A RELAXATION RESULT FOR AUTONOMOUS INTEGRAL FUNCTIONALS WITH DISCONTINUOUS NON-COERCIVE INTEGRAND

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Abstract. Let $L : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Borelian function and consider the following problems

$$\inf \left\{ F(y) = \int_a^b L(y(t), y'(t)) \, dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P)$$

$$\inf \left\{ F^{**}(y) = \int_a^b L^{**}(y(t), y'(t)) \, dt : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P^{**})$$

We give a sufficient condition, weaker than superlinearity, under which $\inf F = \inf F^{**}$ if $L$ is just continuous in $x$. We then extend a result of Cellina on the Lipschitz regularity of the minima of $(P)$ when $L$ is not superlinear.

Mathematics Subject Classification. 37N35.

Received June 27, 2003.

1. INTRODUCTION

We consider the relationships between the problems

$$\inf \left\{ F(y) = \int_a^b L(y(t), y'(t)) \, dt : y \in AC ([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P)$$

$$\inf \left\{ F^{**}(y) = \int_a^b L^{**}(y(t), y'(t)) \, dt : y \in AC ([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} \quad (P^{**})$$

It is well known that $\inf F = \inf F^{**}$ if $L$ is super-linear and continuous. Recently Cellina in [5] proved that the same conclusion holds true assuming, instead of superlinearity, a weaker growth condition that we will call $(GA)$. Roughly, a convex function $L(x, \xi)$ satisfies $(GA)$ if the intersection of the supporting hyperplane to its epigraph at $(\xi, L(x, \xi))$ with the ordinate axis tends to $-\infty$ as $|\xi|$ tends to $+\infty$, uniformly with respect to $x$ in compact sets. This condition implies, but is not equivalent to, a sort of conical growth: we say that $L$...
satisfies (CGA) if for every $\xi_0$ there exist $\varepsilon, R > 0$ such that, for every $|\xi| \geq R$,

$$L(x, \xi) \geq L(x, \xi_0) + p(x, \xi_0) \cdot (\xi - \xi_0) + \varepsilon |\xi| + \text{const.}$$  \hspace{1cm} (CGA)$$

whenever $x$ belongs to a prescribed compact set and $p(x, \xi_0)$ belongs to the subdifferential of $\xi \mapsto L(x, \xi)$ in $\xi_0$.

We weaken here the continuity assumption of $L$ in both variables and we prove that, if $L(x, \xi)$ is just continuous in $x$ and satisfies (CGA), then $\inf F = \inf F^{**}$.

The proof of the result is based on Theorem 3.2, a uniform approximation of the bipolar of a (discontinuous) function $L(\xi)$ satisfying (CGA) in terms of the convex hull of the graph of $L$; this kind of result is classical when $L$ is supposed to be lower semi-continuous and superlinear in $\xi$ [7].

In the last part of the paper we are concerned with an application to the Lipschitz regularity of the minima of $(P)$. It is well known that, if $L(x, \xi)$ is superlinear and convex in $\xi$, then every minimizer of $(P)$ is Lipschitz. The same result was obtained recently by dropping some of the assumptions: no continuity and no convexity but superlinearity is assumed in [6], continuity, no convexity and assumption (GA) instead of superlinearity is assumed in [5], no continuity and no convexity but the requirement that every section $\lambda \mapsto L(x, \lambda u)$ ($\lambda \geq 0$, $|u| = 1$) satisfies (GA) in [8], extending [6].

As a consequence of our relaxation result we prove that the minima of $(P)$ are Lipschitz if $L(x, \xi)$ is just continuous in $x$ and satisfies (GA), thus extending the main result in [5].

We point out that there are several results concerning the representation of the lower semi-continuous envelope of integral functionals; we just mention [2, 4] for some recent results and references. Here we are interested in comparing the values of the infima of problems $(P)$ and $(P^{**})$ instead of establishing a representation formula.

2. Notation and Preliminary Results

In this paper $|.|$ is the Euclidean norm and ‘‘’’ the scalar product in $\mathbb{R}^N$. For a function $L(x, \xi) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ we denote by $L^**(x, \xi)$ (resp. $\partial L^**(x, \xi)$) the bipolar (resp. the subdifferential of the bipolar) of $\xi \mapsto L(x, \xi)$.

Finally, $AC([a, b], \mathbb{R}^N)$ is the space of absolutely continuous functions on $[a, b]$ with values in $\mathbb{R}^N$.

Here $L : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is just a Borelian function. We assume moreover that $L^**(x, \xi) \neq -\infty$ for every $x$ and $\xi$; this is the case, for instance, if $L$ is bounded below by an affine function of $\xi$.

The following growth condition will be assumed in the main result.

Conical growth assumption (CGA). For every compact subset $C$ of $\mathbb{R}^N$ and $R_0 \geq 0$ there exist $\varepsilon > 0$, $R > 0$ and $c \in \mathbb{R}$ such that

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R \quad L^**(x, \xi) \geq L^**(x, \xi_0) + p(x, \xi_0) \cdot (\xi - \xi_0) + \varepsilon |\xi| + c$$

for every $x \in C$, $|\xi_0| \leq R_0$ and $p(x, \xi_0)$ in $\partial L^**(x, \xi_0)$.

The following growth assumption was introduced by Cellina in [5] in the case where $L$ is continuous.

Growth assumption (GA).

We say that $L$ satisfies (GA) if there exist $p(x, \xi)$ in $\partial L^**(x, \xi)$ such that

$$\lim_{|\xi| \rightarrow +\infty} p(x, \xi) \cdot \xi - L^**(x, \xi) = +\infty$$  \hspace{1cm} (2.1)$$

uniformly for $x$ in a compact set.

Remark 2.1.

i) We point out that, in [5], the definition of (GA) is slightly different: it is formulated in an equivalent way in terms of the polar of $L$ in $(x, p(x, \xi))$; moreover the uniformity with respect to the first variable is not required since it is a consequence of the continuity of $L$. We use it here since we drop the continuity assumption.
ii) Assumption (GA) is fulfilled if, for instance, \( L(x, \xi) \) is superlinear with respect to \( \xi \); the proof can be easily done following the lines of [5].

We refer to [8] for a survey on the properties of the functions that satisfy (GA).

**Theorem 2.2.** [8, Cor 4.4] Assume that \( L \) is bounded on compact sets and satisfies the Growth Assumption (GA). Then \( L \) satisfies (CGA).

3. Relaxation

It is well known that if \( L : \mathbb{R}^N \to \mathbb{R} \) is a function whose bipolar is finite, then, for every \( \varepsilon > 0 \) and \( \xi \) in \( \mathbb{R}^N \), there exists \( \xi_1, ..., \xi_m (m \leq N + 1) \) in \( \mathbb{R}^N \) and coefficients of a convex combination \( \alpha_1, ..., \alpha_m \) such that \( \sum_i \alpha_i L(\xi_i) \leq L^{**}(\xi) + \varepsilon \) and \( \sum_i \alpha_i \xi_i = \xi \). We prove in the next Theorem 3.2 that if \( L \) satisfies (CGA) then, allowing \( m \leq 2N + 2 \), the points \( \xi \) may be bounded uniformly with respect to \( \xi \) in compact sets. For this purpose we first quote, in a more general setting, a powerful consequence of (CGA) that was established in [5] in the continuous case. For every \( (x, \xi) \) we set

\[ \mathcal{T}(x, \xi) = \liminf_{\eta \to \xi} L(x, \eta), \]

i.e. \( L(x, \xi) \) denotes the lower semi-continuous envelope of the map \( \eta \mapsto L(x, \eta) \). The proof of the following result is based on the fact that if \( f : \mathbb{R}^N \to \mathbb{R} \) is convex and satisfies (CGA) then the intersection of its epigraph with any supporting hyperplane is bounded. This condition is referred in [5] as the Bounded Intersection Property.

**Theorem 3.1.** Assume that \( L \) satisfies (CGA) and let \( p(x, \xi) \in \partial L^{**}(x, \xi) \). Then given \( R_0 > 0 \) and a compact subset \( C \) of \( \mathbb{R}^N \) there exists \( R > 0 \) (depending only on \( R_0 \) and \( C \)) such that for every \( x \in C \); for every \( \xi \), with \( |\xi| \leq R_0 \), there exist at most \( \nu \leq N + 1 \) points \( \xi_i \), with \( |\xi_i| \leq R \), and coefficients of a convex combination \( \alpha_i \), such that

\[ \left( L^{**}(x, \xi) \right) = \sum_{i=1}^{\nu} \alpha_i \left( \xi \right) \]

and \( L^{**}(x, \xi) = \mathcal{T}(x, \xi) = L^{**}(x, \xi) + p(x, \xi) \cdot (\xi - \xi_i) \).

Proof. It is enough to remark that Theorem 1 in [5] holds for functions that are lower semi-continuous instead of continuous and that the bipolar of a function coincides with the bipolar of its lower semi-continuous envelope. \( \square \)

We are now ready to state a version of Theorem 3.1 that does not involve the lower semi-continuous envelope of \( L \).

**Theorem 3.2.** Assume that \( L(x, \xi) \) satisfies (CGA) and that \( L \) is bounded on the compact sets. Then given \( R_0 > 0 \) and a compact subset \( C \) of \( \mathbb{R}^N \), there exists \( R > 0 \) (depending only on \( R_0 \) and \( C \)) such that for every \( x \in C \), for every \( \xi \), with \( |\xi| \leq R_0 \) and \( \varepsilon > 0 \), there exist at most \( m \leq 2N + 2 \) points \( \xi_i \), with \( |\xi_i| \leq R \), and coefficients of a convex combination \( \lambda_i \), such that

\[ \left\{ \begin{array}{l} \xi = \sum_{i=1}^{m} \lambda_i \xi_i \\ \sum_{i=1}^{m} \lambda_i L(x, \xi_i) \leq L^{**}(x, \xi) + \varepsilon. \end{array} \right. \]

The proof of the result needs several preliminary steps. For the convenience of the reader we first give a sketch of the proof in the case where \( L \) does not depend on \( x \).

By Theorem 3.1, for \( |\xi| \leq R_0 \), the point \( (\xi, L^{**}(\xi)) \) can be written as a convex combination of points \( (\zeta_i, \mathcal{T}(\zeta_i)) \) of the epigraph of the lower semi-continuous envelope of \( L(\cdot) \); moreover the \( \zeta_i \) are uniformly bounded, so that they all lie in a simplex generated by \( N + 1 \) affinely independent points \( \eta_1, ..., \eta_{N+1} \). Now each value \( \mathcal{T}(\zeta_i) \) can be approximated with \( L(\eta_i') \) for some \( \eta_i' \) arbitrarily near to \( \zeta_i \); actually it turns out that for \( \varepsilon > 0 \), if \( |\eta_i' - \zeta_i| \) is sufficiently small, then there is a convex combination of \( (\eta_i', L(\eta_i')) \) and \( N \) points among the \( (\eta_i, L(\eta_i)) \)'s whose
projection on $\mathbb{R}^N$ is $\zeta_j$ and whose last coordinate is less than $\bar{T}(\zeta_j) + \varepsilon$. The conclusion follows by writing $\xi$ as a convex combinations of the points $\eta_i$ and the $\eta_i'$ constructed as above.

We first need two technical lemmas. Let, if $S$ is a subset of $\mathbb{R}^N$, $\text{int} S$ denote its interior and $\text{conv} S$ its convex hull.

**Lemma 3.3.** Let $\eta_1, \ldots, \eta_{N+1}$ be $N+1$ affinely independent points of $\mathbb{R}^N$ and $\eta$ in $\text{int} (\text{conv}\{\eta_1, \ldots, \eta_{N+1}\})$, the interior of the simplex whose vertices are $\eta_1, \ldots, \eta_{N+1}$. Then:

i) for every $I \subset \{1, \ldots, N+1\}$ of cardinality $|I| \leq N$ the set of points $\{\eta_i : i \in I\}$ is affinely independent;

ii) for every $\xi \in \text{int} (\text{conv}\{\eta_1, \ldots, \eta_{N+1}\})$ there exists a subset $I$ of $\{1, \ldots, N+1\}$ of cardinality $N$ such that $\xi \in \text{conv}\{\eta_i : i \in I\}$.

**Proof of Lemma 3.3.** i) It is not restrictive to assume that $I = \{1, \ldots, N\}$. Let

$$\eta = \sum_{j=1}^{N+1} \lambda_j \eta_j \quad \lambda_j > 0 \quad \sum_{j=1}^{N+1} \lambda_j = 1.$$ 

For every $i \in \{1, \ldots, N\}$ we have

$$\eta - \eta_i = \sum_{j \neq i} \lambda_j \eta_j + (\lambda_i - 1) \eta_i$$

$$= \sum_{j \neq i} \lambda_j (\eta_j - \eta_{N+1}) + (\lambda_i - 1)(\eta_i - \eta_{N+1})$$

so that, in a matrix notation,

$$[\eta - \eta_1, \ldots, \eta - \eta_N] = [\eta_1 - \eta_{N+1}, \ldots, \eta_N - \eta_{N+1}](\Lambda - I)$$

where $I$ is the identity and

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdots & \lambda_N \\ \cdots & \cdots & \cdots \\ \lambda_1 & \cdots & \lambda_N \end{pmatrix}.$$ 

Now $\det(\Lambda - I) \neq 0$ since the eigenvalues of $\Lambda$ are $\lambda_1, \ldots, \lambda_N$ and $\lambda_i < 1$ for every $i$, proving i).

Proof of ii). Let

$$\xi = \alpha_1 \eta_1 + \cdots + \alpha_{N+1} \eta_{N+1} \quad \eta = \mu_1 \eta_1 + \cdots + \mu_{N+1} \eta_{N+1}$$

and we may assume that $\alpha_{N+1}/\mu_{N+1} = \min\{\alpha_i/\mu_i : i = 1, \ldots, N+1\}$ (notice that all the $\mu_i$ are strictly positive). Set $c_{N+1} = \alpha_{N+1}/\mu_{N+1}$ and, for $i \in \{1, \ldots, N\}$, $c_i = \alpha_i - c_{N+1}$; then, for every $i$, $c_i \geq 0$; moreover

$$\sum_{i=1}^{N+1} c_i = \sum_{i=1}^{N} \alpha_i - c_{N+1} \sum_{i=1}^{N} \mu_i + c_{N+1}$$

$$= 1 - \alpha_{N+1} - (1 - \mu_{N+1})c_{N+1} + c_{N+1}$$

$$= 1 - \alpha_{N+1} + \mu_{N+1}c_{N+1} = 1.$$
and
\[
\sum_{i=1}^{N} c_i \eta_i + c_{N+1} \eta = \sum_{i=1}^{N} (\alpha_i - \mu_i c_{N+1}) \eta_i + c_{N+1} \sum_{i=1}^{N+1} \mu_i \eta_i \\
= \sum_{i=1}^{N} (\alpha_i - \mu_i c_{N+1} + \mu_i c_{N+1}) \eta_i + c_{N+1} \mu_i \eta_{N+1} \\
= \sum_{i=1}^{N} \alpha_i \eta_i + \alpha_{N+1} \eta_{N+1} = \xi
\]
so that \( \xi \in \text{conv}\{\eta, \eta_i : i \in \{1, \ldots, N\}\} \).

**Lemma 3.4.** Let \( \eta_1, \ldots, \eta_{N+1} \) be \( N + 1 \) affinely independent points of \( \mathbb{R}^N \) and let \( y_1, \ldots, y_{N+1} \) be real numbers and \( K > 0 \). For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( \eta, \xi \in \text{int} (\text{conv}\{\eta_1, \ldots, \eta_{N+1}\}) \) and \( y, \beta \) in \([-K, K] \), with \( |\eta - \xi| < \delta \) and \( |y - \beta| < \delta \), there exists a subset \( I \) of \( \{1, \ldots, N+1\} \) of cardinality \( N \) and coefficients \( \lambda, \lambda_i \) \((i \in I)\) of a convex combination satisfying

\[
\begin{align*}
\xi &= \lambda \eta + \sum_{i \in I} \lambda_i \eta_i \\
\beta + \varepsilon &= \lambda y + \sum_{i \in I} \lambda_i y_i.
\end{align*}
\]

**Remark 3.5.** Geometrically Lemma 3.4 states that given \( N+1 \) points \((\eta_i, y_i)\) of \( \mathbb{R}^N \times \mathbb{R} \) and \((\xi, \beta)\) in \( \mathbb{R}^N \times \mathbb{R} \) such that \( \xi \) lies in the interior of the convex hull \( \Lambda \) of the \( \eta_i \)'s, then, given a positive \( \varepsilon \), for every point \( (\eta, y) \) that is sufficiently near to \((\xi, \beta)\) with \( \eta \in \Lambda \) there exist \( N \) points among the \((\eta_i, y_i)\)'s which, together with \((\eta, y)\), generate a \( N \)-dimensional simplex in \( \mathbb{R}^N \times \mathbb{R} \) whose projection in \( \mathbb{R}^N \) contains \( \xi \) and such that \((\xi, \beta + \varepsilon)\) lies above it.

**Proof of Lemma 3.4.** For every \( I \subset \{1, \ldots, N+1\} \), \(|I| = N\), \( y \in \mathbb{R} \) and \( \eta \in \Lambda := \text{int} \text{conv}\{\eta_1, \ldots, \eta_{N+1}\} \) by Lemma 3.3i) there exists a unique hyperplane \( z = a^I(\eta, y) \cdot \xi + b^I(\eta, y) \) containing the points \((\eta, y)\) and \((\eta_i, y_i)\) \((i \in I)\). Moreover the coefficients \( a^I(\eta, y), b^I(\eta, y) \) are continuous functions of \((\eta, y)\); in fact from the equations

\[
\begin{align*}
\left\{ \begin{array}{l}
a^I(\eta, y) \cdot \eta + b^I(\eta, y) = y \\
a^I(\eta, y) \cdot \eta_i + b^I(\eta, y) = y_i \quad (i \in I)
\end{array} \right.
\]

we deduce that the vector \( a^I(\eta, y) \) solves the system

\[
a^I(\eta, y) \cdot (\eta - \eta_i) = y - y_i \quad (i \in I);
\]

again by Lemma 3.3i) the vectors \( \eta - \eta_i \) \((i \in I)\) are independent so that the latter system has a unique solution \( a^I(\eta, y) \) given by Cramer's rule which is a continuous function of \( \eta \) and \( y \); the continuity of \( b^I \) follows from the equality \( b^I(\eta, y) = y - a^I(\eta, y) \cdot \eta \).

Set, for every \( I \subset \{1, \ldots, N+1\} \),

\[
\varphi^I(\eta, y; \zeta) = a^I(\eta, y) \cdot \zeta + b^I(\eta, y);
\]

we point out that, by construction, for a fixed \((\eta, y)\) the point \((\zeta, \varphi^I(\eta, y; \zeta))\) belongs to the (unique) hyperplane containing the points \((\eta, y)\) and \((\eta_i, y_i)\) \((i \in I)\). Since, for every \( \zeta \in \Lambda \) and \( \beta \in \mathbb{R} \),

\[
\varphi^I(\zeta, \beta; \zeta) = \beta,
\]
then, by the uniform continuity of \( \varphi^I \) on \( \Lambda \times [-K,K] \times \Lambda \), for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for every subset \( I \) of \( \{1, \ldots, N+1\} \) of cardinality \( N \),

\[
\varphi^I(\eta, y; \xi) < \beta + \varepsilon \quad \text{whenever } \eta, \xi \in \Lambda \quad |\eta - \xi| < \delta \quad \text{and } y, \beta \in [-K,K] \quad |y - \beta| < \delta.
\]  

(3.1)

Now fix \( \xi \in \Lambda \) and \( \varepsilon > 0 \). Let \( I \subset \{1, \ldots, N+1\} \) be such that \( \xi \) belongs to conv\{\( \eta, \eta_i : i \in I \)\}; such a set exists by Lemma 3.3ii). Let \( \delta \) be such as in (3.1) and \( |\eta - \xi| < \delta \), \( |y - \beta| < \delta \) so that \( \varphi^I(\eta, y; \xi) < \beta + \varepsilon \). Then, if we set

\[
\xi = \lambda \eta + \sum_{i \in I} \lambda_i \eta_i
\]

for some coefficients \( \lambda, \lambda_i \) (\( i \in I \)) of a convex combination, the linearity of \( \varphi^I \) in the third variable yields

\[
\lambda \varphi^I(\eta, y; \eta) + \sum_{i \in I} \lambda_i \varphi^I(\eta, y; \eta_i) < \beta + \varepsilon,
\]

proving the claim since \( \varphi^I(\eta, y; \eta) = y \) and \( \varphi^I(\eta, y; \eta_i) = y_i \). \( \square \)

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Fix \( x \) in \( C \). By Theorem 3.1 there exists \( R_1 > 0 \) (depending only on \( R_0 \) and \( C \)), \( \zeta_1, \ldots, \zeta_\nu \) (\( \nu \leq N+1 \)), with \( |\zeta_j| \leq R_1 \), and coefficients \( \alpha_j \) of a convex combination satisfying

\[
\begin{align*}
\{ \zeta \in \mathbb{R}^N : |\zeta| \leq R_1 \} & \subset \text{int} \left( \text{conv} \{ \eta_1, \ldots, \eta_{N+1} \} \right) \\
\{ \zeta \in \mathbb{R}^N : |\zeta| \leq R_1 \} & \subset \text{int} \left( \text{conv} \{ \eta_1, \ldots, \eta_{N+1} \} \right)
\end{align*}
\]

and set

\[
y_i = L(\eta_i), \quad i = 1, \ldots, N+1, \quad K = \sup \{ L(\zeta) : |\zeta| \leq R_1 \}.
\]

Fix \( \varepsilon > 0 \) and \( j \in \{1, \ldots, \nu\} \); set \( \beta = \overline{L}(\zeta_j) \). Correspondingly, let \( \delta > 0 \) satisfy the property stated in Lemma 3.4. By the definition of \( \overline{L} \) there exist \( \eta' \in \text{int} \left( \text{conv} \{ \eta_1, \ldots, \eta_{N+1} \} \right) \) such that

\[
|\eta' - \zeta_j| < \delta \quad \text{and} \quad L(\eta') \leq \overline{L}(\zeta_j) + \delta.
\]

We apply Lemma 3.4 with \( \eta = \eta', \xi = \zeta_j \) and \( y = L(\eta') \): there exists a subset \( I_j \) of \( \{1, \ldots, N+1\} \) of cardinality \( N \) and coefficients \( \lambda^j, \lambda_i^j \), \( (i \in I_j) \), such that

\[
\begin{align*}
\zeta_j & = \lambda^j \eta' + \sum_{i \in I_j} \lambda_i^j \eta_i \\
\lambda^j L(\eta') + \sum_{i \in I_j} \lambda_i^j L(\eta_i) & \leq \overline{L}(\zeta_j) + \varepsilon.
\end{align*}
\]

Therefore we obtain that

\[
\begin{align*}
\xi & = \sum_{j=1}^\nu \alpha_j \zeta_j = \sum_{j=1}^\nu \alpha_j \left( \lambda^j \eta' + \sum_{i \in I_j} \lambda_i^j \eta_i \right) \\
& = \sum_{j=1}^\nu \alpha_j \lambda^j \eta' + \sum_{i \in I_j} \left( \sum_{j=1}^\nu \alpha_j \lambda_i^j \right) \eta_i.
\end{align*}
\]
and moreover
\[
L^{**}(\xi) + \varepsilon = \sum_{j=1}^{\nu} \alpha_j (L(\xi_j) + \varepsilon) \\
\geq \sum_{j=1}^{\nu} \alpha_j \left( \lambda_i L(\eta^j) + \sum_{i \in I_j} \lambda_i^j L(\eta_i) \right) \\
= \sum_{j=1}^{\nu} \alpha_j \lambda_i^j L(\eta^j) + \sum_{i \in I_j} \left( \sum_{j=1}^{\nu} \alpha_j \lambda_i^j \right) L(\eta_i).
\]

If we set
\[
\begin{align*}
\lambda_i &= \alpha_i \lambda^i \quad \text{if } i \in \{1, \ldots, \nu\} \\
\lambda_i &= \sum_{j} \alpha_j \lambda^j_i - \nu \quad \text{if } i \in \{\nu + 1, \ldots, \nu + (N+1)\}
\end{align*}
\]

the above formulae can be rewritten as
\[
\begin{align*}
\lambda &= \sum_{i \leq \nu} \lambda_i \eta^i + \sum_{i > \nu} \lambda_i \eta_i \\
\sum_{i \leq \nu} \lambda_i L(\eta^j) + \sum_{i > \nu} \lambda_i L(\eta_i) &\geq L^{**}(\xi) + \varepsilon.
\end{align*}
\]

Moreover $|\eta^i| \leq R$ and $|\eta_i| \leq R$, where $R = \max\{|\eta_i| : i = 1, \ldots, N+1\}$ (which depends only on $R_i$ and therefore only on $R_0$ and $C$); proving the claim.

We consider here the problems
\[
\begin{align*}
\inf \left\{ F(y) : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} &\quad (P) \\
\inf \left\{ \mathcal{F}^{**}(y) : y \in AC([a, b], \mathbb{R}^N), y(a) = A, y(b) = B \right\} &\quad (P^{**})
\end{align*}
\]

It is well known that $\inf F = \inf \mathcal{F}^{**}$ if $L$ is continuous and superlinear ([7], Th. IX.3.1); actually in this case $\mathcal{F}^{**}$ is the relaxed functional of $F$. In [5] Cellina proved that $\inf F = \inf \mathcal{F}^{**}$ if $L$ is just continuous and satisfies $(GA)$. We examine here the case where $L$ is just continuous in the first variable, focusing our attention on the infima of the functionals $F$ and $\mathcal{F}^{**}$ instead on the relaxed functional of $F$.

**Theorem 3.6.** Assume that $L$ is bounded on compact sets and that $x \mapsto L(x, \xi)$ is continuous for every $\xi \in \mathbb{R}^N$.

If $L$ satisfies $(CGA)$ then $\inf F = \inf \mathcal{F}^{**}$.

**Proof.** We follow the lines of the proof of the analogous result ([5], Th. 3) in the case where $L$ is continuous in both variables, but instead of Theorem 3.1 we use Theorem 3.2. Let $x \in AC([a, b], \mathbb{R}^N)$ and $\varepsilon > 0$. From Theorem 2.4 and Remark 2.8 of [1] applied to $L^{**}$ there exists a Lipschitz function $x_{R_0}$ of Lipschitz constant $R_0$ satisfying the boundary conditions and such that

\[
\int_a^b L^{**}(x_{R_0}(t), x'_{R_0}(t)) \, dt \leq \int_a^b L^{**}(x(t), x'(t)) \, dt + \varepsilon/3.
\]

Set $C = \{x_{R_0}(t) : t \in [a, b]\}$. Since $|x'_{R_0}(t)| \leq R_0$ for a.e. $t$ then, by Theorem 3.2, there exists $R$ (depending only on $R_0$ and $C$), $m \leq 2N + 2$ coefficients $\lambda_i(t)$ of a convex combination and vectors $y_i(t)$ ($i = 1, \ldots, m$) with
By a standard selection argument, we may assume that the maps \( y_i \) and \( \lambda_i \) are measurable. Fix an integer \( k \) and consider the intervals \( I_j = [t_j, t_{j+1}] \), where \( t_j = a + j \frac{k}{n} \) \( (j = 0, \ldots, k-1) \) and call \( \chi_j \), their characteristic function. By Lyapunov’s Theorem on the range of vector measures [9] there exists a partition of \([a, b]\) into \( m \) measurable subsets \( E_i \), with characteristic functions \( \chi_{E_i} \), such that, for \( j = 0, \ldots, k-1 \), one has

\[
\int_{I_j} \sum_{i=1}^{m} \lambda_i(t) y_i(t) \, dt = \int_{I_j} \sum_{i=1}^{m} \chi_{E_i}(t) y_i(t) \, dt \\
\int_{I_j} \sum_{i=1}^{m} \lambda_i(t) L(x_{R_0}(t), y_i(t)) \, dt = \int_{I_j} \sum_{i=1}^{m} \chi_{E_i}(t) L(x_{R_0}(t), y_i(t)) \, dt.
\]

Denote by \( x_k \) the absolutely continuous defined by \( x_k(a) = A \) and

\[
x_k(t) = \int_a^t \sum_{i,j} y_i(s) \chi_{I_j \cap E_i}(s) \, ds;
\]

in particular for every \( k \) and every \( j = 1, \ldots, k \), we have

\[
\int_{I_j} x'_{R_0}(t) \, dt = \int_{I_j} x'_k(t) \, dt,
\]

so that the functions \( x_{R_0} \) and \( x_k \) coincide at each point \( t_j \). Since

\[
L(x_{R_0}(t), x'_k(t)) = \sum_{i,j} \chi_{I_j \cap E_i}(t) L(x_{R_0}(t), y_i(t))
\]

we also have that

\[
\int_a^b L(x_{R_0}(t), x'_k(t)) \, dt = \int_a^b \sum_{i=1}^{m} \lambda_i(t) L(x_{R_0}(t), y_i(t)) \, dt;
\]

so that, from \( \approx \), it follows that

\[
\int_a^b L(x_{R_0}(t), x'_k(t)) \, dt \leq \int_a^b L^*(x_{R_0}(t), x'_{R_0}(t)) \, dt + \varepsilon/3.
\]

Now

\[
\int_a^b L(x_{R_0}(t), x'_k(t)) \, dt = \int_a^b L(x_k(t), x'_k(t)) \, dt + \int_a^b L(x_{R_0}(t), x'_k(t)) - L(x_k(t), x'_k(t)) \, dt;
\]

moreover, \( x_{R_0} \) is uniformly continuous, the functions \( x_k \) are equi-Lipschitz, \( x_k(t_j) = x_{R_0}(t_j) \) \( (j = 0, \ldots, k-1) \). Hence, if \( t \in [a, b] \) and \( t_j \leq t \leq t_{j+1} \),

\[
|x_k(t) - x_{R_0}(t)| \leq |x_k(t) - x_k(t_j)| + |x_k(t_j) - x_{R_0}(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)|
\]

\[
= |x_k(t) - x_k(t_j)| + |x_{R_0}(t_j) - x_{R_0}(t)| \leq (R + R_0)(b - a) / k
\]
so that $x_k$ converges uniformly to $x_{R_0}$ as $k$ tends to $+\infty$. By our assumption the function $L(x_{R_0}, x_k') - L(x_k, x_k')$ is bounded a.e. by a constant that does not depend on $k$. The continuity of $L$ with respect to the first variable together with the dominated convergence theorem imply that

$$\lim_{k \to +\infty} \int_a^b L(x_{R_0}(t), x_k'(t)) - L(x_k(t), x_k'(t)) \, dt = 0.$$ 

It follows that for $k$ sufficiently large,

$$\int L(x_k(t), x_k'(t)) \, dt \leq \int L(x_{R_0}(t), x_k'(t)) \, dt + \varepsilon/3 \leq \int L(x(t), x'(t)) \, dt + \varepsilon$$

proving that $F \leq F^{**}$. \hfill \Box

We point out that, under the assumptions of Theorem 3.6, the functional $F^{**}$ is not in general the relaxed functional of $F$; we refer to [2] for some recent results in this direction. This is the case in the forthcoming example, where we also show that the conclusion of Theorem 3.6 does not hold if $L$ is not continuous in $x$.

**Example 3.7.** Let $g$ be the characteristic function of $\mathbb{R} \setminus \{0\}$ and $h(\xi) = \xi^2$ if $\xi \neq 0$, $h(0) = 1$ and set $L(x, \xi) = g(x) + h(\xi)$. Let $(P)$, $(P^{**})$ be the problems

$$\inf \left\{ F(y) = \int_0^1 L(y(t), y'(t)) \, dt ; \quad y(0) = 0, y(1) = 0, y \in AC([0,1], \mathbb{R}) \right\} \quad (P)$$

$$\inf \left\{ F^{**}(y) = \int_0^1 L^{**}(y(t), y'(t)) \, dt ; \quad y(0) = 0, y(1) = 0, y \in AC([0,1], \mathbb{R}) \right\} \quad (P^{**})$$

For every $x$ in $\mathbb{R}$ we have $L^{**}(x, \xi) = g(x) + \xi^2$, so that the minimum of the problem $(P^{**})$ is equal to 0 and it is obviously assumed for $y(t) = 0$. However $\inf F \geq 1$; in fact let $y \in AC([0,1], \mathbb{R})$ and set $E = \{ t \in [0,1] : y(t) = 0 \}$, then $y'(t) = 0$ a.e. on $E$, so that

$$\int_0^1 L(y(t), y'(t)) \, dt = \int_E L(0,0) \, dt + \int_{[0,1]\setminus E} L(y(t), y'(t)) \, dt \geq \int_E \, dt + \int_{[0,1]\setminus E} g(y(t)) \, dt \geq |E| + |\{0,1\} \setminus E| = 1.$$ 

Notice that nevertheless, from [3], the minima of $F$ are Lipschitz.

4. **Lipschitz regularity of the minima of $(P)$**

In this section we apply our result to the problem of the Lipschitz regularity of the minima of $(P)$. It is well known that if $L(x, \xi)$ is continuous, convex and superlinear in $\xi$ then every minimum of $(P)$ is Lipschitz. In some recent papers the same conclusion is proved under weaker assumptions; we just mention [5, 6, 8]. Our result is in the same spirit of the last two that we recall here.

**Theorem 4.1.** [5] Assume that $L(x, \xi)$ is continuous in both variables and satisfies (GA). Then every minimizer of $(P)$ in $AC([a,b], \mathbb{R}^N)$ is Lipschitz.

**Theorem 4.2.** [8] Assume that $L(x, \xi)$ is convex in $\xi$ and satisfies (GA). Then every minimizer of $(P)$ in $AC([a,b], \mathbb{R}^N)$ is Lipschitz.

The following theorem weakens the continuity assumption of Theorem 4.1.
Theorem 4.3. Assume that \( x \mapsto L(x, \xi) \) is continuous for every \( \xi \) and that \( L \) satisfies (GA). Then every minimizer of \((P)\) in \( AC([a, b], \mathbb{R}^N) \) is Lipschitz.

Proof. By Theorem 3.6, \( \inf F = \inf F^{**} \); therefore every minimum of \( F \) is a minimum of \( F^{**} \). The function \( L^{**}(x, \xi) \) is convex in \( \xi \) and satisfies (GA): Theorem 4.2 yields the conclusion. \( \Box \)

References