}(x,\xi) \end{pmatrix} = \sum_{i=1}^{\nu} \alpha_i \begin{pmatrix} \xi_i \\ \overline{L}(x,\xi_i) \end{pmatrix}$$

and $L^{**}(x,\xi) = \overline{L}(x,\xi_i) = L^{**}(x,\xi) + p(x,\xi) \cdot (\xi - \xi_i)$.

Proof. It is enough to remark that Theorem 1 in [5] holds for functions that are lower semi-continuous instead of continuous and that the bipolar of a function coincides with the bipolar of its lower semi-continuous envelope. □

We are now ready to state a version of Theorem 3.1 that does not involve the lower semi-continuous envelope of L.

Theorem 3.2. *Assume that $L(x,\xi)$ satisfies (CGA) and that L is bounded on the compact sets. Then given $R_0 > 0$ and a compact subset C of R^N, there exists $R > 0$ (depending only on R_0 and C) such that for every x in C, for every ξ, with $|\xi| \leq R_0$ and $\varepsilon > 0$, there exist at most $m \leq 2N+2$ points ξ_i, with $|\xi_i| \leq R$, and coefficients of a convex combination λ_i, such that*

$$\begin{cases} \xi = \sum_{i=1}^m \lambda_i \xi_i \\ \sum_{i=1}^m \lambda_i L(x,\xi_i) \leq L^{**}(x,\xi) + \varepsilon. \end{cases}$$

The proof of the result needs several preliminary steps. For the convenience of the reader we first give a sketch of the proof in the case where L does not depend on x.

By Theorem 3.1, for $|\xi| \leq R_0$, the point $(\xi, L^{**}(\xi))$ can be written as a convex combination of points $(\zeta_i, \overline{L}(\zeta_i))$ of the epigraph of the lower semi-continuous envelope of $L(\cdot)$; moreover the ζ_i are uniformly bounded, so that they all lie in a simplex generated by $N+1$ affinely independent points $\eta_1, \ldots, \eta_{N+1}$. Now each value $\overline{L}(\zeta_j)$ can be approximated with $L(\eta^j)$ for some η^j arbitrarily near to ζ_j; actually it turns out that for $\varepsilon > 0$, if $|\eta^j - \zeta_j|$ is sufficiently small, then there is a convex combination of $(\eta^j, L(\eta^j))$ and N points among the $(\eta_i, L(\eta_i))$'s whose projection on $\mathbb{R}^N$ is ζ_j and whose last coordinate is less than $\overline{L}(\zeta_j)+\varepsilon$. The conclusion follows by writing ξ as a convex combinations of the points η_i and the η^j constructed as above.

We first need two technical lemmas. Let, if S is a subset of $\mathbb{R}^N$, $\operatorname{int} S$ denote its interior and $\operatorname{conv} S$ its convex hull.

Lemma 3.3. *Let $\eta_1,\dots,\eta_{N+1}$ be $N+1$ affinely independent points of $\mathbb{R}^N$ and η in $\operatorname{int}(\operatorname{conv}\{\eta_1,\dots,\eta_{N+1}\})$, the interior of the simplex whose vertices are $\eta_1,\dots,\eta_{N+1}$. Then:*

i) *for every $I\subset\{1,\dots,N+1\}$ of cardinality $|I|\le N$ the set of points $\{\eta,\,\eta_i : i\in I\}$ is affinely independent;*
ii) *for every $\xi\in\operatorname{int}(\operatorname{conv}\{\eta_1,\dots,\eta_{N+1}\})$ there exists a subset I of $\{1,\dots,N+1\}$ of cardinality N such that $\xi\in\operatorname{conv}\{\eta,\,\eta_i : i\in I\}$.*

Proof of Lemma 3.3. i) It is not restrictive to assume that $I=\{1,\dots,N\}$. Let

$$\eta=\sum_{j=1}^{N+1}\lambda_j\eta_j \qquad \lambda_j>0 \qquad \sum_{j=1}^{N+1}\lambda_j=1.$$

For every $i\in\{1,\dots,N\}$ we have

$$\begin{aligned}\eta-\eta_i&=\sum_{j\neq i}\lambda_j\eta_j+(\lambda_i-1)\eta_i\\ &=\sum_{j\neq i}\lambda_j(\eta_j-\eta_{N+1})+(\lambda_i-1)(\eta_i-\eta_{N+1})\end{aligned}$$

so that, in a matrix notation,

$$[\eta-\eta_1,\dots,\eta-\eta_N]=[\eta_1-\eta_{N+1},\dots,\eta_N-\eta_{N+1}](\Lambda-I)$$

where I is the identity and

$$\Lambda=\begin{pmatrix}\lambda_1 & \dots & \lambda_N\\ \dots & \dots & \dots\\ \lambda_1 & \dots & \lambda_N\end{pmatrix}.$$

Now $\det(\Lambda-I)\neq 0$ since the eigenvalues of Λ are $\lambda_1,\dots,\lambda_N$ and $\lambda_i<1$ for every i, proving i).
Proof of ii). Let

$$\xi=\alpha_1\eta_1+\dots+\alpha_{N+1}\eta_{N+1} \qquad \eta=\mu_1\eta_1+\dots+\mu_{N+1}\eta_{N+1}$$

and we may assume that $\alpha_{N+1}/\mu_{N+1}=\min\{\alpha_i/\mu_i : i=1,\dots,N+1\}$ (notice that all the μ_i are strictly positive). Set $c_{N+1}=\alpha_{N+1}/\mu_{N+1}$ and, for $i\in\{1,\dots,N\}$, $c_i=\alpha_i-\mu_i c_{N+1}$: then, for every i, $c_i\ge 0$; moreover

$$\begin{aligned}\sum_{i=1}^{N+1}c_i&=\sum_{i=1}^{N}\alpha_i-c_{N+1}\sum_{i=1}^{N}\mu_i+c_{N+1}\\ &=1-\alpha_{N+1}-(1-\mu_{N+1})c_{N+1}+c_{N+1}\\ &=1-\alpha_{N+1}+\mu_{N+1}c_{N+1}=1\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{N} c_i\eta_i + c_{N+1}\eta &= \sum_{i=1}^{N}(\alpha_i - \mu_i c_{N+1})\eta_i + c_{N+1}\sum_{i=1}^{N+1}\mu_i\eta_i \\ &= \sum_{i=1}^{N}(\alpha_i - \mu_i c_{N+1} + \mu_i c_{N+1})\eta_i + c_{N+1}\mu_{N+1}\eta_{N+1} \\ &= \sum_{i=1}^{N}\alpha_i\eta_i + \alpha_{N+1}\eta_{N+1} = \xi\end{aligned}$$

so that $\xi \in \operatorname{conv}\{\eta, \eta_i : i \in \{1, \dots, N\}\}$. □

Lemma 3.4. *Let $\eta_1, \dots, \eta_{N+1}$ be $N+1$ affinely independent points of $\mathbb{R}^N$ and let $y_1, \dots, y_{N+1}$ be real numbers and $K > 0$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\eta, \xi \in \operatorname{int}(\operatorname{conv}\{\eta_1, \dots, \eta_{N+1}\})$ and y, β in $[-K, K]$, with $|\eta - \xi| < \delta$ and $|y - \beta| < \delta$, there exists a subset I of $\{1, \dots, N+1\}$ of cardinality N and coefficients λ, λ_i $(i \in I)$ of a convex combination satisfying*

$$\begin{cases} \xi = \lambda\eta + \sum_{i\in I}\lambda_i\eta_i \\ \beta + \varepsilon > \lambda y + \sum_{i\in I}\lambda_i y_i. \end{cases}$$

Remark 3.5. Geometrically Lemma 3.4 states that given $N+1$ points (η_i, y_i) of $\mathbb{R}^N \times \mathbb{R}$ and (ξ, β) in $\mathbb{R}^N \times \mathbb{R}$ such that ξ lies in the interior of the convex hull Λ of the η_is then, given a positive ε, for every point (η, y) that is sufficiently near to (ξ, β) with $\eta \in \Lambda$ there exist N points among the (η_i, y_i)s which, together with (η, y), generate a N- dimensional simplex in $\mathbb{R}^N \times \mathbb{R}$ whose projection in $\mathbb{R}^N$ contains ξ and such that $(\xi, \beta + \varepsilon)$ lies above it.

Proof of Lemma 3.4. For every $I \subset \{1, \dots, N+1\}$, $|I| = N$, $y \in \mathbb{R}$ and $\eta \in \Lambda := \operatorname{int}\operatorname{conv}\{\eta_1, \dots, \eta_{N+1}\}$ by Lemma 3.3i) there exists a unique hyperplane $z = a^I(\eta, y)\cdot\xi + b^I(\eta, y)$ containing the points (η, y) and (η_i, y_i) $(i \in I)$. Moreover the coefficients $a^I(\eta, y)$,$b^I(\eta, y)$ are continuous functions of (η, y); in fact from the equations

$$\begin{cases} a^I(\eta, y)\cdot\eta + b^I(\eta, y) = y \\ a^I(\eta, y)\cdot\eta_i + b^I(\eta, y) = y_i \,(i \in I) \end{cases}$$

we deduce that the vector $a^I(\eta, y)$ solves the system

$$a^I(\eta, y)\cdot(\eta - \eta_i) = y - y_i \qquad (i \in I);$$

again by Lemma 3.3i) the vectors $\eta - \eta_i$ $(i \in I)$ are independent so that the latter system has a unique solution $a^I(\eta, y)$ given by Cramer's rule which is a continuous function of η and y; the continuity of b^I follows from the equality $b^I(\eta, y) = y - a^I(\eta, y)\cdot\eta$.
Set, for every $I \subset \{1, \dots, N+1\}$,

$$\varphi^I(\eta, y; \zeta) = a^I(\eta, y)\cdot\zeta + b^I(\eta, y);$$

we point out that, by construction, for a fixed (η, y) the point $(\zeta, \varphi^I(\eta, y; \zeta))$ belongs to the (unique) hyperplane containing the points (η, y) and (η_i, y_i) $(i \in I)$. Since, for every $\zeta \in \Lambda$ and $\beta \in \mathbb{R}$,

$$\varphi^I(\zeta, \beta; \zeta) = \beta,$$

then, by the uniform continuity of φ^I on $\Lambda \times [-K, K] \times \Lambda$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every subset I of $\{1, \ldots, N+1\}$ of cardinality N,

$$\varphi^I(\eta, y; \zeta) < \beta + \varepsilon \text{ whenever } \eta, \zeta \in \Lambda \quad |\eta - \zeta| < \delta \text{ and } y, \beta \in [-K, K] \quad |y - \beta| < \delta. \tag{3.1}$$

Now fix ξ in Λ and $\varepsilon > 0$. Let $I \subset \{1, \ldots, N+1\}$ be such that ξ belongs to $\operatorname{conv}\{\eta, \eta_i : i \in I\}$; such a set exists by Lemma 3.3ii). Let δ be such as in (3.1) and $|\eta - \xi| < \delta$, $|y - \beta| < \delta$ so that $\varphi^I(\eta, y; \xi) < \beta + \varepsilon$. Then, if we set

$$\xi = \lambda\eta + \sum_{i \in I} \lambda_i \eta_i$$

for some coefficients λ, λ_i $(i \in I)$ of a convex combination, the linearity of φ^I in the third variable yields

$$\lambda\varphi^I(\eta, y; \eta) + \sum_{i \in I} \lambda_i \varphi^I(\eta, y; \eta_i) < \beta + \varepsilon,$$

proving the claim since $\varphi^I(\eta, y; \eta) = y$ and $\varphi^I(\eta, y; \eta_i) = y_i$. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Fix x in C. By Theorem 3.1 there exists $R_1 > 0$ (depending only on R_0 and C), $\zeta_1, \ldots, \zeta_\nu$ $(\nu \le N+1)$, with $|\zeta_j| \le R_1$, and coefficients α_j of a convex combination satisfying

$$\begin{cases} \xi = \sum_{j=1}^{\nu} \alpha_j \zeta_j \\ L^{**}(x, \xi) = \sum_{j=1}^{\nu} \alpha_j \overline{L}(x, \zeta_j); \end{cases}$$

where $\overline{L}$ denotes as usual the lower semi-continuous envelope of $L(x, \cdot)$. It is not restrictive at this stage to assume that $L(x, \xi) = L(\xi)$. Let $\eta_1, \ldots, \eta_{N+1}$ be such that

$$\left\{\zeta \in \mathbb{R}^N : |\zeta| \le R_1\right\} \subset \operatorname{int}\left(\operatorname{conv}\{\eta_1, \ldots, \eta_{N+1}\}\right)$$

and set

$$y_i = L(\eta_i), \qquad i = 1, \ldots, N+1, \qquad K = \sup\{L(\zeta) : |\zeta| \le R_1\}.$$

Fix $\varepsilon > 0$ and j in $\{1, \ldots, \nu\}$; set $\beta = \overline{L}(\zeta_j)$. Correspondingly, let $\delta > 0$ satisfy the property stated in Lemma 3.4. By the definition of $\overline{L}$ there exist $\eta^j \in \operatorname{int}(\operatorname{conv}\{\eta_1, \ldots, \eta_{N+1}\})$ such that

$$|\eta^j - \zeta_j| < \delta \quad \text{ and } \quad L(\eta^j) \le \overline{L}(\zeta_j) + \delta.$$

We apply Lemma 3.4 with $\eta = \eta^j$, $\xi = \zeta_j$ and $y = L(\eta^j)$: there exists a subset I_j of $\{1, \ldots, N+1\}$ of cardinality N and coefficients λ^j, λ_i^j, $(i \in I_j)$, such that

$$\begin{cases} \zeta_j = \lambda^j \eta^j + \sum_{i \in I_j} \lambda_i^j \eta_i \\ \lambda^j L(\eta^j) + \sum_{i \in I_j} \lambda_i^j L(\eta_i) \le \overline{L}(\zeta_j) + \varepsilon. \end{cases}$$

Therefore we obtain that

$$\begin{aligned} \xi = \sum_{j=1}^{\nu} \alpha_j \zeta_j &= \sum_{j=1}^{\nu} \alpha_j \left(\lambda^j \eta^j + \sum_{i \in I_j} \lambda_i^j \eta_i\right) \\ &= \sum_{j=1}^{\nu} \alpha_j \lambda^j \eta^j + \sum_{i \in I_j} \left(\sum_{j=1}^{\nu} \alpha_j \lambda_i^j\right) \eta_i \end{aligned}$$

and moreover

$$\begin{aligned} L^{**}(\xi)+\varepsilon &= \sum_{j=1}^{\nu}\alpha_j\big(\overline{L}(\zeta_j)+\varepsilon\big) \\ &\geq \sum_{j=1}^{\nu}\alpha_j\left(\lambda^j L(\eta^j)+\sum_{i\in I_j}\lambda_i^j L(\eta_i)\right) \\ &= \sum_{j=1}^{\nu}\alpha_j\lambda^j L(\eta^j)+\sum_{i\in I_j}\left(\sum_{j=1}^{\nu}\alpha_j\lambda_i^j\right)L(\eta_i). \end{aligned}$$

If we set

$$\begin{cases} \lambda_i=\alpha_i\lambda^i & \text{if } i\in\{1,\dots,\nu\} \\ \lambda_i=\sum_j\alpha_j\lambda_{i-\nu}^j & \text{if } i\in\{\nu+1,\dots,\nu+(N+1)\} \end{cases}$$

the above formulae can be rewritten as

$$\begin{cases} \xi=\sum_{i\leq\nu}\lambda_i\eta^i+\sum_{i>\nu}\lambda_i\eta_i \\ \sum_{i\leq\nu}\lambda_i L(\eta^i)+\sum_{i>\nu}\lambda_i L(\eta_i)\geq L^{**}(\xi)+\varepsilon. \end{cases}$$

Moreover $|\eta^i|\leq R$ and $|\eta_i|\leq R$, where $R=\max\{|\eta_i|\,:\,i=1,\dots,N+1\}$ (which depends only on R_1 and therefore only on R_0 and C); proving the claim. □

We consider here the problems

$$\inf\left\{F(y)=\int_a^b L(y(t),y'(t))\,\mathrm{d}t\,:\,y\in AC([a,b],\mathbb{R}^N),y(a)=A,\,y(b)=B\right\} \tag{P}$$

$$\inf\left\{F^{**}(y)=\int_a^b L^{**}(y(t),y'(t))\,\mathrm{d}t\,:\,y\in AC([a,b],\mathbb{R}^N),y(a)=A,\,y(b)=B\right\}\cdot \tag{P^{**}}$$

It is well known that $\inf F=\inf F^{**}$ if L is continuous and superlinear ([7], Th. IX.3.1); actually in this case F^{**} is the relaxed functional of F. In [5] Cellina proved that $\inf F=\inf F^{**}$ if L is just continuous and satisfies (GA). We examine here the case where L is just continuous in the first variable, focusing our attention on the infima of the functionals F and F^{**} instead on the relaxed functional of F.

Theorem 3.6. *Assume that L is bounded on compact sets and that $x\mapsto L(x,\xi)$ is continuous for every $\xi\in\mathbb{R}^N$. If L satisfies (CGA) then $\inf F=\inf F^{**}$.*

Proof. We follow the lines of the proof of the analogous result ([5], Th. 3) in the case where L is continuous in both variables, but instead of Theorem 3.1 we use Theorem 3.2. Let $x\in AC([a,b],\mathbb{R}^N)$ and $\varepsilon>0$. From Theorem 2.4 and Remark 2.8 of [1] applied to L^{**} there exists a Lipschitz function x_{R_0} of Lipschitz constant R_0 satisfying the boundary conditions and such that

$$\int_a^b L^{**}\left(x_{R_0}(t),x'_{R_0}(t)\right)\,\mathrm{d}t\leq\int_a^b L^{**}\left(x(t),x'(t)\right)\,\mathrm{d}t+\varepsilon/3.$$

Set $C=\{x_{R_0}(t)\,:\,t\in[a,b]\}$. Since $|x'_{R_0}(t)|\leq R_0$ for a.e. t then, by Theorem 3.2, there exists R (depending only on R_0 and C), $m\leq 2N+2$ coefficients $\lambda_i(t)$ of a convex combination and vectors $y_i(t)$ $(i=1,\dots,m)$ with

$|y_i(t)| \le \mathbb{R}$ such that

$$\begin{cases} x'_{R_0}(t) = \sum_{i=1}^m \lambda_i(t) y_i(t) \\ \sum_{i=1}^m \lambda_i(t) L(x_{R_0}(t), y_i(t)) \le L^{**}(x_{R_0}(t), x'_{R_0}(t)) + \dfrac{\varepsilon}{3(b-a)}. \end{cases}$$

By a standard selection argument, we may assume that the maps y_i and λ_i are measurable. Fix an integer k and consider the intervals $I_j = [t_j, t_{j+1}]$, where $t_j = a + j\frac{b-a}{k}$ $(j = 0, \dots, k-1)$ and call χ_{I_j} their characteristic function. By Lyapunov's Theorem on the range of vector measures [9] there exists a partition of $[a, b]$ into m measurable subsets E_i, with characteristic functions χ_{E_i}, such that, for $j = 0, \dots, k-1$, one has

$$\int_{I_j} \sum_{i=1}^m \lambda_i(t) y_i(t)\,\mathrm{d}t = \int_{I_j} \sum_{i=1}^m \chi_{E_i}(t) y_i(t)\,\mathrm{d}t$$
$$\int_{I_j} \sum_{i=1}^m \lambda_i(t) L(x_{R_0}(t), y_i(t))\,\mathrm{d}t = \int_{I_j} \sum_{i=1}^m \chi_{E_i}(t) L(x_{R_0}(t), y_i(t))\,\mathrm{d}t.$$

Denote by x_k the absolutely continuous defined by $x_k(a) = A$ and

$$x'_k(t) = \int_a^t \sum_{i,j} y_i(s) \chi_{I_j \cap E_i}(s)\,\mathrm{d}s;$$

in particular for every k and every $j = 1, \dots, k$, we have

$$\int_{I_j} x'_{R_0}(t)\,\mathrm{d}t = \int_{I_j} x'_k(t)\,\mathrm{d}t,$$

so that the functions x_{R_0} and x_k coincide at each point t_j. Since

$$L(x_{R_0}(t), x'_k(t)) = \sum_{i,j} \chi_{I_j \cap E_i}(t) L(x_{R_0}(t), y_i(t))$$

we also have that

$$\int_a^b L(x_{R_0}(t), x'_k(t))\,\mathrm{d}t = \int_a^b \sum_{i=1}^m \lambda_i(t) L(x_{R_0}(t), y_i(t))\,\mathrm{d}t;$$

so that, from $\approx$, it follows that

$$\int_a^b L(x_{R_0}(t), x'_k(t))\,\mathrm{d}t \le \int_a^b L^{**}(x_{R_0}(t), x'_{R_0}(t))\,\mathrm{d}t + \varepsilon/3.$$

Now

$$\int_a^b L(x_{R_0}(t), x'_k(t))\,\mathrm{d}t = \int_a^b L(x_k(t), x'_k(t))\,\mathrm{d}t + \int_a^b L(x_{R_0}(t), x'_k(t)) - L(x_k(t), x'_k(t))\,\mathrm{d}t;$$

moreover, x_{R_0} is uniformly continuous, the functions x_k are equi-Lipschitz, $x_k(t_j) = x_{R_0}(t_j)$ $(j = 0, \dots, k-1)$. Hence, if $t \in [a, b]$ and $t_j \le t \le t_{j+1}$,

$$\begin{aligned} |x_k(t) - x_{R_0}(t)| &\le |x_k(t) - x_k(t_j)| + |x_k(t_j) - x_{R_0}(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)| \\ &= |x_k(t) - x_k(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)| \le (R + R_0)(b-a)/k \end{aligned}$$

so that x_k converges uniformly to x_{R_0} as k tends to $+\infty$. By our assumption the function $L(x_{R_0}, x_k') - L(x_k, x_k')$ is bounded a.e. by a constant that does not depend on k. The continuity of L with respect to the first variable together with the dominated convergence theorem imply that

$$\lim_{k\to+\infty} \int_a^b L(x_{R_0}(t), x_k'(t)) - L(x_k(t), x_k'(t))\,\mathrm{d}t = 0.$$

It follows that for k sufficiently large,

$$\int L(x_k(t), x_k'(t))\,\mathrm{d}t \le \int_a^b L(x_{R_0}(t), x_k'(t))\,\mathrm{d}t + \varepsilon/3 \le \int_a^b L(x(t), x'(t))\,\mathrm{d}t + \varepsilon$$

proving that $\inf F \le \inf F^{**}$. $\square$

We point out that, under the assumptions of Theorem 3.6, the functional F^{**} is not in general the relaxed functional of F; we refer to [2] for some recent results in this direction. This is the case in the forthcoming example, where we also show that the conclusion of Theorem 3.6 does not hold if L is not continuous in x.

Example 3.7. Let g be the characteristic function of $\mathbb{R} \setminus \{0\}$ and $h(\xi) = \xi^2$ if $\xi \neq 0$, $h(0) = 1$ and set $L(x, \xi) = g(x) + h(\xi)$. Let (P), (P^{**}) be the problems

$$\inf \left\{ F(y) = \int_0^1 L(y(t), y'(t))\,\mathrm{d}t; \qquad y(0) = 0,\ y(1) = 0,\ y \in AC([0,1], \mathbb{R}) \right\} \tag{P}$$

$$\inf \left\{ F^{**}(y) = \int_0^1 L^{**}(y(t), y'(t))\,\mathrm{d}t; \qquad y(0) = 0,\ y(1) = 0,\ y \in AC([0,1], \mathbb{R}) \right\}\cdot \tag{P^{**}}$$

For every x in $\mathbb{R}$ we have $L^{**}(x, \xi) = g(x) + \xi^2$, so that the minimum of the problem (P^{**}) is equal to 0 and it is obviously assumed for $y(t) = 0$. However $\inf F \ge 1$; in fact let $y \in AC([0,1], \mathbb{R})$ and set $E = \{t \in [0,1] : y(t) = 0\}$, then $y'(t) = 0$ a.e. on E, so that

$$\begin{aligned}\int_0^1 L(y(t), y'(t))\,\mathrm{d}t &= \int_E L(0,0)\,\mathrm{d}t + \int_{[0,1]\setminus E} L(y(t), y'(t))\,\mathrm{d}t \\ &\ge \int_E 1\,\mathrm{d}t + \int_{[0,1]\setminus E} g(y(t))\,\mathrm{d}t \\ &\ge |E| + |[0,1] \setminus E| = 1.\end{aligned}$$

Notice that nevertheless, from [3], the minima of F are Lipschitz.

4. Lipschitz regularity of the minima of (P)

In this section we apply our result to the problem of the Lipschitz regularity of the minima of (P). It is well known that if $L(x, \xi)$ is continuous, convex and superlinear in ξ then every minimum of (P) is Lipschitz. In some recent papers the same conclusion is proved under weaker assumptions; we just mention [5, 6, 8]. Our result is in the same spirit of the last two that we recall here.

Theorem 4.1. *[5] Assume that $L(x, \xi)$ is continuous in both variables and satisfies (GA). Then every minimizer of (P) in $AC([a,b], \mathbb{R}^N)$ is Lipschitz.*

Theorem 4.2. *[8] Assume that $L(x, \xi)$ is convex in ξ and satisfies (GA). Then every minimizer of (P) in $AC([a,b], \mathbb{R}^N)$ is Lipschitz.*

The following theorem weakens the continuity assumption of Theorem 4.1.

Theorem 4.3. *Assume that $x \mapsto L(x,\xi)$ is continuous for every ξ and that L satisfies (GA). Then every minimizer of (P) in $AC([a,b],\mathbb{R}^N)$ is Lipschitz.*

Proof. By Theorem 3.6, $\inf F = \inf F^{**}$; therefore every minimum of F is a minimum of F^{**}. The function $L^{**}(x,\xi)$ is convex in ξ and satisfies (GA): Theorem 4.2 yields the conclusion. □

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