

A RELAXATION RESULT FOR AUTONOMOUS INTEGRAL FUNCTIONALS WITH DISCONTINUOUS NON-COERCIVE INTEGRAND

CARLO MARICONDA¹ AND GIULIA TREU¹

Abstract. Let $L : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Borelian function and consider the following problems

$$\inf \left\{ F(y) = \int_a^b L(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P)$$

$$\inf \left\{ F^{**}(y) = \int_a^b L^{**}(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\}. \quad (P^{**})$$

We give a sufficient condition, weaker than superlinearity, under which $\inf F = \inf F^{**}$ if L is just continuous in x . We then extend a result of Cellina on the Lipschitz regularity of the minima of (P) when L is not superlinear.

Mathematics Subject Classification. 37N35.

Received June 27, 2003.

1. INTRODUCTION

We consider the relationships between the problems

$$\inf \left\{ F(y) = \int_a^b L(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P)$$

$$\inf \left\{ F^{**}(y) = \int_a^b L^{**}(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\}. \quad (P^{**})$$

It is well known that $\inf F = \inf F^{**}$ if L is super-linear and continuous. Recently Cellina in [5] proved that the same conclusion holds true assuming, instead of superlinearity, a weaker growth condition that we will call (GA) . Roughly, a convex function $L(x, \xi)$ satisfies (GA) if the intersection of the supporting hyperplane to its epigraph at $(\xi, L(x, \xi))$ with the ordinate axis tends to $-\infty$ as $|\xi|$ tends to $+\infty$, uniformly with respect to x in compact sets. This condition implies, but is not equivalent to, a sort of conical growth: we say that L

Keywords and phrases. Lipschitz, regularity, non-coercive, discontinuous, calculus of variations.

¹ Dipartimento di Matematica pura e applicata, Università di Padova, 7 via Belzoni, 35131 Padova, Italy;
e-mail: maricond@math.unipd.it; treu@math.unipd.it

satisfies (CGA) if for every ξ_0 there exist $\varepsilon, R > 0$ such that, for every $|\xi| \geq R$,

$$L(x, \xi) \geq L(x, \xi_0) + p(x, \xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + \text{const.} \tag{CGA}$$

whenever x belongs to a prescribed compact set and $p(x, \xi_0)$ belongs to the subdifferential of $\xi \mapsto L(x, \xi)$ in ξ_0 . We weaken here the continuity assumption of L in both variables and we prove that, if $L(x, \xi)$ is just continuous in x and satisfies (CGA), then $\inf F = \inf F^{**}$.

The proof of the result is based on Theorem 3.2, a uniform approximation of the bipolar of a (discontinuous) function $L(\xi)$ satisfying (CGA) in terms of the convex hull of the graph of L ; this kind of result is classical when L is supposed to be lower semi-continuous and superlinear in ξ [7].

In the last part of the paper we are concerned with an application to the Lipschitz regularity of the minima of (P) . It is well known that, if $L(x, \xi)$ is superlinear and convex in ξ , then every minimizer of (P) is Lipschitz. The same result was obtained recently by dropping some of the assumptions: no continuity and no convexity but superlinearity is assumed in [6], continuity, no convexity and assumption (GA) instead of superlinearity is assumed in [5], no continuity and no convexity but the requirement that every section $\lambda \mapsto L(x, \lambda u)$ ($\lambda \geq 0, |u| = 1$) satisfies (GA) in [8], extending [6].

As a consequence of our relaxation result we prove that the minima of (P) are Lipschitz if $L(x, \xi)$ is just continuous in x and satisfies (GA), thus extending the main result in [5].

We point out that there are several results concerning the representation of the lower semi-continuous envelope of integral functionals; we just mention [2, 4] for some recent results and references. Here we are interested in comparing the values of the infima of problems (P) and (P^{**}) instead of establishing a representation formula.

2. NOTATION AND PRELIMINARY RESULTS

In this paper $|\cdot|$ is the Euclidean norm and “ \cdot ” the scalar product in \mathbb{R}^N . For a function $L(x, \xi) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ we denote by $L^{**}(x, \xi)$ (resp. $\partial L^{**}(x, \xi)$) the bipolar (resp. the subdifferential of the bipolar) of $\xi \mapsto L(x, \xi)$. Finally, $AC([a, b], \mathbb{R}^N)$ is the space of absolutely continuous functions on $[a, b]$ with values in \mathbb{R}^N .

Here $L : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is just a Borelian function. We assume moreover that $L^{**}(x, \xi) \neq -\infty$ for every x and ξ ; this is the case, for instance, if L is bounded below by an affine function of ξ .

The following growth condition will be assumed in the main result.

Conical growth assumption (CGA). For every compact subset C of \mathbb{R}^N and $R_0 \geq 0$ there exist $\varepsilon > 0, R > 0$ and $c \in \mathbb{R}$ such that

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R \quad L^{**}(x, \xi) \geq L^{**}(x, \xi_0) + p(x, \xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + c$$

for every $x \in C, |\xi_0| \leq R_0$ and $p(x, \xi_0)$ in $\partial L^{**}(x, \xi_0)$.

The following growth assumption was introduced by Cellina in [5] in the case where L is continuous.

Growth assumption (GA).

We say that L satisfies (GA) if there exist $p(x, \xi)$ in $\partial L^{**}(x, \xi)$ such that

$$\lim_{|\xi| \rightarrow +\infty} p(x, \xi) \cdot \xi - L^{**}(x, \xi) = +\infty \tag{2.1}$$

uniformly for x in a compact set.

Remark 2.1.

- i) We point out that, in [5], the definition of (GA) is slightly different: it is formulated in an equivalent way in terms of the polar of L in $(x, p(x, \xi))$; moreover the uniformity with respect to the first variable is not required since it is a consequence of the continuity of L . We use it here since we drop the continuity assumption.

- ii) Assumption (GA) is fulfilled if, for instance, $L(x, \xi)$ is superlinear with respect to ξ ; the proof can be easily done following the lines of [5].

We refer to [8] for a survey on the properties of the functions that satisfy (GA).

Theorem 2.2. [8, Cor 4.4] *Assume that L is bounded on compact sets and satisfies the Growth Assumption (GA). Then L satisfies (CGA).*

3. RELAXATION

It is well known that if $L : \mathbb{R}^N \rightarrow \mathbb{R}$ is a function whose bipolar is finite, then, for every $\varepsilon > 0$ and ξ in \mathbb{R}^N , there exists ξ_1, \dots, ξ_m ($m \leq N + 1$) in \mathbb{R}^N and coefficients of a convex combination $\alpha_1, \dots, \alpha_m$ such that $\sum_i \alpha_i L(\xi_i) \leq L^{**}(\xi) + \varepsilon$ and $\sum_i \alpha_i \xi_i = \xi$. We prove in the next Theorem 3.2 that if L satisfies (CGA) then, allowing $m \leq 2N + 2$, the points ξ_i may be bounded uniformly with respect to ξ in compact sets. For this purpose we first quote, in a more general setting, a powerful consequence of (CGA) that was established in [5] in the continuous case. For every (x, ξ) we set

$$\bar{L}(x, \xi) = \liminf_{\eta \rightarrow \xi} L(x, \eta),$$

i.e. $\bar{L}(x, \xi)$ denotes the lower semi-continuous envelope of the map $\eta \mapsto L(x, \eta)$. The proof of the following result is based on the fact that if $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is convex and satisfies (CGA) then the intersection of its epigraph with any supporting hyperplane is bounded. This condition is referred in [5] as the *Bounded Intersection Property*.

Theorem 3.1. *Assume that L satisfies (CGA) and let $p(x, \xi) \in \partial L^{**}(x, \xi)$. Then given $R_0 > 0$ and a compact subset C of \mathbb{R}^N there exists $R > 0$ (depending only on R_0 and C) such that for every $x \in C$; for every ξ , with $|\xi| \leq R_0$, there exist at most $\nu \leq N + 1$ points ξ_i , with $|\xi_i| \leq R$, and coefficients of a convex combination α_i , such that*

$$\begin{pmatrix} \xi \\ L^{**}(x, \xi) \end{pmatrix} = \sum_{i=1}^{\nu} \alpha_i \begin{pmatrix} \xi_i \\ \bar{L}(x, \xi_i) \end{pmatrix}$$

and $L^{**}(x, \xi) = \bar{L}(x, \xi_i) = L^{**}(x, \xi) + p(x, \xi) \cdot (\xi - \xi_i)$.

Proof. It is enough to remark that Theorem 1 in [5] holds for functions that are lower semi-continuous instead of continuous and that the bipolar of a function coincides with the bipolar of its lower semi-continuous envelope. \square

We are now ready to state a version of Theorem 3.1 that does not involve the lower semi-continuous envelope of L .

Theorem 3.2. *Assume that $L(x, \xi)$ satisfies (CGA) and that L is bounded on the compact sets. Then given $R_0 > 0$ and a compact subset C of \mathbb{R}^N , there exists $R > 0$ (depending only on R_0 and C) such that for every x in C , for every ξ , with $|\xi| \leq R_0$ and $\varepsilon > 0$, there exist at most $m \leq 2N + 2$ points ξ_i , with $|\xi_i| \leq R$, and coefficients of a convex combination λ_i , such that*

$$\begin{cases} \xi = \sum_{i=1}^m \lambda_i \xi_i \\ \sum_{i=1}^m \lambda_i L(x, \xi_i) \leq L^{**}(x, \xi) + \varepsilon. \end{cases}$$

The proof of the result needs several preliminary steps. For the convenience of the reader we first give a sketch of the proof in the case where L does not depend on x .

By Theorem 3.1, for $|\xi| \leq R_0$, the point $(\xi, L^{**}(\xi))$ can be written as a convex combination of points $(\zeta_i, \bar{L}(\zeta_i))$ of the epigraph of the lower semi-continuous envelope of $L(\cdot)$; moreover the ζ_i are uniformly bounded, so that they all lie in a simplex generated by $N + 1$ affinely independent points $\eta_1, \dots, \eta_{N+1}$. Now each value $\bar{L}(\zeta_j)$ can be approximated with $L(\eta^j)$ for some η^j arbitrarily near to ζ_j ; actually it turns out that for $\varepsilon > 0$, if $|\eta^j - \zeta_j|$ is sufficiently small, then there is a convex combination of $(\eta^j, L(\eta^j))$ and N points among the $(\eta_i, L(\eta_i))$'s whose

projection on \mathbb{R}^N is ζ_j and whose last coordinate is less than $\bar{L}(\zeta_j) + \varepsilon$. The conclusion follows by writing ξ as a convex combinations of the points η_i and the η^j constructed as above.

We first need two technical lemmas. Let, if S is a subset of \mathbb{R}^N , $\text{int } S$ denote its interior and $\text{conv} S$ its convex hull.

Lemma 3.3. *Let $\eta_1, \dots, \eta_{N+1}$ be $N+1$ affinely independent points of \mathbb{R}^N and η in $\text{int}(\text{conv}\{\eta_1, \dots, \eta_{N+1}\})$, the interior of the simplex whose vertices are $\eta_1, \dots, \eta_{N+1}$. Then:*

- i) *for every $I \subset \{1, \dots, N+1\}$ of cardinality $|I| \leq N$ the set of points $\{\eta, \eta_i : i \in I\}$ is affinely independent;*
- ii) *for every $\xi \in \text{int}(\text{conv}\{\eta_1, \dots, \eta_{N+1}\})$ there exists a subset I of $\{1, \dots, N+1\}$ of cardinality N such that $\xi \in \text{conv}\{\eta, \eta_i : i \in I\}$.*

Proof of Lemma 3.3. i) It is not restrictive to assume that $I = \{1, \dots, N\}$. Let

$$\eta = \sum_{j=1}^{N+1} \lambda_j \eta_j \quad \lambda_j > 0 \quad \sum_{j=1}^{N+1} \lambda_j = 1.$$

For every $i \in \{1, \dots, N\}$ we have

$$\begin{aligned} \eta - \eta_i &= \sum_{j \neq i} \lambda_j \eta_j + (\lambda_i - 1)\eta_i \\ &= \sum_{j \neq i} \lambda_j (\eta_j - \eta_{N+1}) + (\lambda_i - 1)(\eta_i - \eta_{N+1}) \end{aligned}$$

so that, in a matrix notation,

$$[\eta - \eta_1, \dots, \eta - \eta_N] = [\eta_1 - \eta_{N+1}, \dots, \eta_N - \eta_{N+1}](\Lambda - I)$$

where I is the identity and

$$\Lambda = \begin{pmatrix} \lambda_1 & \dots & \lambda_N \\ \dots & \dots & \dots \\ \lambda_1 & \dots & \lambda_N \end{pmatrix}.$$

Now $\det(\Lambda - I) \neq 0$ since the eigenvalues of Λ are $\lambda_1, \dots, \lambda_N$ and $\lambda_i < 1$ for every i , proving i).

Proof of ii). Let

$$\xi = \alpha_1 \eta_1 + \dots + \alpha_{N+1} \eta_{N+1} \quad \eta = \mu_1 \eta_1 + \dots + \mu_{N+1} \eta_{N+1}$$

and we may assume that $\alpha_{N+1}/\mu_{N+1} = \min\{\alpha_i/\mu_i : i = 1, \dots, N+1\}$ (notice that all the μ_i are strictly positive). Set $c_{N+1} = \alpha_{N+1}/\mu_{N+1}$ and, for $i \in \{1, \dots, N\}$, $c_i = \alpha_i - \mu_i c_{N+1}$: then, for every i , $c_i \geq 0$; moreover

$$\begin{aligned} \sum_{i=1}^{N+1} c_i &= \sum_{i=1}^N \alpha_i - c_{N+1} \sum_{i=1}^N \mu_i + c_{N+1} \\ &= 1 - \alpha_{N+1} - (1 - \mu_{N+1})c_{N+1} + c_{N+1} \\ &= 1 - \alpha_{N+1} + \mu_{N+1} c_{N+1} = 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^N c_i \eta_i + c_{N+1} \eta &= \sum_{i=1}^N (\alpha_i - \mu_i c_{N+1}) \eta_i + c_{N+1} \sum_{i=1}^{N+1} \mu_i \eta_i \\ &= \sum_{i=1}^N (\alpha_i - \mu_i c_{N+1} + \mu_i c_{N+1}) \eta_i + c_{N+1} \mu_{N+1} \eta_{N+1} \\ &= \sum_{i=1}^N \alpha_i \eta_i + \alpha_{N+1} \eta_{N+1} = \xi \end{aligned}$$

so that $\xi \in \text{conv}\{\eta, \eta_i : i \in \{1, \dots, N\}\}$. □

Lemma 3.4. *Let $\eta_1, \dots, \eta_{N+1}$ be $N + 1$ affinely independent points of \mathbb{R}^N and let y_1, \dots, y_{N+1} be real numbers and $K > 0$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\eta, \xi \in \text{int}(\text{conv}\{\eta_1, \dots, \eta_{N+1}\})$ and y, β in $[-K, K]$, with $|\eta - \xi| < \delta$ and $|y - \beta| < \delta$, there exists a subset I of $\{1, \dots, N + 1\}$ of cardinality N and coefficients λ, λ_i ($i \in I$) of a convex combination satisfying*

$$\begin{cases} \xi = \lambda \eta + \sum_{i \in I} \lambda_i \eta_i \\ \beta + \varepsilon > \lambda y + \sum_{i \in I} \lambda_i y_i. \end{cases}$$

Remark 3.5. Geometrically Lemma 3.4 states that given $N + 1$ points (η_i, y_i) of $\mathbb{R}^N \times \mathbb{R}$ and (ξ, β) in $\mathbb{R}^N \times \mathbb{R}$ such that ξ lies in the interior of the convex hull Λ of the η_i s then, given a positive ε , for every point (η, y) that is sufficiently near to (ξ, β) with $\eta \in \Lambda$ there exist N points among the (η_i, y_i) s which, together with (η, y) , generate a N - dimensional simplex in $\mathbb{R}^N \times \mathbb{R}$ whose projection in \mathbb{R}^N contains ξ and such that $(\xi, \beta + \varepsilon)$ lies above it.

Proof of Lemma 3.4. For every $I \subset \{1, \dots, N + 1\}$, $|I| = N$, $y \in \mathbb{R}$ and $\eta \in \Lambda := \text{int conv}\{\eta_1, \dots, \eta_{N+1}\}$ by Lemma 3.3i) there exists a unique hyperplane $z = a^I(\eta, y) \cdot \xi + b^I(\eta, y)$ containing the points (η, y) and (η_i, y_i) ($i \in I$). Moreover the coefficients $a^I(\eta, y), b^I(\eta, y)$ are continuous functions of (η, y) ; in fact from the equations

$$\begin{cases} a^I(\eta, y) \cdot \eta + b^I(\eta, y) = y \\ a^I(\eta, y) \cdot \eta_i + b^I(\eta, y) = y_i \quad (i \in I) \end{cases}$$

we deduce that the vector $a^I(\eta, y)$ solves the system

$$a^I(\eta, y) \cdot (\eta - \eta_i) = y - y_i \quad (i \in I);$$

again by Lemma 3.3i) the vectors $\eta - \eta_i$ ($i \in I$) are independent so that the latter system has a unique solution $a^I(\eta, y)$ given by Cramer's rule which is a continuous function of η and y ; the continuity of b^I follows from the equality $b^I(\eta, y) = y - a^I(\eta, y) \cdot \eta$.

Set, for every $I \subset \{1, \dots, N + 1\}$,

$$\varphi^I(\eta, y; \zeta) = a^I(\eta, y) \cdot \zeta + b^I(\eta, y);$$

we point out that, by construction, for a fixed (η, y) the point $(\zeta, \varphi^I(\eta, y; \zeta))$ belongs to the (unique) hyperplane containing the points (η, y) and (η_i, y_i) ($i \in I$). Since, for every $\zeta \in \Lambda$ and $\beta \in \mathbb{R}$,

$$\varphi^I(\zeta, \beta; \zeta) = \beta,$$

then, by the uniform continuity of φ^I on $\Lambda \times [-K, K] \times \Lambda$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every subset I of $\{1, \dots, N+1\}$ of cardinality N ,

$$\varphi^I(\eta, y; \zeta) < \beta + \varepsilon \text{ whenever } \eta, \zeta \in \Lambda \quad |\eta - \zeta| < \delta \text{ and } y, \beta \in [-K, K] \quad |y - \beta| < \delta. \tag{3.1}$$

Now fix ξ in Λ and $\varepsilon > 0$. Let $I \subset \{1, \dots, N+1\}$ be such that ξ belongs to $\text{conv}\{\eta, \eta_i : i \in I\}$; such a set exists by Lemma 3.3ii). Let δ be such as in (3.1) and $|\eta - \xi| < \delta, |y - \beta| < \delta$ so that $\varphi^I(\eta, y; \xi) < \beta + \varepsilon$. Then, if we set

$$\xi = \lambda\eta + \sum_{i \in I} \lambda_i \eta_i$$

for some coefficients λ, λ_i ($i \in I$) of a convex combination, the linearity of φ^I in the third variable yields

$$\lambda\varphi^I(\eta, y; \eta) + \sum_{i \in I} \lambda_i \varphi^I(\eta, y; \eta_i) < \beta + \varepsilon,$$

proving the claim since $\varphi^I(\eta, y; \eta) = y$ and $\varphi^I(\eta, y; \eta_i) = y_i$. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Fix x in C . By Theorem 3.1 there exists $R_1 > 0$ (depending only on R_0 and C), $\zeta_1, \dots, \zeta_\nu$ ($\nu \leq N+1$), with $|\zeta_j| \leq R_1$, and coefficients α_j of a convex combination satisfying

$$\begin{cases} \xi = \sum_{j=1}^{\nu} \alpha_j \zeta_j \\ L^{**}(x, \xi) = \sum_{j=1}^{\nu} \alpha_j \bar{L}(x, \zeta_j); \end{cases}$$

where \bar{L} denotes as usual the lower semi-continuous envelope of $L(x, \cdot)$. It is not restrictive at this stage to assume that $L(x, \xi) = L(\xi)$. Let $\eta_1, \dots, \eta_{N+1}$ be such that

$$\{\zeta \in \mathbb{R}^N : |\zeta| \leq R_1\} \subset \text{int}(\text{conv}\{\eta_1, \dots, \eta_{N+1}\})$$

and set

$$y_i = L(\eta_i), \quad i = 1, \dots, N+1, \quad K = \sup\{L(\zeta) : |\zeta| \leq R_1\}.$$

Fix $\varepsilon > 0$ and j in $\{1, \dots, \nu\}$; set $\beta = \bar{L}(\zeta_j)$. Correspondingly, let $\delta > 0$ satisfy the property stated in Lemma 3.4. By the definition of \bar{L} there exist $\eta^j \in \text{int}(\text{conv}\{\eta_1, \dots, \eta_{N+1}\})$ such that

$$|\eta^j - \zeta_j| < \delta \quad \text{and} \quad L(\eta^j) \leq \bar{L}(\zeta_j) + \delta.$$

We apply Lemma 3.4 with $\eta = \eta^j, \xi = \zeta_j$ and $y = L(\eta^j)$: there exists a subset I_j of $\{1, \dots, N+1\}$ of cardinality N and coefficients $\lambda^j, \lambda_i^j, (i \in I_j)$, such that

$$\begin{cases} \zeta_j = \lambda^j \eta^j + \sum_{i \in I_j} \lambda_i^j \eta_i \\ \lambda^j L(\eta^j) + \sum_{i \in I_j} \lambda_i^j L(\eta_i) \leq \bar{L}(\zeta_j) + \varepsilon. \end{cases}$$

Therefore we obtain that

$$\begin{aligned} \xi &= \sum_{j=1}^{\nu} \alpha_j \zeta_j = \sum_{j=1}^{\nu} \alpha_j \left(\lambda^j \eta^j + \sum_{i \in I_j} \lambda_i^j \eta_i \right) \\ &= \sum_{j=1}^{\nu} \alpha_j \lambda^j \eta^j + \sum_{i \in I_j} \left(\sum_{j=1}^{\nu} \alpha_j \lambda_i^j \right) \eta_i \end{aligned}$$

and moreover

$$\begin{aligned} L^{**}(\xi) + \varepsilon &= \sum_{j=1}^{\nu} \alpha_j (\bar{L}(\zeta_j) + \varepsilon) \\ &\geq \sum_{j=1}^{\nu} \alpha_j \left(\lambda^j L(\eta^j) + \sum_{i \in I_j} \lambda_i^j L(\eta_i) \right) \\ &= \sum_{j=1}^{\nu} \alpha_j \lambda^j L(\eta^j) + \sum_{i \in I_j} \left(\sum_{j=1}^{\nu} \alpha_j \lambda_i^j \right) L(\eta_i). \end{aligned}$$

If we set

$$\begin{cases} \lambda_i = \alpha_i \lambda^i & \text{if } i \in \{1, \dots, \nu\} \\ \lambda_i = \sum_j \alpha_j \lambda_{i-\nu}^j & \text{if } i \in \{\nu + 1, \dots, \nu + (N+1)\} \end{cases}$$

the above formulae can be rewritten as

$$\begin{cases} \xi = \sum_{i \leq \nu} \lambda_i \eta^i + \sum_{i > \nu} \lambda_i \eta_i \\ \sum_{i \leq \nu} \lambda_i L(\eta^i) + \sum_{i > \nu} \lambda_i L(\eta_i) \geq L^{**}(\xi) + \varepsilon. \end{cases}$$

Moreover $|\eta^i| \leq R$ and $|\eta_i| \leq R$, where $R = \max\{|\eta_i| : i = 1, \dots, N+1\}$ (which depends only on R_1 and therefore only on R_0 and C); proving the claim. \square

We consider here the problems

$$\inf \left\{ F(y) = \int_a^b L(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \tag{P}$$

$$\inf \left\{ F^{**}(y) = \int_a^b L^{**}(y(t), y'(t)) dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\}. \tag{P^{**}}$$

It is well known that $\inf F = \inf F^{**}$ if L is continuous and superlinear ([7], Th. IX.3.1); actually in this case F^{**} is the relaxed functional of F . In [5] Cellina proved that $\inf F = \inf F^{**}$ if L is just continuous and satisfies (GA). We examine here the case where L is just continuous in the first variable, focusing our attention on the infima of the functionals F and F^{**} instead on the relaxed functional of F .

Theorem 3.6. *Assume that L is bounded on compact sets and that $x \mapsto L(x, \xi)$ is continuous for every $\xi \in \mathbb{R}^N$. If L satisfies (CGA) then $\inf F = \inf F^{**}$.*

Proof. We follow the lines of the proof of the analogous result ([5], Th. 3) in the case where L is continuous in both variables, but instead of Theorem 3.1 we use Theorem 3.2. Let $x \in AC([a, b], \mathbb{R}^N)$ and $\varepsilon > 0$. From Theorem 2.4 and Remark 2.8 of [1] applied to L^{**} there exists a Lipschitz function x_{R_0} of Lipschitz constant R_0 satisfying the boundary conditions and such that

$$\int_a^b L^{**}(x_{R_0}(t), x'_{R_0}(t)) dt \leq \int_a^b L^{**}(x(t), x'(t)) dt + \varepsilon/3.$$

Set $C = \{x_{R_0}(t) : t \in [a, b]\}$. Since $|x'_{R_0}(t)| \leq R_0$ for a.e. t then, by Theorem 3.2, there exists R (depending only on R_0 and C), $m \leq 2N + 2$ coefficients $\lambda_i(t)$ of a convex combination and vectors $y_i(t)$ ($i = 1, \dots, m$) with

$|y_i(t)| \leq \mathbb{R}$ such that

$$\begin{cases} x'_{R_0}(t) = \sum_{i=1}^m \lambda_i(t)y_i(t) \\ \sum_{i=1}^m \lambda_i(t)L(x_{R_0}(t), y_i(t)) \leq L^{**}(x_{R_0}(t), x'_{R_0}(t)) + \frac{\varepsilon}{3(b-a)}. \end{cases}$$

By a standard selection argument, we may assume that the maps y_i and λ_i are measurable. Fix an integer k and consider the intervals $I_j = [t_j, t_{j+1}]$, where $t_j = a + j\frac{b-a}{k}$ ($j = 0, \dots, k-1$) and call χ_{I_j} their characteristic function. By Lyapunov's Theorem on the range of vector measures [9] there exists a partition of $[a, b]$ into m measurable subsets E_i , with characteristic functions χ_{E_i} , such that, for $j = 0, \dots, k-1$, one has

$$\begin{aligned} \int_{I_j} \sum_{i=1}^m \lambda_i(t)y_i(t) dt &= \int_{I_j} \sum_{i=1}^m \chi_{E_i}(t)y_i(t) dt \\ \int_{I_j} \sum_{i=1}^m \lambda_i(t)L(x_{R_0}(t), y_i(t)) dt &= \int_{I_j} \sum_{i=1}^m \chi_{E_i}(t)L(x_{R_0}(t), y_i(t)) dt. \end{aligned}$$

Denote by x_k the absolutely continuous defined by $x_k(a) = A$ and

$$x'_k(t) = \int_a^t \sum_{i,j} y_i(s)\chi_{I_j \cap E_i}(s) ds;$$

in particular for every k and every $j = 1, \dots, k$, we have

$$\int_{I_j} x'_{R_0}(t) dt = \int_{I_j} x'_k(t) dt,$$

so that the functions x_{R_0} and x_k coincide at each point t_j . Since

$$L(x_{R_0}(t), x'_k(t)) = \sum_{i,j} \chi_{I_j \cap E_i}(t)L(x_{R_0}(t), y_i(t))$$

we also have that

$$\int_a^b L(x_{R_0}(t), x'_k(t)) dt = \int_a^b \sum_{i=1}^m \lambda_i(t)L(x_{R_0}(t), y_i(t)) dt;$$

so that, from \approx , it follows that

$$\int_a^b L(x_{R_0}(t), x'_k(t)) dt \leq \int_a^b L^{**}(x_{R_0}(t), x'_{R_0}(t)) dt + \varepsilon/3.$$

Now

$$\int_a^b L(x_{R_0}(t), x'_k(t)) dt = \int_a^b L(x_k(t), x'_k(t)) dt + \int_a^b L(x_{R_0}(t), x'_k(t)) - L(x_k(t), x'_k(t)) dt;$$

moreover, x_{R_0} is uniformly continuous, the functions x_k are equi-Lipschitz, $x_k(t_j) = x_{R_0}(t_j)$ ($j = 0, \dots, k-1$). Hence, if $t \in [a, b]$ and $t_j \leq t \leq t_{j+1}$,

$$\begin{aligned} |x_k(t) - x_{R_0}(t)| &\leq |x_k(t) - x_k(t_j)| + |x_k(t_j) - x_{R_0}(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)| \\ &= |x_k(t) - x_k(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)| \leq (R + R_0)(b-a)/k \end{aligned}$$

so that x_k converges uniformly to x_{R_0} as k tends to $+\infty$. By our assumption the function $L(x_{R_0}, x'_k) - L(x_k, x'_k)$ is bounded a.e. by a constant that does not depend on k . The continuity of L with respect to the first variable together with the dominated convergence theorem imply that

$$\lim_{k \rightarrow +\infty} \int_a^b L(x_{R_0}(t), x'_k(t)) - L(x_k(t), x'_k(t)) dt = 0.$$

It follows that for k sufficiently large,

$$\int L(x_k(t), x'_k(t)) dt \leq \int_a^b L(x_{R_0}(t), x'_k(t)) dt + \varepsilon/3 \leq \int_a^b L(x(t), x'(t)) dt + \varepsilon$$

proving that $\inf F \leq \inf F^{**}$. □

We point out that, under the assumptions of Theorem 3.6, the functional F^{**} is not in general the relaxed functional of F ; we refer to [2] for some recent results in this direction. This is the case in the forthcoming example, where we also show that the conclusion of Theorem 3.6 does not hold if L is not continuous in x .

Example 3.7. Let g be the characteristic function of $\mathbb{R} \setminus \{0\}$ and $h(\xi) = \xi^2$ if $\xi \neq 0$, $h(0) = 1$ and set $L(x, \xi) = g(x) + h(\xi)$. Let (P) , (P^{**}) be the problems

$$\begin{aligned} \inf \left\{ F(y) = \int_0^1 L(y(t), y'(t)) dt; \quad y(0) = 0, y(1) = 0, y \in AC([0, 1], \mathbb{R}) \right\} & \quad (P) \\ \inf \left\{ F^{**}(y) = \int_0^1 L^{**}(y(t), y'(t)) dt; \quad y(0) = 0, y(1) = 0, y \in AC([0, 1], \mathbb{R}) \right\} & \quad (P^{**}) \end{aligned}$$

For every x in \mathbb{R} we have $L^{**}(x, \xi) = g(x) + \xi^2$, so that the minimum of the problem (P^{**}) is equal to 0 and it is obviously assumed for $y(t) = 0$. However $\inf F \geq 1$; in fact let $y \in AC([0, 1], \mathbb{R})$ and set $E = \{t \in [0, 1] : y(t) = 0\}$, then $y'(t) = 0$ a.e. on E , so that

$$\begin{aligned} \int_0^1 L(y(t), y'(t)) dt &= \int_E L(0, 0) dt + \int_{[0,1] \setminus E} L(y(t), y'(t)) dt \\ &\geq \int_E 1 dt + \int_{[0,1] \setminus E} g(y(t)) dt \\ &\geq |E| + |[0, 1] \setminus E| = 1. \end{aligned}$$

Notice that nevertheless, from [3], the minima of F are Lipschitz.

4. LIPSCHITZ REGULARITY OF THE MINIMA OF (P)

In this section we apply our result to the problem of the Lipschitz regularity of the minima of (P) . It is well known that if $L(x, \xi)$ is continuous, convex and superlinear in ξ then every minimum of (P) is Lipschitz. In some recent papers the same conclusion is proved under weaker assumptions; we just mention [5, 6, 8]. Our result is in the same spirit of the last two that we recall here.

Theorem 4.1. [5] *Assume that $L(x, \xi)$ is continuous in both variables and satisfies (GA). Then every minimizer of (P) in $AC([a, b], \mathbb{R}^N)$ is Lipschitz.*

Theorem 4.2. [8] *Assume that $L(x, \xi)$ is convex in ξ and satisfies (GA). Then every minimizer of (P) in $AC([a, b], \mathbb{R}^N)$ is Lipschitz.*

The following theorem weakens the continuity assumption of Theorem 4.1.

Theorem 4.3. *Assume that $x \mapsto L(x, \xi)$ is continuous for every ξ and that L satisfies (GA). Then every minimizer of (P) in $AC([a, b], \mathbb{R}^N)$ is Lipschitz.*

Proof. By Theorem 3.6, $\inf F = \inf F^{**}$; therefore every minimum of F is a minimum of F^{**} . The function $L^{**}(x, \xi)$ is convex in ξ and satisfies (GA): Theorem 4.2 yields the conclusion. \square

REFERENCES

- [1] G. Alberti and F. Serra Cassano, Non-occurrence of gap for one-dimensional autonomous functionals. *Ser. Adv. Math. Appl. Sci. Calculus of variations, homogenization and continuum mechanics* **18** (1993) 1-17.
- [2] M. Amar, G. Bellettini and S. Venturini, Integral representation of functionals defined on curves of $W^{1,p}$. *Proc. R. Soc. Edinb. Sect. A* **128** (1998) 193-217.
- [3] L. Ambrosio, O. Ascenzi and G. Buttazzo, Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands. *J. Math. Anal. Appl.* **142** (1989) 301-316.
- [4] G. Buttazzo, Semicontinuity, relaxation and integral representation in the calculus of variations. *Pitman Res. Notes Math. Ser.* **207** (1989).
- [5] A. Cellina, *The classical problem of the calculus of variations in the autonomous case: Relaxation and lipschitzianity of solutions*. Preprint (2001).
- [6] G. Dal Maso and H. Frankowska, *Autonomous Integral Functionals with Discontinuous Nonconvex Integrands: Lipschitz Regularity of Minimizers, DuBois-Reymond Necessary Conditions, and Hamilton-Jacobi Equations*. Preprint (2002).
- [7] I. Ekeland and R. Témam, Convex analysis and variational problems. *Classics Appl. Math.* **28** (1999).
- [8] C. Mariconda and G. Treu, *Lipschitz regularity of the minimizers of autonomous integral functionals with discontinuous non-convex integrands of slow growth*. Dipartimento di Matematica pura e applicata, Università di Padova **10** (2003) preprint.
- [9] W. Rudin, *Functional analysis*. International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York (1991).