ESAIM: COCV
July 2004, Vol. 10, 346–380
DOI: 10.1051/cocv:2004012

**ESAIM: Control, Optimisation and Calculus of Variations**

# CONTROL OF THE SURFACE OF A FLUID BY A WAVEMAKER

Lionel Rosier[1]

**Abstract.** The control of the surface of water in a long canal by means of a wavemaker is investigated. The fluid motion is governed by the Korteweg-de Vries equation in Lagrangian coordinates. The null controllability of the elevation of the fluid surface is obtained thanks to a Carleman estimate and some weighted inequalities. The global uncontrollability is also established.

**Mathematics Subject Classification.** 35B37, 49J20, 76B15, 93B05, 93C20.

Received April 10, 2003. Revised November 12, 2003.

## 1. Introduction

Canals with moving boundary are simple devices frequently used to investigate the motion of water waves. In [5], the authors described a device composed of a canal with a moving wall at the left (the so-called *wavemaker*) and a fixed wall at the right, and a series of sensors along the canal which measure the deflection of the fluid surface from rest position (see Fig. 1). Such a device proves to be useful to validate/invalidate any mathematical model for the propagation of unidirectional water waves (*e.g.*, the KdV or the BBM equations). Indeed, any small displacement of the wavemaker gives rise to a set of travelling waves moving from the left to the right. Certain displacements of the wavemaker have been recognized to generate pure solitons. However, if we take as input (resp., output) the speed of the wavemaker (resp., the elevation of the water surface), then the input/output behaviour has not yet been investigated from a mathematical point of view. The purpose of this paper is to see to what extent one may control the surface of a fluid in a uniform canal by means of one wavemaker. In particular, we have in mind to design a control input allowing to create/destroy any set of (sufficiently small) solitons in finite time.

The control of the fluid surface in a canal or in a moving tank has been recently investigated in situations where solitons do not emerge, that is when dispersive effects have no time to develop and balance nonlinear effects. A linear wave equation without dispersive term, which serves as a linear model for the motion of the fluid surface in a bounded canal with a (rigid or flexible) wavemaker, is derived and studied in [16]. It is proved there that the system fails to be approximately controllable. The same kind of result is obtained in [17] for the moving tank. It should be emphasized that in this model both dispersive and nonlinear terms have been neglected. If a nonlinear convection term is incorporated into the model, then we obtain the famous shallow

*Keywords and phrases.* Korteweg-de Vries equation, Lagrangian coordinates, exact boundary controllability, Carleman estimate.

[1] Institut Elie Cartan, Université Henri Poincaré Nancy 1, BP 239, 54506 Vandœuvre-lès-Nancy Cedex, France;
e-mail: `rosier@iecn.u-nancy.fr`

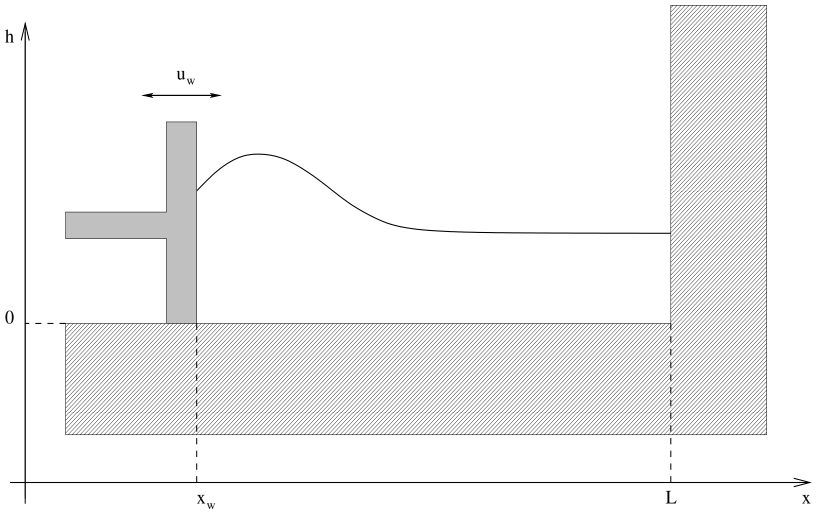


FIGURE 1. Canal with one wavemaker.

water equations [25] (p. 456), which read

$$\begin{aligned} h_t + (uh)_x &= 0, \\ u_t + uu_x + gh_x &= 0. \end{aligned} \tag{1}$$

In these equations, $h = h(t,x)$ is the height of the fluid at time $t$ and at the position $x$, $u = u(t,x)$ is the horizontal component of the fluid velocity, and $g$ denotes the gravitation constant. The optimal control of (1) has been intensively studied in [13] from a numerical viewpoint for a canal with one (or two) moving wall(s). In the case of a moving tank containing a fluid whose motion is again described by (1), Coron proved in [7] the local controllability of the full system (tank+liquid) thanks to a subtle analysis and the so-called return method, developed earlier by himself for Euler equations [6]. That result is in sharp contrast with the one given in [17], and it demonstrates that the nonlinear terms are sometimes useful to derive the controllability of the system.

In the physical system considered here we assume that

- the fluid is inviscid, incompressible, and irrotational;
- the canal is sufficiently long, so that dispersive effects may develop during the waves propagation;
- the surface tension may be neglected.

In this context, the Boussinesq system [25] (p. 462)

$$\begin{aligned} h_t + (uh)_x &= 0, \\ u_t + uu_x + gh_x + \frac{1}{3}h_0 h_{xtt} &= 0 \end{aligned} \tag{2}$$

is commonly recognized as a convenient model for the two-way propagation of small-amplitude, long wavelength gravity waves on the fluid surface in a canal. ($h_0$ is the fluid height at rest.) Recently, whole a family of different (although formally equivalent) Boussinesq systems has been derived and studied in [3]. Restricting our attention to waves moving from the left to the right, we obtain the popular Korteweg-de Vries (KdV) equation (see [25], (p. 463))

$$\eta_t + c_0\left(1+\frac{3}{2}\frac{\eta}{h_0}\right)\eta_x + \frac{c_0h_0^2}{6}\eta_{xxx} = 0, \qquad t>0,\ 0<x<L, \tag{3}$$

where $\eta = h - h_0$, and $c_0 = \sqrt{gh_0}$. The equation relating $u$ to $\eta$ reads then

$$u = \frac{g}{c_0}\left(\eta - \frac{1}{4}\frac{\eta^2}{h_0} + \frac{1}{3}h_0^2\eta_{xx}\right).$$

The main advantage of the KdV equation (when compared to Boussinesq system) is its relative simplicity: (3) involves only $\eta$ and its derivatives, not $u$.

The boundary controllability of (3) has been extensively studied in the last decade (see [8,19–23,27], and [15] for a nonlinear system of two KdV equations). In all these papers the boundary control involves a linear combination of the traces $\eta_{|x=L}$, $\eta_{x|x=L}$ and $\eta_{xx|x=L}$. This choice, although leading to some nice mathematical results, is not convenient here for two reasons.

(1) What is indeed controlled here is the speed of the wavemaker (in fact, the force applied to the moving wall), that is $u = \frac{g}{c_0}\left(\eta - \frac{1}{4}\frac{\eta^2}{h_0} + \frac{1}{3}h_0^2\eta_{xx}\right)$.

(2) The boundary control should be applied at the left endpoint, not at the right endpoint. Indeed, among the numerous waves solving the KdV equation, the only physically acceptable ones are those propagating from the left to the right. It turns out that only the null-controllability holds for the linear KdV equation when the control is applied at the left.

As it has been pointed out to the author by Bona, for the linearized KdV equation (without coefficient)

$$\eta_t + \eta_x + \eta_{xxx} = 0$$

high wavenumber exponential solutions (that is, of the form $\eta(t,x) = \mathrm{e}^{i(kx+\omega t)}$ with $k \gg 1$) propagate from the right to the left. Indeed, the dispersion relation

$$\omega = k^3 - k$$

implies that $\omega > 0$ for $k > 1$. This is probably the reason for which the linearized KdV equation may be exactly controllable with a right boundary control.

Let us now describe the content of the paper. Since the length of the canal changes as the wavemaker is moving, we are led to adopt a Lagrangian formalism, as in [13]. The KdV system in Lagrangian coordinates reads

$$\begin{aligned} &y_t + y_x + yy_x + y_{xxx} = 0,\\ &v = y - \tfrac{1}{6}y^2 + y_{xx}. \end{aligned}$$

The derivation of this system, which to the author's knowledge has not been reported elsewhere, is provided in the appendix for the sake of completeness. Here, the dimensionless and scaled variables $y$, $t$, $x$, and $v$ stand for the deflection from rest position, the time, the space variable and the velocity, respectively. The main result is this paper asserts that any (smooth) trajectory for the KdV equation may be (locally) reached in finite time. In particular, the KdV equation is locally null controllable when using a boundary control of the type described above.

**Theorem 1.1.** *Let $L, T$ be positive numbers, and let*

$$\overline{y} \in \mathrm{C}^0([0,T],\mathrm{H}^3(0,L)) \cap \mathrm{C}^1([0,T],\mathrm{L}^2(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$$

*be a function such that*

$$\begin{cases} \overline{y}_t + \overline{y}_x + \overline{y}\,\overline{y}_x + \overline{y}_{xxx} = 0 \quad \text{for} \quad 0 < x < L,\ 0 < t < T, \\ \overline{y}_{|x=L} = \overline{y}_{x|x=L} = 0. \end{cases} \tag{4}$$

*Then there exists a number $r_0 > 0$ such that for any initial state $y_0 \in \mathrm{H}^3(0, L)$ fulfilling $y_0(L) = y_0'(L) = 0$ and*

$$\|y_0 - \overline{y}(0)\|_{\mathrm{H}^3(0,L)} < r_0,$$

*there exists a control input $h \in \mathrm{C}^0([0, T])$ such that the following initial-boundary-value problem*

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, & 0 < x < L,\ 0 < t < T, \\ (y - \frac{1}{6}y^2 + y_{xx})_{|x=0} = h, & \\ y_{|x=L} = y_{x|x=L} = 0, & \\ y(0) = y_0 & \end{cases} \tag{5}$$

*possesses a solution $y \in \mathrm{L}^2(0, T, \mathrm{H}^3(0, L)) \cap \mathrm{H}^1(0, T, \mathrm{H}^1(0, L))$ such that $y(T) = \overline{y}(T)$. Furthermore, the solution of (5) is* unique *if $\|\overline{y}_{|x=0}\|_{\mathrm{L}^\infty(0,T)} < \frac{3}{5}$ and $r_0$ is small enough.*

The fact that the speed of the wavemaker is indeed controlled is expressed through the boundary condition $(y - \frac{1}{6}y^2 + y_{xx})_{|x=0} = h$, where $h$ denotes the control input. The other boundary conditions $y_{|x=L} = y_{x|x=L} = 0$ guarantee that

(1) there is no (artificial) control applied at the right endpoint;
(2) waves propagate from the left to the right and remain flat at the right endpoint, so that their dynamics is well described by the Korteweg-de Vries equation.

As a consequence of Theorem 1.1, a small (hence slow) soliton moving from the left to the right may be caught up and annihilated by a set of waves generated by the wavemaker. The main difficulty in proving this result is that the exact controllability of the linearized KdV equation fails to be true with a left boundary control (see below Appendix B). However, the approximate controllability may be easily obtained by using Holmgren uniqueness theorem. Let us point out that the linearized control system for the water-tank problem studied in [7] is not approximately controllable (see [18]). The remarkable gain in the controllability property when going back to the original *nonlinear* system (compare [7] to [18]) is not expected for the KdV equation, due to the presence of high order derivatives in the linear part.

Theorem 1.1 is proved in following the method developed in [10] for proving the null-controllability of Burgers equation. The proof rests mainly on some global Carleman inequality, which is quite sharp since no control is applied at the right endpoint ($y_{|x=L} = y_{x|x=L} = 0$). A linearized control system is first proved to possess a square integrable trajectory connecting some initial state to 0. The fact that this trajectory has the regularity depicted in Theorem 1.1 rests on two weighted estimates proved by means of the multiplier method. The first one (Prop. 2.6) is just a variant of the classical Kato smoothing effect, the second one (Prop. 2.7), which asserts that the time derivative of the trajectory is square integrable, rests on a clever choice of multipliers. Finally, the existence of a trajectory for the nonlinear control problem (5) is proved by means of a standard fixed point argument. An application of Gronwall Lemma provides the uniqueness of the trajectory.

The second main result is this paper (Th. 3.1) asserts that the *global* controllability of (5) fails to be true in finite time. This result rests on the observation that (large) solutions of the KdV equation behave like solutions of the Hopf equation

$$y_t + yy_x = 0.$$

Roughly speaking, (large) negative waves propagate from the right to the left. Therefore, a negative wave cannot be generated by a left boundary control.

The paper is outlined as follows. The proof of the main result (Th. 1.1) is given in Section 2. A global Carleman estimate is provided in Section 2.2, two weighted estimates are given in Section 2.3, and the fixed point argument is developed in Section 2.4. The smoothness (resp., the uniqueness) of the trajectory are studied in Sections 2.5 and 2.6, respectively. The Section 3 is devoted to the proof of the global uncontrollability of (5). Finally, the derivation of the Korteweg-de Vries equation in Lagrangian coordinates is given in Appendix A, and the uncontrollability of the linear KdV equation with only one boundary control applied to the left is established in Appendix B.

## 2. Proof of the main result

Throughout this section $L$ and $T$ denote respectively the length of the domain and the final time of the control process. We set

$$Q = (0,T) \times (-L,L)$$

and we introduce the space

$$V = \left\{ z \in \mathrm{L}^2(0,T,\mathrm{H}^3(-L,L)),\ z_t \in \mathrm{L}^2(0,T,\mathrm{H}^1(-L,L)) \right\} \cdot$$

$V$ is endowed with the natural Hilbertian norm

$$\|z\|_V = \left( \|z\|^2_{\mathrm{L}^2(0,T,\mathrm{H}^3(-L,L))} + \|z_t\|^2_{\mathrm{L}^2(0,T,\mathrm{H}^1(-L,L))} \right)^{\frac{1}{2}}.$$

Recall (see [12]) that

$$V \subset \mathrm{C}^0\left([0,T],\mathrm{H}^2(-L,L)\right). \tag{6}$$

For the sake of shortness we shall write $\|z\|_{L^p_t L^q_x}$ instead of $\|z\|_{L^p(0,T,L^q(-L,L))}$ for any $1 \le p,q \le +\infty$, etc. With these abbreviated notations, the variables $t$ and $x$ are assumed to range over $(0,T)$ and $(-L,L)$, respectively. In what follows, $c(s)$ (resp. $K$) will denote a positive nondecreasing function of $s \in \mathbb{R}_+$ (resp., a positive constant depending *only* on $L$ and $T$). Let us point out that both of them may vary from line to line. In the first step we focus on the existence of a trajectory for a simplified control problem, in which the control input is not specified:

**Theorem 2.1.** *Let $L,T$ be positive numbers, and let $\overline{y} \in \mathrm{C}^0([0,T],\mathrm{H}^3(0,L)) \cap \mathrm{C}^1([0,T],\mathrm{L}^2(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$ be a function such that*

$$\begin{cases} \overline{y}_t + \overline{y}_{xxx} + \overline{y}_x + \overline{y}\,\overline{y}_x = 0 \quad \text{for} \quad 0<x<L,\ 0<t<T, \\ \overline{y}_{|x=L} = \overline{y}_{x\,|x=L} = 0. \end{cases} \tag{7}$$

*Then there exists a number $r_0 > 0$ such that for any initial state $y_0 \in \mathrm{H}^3(0,L)$ fulfilling $y_0(L) = y_0'(L) = 0$ and*

$$\|y_0 - \overline{y}(0)\|_{\mathrm{H}^3(0,L)} < r_0, \tag{8}$$

*there exists a function $y \in \mathrm{L}^2(0,T,\mathrm{H}^3(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$ satisfying*

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0, \qquad 0<x<L,\ 0<t<T, \\ y_{|x=L} = y_{x\,|x=L} = 0, \\ y(0) = y_0, \\ y(T) = \overline{y}(T). \end{cases} \tag{9}$$

*Furthermore, $y \in \mathrm{C}^0([0,T],\mathrm{H}^s(0,L))$ for any $s < 3$.*

To prove Theorem 2.1, we need first to establish the null-controllability of the linearized equation.

### 2.1. **Null controllability of the linearized equation**

For each $z \in V$ we consider the following control problem: for any initial state $u_0 \in \mathrm{H}^3(-L,L)$ with $u_0(L) = u_0'(L) = 0$, find a control function $h$ such that the solution $u = u(t,x)$ of

$$\begin{cases} u_t + u_{xxx} + u_x + \frac{1}{2}(zu)_x = 0, & -L < x < L,\ 0 < t < T, \\ u_{|x=L} = u_{x|x=L} = 0, & \\ u_{|x=-L} = h, & \\ u_{|t=0} = u_0 & \end{cases} \tag{10}$$

satisfies

$$u(T) = 0. \tag{11}$$

This problem will be solved by adapting the method developed in [10] for proving the null-controllability of Burgers' equation. Let us consider a function $u_0 \in \mathrm{H}^3(-L,L)$ such that $u_0(-L) = u_0(L) = u_0'(L) = 0$. Let $v$ denote the solution of the boundary-initial value problem:

$$\begin{cases} v_t + v_{xxx} + v_x + \frac{1}{2}(zv)_x = 0, & -L < x < L,\ 0 < t < T, \\ v_{|x=-L} = v_{|x=L} = v_{x|x=L} = 0, & \\ v_{|t=0} = u_0. & \end{cases} \tag{12}$$

We need the following elementary result.

**Lemma 2.2.** *Let $f \in W^{1,1}(0,T,\mathrm{L}^2(-L,L))$. Then there exists a unique solution*

$$v \in \mathrm{C}^0([0,T],\mathrm{H}^3(-L,L)) \cap \mathrm{C}^1([0,T],\mathrm{L}^2(-L,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(-L,L))$$

*of the following forced boundary-initial value problem*

$$\begin{cases} v_t + v_{xxx} + v_x + \frac{1}{2}(zv)_x = f, & -L < x < L,\ 0 < t < T, \\ v_{|x=-L} = v_{|x=L} = v_{x|x=L} = 0, & \\ v_{|t=0} = u_0. & \end{cases} \tag{13}$$

*Furthermore,*

$$\|v\|_{\mathrm{L}^\infty_t\mathrm{H}^3_x} + \|v_t\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x} + \|v\|_{\mathrm{H}^1_t\mathrm{H}^1_x} \le c(\|z\|_V)\left(\|f\|_{\mathrm{W}^{1,1}_t\mathrm{L}^2_x} + \|u_0\|_{\mathrm{H}^3(-L,L)}\right). \tag{14}$$

*Proof.* We set $B = \mathrm{L}^2(0,T,\mathrm{H}^1(-L,L)) \cap \mathrm{C}^0([0,T],\mathrm{L}^2(-L,L))$ and $\|v\|_B = \|v\|_{L^2_tH^1_x} + \|v\|_{L^\infty_tL^2_x}$ for any $v \in B$. For any $\hat{v} \in B$, let $v$ denote the *mild* solution of the problem

$$\begin{cases} v_t + v_{xxx} + v_x = f - \frac{1}{2}(z\hat{v})_x =: \hat{f}, & -L < x < L,\ 0 < t < T, \\ v_{|x=-L} = v_{|x=L} = v_{x|x=L} = 0, & \\ v_{|t=0} = u_0. & \end{cases}$$

Observe that $\hat{f} \in \mathrm{L}^1(0,T,\mathrm{L}^2(-L,L))$, since $z$ and $\hat{v}$ belong to $\mathrm{L}^2(0,T,\mathrm{H}^1(-L,L))$. It follows from [19] (Props. 3.2 and 4.1) that $v \in B$ and that

$$\begin{aligned} \|v\|_B &\le K\left(\|\hat{f}\|_{L^1_tL^2_x} + \|u_0\|_{\mathrm{L}^2(-L,L)}\right) \\ &\le K\left(\|z\|_{L^2_tH^1_x}\|\hat{v}\|_B + \|f\|_{L^1_tL^2_x} + \|u_0\|_{\mathrm{L}^2(-L,L)}\right). \end{aligned}$$

(We stress that the constant $K$ varies from line to line.) We take $R = 2K\left(\|f\|_{L^1_t L^2_x} + \|u_0\|_{\mathrm{L}^2(-L,L)}\right)$ and we pick any $\hat{v}$ with $\|\hat{v}\|_B \le R$. If $T$ is small enough then $K\|z\|_{L^2_t H^1_x} < \frac{1}{2}$, hence $\|v\|_B \le R$. On the other hand, for any given functions $\hat{v}_1$, $\hat{v}_2$ in $B$ with $\|\hat{v}_1\|_B, \|\hat{v}_2\|_B \le R$ we have

$$\|v_1 - v_2\|_B \le K\|z\|_{L^2_t H^1_x}\|\hat{v}_1 - \hat{v}_2\|_B \le \frac{1}{2}\|\hat{v}_1 - \hat{v}_2\|_B.$$

Therefore, the map $\hat{v} \mapsto v$ is a contraction in the closed ball $\{v \in B : \|v\|_B \le R\}$ of $B$. It admits a unique fixed point, according to the contraction principle, provided that $T$ is small enough. If $T$ is large, applying the previous result on the intervals $\left[0, \frac{T}{N}\right]$, $\left[\frac{T}{N}, \frac{2T}{N}\right]$, etc. with $N \gg 1$ we obtain the existence and uniqueness in $B$ of the solution to (13) on $[0,T]$. Notice that the above proof remains valid when $u_0 \in \mathrm{L}^2(-L,L)$ and $f \in \mathrm{L}^1(0,T,\mathrm{L}^2(-L,L))$. To establish more regularity for the solution, we formally derive with respect to time in (13). We obtain that $w = v_t$ solves

$$\begin{cases} w_t + w_{xxx} + w_x + \frac{1}{2}(zw)_x = f_t - \frac{1}{2}(z_t v)_x, \\ w_{|x=-L} = w_{|x=L} = w_{x|x=L} = 0, \\ w_{|t=0} = f_{|t=0} - u_{0xxx} - u_{0x} - \frac{1}{2}(zv)_{x|t=0}. \end{cases} \tag{15}$$

Let $w$ be the unique solution of (15) in $B$, and let $\bar{v} = u_0 + \int_0^t w(\tau)\,\mathrm{d}\tau$. Clearly, $\bar{v} \in \mathrm{H}^1(0,T,\mathrm{H}^1(-L,L)) \cap \mathrm{C}^1([0,T],\mathrm{L}^2(-L,L))$. It remains to check that $\bar{v} = v$. Let $\varepsilon = \bar{v} - v$. Then $\varepsilon_{|t=0} = 0$, $\varepsilon_{|x=-L} = \varepsilon_{|x=L} = \varepsilon_{x|x=L} = 0$ and

$$\begin{aligned} \varepsilon_t + \varepsilon_{xxx} + \varepsilon_x &= u_{0xxx} + u_{0x} + \int_0^t (w_t + w_{xxx} + w_x)\,\mathrm{d}\tau + w_{|t=0} - \left(f - \frac{1}{2}(zv)_x\right) \\ &= f_{|t=0} - \frac{1}{2}(zv)_{x|t=0} + \int_0^t \left(f_t - \frac{1}{2}(zw + z_t v)_x\right)\mathrm{d}\tau - f + \frac{1}{2}(zv)_x \\ &= -\frac{1}{2}\int_0^t (zv)_{tx}\,\mathrm{d}\tau + \frac{1}{2}\left((zv)_x - (zv)_{x|t=0}\right) \\ &= 0. \end{aligned}$$

It follows that $\varepsilon = 0$, that is $v = \bar{v}$. The fact that $v \in \mathrm{C}^0\left([0,T],\mathrm{H}^3(-L,L)\right)$ follows from the first equation in (12). We now turn to the proof of (14). We first check that

$$\|v\|_B \le c(\|z\|_V)\left(\|u_0\|_{\mathrm{L}^2(-L,L)} + \|f\|_{L^1_t L^2_x}\right). \tag{16}$$

Multiplying the first equation in (13) by $2v$ and integrating over $(0,t)\times(-L,L)$ we obtain, after some integrations by parts

$$\int_{-L}^{L} v(t,x)^2\,\mathrm{d}x - \int_{-L}^{L} u_0(x)^2\,\mathrm{d}x + \int_0^t v_x^2(\tau,-L)\,\mathrm{d}\tau + \frac{1}{2}\int_0^t\!\!\int_{-L}^{L} z_x\, v^2\,\mathrm{d}x\mathrm{d}\tau = 2\int_0^t\!\!\int_{-L}^{L} vf\,\mathrm{d}x\mathrm{d}\tau. \tag{17}$$

Set $h(t) = \max_{\tau\in[0,t]} \|v(\tau,.)\|^2_{\mathrm{L}^2(-L,L)}$ (recall that $v \in \mathrm{C}^0([0,T],\mathrm{L}^2(-L,L))$). Let $t \in [0,T]$ and pick any $t' \in [0,t]$ such that $h(t) = \|v(t',.)\|^2_{\mathrm{L}^2(-L,L)}$. Then, by (17)

$$\begin{aligned} h(t) &= \int_{-L}^{L} v(t',x)^2\,\mathrm{d}x \\ &\le \int_{-L}^{L} u_0(x)^2\,\mathrm{d}x + \frac{1}{2}\int_0^{t'}\int_{-L}^{L} |z_x| v^2\,\mathrm{d}x\mathrm{d}\tau + 2\int_0^{t'}\int_{-L}^{L} |vf|\,\mathrm{d}x\mathrm{d}\tau \\ &\le \int_{-L}^{L} u_0(x)^2\,\mathrm{d}x + \frac{1}{2}\|z_x\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty_x}\int_0^t h(\tau)\,\mathrm{d}\tau + 2\|f\|_{\mathrm{L}^1(0,t,\mathrm{L}^2(-L,L))}\cdot\|v\|_{\mathrm{L}^\infty(0,t,\mathrm{L}^2(-L,L))} \\ &\le \int_{-L}^{L} u_0(x)^2\,\mathrm{d}x + 2\|f\|^2_{\mathrm{L}^1_t\mathrm{L}^2_x} + \frac{1}{2}h(t) + \frac{1}{2}\|z_x\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty_x}\int_0^t h(\tau)\,\mathrm{d}\tau. \end{aligned}$$

An application of Gronwall lemma yields

$$\|v(t,.)\|^2_{\mathrm{L}^2(-L,L)} \le h(t) \le \left(2\|u_0\|^2_{\mathrm{L}^2(-L,L)} + 4\|f\|^2_{L^1_tL^2_x}\right)\exp(T\|z_x\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty_x}),$$

hence

$$\|v\|_{L^\infty_tL^2_x} \le c(\|z\|_V)\left(\|u_0\|_{\mathrm{L}^2(-L,L)} + \|f\|_{\mathrm{L}^1_t\mathrm{L}^2_x}\right). \tag{18}$$

Multiplying the first equation in (13) by $(L+x)v$ and integrating the result over $Q$ we find after using some integrations by parts that

$$\frac{1}{2}\left(\int_{-L}^{L}(L+x)v^2(T,x)\,\mathrm{d}x - \int_{-L}^{L}(L+x)u_0^2(x)\,\mathrm{d}x\right) + \frac{3}{2}\iint_Q v_x^2\,\mathrm{d}x\mathrm{d}t \\ - \frac{1}{2}\iint_Q v^2 + \frac{1}{4}\iint_Q((L+x)z_x - z)\,v^2 = \iint_Q (L+x)vf\,\mathrm{d}x\mathrm{d}t.$$

Therefore

$$\frac{3}{2}\iint_Q v_x^2\,\mathrm{d}x\mathrm{d}t \le 2L\|v\|_{\mathrm{L}^\infty_t\mathrm{L}^2_x}\|f\|_{\mathrm{L}^1_t\mathrm{L}^2_x} + L\int_{-L}^{L}u_0^2(x)\,\mathrm{d}x + \frac{1}{2}\iint_Q v^2\,\mathrm{d}x\mathrm{d}t \\ + \frac{1}{4}\left(2L\|z_x\|_{L^\infty_tL^\infty_x} + \|z\|_{L^\infty_tL^\infty_x}\right)\iint_Q v^2\,\mathrm{d}x\mathrm{d}t.$$

Hence, using (18), we obtain

$$\|v\|_{\mathrm{L}^2_t\mathrm{H}^1_x} \le c(\|z\|_V)\left(\|u_0\|_{\mathrm{L}^2(-L,L)} + \|f\|_{\mathrm{L}^1_t\mathrm{L}^2_x}\right). \tag{19}$$

(16) follows from (18) and (19). Applying (16) to $v$ and to $w = v_t$, using the first equation in (13) and the fact that $f \in \mathrm{C}^0([0,T],\mathrm{L}^2(-L,L))$, and that $z, z_x \in \mathrm{L}^\infty(Q)$, we obtain (14). □

Let $\psi$ be any function of class $C^\infty$ on $[0,T]$ such that

$$\psi(t) = \begin{cases} 1 & \text{for} \quad t \le \dfrac{T}{3}, \\ 0 & \text{for} \quad t \ge \dfrac{2T}{3}. \end{cases} \tag{20}$$

We seek a solution $u$ of (10) and (11) in the form

$$u(t,x) = \psi(t)v(t,x) + w(t,x),$$

where $v$ denotes the solution of (12). We set

$$L = L(z) = \partial_t + \partial_x^3 + \partial_x + \frac{1}{2}\partial_x(z\cdot).$$

Then $0 = L\,u = \psi'(t)v + L\,w$, hence $w$ has to solve

$$\begin{cases} L\,w = w_t + w_{xxx} + w_x + \frac{1}{2}(zw)_x = -\psi'(t)v =: f(t,x), \\ w_{|_{x=L}} = w_{x\,|_{x=L}} = 0, \\ w_{|_{t=0}} = w_{|_{t=T}} = 0. \end{cases} \tag{21}$$

Conversely, if $w$ fulfils (21), then $u$ fulfils (10) and (11). (The boundary control $h$ is defined as being the trace $u_{|_{x=-L}}$. The issues of the existence of the trace, of the uniqueness and of the smoothness of the solution will be investigated for the nonlinear equation only.) Notice that $f \in \mathrm{H}^1(0,T,\mathrm{H}^1(-L,L)) \cap \mathrm{C}^0([0,T],\mathrm{H}^3(-L,L)) \cap \mathrm{C}^1([0,T],\mathrm{L}^2(-L,L))$, according to Lemma 2.2. We infer from (20) that

$$\mathrm{Supp}\ f \subset \left[\frac{T}{3}, \frac{2T}{3}\right] \times (-L,L). \tag{22}$$

To prove the existence of a solution $w \in \mathrm{L}^2(Q)$ to (21) we need some Carleman estimate, which is stated and proved in the next section.

### 2.2. **A Carleman estimate**

We introduce the space

$$\mathcal{Z} = \left\{q \in C^3([0,T]\times[-L,L]);\ q_{|_{x=-L}} = q_{x\,|_{x=-L}} = q_{xx\,|_{x=-L}} = q_{|_{x=L}} = 0\right\}\cdot$$

The following proposition improves a result given in [21], namely [21] (Prop. 3.1). Indeed, here it is no longer assumed that $q_{x\,|_{x=L}} = q_{xx\,|_{x=L}} = 0$.

**Proposition 2.3.** *There exists a smooth positive function $\psi$ on $[-L,L]$, and for any $R>0$ there exist some constants $s_0 = s_0(L,T,R)$ and $C_0 = C_0(L,T,R)$ such that for all $s \geq s_0$, all $\zeta \in V$ with $\|\zeta\|_V \leq R$ and all $q \in \mathcal{Z}$ we have*

$$\int_0^T\!\!\int_{-L}^L \left\{\frac{s^5}{t^5(T-t)^5}|q|^2 + \frac{s^3}{t^3(T-t)^3}|q_x|^2 + \frac{s}{t(T-t)}|q_{xx}|^2\right\}\exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t$$
$$\leq C_0\int_0^T\!\!\int_{-L}^L |q_t + q_{xxx} + \zeta\, q_x|^2 \exp\left(-\frac{2s\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t. \tag{23}$$

*Proof.* Let $R>0$ and $\zeta \in V$ with $\|\zeta\|_V \leq R$. Let $\psi = \psi(x)$ be a positive function (to be specified later) of class $C^3$ in $[-L,L]$ and let $\varphi(t,x) = \frac{\psi(x)}{t(T-t)}\cdot$ Let $q$ be given in $\mathcal{Z}$ and let $s>0$. Set $u = \mathrm{e}^{-s\varphi}q$ and $w = \mathrm{e}^{-s\varphi}P(\mathrm{e}^{s\varphi}u)$, where

$$P = \partial_t + \partial_{xxx} + \zeta\partial_x.$$

We readily get

$$w = Au + Bu_x + Cu_{xx} + u_{xxx} + u_t, \tag{24}$$

with

$$\begin{aligned} A &= s(\varphi_t + \zeta\varphi_x + \varphi_{xxx}) + 3s^2\varphi_x\varphi_{xx} + s^3\varphi_x^3, \\ B &= \zeta + 3s\varphi_{xx} + 3s^2\varphi_x^2, \\ C &= 3s\varphi_x. \end{aligned}$$

Setting

$$\begin{aligned} \tilde{A} &= s(\varphi_t + \varphi_{xxx}) + s^3\varphi_x^3, \\ \tilde{B} &= (3-\delta)s\varphi_{xx} + 3s^2\varphi_x^2, \\ M_1(u) &= u_t + u_{xxx} + \tilde{B}u_x, \\ M_2(u) &= \tilde{A}u + Cu_{xx} \end{aligned}$$

where $\delta \in (0,1)$ is a small number to be specified later, we have

$$M_1(u) + M_2(u) = w - (s\zeta\varphi_x + 3s^2\varphi_x\varphi_{xx})u - (\zeta + \delta s\varphi_{xx})u_x,$$

hence

$$\begin{aligned} \left\|M_1(u) + M_2(u)\right\|^2_{\mathrm{L}^2(Q)} &\le 3\left(\|w\|^2_{\mathrm{L}^2(Q)} + s^2\left\|(\zeta + 3s\varphi_{xx})\varphi_x u\right\|^2_{\mathrm{L}^2(Q)} + \left\|(\zeta + \delta s\varphi_{xx})u_x\right\|^2_{\mathrm{L}^2(Q)}\right) \\ &\le 3\left(\|w\|^2_{\mathrm{L}^2(Q)} + (3+\delta)^2 s^4\left\|\varphi_x\varphi_{xx}u\right\|^2_{\mathrm{L}^2(Q)} + 4\delta^2 s^2\left\|\varphi_{xx}u_x\right\|^2_{\mathrm{L}^2(Q)}\right) \end{aligned} \tag{25}$$

if $|\zeta(t,x)| \le \delta \cdot s \cdot |\varphi_{xx}(t,x)|$ for all $(t,x) \in Q$. This is possible if $s$ is large enough and $|\psi''(x)| > 0$ on $[-L,L]$, since $\|\zeta\|_{\mathrm{L}^\infty} \le K\|\zeta\|_V \le K\,R$ for some constant $K > 0$. On the other hand

$$\|M_1(u) + M_2(u)\|^2_{\mathrm{L}^2(Q)} = \|M_1(u)\|^2_{\mathrm{L}^2(Q)} + \|M_2(u)\|^2_{\mathrm{L}^2(Q)} + 2\iint M_1(u)M_2(u). \tag{26}$$

(From now on, for the sake of brevity, we write $\iint u$ instead of $\int_0^T\int_{-L}^L u(t,x)\,\mathrm{d}x\mathrm{d}t$ and $\int u$ instead of $\int_0^T u(t,L)\,\mathrm{d}t$.) It remains to estimate the term $2\iint M_1(u)M_2(u)$.

$$\begin{aligned} 2\iint M_1(u)M_2(u) &= 2\iint M_1(u)\tilde{A}u + 2\iint M_1(u)Cu_{xx} \\ &= \iint 2M_1(u)\tilde{A}u + \iint 2u_tCu_{xx} + \iint 2(u_{xxx} + \tilde{B}u_x)Cu_{xx} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{27}$$

To compute the integral terms $I_1$, $I_2$ and $I_3$ we perform integrations by parts with respect to $t$ or $x$. We readily get

$$\begin{aligned} I_1 &= \iint (u_t + u_{xxx} + \tilde{B}u_x)\tilde{A}\cdot 2u \\ &= -\iint (\tilde{A}_t + \tilde{A}_{xxx} + (\tilde{A}\tilde{B})_x)u^2 + 3\iint \tilde{A}_x u_x^2 - \int \tilde{A}u_x^2 \end{aligned} \tag{28}$$

and

$$I_3 = -\iint (\tilde{B}C)_x u_x^2 + \int \tilde{B}Cu_x^2 - \iint C_x u_{xx}^2 + \int Cu_{xx}^2. \tag{29}$$

Finally, using (24), we get

$$\begin{aligned} I_2 &= -\iint 2u_{tx}Cu_x - \iint 2u_tC_xu_x \qquad (\text{for } u_{t|x=L} = 0) \\ &= \iint C_t u_x^2 + \iint 2C_x\left\{Au + Bu_x + Cu_{xx} + u_{xxx} - w\right\}u_x \\ &= -\iint (C_xA)_x u^2 + \iint \left\{C_t + 2C_xB - (CC_x)_x + C_{xxx}\right\}u_x^2 \\ &\quad + \int \left(CC_xu_x^2 + 2C_xu_xu_{xx} - C_{xx}u_x^2\right) - \iint 2C_xu_{xx}^2 - \iint 2C_xwu_x. \end{aligned} \tag{30}$$

Combining (27) and (30) we obtain

$$\begin{aligned} 2\iint M_1(u)M_2(u) &= -\iint \left\{\tilde{A}_t + \tilde{A}_{xxx} + (\tilde{A}\tilde{B})_x + (C_xA)_x\right\}u^2 \\ &\quad + \iint \left\{3\tilde{A}_x + C_t + 2C_xB - (CC_x)_x + C_{xxx} - (\tilde{B}C)_x\right\}u_x^2 \\ &\quad - 3\iint C_xu_{xx}^2 - 2\iint wC_xu_x + \int \left(-\tilde{A} + CC_x - C_{xx} + \tilde{B}C\right)u_x^2 \\ &\quad + \int Cu_{xx}^2 + 2\int C_xu_xu_{xx} \\ &=: \iint Du^2 + \iint Eu_x^2 - 3\iint C_xu_{xx}^2 - 2\iint wC_xu_x \\ &\quad + \int Fu_x^2 + \int Cu_{xx}^2 + 2\int C_xu_xu_{xx}. \end{aligned} \tag{31}$$

If $\varepsilon > 0$ is any number in $(0,1)$, then by the Cauchy-Schwarz inequality

$$2\left|\iint wC_xu_x\right| \le \varepsilon\iint C_x^2u_x^2 + \varepsilon^{-1}\iint w^2. \tag{32}$$

On the other hand

$$2\left|\int C_x u_x u_{xx}\right| \leq \int u_{xx}^2 + \int C_x^2 u_x^2. \tag{33}$$

Using (25) and (26) and (31) and (33) we obtain

$$\begin{aligned}\iint Du^2 + \iint Eu_x^2 + \iint (-3C_x)u_{xx}^2 + \int Fu_x^2 + \int Cu_{xx}^2 + \|M_1(u)\|^2_{\mathrm{L}^2(Q)} + \|M_2(u)\|^2_{\mathrm{L}^2(Q)} \\ = 2\iint wC_x u_x - 2\int C_x u_x u_{xx} + \|M_1(u) + M_2(u)\|^2_{\mathrm{L}^2(Q)} \\ \leq \varepsilon \iint C_x^2 u_x^2 + \varepsilon^{-1}\iint w^2 + \int u_{xx}^2 + \int C_x^2 u_x^2 \\ + 3\iint w^2 + 3(3+\delta)^2 s^4 \iint \varphi_x^2 \varphi_{xx}^2 u^2 + 12\delta^2 s^2 \iint \varphi_{xx}^2 u_x^2,\end{aligned}$$

hence

$$\begin{aligned}\iint \left\{D - 3(3+\delta)^2 s^4 \varphi_x^2 \varphi_{xx}^2\right\} u^2 + \iint \left\{E - \varepsilon C_x^2 - 12\delta^2 s^2 \varphi_{xx}^2\right\} u_x^2 \\ + \iint (-3C_x)u_{xx}^2 + \int \left(F - C_x^2\right) u_x^2 + \int (C-1)u_{xx}^2 \leq (3+\varepsilon^{-1}) \iint w^2.\end{aligned} \tag{34}$$

The function $\psi$ and the constants $\delta$, $\varepsilon$ and $s_0$ are chosen in such a way that the functions between brackets in the left hand side of (34) are positive. Since the function $\zeta$ appears in $A$ and $B$, it appears also in $D$ and in $E$ together with $\zeta_x$. These functions are uniformly bounded, since

$$\|\zeta\|_{\mathrm{L}^\infty(Q)} + \|\zeta_x\|_{\mathrm{L}^\infty(Q)} \leq K\|\zeta\|_V \leq KR.$$

Clearly

$$D = -15s^5 \frac{\psi'(x)^4 \psi''(x)}{t^5(T-t)^5} + \frac{O(s^4)}{t^4(T-t)^4} \qquad \text{as } s \to +\infty.$$

It follows that for $s$ large enough, if

$$|\psi'(x)| > 0 \text{ and } \psi''(x) < 0 \text{ for } x \in [-L, L],$$

then we have

$$D - 3(3+\delta)^2 s^4 \varphi_x^2 \varphi_{xx}^2 \geq K_1 \frac{s^5}{t^5(T-t)^5} \tag{35}$$

for some constant $K_1 > 0$. On the other hand

$$\begin{aligned}E &= 9s^3 \varphi_x^2 \varphi_{xx} + 6s\varphi_{xx}\left(3s\varphi_{xx} + 3s^2\varphi_x^2\right) - (9s^2 \varphi_x \varphi_{xx})_x \\ &\quad - \left\{\left((3-\delta)s\varphi_{xx} + 3s^2\varphi_x^2\right) 3s\varphi_x\right\}_x + O\left(\frac{s}{t(T-t)}\right) \\ &= 3\delta s^2 \varphi_{xx}^2 + (3\delta - 18)s^2 \varphi_x \varphi_{xxx} + O\left(\frac{s}{t(T-t)}\right)\end{aligned}$$

hence

$$E - \varepsilon C_x^2 - 12\delta^2 s^2 \varphi_{xx}^2 = s^2\left(3\delta - 9\varepsilon - 12\delta^2\right)\frac{\psi''(x)^2}{t^2(T-t)^2} + (3\delta - 18)s^2\frac{\psi'(x)\psi'''(x)}{t^2(T-t)^2} + O\left(\frac{s}{t(T-t)}\right)\cdot$$

We take $\delta = 10^{-1}$, $\varepsilon = 10^{-2}$ so that $3\delta - 9\varepsilon - 12\delta^2 > 0$. Then, if $\psi'' \neq 0$ and $\psi'\psi''' \leq 0$ on $[-L, L]$, we get for some constant $K_2 > 0$ and for $s$ large enough

$$E - \varepsilon C_x^2 - 12\delta^2 s^2 \varphi_{xx}^2 \geq K_2 \frac{s^2}{t^2(T-t)^2}\cdot \tag{36}$$

Finally

$$-3C_x = -9s\frac{\psi''(x)}{t(T-t)} \geq K_3\frac{s}{t(T-t)}, \tag{37}$$

$$F - C_x^2 = -s^3\varphi_x^3 + 9s^3\varphi_x^3 + O\left(\frac{s^2}{t^2(T-t)^2}\right) \geq K_4\frac{s^3}{t^3(T-t)^3} \tag{38}$$

$$C - 1 = 3s\varphi_x - 1 \geq K_5\frac{s}{t(T-t)} \tag{39}$$

for some positive constants $K_3, K_4$ and $K_5$, provided that $\psi''(x) < 0$ and $\psi'(x) > 0$ for all $x \in [-L, L]$ and that $s$ is large enough. To summarize, the function $\psi$ has to fulfill the following conditions: $\psi \in C^3([-L, L])$, $\psi > 0$, $\psi' > 0$, $\psi'' < 0$ and $\psi'\psi''' \leq 0$ on $[-L, L]$. $\psi(x) := 1 + 4L^2 + x(3L - x)$ is clearly convenient. We infer from (34)–(39) that for $s$ large enough

$$\iint\left\{\frac{s^5}{t^5(T-t)^5}u^2 + \frac{s^2}{t^2(T-t)^2}u_x^2 + \frac{s}{t(T-t)}u_{xx}^2\right\} \leq K_6\iint w^2$$

for some constant $K_6 > 0$. As it has been noticed in [21]

$$\iint \frac{s^3}{t^3(T-t)^3}u_x^2 = -\iint \frac{s^3}{t^3(T-t)^3}uu_{xx} \leq \frac{K_6}{2}\iint w^2,$$

hence for $s$ large enough

$$\iint\left\{\frac{s^5}{t^5(T-t)^5}u^2 + \frac{s^3}{t^3(T-t)^3}u_x^2 + \frac{s}{t(T-t)}u_{xx}^2\right\} \leq \frac{3}{2}K_6\iint w^2. \tag{40}$$

Replacing $u$ by $\mathrm{e}^{-s\varphi}q$ in (40), we readily get (23) for some constants $C_0 = C_0(L, T, R)$ and $s_0 = s_0(L, T, R)$. The proof of Proposition 2.3 is complete. □

The (formal) adjoint to the operator $L = \partial_t + \partial_x^3 + \partial_x + \frac{1}{2}\partial_x(z\cdot)$ is

$$L^* = -\left(\partial_t + \partial_x^3 + \partial_x + \frac{1}{2}z\partial_x\right).$$

Let $H$ denote the completion of the space $\mathcal{Z}$ for the Hilbertian norm $\|\cdot\|_H$ defined as

$$\|q\|_H^2 := \|L^*q\|_{\mathrm{L}^2(Q)}^2 = \iint_Q \left|q_t + q_{xxx} + \left(1 + \frac{1}{2}z\right)q_x\right|^2 \,\mathrm{d}x\mathrm{d}t. \tag{41}$$

The proof of the next result is only sketched.

**Lemma 2.4.** *Let $z \in V$ with $\|z\|_V \le R$, and let $s_0 = s_0(L,T,R'), C_0 = C_0(L,T,R')$ and $\psi$ be as given in Proposition 2.3, with $R' = \sqrt{2LT} + \frac{R}{2}$. Then $H$ is made of the functions $q \in \mathrm{L}^1_{loc}(Q)$ such that*

$$\|q\|_H^2 + \iint_Q \left\{ \frac{1}{t^5(T-t)^5}|q|^2 + \frac{1}{t^3(T-t)^3}|q_x|^2 + \frac{1}{t(T-t)}|q_{xx}|^2 \right\} \exp\left(-\frac{2s_0\psi(x)}{t(T-t)}\right) \mathrm{d}x\mathrm{d}t < \infty \tag{42}$$

*and*

$$0 = q_{|x=-L} = q_{x|_{x=-L}} = q_{xx|_{x=-L}} = q_{|x=L}. \tag{43}$$

*Proof.* Let $\zeta = 1 + \frac{1}{2}z$, hence $\|\zeta\|_V \le \|1\|_V + \frac{1}{2}\|z\|_V \le \sqrt{2LT} + \frac{R}{2}$. (42) follows readily from Proposition 2.3. (Notice that (23) is still valid for any $q \in H$.) We now turn to the boundary conditions. Fix any $\varepsilon > 0$. Then, by (23),

$$\|q\|_{\mathrm{L}^2(\varepsilon,T-\varepsilon,\mathrm{H}^2(-L,L))} \le K\|q\|_H.$$

Hence, for any $q \in H$, the traces $q_{|x=-L}, q_{x|_{x=-L}}, q_{|x=L}$ exist in $\mathrm{L}^2(\varepsilon, T-\varepsilon)$. Since they vanish when $q \in \mathcal{Z}$, they vanish also for $q \in H$. Finally, we notice that

$$q_{xxx} = -\left(q_t + q_x\left(1 + \frac{1}{2}z\right)\right) - L^*q \in \mathrm{L}^2(-L,L,\mathrm{H}^{-1}(\varepsilon,T-\varepsilon))$$

hence $q \in \mathrm{H}^3(-L,L,\mathrm{H}^{-1}(\varepsilon,T-\varepsilon))$ and

$$\|q_{xx|_{x=-L}}\|_{\mathrm{H}^{-1}(\varepsilon,T-\varepsilon)} \le K\|q\|_H.$$

Hence, we also have that $q_{xx|_{x=-L}} = 0$. The proof of the fact that $\mathcal{Z}$ is dense in the space of the functions $q \in \mathrm{L}^1_{loc}(Q)$ fulfilling (42) and (43) is left to the reader. □

We go back to the existence of a solution to (21).

**Theorem 2.5.** *Let $z \in V$. Then there exists a function $w \in \mathrm{L}^2(Q)$ fulfilling (21) and such that*

$$\|w\|_{\mathrm{L}^2(Q)} \le c(\|z\|_V) \cdot \|f\|_{\mathrm{L}^2(Q)}. \tag{44}$$

*Proof.* The linear form

$$l(q) = \iint_Q fq\,\mathrm{d}x\mathrm{d}t$$

is well defined and continuous on $H$. Indeed, using (22) and (23), we get

$$|l(q)| \le \int_{\frac{T}{3}}^{\frac{2T}{3}} \int_{-L}^{L} |fq|\,\mathrm{d}x\mathrm{d}t \le c(\|z\|_V)\|f\|_{\mathrm{L}^2(\frac{T}{3},\frac{2T}{3},\,\mathrm{L}^2(-L,L))} \cdot \|q\|_H. \tag{45}$$

It follows from Riesz representation theorem that there exists a unique $p \in H$ such that

$$(L^*p, L^*q)_{\mathrm{L}^2(Q)} = l(q) \quad \forall q \in H. \tag{46}$$

We set $w = L^*p$, hence $w \in \mathrm{L}^2(Q)$. Taking $q = p$ in (46) and using (45) we obtain

$$\|w\|^2_{\mathrm{L}^2(Q)} = \|p\|_H^2 = l(p) \le c(\|z\|_V)\|f\|_{\mathrm{L}^2(Q)}\|p\|_H$$

hence (44) holds true. Choosing any $q \in C_0^\infty(Q)$ as a test function in (46) we get

$$\langle Lw, q\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} = \langle f, q\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)}$$

and then

$$Lw = w_t + w_{xxx} + w_x + \frac{1}{2}(zw)_x = f. \tag{47}$$

Notice that $w \in \mathrm{H}^1(0,T,\mathrm{H}^{-3}(-L,L))$, since $w$ and $w_t = f-(w_{xxx}+w_x+\frac{1}{2}(zw)_x)$ belongs to $\mathrm{L}^2(0,T,\mathrm{H}^{-3}(-L,L))$. Hence $w_{|t=0}$ and $w_{|t=T}$ are meaningful in $\mathrm{H}^{-3}(-L,L)$. Pick now any $q \in \mathrm{H}^1(0,T,\mathrm{H}_0^3(-L,L)) \subset H$. It follows from (46) that

$$\begin{aligned}\iint_Q fq\,\mathrm{d}x\mathrm{d}t &= -\iint_Q w\left(q_t + q_{xxx} + \left(1+\frac{1}{2}z\right)q_x\right)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \left\langle w_t + w_{xxx} + w_x + \frac{1}{2}(zw)_x, q\right\rangle \mathrm{d}t + [\langle w,q\rangle]_0^T \\ &= \iint_Q fq\,\mathrm{d}x\mathrm{d}t + [\langle w,q\rangle]_0^T \end{aligned}\tag{48}$$

where $\langle .,.\rangle$ denotes the duality pairing $\langle .,.\rangle_{\mathrm{H}^{-3}(-L,L),\mathrm{H}_0^3(-L,L)}$. Since $q_{|t=0}$ and $q_{|t=T}$ may be arbitrary chosen in $\mathrm{H}_0^3(-L,L)$, we infer that $w_{|t=0} = w_{|t=T} = 0$. Since

$$\left(w_{xx} + \left(1+\frac{1}{2}z\right)w\right)_x = f - w_t \in \mathrm{L}^2(-L,L,\mathrm{H}^{-1}(0,T))$$

we have that $w_{xx} + \left(1+\frac{1}{2}z\right)w \in \mathrm{H}^1(-L,L,\mathrm{H}^{-2}(0,T))$, hence $w \in \mathrm{H}^3(-L,L,\mathrm{H}^{-2}(0,T))$ and $w_{|x=L}, w_{x|x=L}$ are meaningful in $\mathrm{H}^{-2}(0,T)$. Let $q(t,x) = a(t)b(x)$, with $a \in C_0^\infty(0,T)$, $b \in C^3([-L,L])$ and $0 = b(-L) = b'(-L) = b''(-L) = b(L)$. Then $q \in \mathcal{Z}$ and we may write

$$\begin{aligned}\iint_Q fq &= -\iint_Q w\left(q_t + q_{xxx} + \left(1+\frac{1}{2}z\right)q_x\right)\mathrm{d}x\mathrm{d}t \\ &= \int_{-L}^L \left\langle w_t + w_{xxx} + w_x + \frac{1}{2}(zw)_x, q\right\rangle \\ &\quad + \left[-\langle w, q_{xx}\rangle + \langle w_x, q_x\rangle - \langle w_{xx}, q\rangle - \left\langle \left(1+\frac{1}{2}z\right)w, q\right\rangle\right]_{-L}^L \end{aligned}\tag{49}$$

where $\langle .,.\rangle$ denotes here the duality pairing $\langle .,.\rangle_{\mathrm{H}^{-2}(0,T),\mathrm{H}_0^2(0,T)}$. Since $q \in \mathcal{Z}$ and $Lw = f$, we obtain

$$-b''(L)\,\langle w(.,L), a\rangle + b'(L)\,\langle w_x(.,L), a\rangle = 0.$$

Since $b'(L), b''(L) \in \mathbb{R}$ and $a \in C_0^\infty(0,T)$ are arbitrary, we infer that $w_{|x=L} = w_{x|x=L} = 0$. □

At this stage, we know that the control problem (10) and (11) has (at least) one solution $u \in \mathrm{L}^2((0,T)\times(-L,L))$, namely the function $\psi v + w$. To apply a fixed point argument we need more regularity for $u$. This is done in the following section.

## 2.3. **Weighted estimates**

We aim to prove that the solution $w$ to (21) is more than square integrable. Let us stress that (21) is not a classical initial-boundary-value problem, since $w_{|x=-L}$ is not prescribed. On the other hand, we have at our disposal the final value condition $w_{|t=T} = 0$. More regularity is proved thanks to weighted estimates. Roughly speaking, we multiply each term in $Lw = f$ by a function $(L+x)^k$, $k \geq 1$ and integrate over $Q$, and we pick $k$

so that all the integrals are convergent near $x = -L$. However, since $w$ is (at this stage) only known to belong to $\mathrm{L}^2(Q)$, we cannot do such computations. For this reason we need to replace $w$ by a more regular function $w^h$ obtained in convolving $w$ with $h^{-1}\chi_{(-h,0)}(t) \otimes \delta_0(x)$, with $h > 0$. More generally, for any $g \in \mathrm{L}^2(\mathbb{R}_t \times (-L, L)_x)$ we set

$$g^h(t,x) = \frac{1}{h}\int_t^{t+h} g(\tau, x)\,\mathrm{d}\tau \qquad \text{for any } (t,x) \in \mathbb{R} \times (-L, L).$$

Straightforward computations show that $g^h \in \mathrm{L}^2(\mathbb{R}_t \times (-L, L)_x)$ with

$$\|g^h\|_{\mathrm{L}^2(\mathbb{R}\times(-L,L))} \leq \|g\|_{\mathrm{L}^2(\mathbb{R}\times(-L,L))} \tag{50}$$

and that $g^h \to g$ in $\mathrm{L}^2_{loc}(\mathbb{R}_t \times (-L, L)_x)$ as $h \searrow 0$. In what follows computations will be performed with the regularized function $w^h$, $w$ being extended by 0 for $t \notin (0, T)$. We also extend $f$ by 0 for $t \notin (0, T)$, and set

$$z(t,x) = \begin{cases} z(0,x) & \text{for } \; t \leq 0, \\ z(T,x) & \text{for } \; t \geq T. \end{cases}$$

Notice first that $w^h \in \mathrm{H}^1(\mathbb{R}, \mathrm{L}^2(-L, L))$, for

$$(w^h)_t = \frac{1}{h}(w(t+h) - w(t)) \in \mathrm{L}^2(\mathbb{R}_t \times (-L, L)_x).$$

On the other hand, we infer from the equation

$$(w^h)_{xxx} = f^h - \left((w^h)_t + (w^h)_x + \frac{1}{2}((zw)^h)_x\right)$$

that $w^h \in \mathrm{L}^2(\mathbb{R}, \mathrm{H}^3(-L, L))$. Finally $w^h$ solves the system

$$w^h_t + w^h_{xxx} + w^h_x + \frac{1}{2}(zw)^h_x = f^h, \quad -L < x < L,\ t \in \mathbb{R} \tag{51}$$

$$w^h{}_{|x=L} = w^h_{x\,|x=L} = 0, \tag{52}$$

$$w^h{}_{|t\leq -h} = w^h{}_{|t\geq T} = 0. \tag{53}$$

The first result reveals a boundary smoothing effect of Kato type.

**Proposition 2.6.** *Let $z \in \mathrm{L}^\infty(Q)$, $f \in \mathrm{L}^1(0, T, \mathrm{L}^2(-L, L))$ and let $w \in \mathrm{L}^2(Q)$ satisfy (21). Then $(L + x)^{\frac{3}{2}} w \in \mathrm{L}^\infty(0, T, \mathrm{L}^2(-L, L))$, $(L + x)w_x \in \mathrm{L}^2(Q)$ and*

$$\|(L+x)^{\frac{3}{2}}w\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x} + \|(L+x)w_x\|^2_{\mathrm{L}^2(Q)} \leq c\left(\|z\|_{\mathrm{L}^\infty(Q)}\right)\|w\|^2_{\mathrm{L}^2(Q)} + K\|f\|^2_{\mathrm{L}^1_t\mathrm{L}^2_x}. \tag{54}$$

*Proof.* Multiplying each term in (51) by $(L + x)^3 w^h$ and integrating over $D := (-h, T') \times (-L, L)$ where $-h \leq T' \leq T$, we obtain

$$\iint_D (L+x)^3 w^h w^h_t + \iint_D (L+x)^3 w^h w^h_{xxx} + \iint_D (L+x)^3 w^h w^h_x + \frac{1}{2}\iint (L+x)^3 w^h (zw)^h_x$$
$$\equiv I_1 + I_2 + I_3 + I_4 = \iint_D (L+x)^3 w^h f^h.$$

As $w^h \in \mathrm{L}^2(\mathbb{R},\mathrm{H}^3(-L,L))\cap \mathrm{H}^1(\mathbb{R},\mathrm{L}^2(-L,L))$, we may integrate by parts in $I_1$, $I_2$, $I_3$ and $I_4$. Using (52) and (53) we obtain

$$I_1 = \frac{1}{2}\int_{-L}^{L}(L+x)^3 w^h(T',x)^2\mathrm{d}x;$$

$$\begin{aligned}I_2 &= -\iint_D \left\{3(L+x)^2 w^h + (L+x)^3 w_x^h\right\}\cdot w_{xx}^h\\ &= \iint_D \left\{6(L+x)w^h + 3(L+x)^2 w_x^h\right\}\cdot w_x^h + \frac{3}{2}\iint_D (L+x)^2 (w_x^h)^2\\ &= -3\iint_D (w^h)^2 + \frac{9}{2}\iint (L+x)^2 (w_x^h)^2;\end{aligned}$$

$$I_3 = -\frac{3}{2}\iint_D (L+x)^2 (w^h)^2;$$

$$I_4 = -\frac{1}{2}\iint \left\{3(L+x)^2 w^h + (L+x)^3 w_x^h\right\}(zw)^h,$$

hence, by (50) applied to $w^h$ and to $(zw)^h$

$$\begin{aligned}|I_4| &\le K\big(\|w^h\|_{\mathrm{L}^2(D)}\|(zw)^h\|_{\mathrm{L}^2(D)} + \|(L+x)w_x^h\|_{\mathrm{L}^2(D)}\|(zw)^h\|_{\mathrm{L}^2(D)}\big)\\ &\le K\|z\|_{\mathrm{L}^\infty(Q)}\|w\|^2_{\mathrm{L}^2(Q)} + \frac{1}{2}\|(L+x)w_x^h\|^2_{\mathrm{L}^2(D)} + K\|z\|^2_{\mathrm{L}^\infty(Q)}\|w\|^2_{\mathrm{L}^2(Q)}.\end{aligned}$$

Therefore, for all $T' \in [-h,T]$,

$$\begin{aligned}&\frac{1}{2}\int_{-L}^{L}(L+x)^3 w^h(T',x)^2\mathrm{d}x + \frac{9}{2}\iint_D (L+x)^2 (w_x^h)^2\\ &\quad\le 3\iint_D (w^h)^2 + \frac{3}{2}\iint_D (L+x)^2 (w^h)^2 + |I_4| + \iint_D (L+x)^3 |w^h f^h|\\ &\quad\le c\left(\|z\|_{\mathrm{L}^\infty(Q)}\right)\|w\|^2_{\mathrm{L}^2(Q)} + \frac{1}{2}\iint_D (L+x)^2 (w_x^h)^2\\ &\qquad + \frac{1}{4}\left\|(L+x)^{\frac{3}{2}} w^h\right\|^2_{\mathrm{L}^\infty(-h,T',\mathrm{L}^2(-L,L))} + K\|f\|^2_{\mathrm{L}^1_t\mathrm{L}^2_x}.\end{aligned}$$

Picking for $T'$ the instant at which $\|(L+x)^{\frac{3}{2}}w^h\|_{\mathrm{L}^2(-L,L)}$ assumes its largest value on $[-h,T]$, and then letting $T'=T$, we obtain

$$\|(L+x)^{\frac{3}{2}}w^h\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x} + \|(L+x)w_x^h\|^2_{\mathrm{L}^2(Q)} \le c(\|z\|_{\mathrm{L}^\infty(Q)})\|w\|^2_{\mathrm{L}^2(Q)} + K\|f\|^2_{\mathrm{L}^1_t\mathrm{L}^2_x}$$

and this implies (54) by letting $h\to 0$. $\square$

Notice that (54) means that $w(t,.)$ is $\mathrm{H}^1$ *away from* $x=-L$ for almost every $t$. (More precisely, we have that $w\in \mathrm{L}^2(0,T,\mathrm{H}^1(-L+\varepsilon,L))$ for any $\varepsilon\in(0,L)$.) The next result (more difficult to establish) asserts that $w_t$ is square integrable away from $x=-L$.

**Proposition 2.7.** *Let* $z\in \mathrm{W}^{1,\infty}(-L,L,\mathrm{L}^\infty(0,T))\cap \mathrm{H}^1(0,T,\mathrm{L}^\infty(-L,L))$, $f\in \mathrm{L}^2(0,T,\mathrm{H}^1(-L,L))$ *and let* $w\in \mathrm{L}^2(Q)$ *satisfy (21). Then* $(L+x)^3w_t\in \mathrm{L}^2(Q)$ *and*

$$\|(L+x)^3w_t\|^2_{\mathrm{L}^2(Q)}\le c\left(\|z\|_{\mathrm{L}^\infty(Q)}+\|z_x\|_{\mathrm{L}^\infty(Q)}+\|z_t\|_{\mathrm{L}^2_t\mathrm{L}^\infty_x}\right)\left(\|w\|^2_{\mathrm{L}^2(Q)}+\|f\|^2_{\mathrm{L}^2_t\mathrm{H}^1_x}\right). \tag{55}$$

*Proof.* Let $l\in(0,L)$, and set $Q_l:=(0,T)\times(-l,L)$ (hence $Q=Q_L$). It follows from (54) that

$$w^h\in \mathrm{L}^2(\mathbb{R},\mathrm{H}^3(-l,L))\cap \mathrm{H}^1(\mathbb{R},\mathrm{H}^1(-l,L)).$$

We scale each term in (51) by $(l+x)^7w^h_{xt}$ and integrate over $D_l:=(-h,T)\times(-l,L)$. In what follows, the integrals are extended to $D_l$, $K$ denotes again a positive number which 0 varies from line to line and does not depend on $\varepsilon$ or on $l$, and $C(r,\varepsilon)$ denotes a positive function (varying from line to line), which is increasing with respect to $r$. We get

$$\begin{aligned}0=\iint(l+x)^7w^h_{xt}w^h_t+\iint(l+x)^7w^h_{xt}w^h_{xxx}+\iint(l+x)^7w^h_{xt}w^h_x+\iint(l+x)^7w^h_{xt}\cdot\frac12(zw)^h_x&\\ -\iint(l+x)^7w^h_{xt}f^h\equiv I_1+I_2+I_3+I_4-I_5.&\end{aligned}$$

An integration by parts yields

$$I_1=-7\iint(l+x)^6\frac{(w^h_t)^2}{2}\qquad(\text{ since } w^h_t=0 \text{ for } x=L). \tag{56}$$

For any $v\in C^\infty(\bar D)$ fulfilling the boundary conditions

$$v_{|t=-h}=v_{|t=T}=0 \text{ and } v_{|x=L}=v_{x\,|x=L}=0, \tag{57}$$

we have (using integrations by parts)

$$\begin{aligned}\iint(l+x)^7v_{xt}v_{xxx}&=-\iint\left\{7(l+x)^6v_{xt}+(l+x)^7v_{xxt}\right\}v_{xx}\quad(\text{since } v_{xt|_{x=L}}=0)\\&=7\iint v_t\left\{6(l+x)^5v_{xx}+(l+x)^6v_{xxx}\right\}\quad(\text{since } v_{xx|_{t=-h}}=v_{xx|_{t=T}}=0).\end{aligned} \tag{58}$$

Let $(v_n)$ be a sequence in $C^\infty(\overline{D})$ fulfilling (57) and converging to $w^h$ in $\mathrm{L}^2(-h,T,\mathrm{H}^3(-l,L))\cap\mathrm{H}^1(-h,T,\mathrm{H}^1(-l,L))$. Applying (58) to $v=v_n$, letting $n\to\infty$ and integrating by parts, we get

$$\begin{aligned}
I_2 &= 7\iint w_t^h\left\{6(l+x)^5 w_{xx}^h+(l+x)^6 w_{xxx}^h\right\}\\
&= -42\iint w_x^h\left\{w_{tx}^h(l+x)^5+5(l+x)^4 w_t^h\right\}\\
&\quad+7\iint w_t^h(l+x)^6\left\{f^h-w_t^h-w_x^h-\frac{1}{2}(zw)_x^h\right\}\qquad(\text{using }\ (51))\\
&= 0-210\iint(l+x)^4 w_t^h w_x^h+7\iint(l+x)^6 w_t^h f^h-7\iint(l+x)^6(w_t^h)^2\\
&\quad-7\iint(l+x)^6 w_t^h w_x^h-\frac{7}{2}\iint(l+x)^6 w_t^h(zw)_x^h.
\end{aligned}\tag{59}$$

Since $w_x^h=0$ for $t=-h$ or $t=T$, we get at once

$$I_3=0.\tag{60}$$

An integration by parts in $I_5$ yields

$$I_5=-\iint w_t^h\left\{7(l+x)^6 f^h+(l+x)^7 f_x^h\right\}$$

hence

$$\begin{aligned}
|I_5| &\le \varepsilon\iint(l+x)^6(w_t^h)^2+\frac{1}{4\varepsilon}\iint(l+x)^6\left\{7f^h+(l+x)f_x^h\right\}^2\\
&\le \varepsilon\iint(l+x)^6(w_t^h)^2+K\,\varepsilon^{-1}\,\|f\|^2_{\mathrm{L}^2_t\mathrm{H}^1_x}.
\end{aligned}\tag{61}$$

It remains to estimate $I_4$. An integration by parts gives

$$I_4=-\frac{1}{2}\iint(l+x)^7 w_x^h(zw)_{xt}^h.\tag{62}$$

But

$$\begin{aligned}
(zw)_{xt}^h &= \frac{(zw)_x(t+h)-(zw)_x(t)}{h}\\
&= \frac{z_x(t+h)-z_x(t)}{h}w(t+h)+z_x(t)\frac{w(t+h)-w(t)}{h}\\
&\quad+\frac{z(t+h)-z(t)}{h}w_x(t+h)+z(t)\frac{w_x(t+h)-w_x(t)}{h}\\
&= z_{xt}^h w(t+h)+z_x w_t^h+z_t^h w_x(t+h)+z w_{xt}^h,
\end{aligned}$$

hence

$$\begin{aligned} I_4 &= -\frac{1}{2}\iint (l+x)^7 w_x^h z_x w_t^h - \frac{1}{2}\iint (l+x)^7 w_x^h z w_{xt}^h \\ &\quad - \frac{1}{2}\iint (l+x)^7 w_x^h (z_t^h w(t+h))_x \\ &\equiv I_6 + I_7 + I_8. \end{aligned} \tag{63}$$

We estimate successively $I_6$, $I_7$ and $I_8$.

$$\begin{aligned} |I_6| &\le \varepsilon \iint (l+x)^6 (w_t^h)^2 + K\,\varepsilon^{-1}\, \|z_x\|^2_{\mathrm{L}^\infty(Q)} \iint (l+x)^2 (w_x^h)^2 \\ &\le \varepsilon \iint (l+x)^6 (w_t^h)^2 + K\,\varepsilon^{-1}\, \|z_x\|^2_{\mathrm{L}^\infty(Q)} \|(l+x) w_x\|^2_{\mathrm{L}^2(Q_l)}. \end{aligned} \tag{64}$$

$$I_7 = \frac{1}{2}\iint \left\{7(l+x)^6 w_x^h z + (l+x)^7 w_{xx}^h z + (l+x)^7 w_x^h z_x\right\} w_t^h$$

hence

$$\begin{aligned} |I_7| &\le \varepsilon \iint (l+x)^6 (w_t^h)^2 + K\,\varepsilon^{-1} \left( \|z\|^2_{\mathrm{L}^\infty(Q)} \iint (l+x)^2 (w_x^h)^2 + \|z\|^2_{\mathrm{L}^\infty(Q)} \iint (l+x)^4 (w_{xx}^h)^2 \right. \\ &\qquad \left. + \|z_x\|^2_{\mathrm{L}^\infty(Q)} \iint (l+x)^2 (w_x^h)^2 \right) \\ &\le \varepsilon \iint (l+x)^6 (w_t^h)^2 + K\,\varepsilon^{-1} \left( \|z\|^2_{\mathrm{L}^\infty(Q)} + \|z_x\|^2_{\mathrm{L}^\infty(Q)} \right) \|(l+x) w_x\|^2_{\mathrm{L}^2(Q_l)} \\ &\qquad + K\,\varepsilon^{-1}\, \|z\|^2_{\mathrm{L}^\infty(Q)} \iint (l+x)^4 (w_{xx}^h)^2. \end{aligned} \tag{65}$$

We now have to estimate the $L^2(D_l)$-norm of $(l+x)^2 w_{xx}^h$. This is done in the following lemma.

**Lemma 2.8.** *For any $\eta > 0$, there exists a constant $C_\eta > 0$ such that*

$$\begin{aligned} \iint (l+x)^4 (w_{xx}^h)^2 &\le \eta \iint (l+x)^6 (w_t^h)^2 + K\|f\|^2_{\mathrm{L}^2(Q)} \\ &\quad + C_\eta \left(1 + \|z\|^2_{\mathrm{L}^\infty(Q)} + \|z_x\|^2_{\mathrm{L}^\infty(Q)}\right) \cdot \left( \|(l+x) w_x\|^2_{\mathrm{L}^2(Q_l)} + \|w\|^2_{\mathrm{L}^2(Q)} \right). \end{aligned} \tag{66}$$

*Proof of Lemma 2.8.* Scaling (51) by $(l+x)^4 w_x^h$ and integrating over $D_l$, we obtain

$$\begin{aligned} 0 &= \iint (l+x)^4 w_x^h w_t^h + \iint (l+x)^4 w_x^h w_{xxx}^h + \iint (l+x)^4 (w_x^h)^2 + \iint (l+x)^4 w_x^h g \\ &\equiv I_1' + I_2' + I_3' + I_4', \end{aligned} \tag{67}$$

where $g = \frac{1}{2}(zw)_x^h - f^h$. Clearly,

$$|I_1'| \le \eta \iint (l+x)^6 (w_t^h)^2 + C_\eta \iint (l+x)^2 (w_x^h)^2, \tag{68}$$

$$I_2' = -\iint \{4(l+x)^3 w_x^h + (l+x)^4 w_{xx}^h\} w_{xx}^h$$
$$= 6\iint (l+x)^2 (w_x^h)^2 - \iint (l+x)^4 (w_{xx}^h)^2, \tag{69}$$

$$I_3' \leq K \iint (l+x)^2 (w_x^h)^2, \tag{70}$$

$$|I_4'| = \left| -\iint (l+x)^4 w_x^h f^h + \frac{1}{2}\iint (l+x)^4 w_x^h (z_x w)^h + \frac{1}{2}\iint (l+x)^4 w_x^h (z w_x)^h \right|$$
$$\leq K \left( \iint (l+x)^2 (w_x^h)^2 + \iint (f^h)^2 + \|z_x\|^2_{\mathrm{L}^\infty(Q)} \iint_{Q_l} w^2 + \|z\|^2_{\mathrm{L}^\infty(Q)} \iint_{Q_l} (l+x)^2 w_x^2 \right). \tag{71}$$

Gathering together (67) and (71) and using (50), we readily get (66). The proof of Lemma 2.8 is complete.

Choosing $\eta = \left(1 + K\|z\|^2_{\mathrm{L}^\infty(Q)}\right)^{-1} \varepsilon^2$ in (66) and using (65), we obtain

$$|I_7| \leq 2\varepsilon \iint (l+x)^6 (w_t^h)^2 + C\left(\|z\|_{\mathrm{L}^\infty(Q)} + \|z_x\|_{\mathrm{L}^\infty(Q)}, \varepsilon\right)\left(\|(L+x)w_x\|^2_{\mathrm{L}^2(Q)} + \|f\|^2_{\mathrm{L}^2(Q)} + \|w\|^2_{\mathrm{L}^2(Q)}\right). \tag{72}$$

It remains to estimate $I_8$ in (63). We obtain after some integration by parts

$$I_8 = \frac{1}{2}\iint \left\{7(l+x)^6 w_x^h + (l+x)^7 w_{xx}^h\right\} z_t^h w(t+h),$$

hence

$$|I_8| \leq K\left(\iint (l+x)^2 (w_x^h)^2 + \iint (l+x)^4 (w_{xx}^h)^2\right) + \iint (l+x)^3 \left(z_t^h w(t+h)\right)^2$$
$$\leq K\left(\iint (l+x)^2 (w_x)^2 + \iint (l+x)^4 (w_{xx}^h)^2\right) + \|z_t^h\|^2_{\mathrm{L}^2_t\mathrm{L}^\infty_x} \|(l+x)^{\frac{3}{2}} w\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}$$
$$\leq K\left(\|(L+x)w_x\|^2_{\mathrm{L}^2(Q)} + \iint (l+x)^4 (w_{xx}^h)^2\right) + \|z_t\|^2_{\mathrm{L}^2_t\mathrm{L}^\infty_x} \|(L+x)^{\frac{3}{2}} w\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}$$
$$\leq \varepsilon \iint (l+x)^6 (w_t^h)^2 + C\left(\|z\|_{\mathrm{L}^\infty(Q)} + \|z_x\|_{\mathrm{L}^\infty(Q)} + \|z_t\|_{\mathrm{L}^2_t\mathrm{L}^\infty_x}, \varepsilon\right)\left(\|(L+x)w_x\|^2_{\mathrm{L}^2(Q)}\right.$$
$$\left. + \|f\|^2_{\mathrm{L}^2(Q)} + \|w\|^2_{\mathrm{L}^2(Q)} + \|(L+x)^{\frac{3}{2}} w\|^2_{\mathrm{L}^\infty_t\mathrm{L}^2_x}\right), \text{by (66) applied with } \eta = \varepsilon/K. \tag{73}$$

Combining (56)–(73) and using (44) and (54) we obtain

$$\int_{-\infty}^{+\infty}\int_{-l}^{L} (l+x)^6 (w_t^h)^2 \,\mathrm{d}x\mathrm{d}t \leq 5\varepsilon \int_{-\infty}^{+\infty}\int_{-l}^{L} (l+x)^6 (w_t^h)^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ C\left(\|z\|_{\mathrm{L}^\infty(Q)} + \|z_x\|_{\mathrm{L}^\infty(Q)} + \|z_t\|_{\mathrm{L}^2_t\mathrm{L}^\infty_x}, \varepsilon\right)\left(\|w\|^2_{\mathrm{L}^2(Q)} + \|f\|^2_{\mathrm{L}^2_t\mathrm{H}^1_x}\right). \tag{74}$$

We infer that $(l+x)^3 w_t \in \mathrm{L}^2((0,T)\times(-l,L))$ by choosing $\varepsilon < 5^{-1}$ in (74) and letting $h \searrow 0$. Finally, since the function $C$ does not depend on $l$, we obtain (55) by letting $l \nearrow L$ and using Beppo-Levi theorem. □

**Corollary 2.9.** *Let $z$ and $w$ be as in Theorem 2.5. Then* $w \in \mathrm{L}^2(0,T,\mathrm{H}^3(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$ *with*

$$\|w\|_{\mathrm{L}^2(0,T,\mathrm{H}^3(0,L))} + \|w_t\|_{\mathrm{L}^2(0,T,\mathrm{H}^1(0,L))} \le c(\|z\|_V)\|f\|_{\mathrm{H}^1_t\mathrm{H}^1_x}. \tag{75}$$

*Proof.* Recall that $f$ and $w$ have been extended by 0 for $t \notin (0,T)$, and that

$$z(t,x) = \begin{cases} z(0,x) & \text{for } t \le 0, \\ z(T,x) & \text{for } t \ge T. \end{cases}$$

Then $z, z_x \in \mathrm{L}^\infty((-1,T+1)\times(-L,L))$, since $z(t,.) \in \mathrm{H}^2(-L,L)$ for $t = 0,T$. On the other hand, $z_t \in \mathrm{L}^2(-1,T+1,\mathrm{L}^\infty(-L,L))$. As $w \in \mathrm{C}^0([0,T],\mathrm{L}^2(-l,L))$ for any $l \in (0,L)$ (by virtue of Prop. 2.7) and $w_{|t=0} = w_{|t=T} = 0$, we see that the equation

$$w_t + w_{xxx} + w_x + \frac{1}{2}(wz)_x = f$$

holds true in $\mathcal{D}'(\mathbb{R}_t \times (-L,L))$. We set $\omega = w_t$, hence Supp $\omega \subset [0,T]\times[-L,L]$. Applying Proposition 2.7 to $w$ on $(-1,T+1)\times(-L,L)$ (instead of $(0,T)\times(-L,L)$), we infer that $\omega \in \mathrm{L}^2((-1,T+1)\times(-\frac{L}{2},L))$ with

$$\|\omega\|_{\mathrm{L}^2\left((-1,T+1)\times(-\frac{L}{2},L)\right)} \le c(\|z\|_V)\left\{\|w\|^2_{\mathrm{L}^2(Q)} + \|f\|^2_{\mathrm{L}^2_t\mathrm{H}^1_x}\right\}$$

and

$$\begin{cases} \omega_t + \omega_{xxx} + \omega_x + \dfrac{1}{2}(z\omega)_x = f_t - \dfrac{1}{2}(z_t w)_x, & -\dfrac{L}{2} < x < L,\ t \in \mathbb{R}, \\ \omega_{|t\le 0} = \omega_{|t\ge T} = 0, \\ \omega_{|x=L} = \omega_{x|x=L} = 0. \end{cases} \tag{76}$$

(Notice that $z_t w \in \mathrm{L}^2\left((-1,T+1)\times(-\frac{L}{2},L)\right)$, since $z_t \in \mathrm{L}^2(-1,T+1,\mathrm{H}^1(-\frac{L}{2},L))$ and $w \in \mathrm{H}^1(-1,T+1,\mathrm{L}^2(-\frac{L}{2},L))$. It follows that $-\frac{1}{2}(z_t w)_x \in \mathrm{L}^2(-1,T+1,\mathrm{H}^{-1}(-\frac{L}{2},L))$ and $\omega \in \mathrm{H}^1(-1,T+1,\mathrm{H}^{-3}(-\frac{L}{2},L))$, which yields $\omega \in \mathrm{C}^0(\mathbb{R},\mathrm{H}^{-3}(0,L))$.) Applying Proposition 2.6 to $\omega$ on $(0,T)\times(-\frac{L}{2},L)$ (instead of $Q$) and with the weight $\frac{L}{2}+x$ substituted to $L+x$, we get

$$\|\omega\|^2_{\mathrm{L}^2(0,T,\mathrm{H}^1(0,L))} \le c(\|z\|_V)\|\omega\|^2_{\mathrm{L}^2\left((0,T)\times(-\frac{L}{2},L)\right)} + K\left\{\|f_t\|^2_{\mathrm{L}^1_t\mathrm{L}^2_x} + \|(z_t w)_x\|^2_{\mathrm{L}^1(0,T,\mathrm{L}^2(-L/2,L))}\right\}. \tag{77}$$

Recall that $z_t, w \in \mathrm{L}^2(0,T,\mathrm{H}^1(-L/2,L))$, hence

$$\|(z_t w)_x\|^2_{\mathrm{L}^1(0,T,\mathrm{L}^2(-L/2,L))} \le K\|z_t\|_{\mathrm{L}^2(0,T,\mathrm{H}^1(-L/2,L))}\|w\|_{\mathrm{L}^2(0,T,\mathrm{H}^1(-L/2,L))}. \tag{78}$$

Combining (44), (54), (55) and (77), (78), we obtain

$$\|w\|_{\mathrm{H}^1(0,T,\mathrm{H}^1(0,L))} \le c(\|z\|_V)\|f\|_{\mathrm{H}^1_t\mathrm{H}^1_x}. \tag{79}$$

On the other hand,

$$w_{xxx} = f - w_t - w_x - \frac{1}{2}(zw)_x \in \mathrm{L}^2\big((0,T)\times(0,L)\big) \quad (\text{since } z,w \in \mathrm{L}^\infty\big((0,T)\times(0,L)\big).)$$

It follows that $w \in \mathrm{L}^2(0,T,\mathrm{H}^3(0,L))$, with

$$\|w\|_{\mathrm{L}^2(0,T,\mathrm{H}^3(0,L))} \le c(\|z\|_V)\|f\|_{\mathrm{H}^1_t\mathrm{H}^1_x}.$$

□

### 2.4. **The fixed-point argument**

Let $\gamma$ denote a prolongation operator mapping continuously $\mathrm{H}^k(0,L)$ into $\mathrm{H}^k(-L,L)$ for each $k\in\{0,...,3\}$; *e.g.*

$$\gamma(z)(x)=\begin{cases} z(x) & \text{for } x\geq 0,\\ 6z(-x)-32z\left(-\dfrac{x}{2}\right)+27z\left(-\dfrac{x}{3}\right) & \text{for } x<0.\end{cases} \tag{80}$$

Let $\bar V=\{z\in \mathrm{L}^2(0,T,\mathrm{H}^3(0,L)),\ z_t\in \mathrm{L}^2(0,T,\mathrm{H}^1(0,L)\}$ be endowed with its natural norm $\|\cdot\|_{\bar V}$, and let $\Lambda$ denote the map from $\bar V$ into itself, defined as follows: for any $z\in\bar V$, $\Lambda(z)$ is the restriction to $(0,T)\times(0,L)$ of the function $u(t,x)=\psi(t)v(t,x)+w(t,x)$, where $v$ and $w$ are respectively defined in (12) and in (21), with $z$ replaced by $\gamma(z)$. We intend to prove that $\Lambda$ has a fixed point in some ball of $\bar V$. We first need the following weak sequential continuity result.

**Proposition 2.10.** *If $z^n\rightharpoonup z$ in $\bar V$, then $\Lambda(z^n)\rightharpoonup\Lambda(z)$ in $\bar V$.*

*Proof.* For the sake of brevity, we write $z$ instead of $\gamma(z)$ in what follows. For any $z\in\bar V$, $\gamma(z)\in V$ and $u=\Lambda(z)=(\psi v+w)_{|(0,T)\times(0,L)}\in\bar V$, by virtue of Lemma 2.2 and Corollary 2.9. Let us set $u^n=\Lambda(z^n)=\psi v^n+w^n$ for each $n$. We infer from (14), (21) and (75) that $(u^n)$ is bounded in $\bar V$, hence it possesses a convergent subsequence for the weak topology of $\bar V$. Clearly, we are done if we prove that $u^n\to u$ in $\mathrm{L}^2\big((0,T)\times(0,L)\big)$. We begin with the

**Claim 1.** $v^n\to v$ in $B=\mathrm{L}^2(0,T,\mathrm{H}^1(-L,L))\cap \mathrm{C}^0([0,T],\mathrm{L}^2(-L,L))$.

It follows from [24] (Cor. 4) and (6) that the embedding $V\subset \mathrm{C}^0([0,T],\mathrm{H}^\delta)$ is compact whenever $1\leq\delta<2$. Therefore, $z^n\to z$ in $\mathrm{L}^\infty(Q)$ and in $\mathrm{L}^2(0,T,\mathrm{H}^1(-L,L))$. Let $\varepsilon^n=v^n-v$. Then $\varepsilon^n$ solves the system

$$\begin{cases}\varepsilon^n_t+\varepsilon^n_{xxx}+\varepsilon^n_x+\frac12(z\varepsilon^n)_x=\frac12(v^n(z-z^n))_x,\\ \varepsilon^n_{|x=-L}=\varepsilon^n_{|x=L}=\varepsilon^n_{x\,|x=L}=0,\\ \varepsilon^n_{|t=0}=u_0-u_0=0.\end{cases}\tag{81}$$

But $\big(v^n(z-z^n)\big)_x=v^n_x(z-z^n)+v^n(z-z^n)_x\to 0$ in $\mathrm{L}^2(Q)$, hence also in $\mathrm{L}^1(0,T,\mathrm{L}^2(-L,L))$, since (by (14)) $\|v^n_x\|_{\mathrm{L}^2(Q)}\leq Const.$, $\|v^n\|_{\mathrm{L}^\infty(Q)}\leq Const.$, $z^n\to z$ in $\mathrm{L}^\infty(Q)$ and $z^n_x\to z_x$ in $\mathrm{L}^2(Q)$. Using (16) we infer that $\varepsilon^n\to 0$ in $B$. The claim is proved.

For any $n$ let $L_n=L(z^n)=\partial_t+\partial_{xxx}+\partial_x+\frac12(z^n\cdot)_x$. Let $H_n$ denote the completion of $\mathcal{Z}$ for the norm defined in (41) (with $z$ replaced by $z^n$), set $s_0=s_0(L,T,\sqrt{2LT}+\frac12\sup_{n\geq0}\|z^n\|_V)$, and set $f^n(t,x)=-\psi'(t)v^n(t,x)$. It follows from Claim 1 that

$$f^n\to f\ \text{ in } \mathrm{L}^2(Q).\tag{82}$$

By construction $w^n=L_n^*p^n$, where $p^n$ is the unique function in $H_n$ fulfilling

$$\iint_Q (L_n^*p^n)(L_n^*q)\,\mathrm{d}x\mathrm{d}t=\iint_Q f^nq\,\mathrm{d}x\mathrm{d}t\qquad\text{for all } q\in\mathcal{Z}.\tag{83}$$

By (44)

$$\|w^n\|_{\mathrm{L}^2(Q)}\leq c(\|z^n\|_V)\|f^n\|_{\mathrm{L}^2(Q)}\leq \text{Const.}$$

hence, using (23),

$$\iint_Q\left\{\frac{1}{t^5(T-t)^5}|p^n|^2+\frac{1}{t^3(T-t)^3}|p^n_x|^2+\frac{1}{t(T-t)}|p^n_{xx}|^2\right\}\exp\left(-2\frac{s_0\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t\leq\text{Const.}$$

Therefore, there exists a function $\bar p\in \mathrm{L}^2_{loc}(0,T,\mathrm{H}^2(-L,L))$ and a sequence $n'\to+\infty$ such that

$$p^{n'}\rightharpoonup\bar p\quad\text{ in } \mathrm{L}^2\left(Q,t^{-5}(T-t)^{-5}\exp\left(-2\frac{s_0\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t\right),\tag{84}$$

$$p_x^{n'} \rightharpoonup \bar{p}_x \quad \text{in } \mathrm{L}^2\left(Q, t^{-3}(T-t)^{-3}\exp\left(-2\frac{s_0\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t\right), \tag{85}$$

$$p_{xx}^{n'} \rightharpoonup \bar{p}_{xx} \quad \text{in } \mathrm{L}^2\left(Q, t^{-1}(T-t)^{-1}\exp\left(-2\frac{s_0\psi(x)}{t(T-t)}\right)\mathrm{d}x\mathrm{d}t\right). \tag{86}$$

Let $\bar{w} = L^*\bar{p} = -(\bar{p}_t + \bar{p}_{xxx} + \bar{p}_x + \frac{1}{2}z\bar{p}_x)$. We proceed to the

**Claim 2.** $w^{n'} \to \bar{w}$ in $\mathcal{D}'(Q)$.

Since $p^{n'} \to \bar{p}$ in $\mathcal{D}'(Q)$, we only have to check that $z^{n'}p_x^{n'} \to z\bar{p}_x$ in $\mathcal{D}'(Q)$, but this last property is true, since $z^{n'} \to z$ in $\mathrm{L}^\infty(Q)$ and $p_x^{n'} \rightharpoonup \bar{p}_x$ in $\mathrm{L}^2(\Omega)$ for any $\Omega \subset\subset Q$.

Since $\|w^n\|_{\mathrm{L}^2(Q)} \le Const.$, we infer that $\bar{w} \in \mathrm{L}^2(Q)$ and that

$$w^{n'} \rightharpoonup \bar{w} \text{ in } \mathrm{L}^2(Q).$$

It remains to show that $\bar{w} = w$. We may write $w = L^*p$, where $p \in H$ is the unique function in $H$ fulfilling

$$\iint L^*p\, L^*q\,\mathrm{d}x\mathrm{d}t = \iint fq\,\mathrm{d}x\mathrm{d}t = -\iint (\psi'v)q \qquad \forall q \in \mathcal{Z}.$$

**Claim 3.** $\bar{p} = p$.

Since in $\mathrm{L}^2(Q)$ $L_{n'}^*p^{n'} = w^{n'} \rightharpoonup \bar{w} = L^*\bar{p}$, $L_{n'}^*q \to L^*q$ (for $z^n \to z$ in $\mathrm{L}^\infty(Q)$), and $f^n \to f$ (according to Claim 1), we can pass to the limit in (83) and we obtain

$$\iint L^*\bar{p}L^*q\,\mathrm{d}x\mathrm{d}t = \iint fq\,\mathrm{d}x\mathrm{d}t \qquad \forall q \in \mathcal{Z}. \tag{87}$$

We must check that $\bar{p} \in H$. The condition (42) is clearly fulfilled. The boundary conditions $\bar{p}_{|x=\pm L} = (\bar{p}_x)_{|x=-L} = 0$ follow from (84)–(86). It remains to show that $(\bar{p}_{xx})_{|x=-L} = 0$. Let $\varepsilon \in (0,T)$. Then

$$L^*p^{n'} = L_{n'}^*p^{n'} + \frac{1}{2}(z^{n'} - z)p_x^{n'} \rightharpoonup L^*\bar{p} \quad \text{in } \mathrm{L}^2\big((\varepsilon, T-\varepsilon)\times(-L,L)\big),$$

since $L_{n'}^*p^{n'} \rightharpoonup L^*\bar{p}$ in $\mathrm{L}^2\big((\varepsilon,T-\varepsilon)\times(-L,L)\big)$, $z^{n'} \to z$ in $\mathrm{L}^\infty(Q)$ and $\|p_x^{n'}\|_{\mathrm{L}^2\left((\varepsilon,T-\varepsilon)\times(-L,L)\right)} \le$ Const. Therefore $0 = p_{xx|x=-L}^{n'} \rightharpoonup (\bar{p}_{xx})_{|x=-L}$ in $\mathrm{H}^{-1}(\varepsilon, T-\varepsilon)$. Thus we have proved that $\bar{p} \in H$, and it follows from (87) that $\bar{p} = p$, hence $\bar{w} = w$. A standard argument shows that the convergences in (84)–(86) hold for the whole sequence $(p^n)$, and that $w^n \rightharpoonup w$ in $\mathrm{L}^2(Q)$. Finally,

$$\|w^n\|_{\mathrm{L}^2(Q)}^2 = \iint_Q |L_n^*p^n|^2 = \iint_Q f^np^n \to \iint fp = \|w\|_{\mathrm{L}^2(Q)}^2$$

thanks to (82), (84), and the fact that Supp $f^n \subset \left[\frac{T}{3}, \frac{2T}{3}\right]\times(-L,L)$. Therefore, $w^n \to w$ and $u^n \to u$ in $\mathrm{L}^2(Q)$. The proof of Proposition 2.10 is complete. □

We are in a position to apply the fixed point argument. Let the nominal trajectory $\bar{y}$ of the KdV equation be as in the statement of Theorem 2.1. We search for the function $y \in \bar{V}$ in the form $y = \bar{y} + u$. It follows that $u$ has to satisfy the system

$$\begin{cases} u_t + u_{xxx} + u_x + \dfrac{1}{2}((u+2\bar{y})u)_x = 0, & 0 < x < L,\ 0 < t < T, \\ u_{|x=L} = u_{x|x=L} = 0, & \\ u_{|t=0} = u_0,\ u_{|t=T} = 0, & \end{cases} \tag{88}$$

where $u_0 := y_0 - \bar{y}(0) \in \mathrm{H}^3(0,L)$ is the initial difference between $y$ and $\bar{y}$. We extend $u_0$ as a function in $\mathrm{H}^3(-L,L)$ by using the prolongation operator $\gamma$ defined in (80). Clearly, $\bar{y} \in \bar{V}$. Take $R > 2\|\bar{y}\|_{\bar{V}}$, and pick any $\zeta \in \bar{V}$ with $\|\zeta\|_{\bar{V}} \leq R$. Set $z = \zeta + 2\bar{y}$ and $u = \Lambda(z) \in \bar{V}$. Then $\|z\|_{\bar{V}} \leq 2R$ and then, by (14), (75),

$$\begin{aligned}\|u\|_{\bar{V}} &\leq \|\psi(t)v\|_{\bar{V}} + \|w\|_{\bar{V}}\\ &\leq c(\|z\|_{\bar{V}}) \cdot \|u_0\|_{\mathrm{H}^3(-L,L)}\\ &\leq K\, c(2R)\|u_0\|_{\mathrm{H}^3(0,L)}.\end{aligned}$$

Therefore, if $\|u_0\|_{\mathrm{H}^3(0,L)}$ is small enough, the closed ball $B_R(0) = \{\zeta \in \bar{V},\ \|\zeta\|_{\bar{V}} \leq R\}$ is mapped into itself by the application $\zeta \mapsto u$. Since this application is weakly sequentially continuous according to Proposition 2.10, we infer from the (second) Schauder fixed point theorem [26] (Cor. 9.7) that this map has a fixed point $\zeta = u$. Hence, (88) is satisfied. We now turn to the regularity of the solution.

## 2.5. Smoothness of the trajectory

The solution $y$ to (9) is decomposed as

$$y(t,x) = \bar{y}(t,x) + u(t,x) = \bar{y}(t,x) + \psi(t)v(t,x) + w(t,x).$$

We infer from the assumptions of Theorem 2.1 and from Lemma 2.2 that $\bar{y},\, v \in \mathrm{C}^0([0,T],\mathrm{H}^3(0,L)) \cap C^1([0,T], \mathrm{L}^2(0,L))$. By construction $w \in \bar{V}$, so $w \in \mathrm{C}^0([0,T],\mathrm{H}^2(0,L)) \subset \mathrm{C}^0([0,T]\times[0,L])$ and $w_x, ww_x \in \mathrm{L}^\infty(0,T, \mathrm{L}^2(0,L))$. On the other hand, we infer from (77) (with $\omega = w_t$) that $w_t \in \mathrm{L}^\infty(0,T,\mathrm{L}^2(0,L))$. Therefore, using (21), $w_{xxx} \in \mathrm{L}^\infty(0,T,\mathrm{L}^2(0,L))$ and $w \in \mathrm{L}^\infty(0,T,\mathrm{H}^3(0,L))$. Since $w \in \mathrm{C}^0([0,T],\mathrm{H}^2(0,L))$, it follows from [12] (Lem. 8.1, Chap. 3) that $w \in C_s([0,T],\mathrm{H}^3(0,L))$, *i.e.*, the map $t \mapsto (g, w(t))_{\mathrm{H}^3(0,L)}$ is continuous on $[0,T]$ for any $g \in \mathrm{H}^3(0,L)$. This implies that $w \in \mathrm{C}^0([0,T],\mathrm{H}^s(0,L))$ for any $1 \leq s < 3$. The same property holds true for $y$. In particular, the trace $[y - \frac{1}{6}y^2 + y_{xx}]_{|x=0} \in \mathrm{C}^0([0,T])$. The proof of Theorem 2.1 is achieved.

## 2.6. Uniqueness of the solution to the control problem

This section is devoted to the proof of the following

**Proposition 2.11.** *Let $\overline{y}$, $r_0$, $y_0$ be as in the statement of Theorem 2.1, and let $y_1, y_2$ be two solutions of (5) associated with the same control input $h$. Assume further that $\|\overline{y}_{|x=0}\|_{\mathrm{L}^\infty(0,T)} < \frac{3}{5}$. Then $y_1 \equiv y_2$.*

*Proof.* Form $\varepsilon = y_1 - y_2$. Then $\varepsilon \in \mathrm{L}^2(0,T,\mathrm{H}^3(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$ and $\varepsilon$ solves

$$\begin{cases}\varepsilon_t + \varepsilon_{xxx} + \varepsilon_x + \frac{1}{2}((y_1+y_2)\varepsilon)_x = 0,\\ \left[\varepsilon - \frac{1}{6}(y_1+y_2)\varepsilon + \varepsilon_{xx}\right]_{|x=0} = 0,\\ \varepsilon_{|x=L} = \varepsilon_{x|x=L} = 0,\\ \varepsilon_{|t=0} = 0.\end{cases}$$

Scaling by $2\varepsilon$ and integrating by parts, we readily obtain

$$\begin{aligned}\int_0^L \varepsilon^2(t,x)\mathrm{d}x - \int_0^L \varepsilon^2(0,x)\mathrm{d}x &= -\frac{1}{2}\int_0^t\!\!\int_0^L (y_1+y_2)_x\varepsilon^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad - \int_0^t \left[2\varepsilon\varepsilon_{xx} - \varepsilon_x^2 + \varepsilon^2 + \frac{1}{2}(y_1+y_2)\varepsilon^2\right]_0^L \mathrm{d}\tau \\ &= -\frac{1}{2}\int_0^t\!\!\int_0^L (y_1+y_2)_x\varepsilon^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t \left(-\varepsilon^2 + \frac{5}{6}(y_1+y_2)\varepsilon^2 - \varepsilon_x^2\right)_{|x=0} \mathrm{d}\tau \\ &\leq -\frac{1}{2}\int_0^t\!\!\int_0^L (y_1+y_2)_x\varepsilon^2\,\mathrm{d}x\mathrm{d}\tau\end{aligned}$$

provided that $y_i(t,0) \leq \frac{3}{5}$ for $i=1,2$ and for any $t\in(0,T)$. (This is the case *e.g.* if $\|\overline{y}_{|x=0}\|_{\mathrm{L}^\infty(0,T)} < \frac{3}{5}$ and $r_0$ is small enough.) Since $y_1, y_2 \in \mathrm{C}^0([0,T],\mathrm{H}^2(0,L))$, we obtain from Gronwall Lemma that

$$\|\varepsilon(t,.)\|^2_{\mathrm{L}^2(0,L)} \leq \|\varepsilon(0,.)\|^2_{\mathrm{L}^2(0,L)} \exp\left(\frac{1}{2}\|(y_1+y_2)_x\|_{\mathrm{L}^\infty\left((0,T)\times(0,L)\right)} t\right) = 0.$$

Thus $y_1 \equiv y_2$. □

Theorem 1.1 follows at once from Theorem 2.1 and from Proposition 2.11.

## 3. Global uncontrollability of the KdV equation

We prove in this section that the KdV equation with a left boundary control is not *globally* controllable in finite time. This result rests on the well known fact that (large) solutions of the KdV equation behave like solutions of the Hopf equation

$$y_t + yy_x = 0.$$

A similar (negative) result has been derived in [10] for Burgers equation.

**Theorem 3.1.** *Let $L,T>0$ and $n>6$. Let $y_T \in \mathrm{H}^3(0,L)$ and $a_T \in (0,L)$ be such that*

$$y_T(a_T) = 0 \text{ and } y_T(x) < 0 \quad \forall x \in (a_T, L).$$

*Let $y \in \mathrm{C}^0([0,T],\mathrm{H}^3(0,L)) \cap \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$ be a function satisfying*

$$\begin{cases} y_t + y_x + y_{xxx} + y\,y_x = 0, \quad 0<x<L,\ 0<t<T, \\ y_{|x=L} = y_{x|x=L} = 0, \\ y(0) = 0, \\ y(T) = y_T. \end{cases} \tag{89}$$

*Let the time $T_0 \in [0,T]$ and the function $a : [T_0,T] \to [0,L]$ be defined by the following properties*

(1) *$y(t,x) < 0$ for $a(t) < x < L$, $T_0 \leq t \leq T$;*
(2) *$a(t)y(t,a(t)) = 0$ for $T_0 \leq t \leq T$;*
(3) *$a(t) < L$ for $T_0 < t \leq T$ and $a(T_0) = L$ if $T_0 > 0$.*

*Assume further that $a \in W^{1,\infty}(T_0,T)$. Then we have the estimate*

$$\int_{a_T}^{L} x^{2n} y_T^2(x)\,\mathrm{d}x \le K$$

*for some positive constant $K = K(L,T,n)$.*

*Proof.* Let $y_T$, $y$ and $a = a(t)$ be as in the statement of Theorem 3.1. Set $y^- = \sup(0,-y)$. Since $y \in \mathrm{H}^1(0,T,\mathrm{H}^1(0,L))$, $y^- \in \mathrm{C}^0([0,T],\mathrm{H}^1(0,L)) \cap \mathrm{C}^0([0,L],\mathrm{H}^1(0,T))$, with $y_x^- = -y_x\chi_{\{y<0\}}$ a.e., $y_t^- = -y_t\chi_{\{y<0\}}$ a.e., where $\chi_{\{y<0\}}(t,x) = 1$ if $y(t,x) < 0$, 0 otherwise. It is then clear that

$$y^- y = -(y^-)^2, \quad y^- y_x = -y^- y_x^-, \quad y^- y_t = -y^- y_t^-, \quad y_x^- y_x = -(y_x^-)^2. \tag{90}$$

Let $h(x) = x^n$. We multiply each term in the first equation of (89) by $h(x)y(t,x)$ and integrate by part. We first notice that for a.e. $t \in (T_0,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{L} h(x)y^2(t,x)\mathrm{d}x &= 2\int_{a(t)}^{L} h(x)y(t,x)y_t(t,x)\,\mathrm{d}x - a'(t)h(a(t))y^2(t,a(t))\\ &= 2\int_{a(t)}^{L} h\,y\,y_t\mathrm{d}x \qquad (\text{since } h(a(t))y(t,a(t)) = 0).\end{aligned}$$

On the other hand, for any $t \in [0,T]$,

$$\int_a^L hyy_x\,\mathrm{d}x = -\int_a^L h\frac{y^2}{2}\,\mathrm{d}x, \qquad \int_a^L hy^2y_x\,\mathrm{d}x = -\int_a^L h_x\frac{y^3}{3}\,\mathrm{d}x$$

and finally

$$\begin{aligned}\int_a^L hyy_{xxx}\,\mathrm{d}x &= -\int_a^L (h_xy + hy_x)y_{xx}\,\mathrm{d}x \qquad (\text{since } h(a(t))y(t,a(t)) = 0 = y(t,L))\\ &= \int_a^L (h_{xx}y + h_xy_x)y_x\,\mathrm{d}x + \int_a^L h_x\frac{y_x^2}{2}\,\mathrm{d}x - \left[h\frac{y_x^2}{2}\right]_a^L\\ &= -\int_a^L h_{xxx}\frac{y^2}{2}\mathrm{d}x + \frac{3}{2}\int_a^L h_xy_x^2\,\mathrm{d}x + h(a)\frac{y_x^2(a)}{2} \qquad (\text{since } y_x(L) = 0).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^L hy^2\,\mathrm{d}x &= \int_a^L h_x\frac{y^2}{2}\,\mathrm{d}x + \int_a^L h_x\frac{y^3}{3}\,\mathrm{d}x + \int_a^L h_{xxx}\frac{y^2}{2}\,\mathrm{d}x - \frac{3}{2}\int_a^L h_xy_x^2\,\mathrm{d}x - \frac{h(a)}{2}y_x^2(a)\\ &\le \frac{1}{2}\int_a^L (h_x + h_{xxx})y^2\,\mathrm{d}x - \frac{1}{3}\int_a^L h_x|y|^3\,\mathrm{d}x.\end{aligned}$$

An application of Hölder inequality yields

$$\begin{aligned}\int_a^L h_xy^2\,\mathrm{d}x &\le \left(\int_a^L h_x\,\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_a^L h_x|y|^3\,\mathrm{d}x\right)^{\frac{2}{3}}\\ &\le L^{\frac{n}{3}}\left(\int_a^L h_x|y|^3\,\mathrm{d}x\right)^{\frac{2}{3}}\end{aligned}$$

and

$$\int_a^L h_{xxx} y^2 \,\mathrm{d}x \le \left(\int_a^L \frac{h_{xxx}^3}{h_x^2}\,\mathrm{d}x\right)^{\frac{1}{3}} \left(\int_a^L h_x |y|^3 \,\mathrm{d}x\right)^{\frac{2}{3}}$$
$$\le \frac{n(n-1)(n-2)}{n^{\frac{2}{3}}} \left(\int_0^L x^{n-7}\,\mathrm{d}x\right)^{\frac{1}{3}} \left(\int_a^L h_x|y|^3\,\mathrm{d}x\right)^{\frac{2}{3}}.$$

Therefore, for some constant $C = C(L,n) > 0$,

$$\frac{1}{2}\int_a^L (h_x + h_{xxx})y^2 \,\mathrm{d}x \le \frac{1}{2}(L^{\frac{n}{3}} + n^{\frac{1}{3}}(n-1)(n-2)\left(\frac{L^{n-6}}{n-6}\right)^{\frac{1}{3}} \left(\int_a^L h_x|y|^3\,\mathrm{d}x\right)^{\frac{2}{3}}$$
$$\le C + \frac{1}{3}\int_a^L h_x|y|^3\,\mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^L hy^2\,\mathrm{d}x \le 2C.$$

Integrating over $(T_0, T)$ and using the fact that $a(T_0) = L$ if $T_0 > 0$ (hence $\int_{a(T_0)}^L h(x)y(T_0,x)\mathrm{d}x = 0$), we arrive to

$$\int_{a_T}^L h^2(x)y^2(T,x)\,\mathrm{d}x \le 2CT + \int_0^L h^2(x)y^2(0,x)\,\mathrm{d}x = 2CT. \qquad \square$$

**Remark 3.2.** The author conjectures that in Theorem 3.1 the technical assumption $a \in W^{1,\infty}(T_0,T)$ should be dropped.

## Appendix A: Derivation of the Korteweg-de Vries equation in Lagrangian coordinates

We start from a *normalised* form of Boussinesq system borrowed from [25] (p. 446):

$$\eta_t + \{(1+\alpha\eta)u\}_x = 0, \tag{91}$$

$$u_t + \alpha u u_x + \eta_x - \frac{1}{3}\beta u_{xxt} = 0. \tag{92}$$

Here, $x$ is the Eulerian coordinate, $t$ is the elapsed time, $\eta$ is the deflection from rest position, $u$ is (up to a factor) the value averaged over the depth of the horizontal velocity. The small parameters $\alpha$ and $\beta$ are defined as $\alpha = a/h_0$, $\beta = h_0^2/l^2$, where $h_0$ is the height of the surface fluid at rest, and $a$ (resp., $l$) is a typical amplitude (resp., wavelength). Because of the normalization adopted here, the velocity of a fluid particle is $\alpha u$, not $u$. Indeed, if we introduce the (normalised) height $h = 1 + \alpha\eta$, then (91) may be rewritten

$$h_t + \alpha(hu)_x = 0. \tag{93}$$

Integrating on some interval $(a,b)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b h(t,x)\,\mathrm{d}x = \alpha u(a)h(a) - \alpha u(b)h(b),$$

hence $\alpha u(a)$ (resp., $\alpha u(b)$) is the fluid velocity at $x = a$ (resp., $x = b$). Proceeding as in [13] (Chap. 1) (see also [1], (p. 39)), we express the Boussinesq system in mass Lagrangian coordinates, keeping only the first order terms in $\alpha$, $\beta$. Let $\xi \in [0, L]$ denote the Lagrangian coordinate. Then the Eulerian coordinate $x = x(\tau, \xi)$, which stands for the position at time $t = \tau$ of the fluid particle issued from $\xi$ at $t = 0$, is obtained by integrating the characteristic system

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha\, u(t, x), \\ x_{|t=0} = \xi. \end{array} \right. \tag{94}$$

Let $\psi : (\tau, \xi) \mapsto (t, x) = (\tau, x(\tau, \xi))$. To express the Boussinesq system in the new variables $\tau, \xi$, we first need to compute the derivatives of $\xi$ with respect to $t$, $x$. The Jacobian matrix of $\psi$ reads

$$J = J(\tau, \xi) = \begin{pmatrix} 1 & 0 \\ \alpha\, u(t, x) & j \end{pmatrix}$$

where $j = \frac{\partial x}{\partial \xi}$. Observe that $j(0, \xi) = 1$ and that

$$\frac{\partial j}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \tau}\right) = \frac{\partial\, [\alpha u(\tau, x(\tau, \xi))]}{\partial \xi} = \alpha u_x(\tau, x)\frac{\partial x}{\partial \xi} = \alpha u_x(\tau, x) j,$$

hence

$$\frac{\partial}{\partial \tau}(\ln |j|) = \alpha u_x(\tau, x)\cdot$$

On the other hand

$$\begin{aligned} \frac{\partial}{\partial \tau}[h(\tau, x(\tau, \xi))] &= h_t(\tau, x) + h_x(\tau, x)\alpha u(\tau, x) \\ &= -\alpha h u_x(\tau, x) \text{ by (93)}, \end{aligned}$$

hence

$$\frac{\partial [\ln |h|]}{\partial \tau} = -\alpha u_x(\tau, x) = -\frac{\partial [\ln |j|]}{\partial \tau}\cdot$$

Therefore, setting $h_0 = h_{|t=0}$, we obtain

$$j(\tau, \xi) = j(0, \xi)\frac{h(0, \xi)}{h(\tau, x(\tau, \xi))} = \frac{h_0(\xi)}{h(t, x)}\cdot$$

Thus

$$J = \begin{pmatrix} 1 & 0 \\ \alpha u(t, x) & \frac{h_0(\xi)}{h(t,x)} \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{h(t,x)}{h_0(\xi)}\alpha u(t, x) & \frac{h(t,x)}{h_0(\xi)} \end{pmatrix}.$$

The mass Lagrangian coordinate $\zeta$ is defined as

$$\zeta = \int_0^\xi h_0(\eta)\, \mathrm{d}\eta.$$

Clearly,

$$\frac{\partial \zeta}{\partial t} = h_0(\xi)\frac{\partial \xi}{\partial t} = -\alpha u(t, x) h(t, x)$$

and

$$\frac{\partial \zeta}{\partial x} = h_0(\xi)\frac{\partial \xi}{\partial x} = h(t, x).$$

Therefore

$$\begin{aligned}\partial_t &= \partial_\tau + \frac{\partial\zeta}{\partial t}\partial_\zeta = \partial_\tau - \alpha uh\partial_\zeta,\\ \partial_x &= \frac{\partial\zeta}{\partial x}\partial_\zeta = h\partial_\zeta.\end{aligned} \tag{95}$$

The first equation in Boussinesq system becomes

$$0 = \alpha^{-1}h_t + h_x u + hu_x = (\alpha^{-1}h_\tau - uhh_\zeta) + hh_\zeta u + h^2 u_\zeta,$$

which yields

$$\eta_\tau + (1+2\alpha\eta)u_\zeta + O(\alpha^2) = 0. \tag{96}$$

We proceed to the second equation in Boussinesq system. We have to express $u_{xxt}$ in the new coordinates $\tau, \zeta$. A direct application of (95) yields $u_x = hu_\zeta$, $u_{xx} = h_x u_\zeta + h^2 u_{\zeta\zeta}$ and

$$u_{xxt} = h_{xt}u_\zeta + h_x(u_{\zeta\tau} - \alpha uhu_{\zeta\zeta}) + 2hh_t u_{\zeta\zeta} + h^2(u_{\zeta\zeta\tau} - \alpha uhu_{\zeta\zeta\zeta}).$$

Since

$$h_{xt} = h_\zeta[h_\tau - \alpha uhh_\zeta] + [h_{\zeta\tau} - \alpha uhh_{\zeta\zeta}]h,$$

we obtain

$$u_{xxt} = h^2(u_{\zeta\zeta\tau} - \alpha uhu_{\zeta\zeta\zeta}) + 2h(h_\tau - \alpha uhh_\zeta)u_{\zeta\zeta} + hh_\zeta(u_{\zeta\tau} - \alpha uhu_{\zeta\zeta}) + [h_\zeta(h_\tau - \alpha uhh_\zeta) + h(h_{\zeta\tau} - \alpha uhh_{\zeta\zeta})]u_\zeta. \tag{97}$$

Combining (92), (95) and (97), we obtain

$$\begin{aligned}u_\tau - \alpha uhu_\zeta + \alpha uhu_\zeta + h\eta_\zeta - \frac{1}{3}\beta\big\{h^2(u_{\zeta\zeta\tau} - \alpha uhu_{\zeta\zeta\zeta}) + 2h(\alpha\eta_\tau - \alpha^2 uh\eta_\zeta)u\\ + h\alpha\eta_\zeta(u_{\zeta\tau} - \alpha uhu_{\zeta\zeta}) + [\alpha\eta_\zeta(\alpha\eta_\tau - \alpha^2 uh\eta_\zeta) + h(\alpha\eta_{\zeta\tau} - \alpha^2 uh\eta_{\zeta\zeta})]u_\zeta\big\} = 0,\end{aligned}$$

hence

$$u_\tau + (1+\alpha\eta)\eta_\zeta - \frac{1}{3}\beta u_{\zeta\zeta\tau} + O(\alpha\beta, \alpha^2) = 0. \tag{98}$$

We infer from (96) and (98) that the Boussinesq system in mass Lagrangian coordinates is (to the first order)

$$\begin{cases}\eta_\tau + (1+2\alpha\eta)u_\zeta = 0,\\ u_\tau + (1+\alpha\eta)\eta_\zeta - \frac{1}{3}\beta u_{\zeta\zeta\tau} = 0.\end{cases} \tag{99}$$

In the last step we derive the Korteweg-de Vries equation (in mass Lagrangian coordinates) by specializing to a wave moving to the right. Following [25], we look for a solution $u$ in the form

$$u = \eta + \alpha A + \beta B + O(\alpha^2 + \beta^2), \tag{100}$$

where $A$ and $B$ are functions of $\eta$ and of its $\zeta$ derivatives. Then (99) becomes

$$\begin{cases}\eta_\tau + \eta_\zeta + \alpha(2\eta\eta_\zeta + A_\zeta) + \beta B_\zeta & + O(\alpha^2 + \beta^2) = 0,\\ \eta_\tau + \eta_\zeta + \alpha(\eta\eta_\zeta + A_\tau) + \beta\left(B_\tau - \frac{1}{3}\eta_{\zeta\zeta\tau}\right) & + O(\alpha^2 + \beta^2) = 0.\end{cases}$$

The two equations are consistent provided that

$$2\eta\eta_\zeta + A_\zeta = \eta\eta_\zeta + A_\tau + O(\alpha,\beta), \quad B_\zeta = B_\tau - \frac{1}{3}\eta_{\zeta\zeta\tau} + O(\alpha,\beta). \tag{101}$$

Then $A_\tau = -A_\zeta + O(\alpha,\beta)$ and $B_\tau = -B_\zeta + O(\alpha,\beta)$ (since $\eta_\tau = -\eta_\zeta + O(\alpha,\beta)$), so (101) may be rewritten as

$$\begin{cases} A_\zeta = -\frac{1}{2}\eta\eta_\zeta + O(\alpha,\beta), \\ B_\zeta = \frac{1}{6}\eta_{\zeta\zeta\zeta} + O(\alpha,\beta). \end{cases}$$

We obtain at once

$$A = -\frac{1}{4}\eta^2 \text{ and } B = \frac{1}{6}\eta_{\zeta\zeta}.$$

(Notice that a first order error in $A$ or $B$ gives rise to a second order error in (100), so that the constants of integration may be omitted in the expressions of $A$ and $B$.) We conclude that the Korteweg-de Vries system in mass Lagrangian coordinates reads

$$\begin{cases} \eta_\tau + \eta_\zeta + \frac{3}{2}\alpha\eta\eta_\zeta + \frac{1}{6}\beta\eta_{\zeta\zeta\zeta} = 0, \\ u = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{6}\beta\eta_{\zeta\zeta}. \end{cases}$$

The system is very similar to the KdV system in Eulerian coordinates (compare [25], Eq. (13.102)), the only difference is in the coefficient in front of $\beta\eta_{\zeta\zeta}$ in the second equation (namely, 1/6 instead of 1/3). We may express these equations without the small parameters $\alpha,\beta$ in using the dimensioned variables

$$\bar\zeta = l\zeta, \quad \bar\eta = a\eta, \ \bar\tau = \frac{l}{c_0}\tau \ \text{ and } \ \bar u = \frac{ga}{c_0}u,$$

where $c_0 = \sqrt{gh_0}$ is the sound speed in the fluid. We obtain the system

$$\begin{cases} \bar\eta_{\bar\tau} + c_0\bar\eta_{\bar\zeta} + \frac{3}{2}\frac{c_0}{h_0}\bar\eta\bar\eta_{\bar\zeta} + \frac{1}{6}c_0h_0^2\bar\eta_{\bar\zeta\bar\zeta\bar\zeta} = 0, \\ \bar u = \frac{g}{c_0}\left(\bar\eta - \frac{1}{4}\frac{\bar\eta^2}{h_0} + \frac{1}{6}h_0^2\bar\eta_{\bar\zeta\bar\zeta}\right). \end{cases}$$

Finally, setting

$$t = \frac{\sqrt{6}c_0}{h_0}\bar\tau, \quad x = \frac{\sqrt{6}}{h_0}\bar\zeta, \quad y = \frac{3}{2h_0}\bar\eta, \quad v = \frac{3}{2}\frac{c_0}{gh_0}\bar u,$$

we obtain at once the system

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \\ v = y - \frac{1}{6}y^2 + y_{xx}. \end{cases}$$

## Appendix B: Uncontrollability of the linearized KdV equation

We have seen in Section 2 that the linear KdV equation is null-controllable when the boundary control is applied to the left endpoint of the domain. Here, we show that the linear KdV equation with a left boundary control fails to be exactly controllable.

The non-homogeneous initial-boundary-value problem for the KdV equation has been extensively investigated in [4]. As far as the linear KdV equation is concerned, we infer from [4] (Prop. 2.6) that the following initial-boundary-value problem

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & 0 < x < L,\ 0 < t < T, \\ y_{|x=0} = h, \\ y_{|x=L} = y_{x|x=L} = 0, \\ y_{|t=0} = y_0 \end{cases}$$

is well-posed in $\mathrm{L}^2(0,L)$ provided that the boundary control $h \in \mathrm{H}^{\frac{1}{3}}(0,T)$. (Notice that this result is sharp.) We are now in a position to state the negative result mentioned above.

**Theorem 3.3.** *For any $L, T > 0$, the system*

$$\begin{cases} y_t + y_x + y_{xxx} = 0, & 0 < x < L,\ 0 < t < T, \\ y_{|x=0} = h, \\ y_{|x=L} = y_{x|x=L} = 0, \\ y_{|t=0} = 0 \end{cases} \tag{102}$$

*with control input $h \in \mathrm{H}^{\frac{1}{3}}(0,T)$ fails to be exactly controllable in $\mathrm{L}^2(0,L)$.*

*Proof.* We argue by contradiction. Assume that (102) is exactly controllable in $\mathrm{L}^2(0,L)$. Then, by a classical argument relying on the closed graph theorem we readily infer the existence of a continuous operator $\Lambda:$ $\mathrm{L}^2(0,L) \to \mathrm{H}^{\frac{1}{3}}(0,L)$ such that the solution $y$ to (102) associated with the boundary control $h = \Lambda(y_T)$ fulfils $y_{|t=T} = y_T$. Let $u \in \mathrm{C}^0([0,T], \mathrm{H}^3(0,L))$ denote a solution of the backward problem

$$\begin{cases} u_t + u_{xxx} + u_x = 0, & 0 < x < L,\ 0 < t < T, \\ u_{|x=0} = u_{x|x=0} = u_{|x=L} = 0, \\ u_{|t=T} = u_T. \end{cases}$$

Integrating by parts in $\displaystyle\int_0^T \int_0^L u(y_t + y_x + y_{xxx})\,\mathrm{d}x\mathrm{d}t = 0$ yields

$$\int_0^L y(T,x)u_T(x)\,\mathrm{d}x = \int_0^T u_{xx}(t,0)h(t)\,\mathrm{d}t. \tag{103}$$

Picking any function $u_T \in \mathrm{H}^3(0,L)$ with $u_T(0) = u_T'(0) = u_T(L) = 0$ and $y_T = u_T$, we infer from (103) that

$$\|u_T\|_{\mathrm{L}^2(0,L)} \leq \|\Lambda\| \cdot \|u_{xx}(\cdot,0)\|_{\mathrm{L}^2(0,T)}. \tag{104}$$

(The fact that $u_{xx}(.,0) \in \mathrm{L}^2(0,T)$ is obvious, for $u \in \mathrm{C}^0([0,T], \mathrm{H}^3(0,L))$.) Changing $x$ in $-x$ and $t$ in $T-t$, we see that (104) is equivalent to the estimate

$$\|v_0\|_{\mathrm{L}^2(-L,0)} \leq \|\Lambda\| \cdot \|v_{xx}(\cdot,0)\|_{\mathrm{L}^2(0,T)} \tag{105}$$

for any $v_0 \in D(A)$, where $A$ denotes the operator $A\,v = -v_x - v_{xxx}$ with domain

$$D(A) = \left\{v_0 \in \mathrm{H}^3(-L,0),\ v_0(0) = v_0'(0) = v_0(-L) = 0\right\}$$

and $v = \mathrm{e}^{tA}v_0$. (We denote by $\mathrm{e}^{tA}$ the semigroup associated with $A$.) We now prove that (105) does not hold in considering a sequence of exponential solutions; *i.e.*, of the form $v(t,x) = \mathrm{e}^{\lambda t}v_0(x)$, with $Av_0 = \lambda v_0$ and $\lambda \in \mathbb{R}$. We first have to determine the eigenvectors of $A$; *i.e.*, the functions $v_0 \in D(A)$ fulfilling

$$v_{0xxx} + v_{0x} + \lambda v_0 = 0.$$

To solve the characteristic equation

$$r^3 + r + \lambda = 0, \tag{106}$$

it is convenient to set $\lambda = -s - s^3$. In what follows, $s \in \mathbb{R}$. Then the roots of (106) are

$$r_1 = s,\ r_2 = \mu := \frac{1}{2}\left(-s + i\sqrt{3s^2+4}\right) \text{ and } r_3 = \bar{\mu} = \frac{1}{2}\left(-s - i\sqrt{3s^2+4}\right).$$

Therefore, $v_0(x) = a\mathrm{e}^{sx} + b\mathrm{e}^{\mu x} + c\mathrm{e}^{\bar{\mu}x}$ for some complex coefficients $a, b, c$. The boundary conditions $v_0(0) = v_0'(0) = 0$ yield the system

$$\begin{cases} a + b + c &= 0, \\ sa + \mu\, b + \bar{\mu}\, c &= 0 \end{cases}$$

which gives $b = \rho a$, $c = \bar{\rho}a$ for

$$\rho := \frac{s - \bar{\mu}}{\bar{\mu} - \mu} = -\frac{1}{2} + i\frac{3s}{2\sqrt{3s^2+4}}\cdot$$

Finally, the condition $v_0(-L) = 0$ gives $a(\mathrm{e}^{-sL} + \rho\mathrm{e}^{-\mu L} + \bar{\rho}\mathrm{e}^{-\bar{\mu}L}) = 0$, or (since $a \neq 0$)

$$\mathrm{e}^{-sL} = -2\mathrm{Re}\,(\rho\mathrm{e}^{-\mu L}) = -2\mathrm{Re}\left(\left(-\frac{1}{2} + i\frac{3s}{2\sqrt{3s^2+4}}\right)\mathrm{e}^{-\frac{L}{2}(-s+i\sqrt{3s^2+4})}\right),$$

or

$$-\frac{1}{2}\mathrm{e}^{-\frac{3}{2}sL} = \mathrm{Re}\left(\left(-\frac{1}{2} + i\frac{3s}{2\sqrt{3s^2+4}}\right)\mathrm{e}^{-\frac{iL}{2}\sqrt{3s^2+4}}\right). \tag{107}$$

Easy calculations give, as $s \to +\infty$,

$$\begin{aligned}\mathrm{Re}\left(\left(-\frac{1}{2} + i\frac{3s}{2\sqrt{3s^2+4}}\right)\mathrm{e}^{-\frac{iL}{2}\sqrt{3s^2+4}}\right) &= \mathrm{Re}\left((\mathrm{e}^{i\frac{2\pi}{3}} + O(s^{-2}))\mathrm{e}^{-i\frac{\sqrt{3}}{2}Ls}(1 + O(s^{-1}))\right) \\ &= \cos\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}Ls\right) + O(s^{-1}).\end{aligned}$$

It is then clear that there exists a sequence $s_n \nearrow +\infty$ such that (107) holds true, and such that $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}Ls_n \sim \frac{\pi}{2} + n\pi$ as $n \to +\infty$. Without any loss of generality, we may assume that $a = 1$. We now estimate $\|(v_{xx})_{|x=0}\|_{\mathrm{L}^2(0,T)}$ and $\|v_0\|_{\mathrm{L}^2(-L,L)}$ as $s = s_n \to +\infty$. We have

$$\int_0^T |v_{xx}(t,0)|^2\,\mathrm{d}t = \int_0^T \left|(s^2 + 2\mathrm{Re}\,(\rho\mu^2))\mathrm{e}^{\lambda t}\right|^2\mathrm{d}t = \left|s^2 + 2\mathrm{Re}\,(\rho\mu^2)\right|^2\frac{1 - \mathrm{e}^{-2(s+s^3)T}}{2(s+s^3)}\cdot$$

Since

$$\mathrm{Re}\,(\rho\mu^2) = -\frac{1}{2}\left(\frac{s^2 - (3s^2+4)}{4}\right) + \frac{3s}{2\sqrt{3s^2+4}}\,\frac{s}{2}\sqrt{3s^2+4} \sim s^2,$$

we infer that

$$\int_0^T |v_{xx}(t,0)|^2\,\mathrm{d}t \sim 2s \qquad \text{as} \qquad s \to +\infty. \tag{108}$$

On the other hand,

$$\begin{aligned}\int_{-L}^{0} |v_0(x)|^2 \,\mathrm{d}x &= \int_{-L}^{0} (\mathrm{e}^{sx} + \rho \mathrm{e}^{\mu x} + \bar{\rho}\mathrm{e}^{\bar{\mu}x})^2 \mathrm{d}x \\ &= \int_{-L}^{0} \left(4\mathrm{Re}\left(\rho \mathrm{e}^{(s+\mu)x}\right) + 2\mathrm{Re}(\rho^2 \mathrm{e}^{2\mu x}) + \mathrm{e}^{2sx} + 2|\rho|^2 \mathrm{e}^{(\mu+\bar{\mu})x}\right) \mathrm{d}x\end{aligned}$$

with $\mathrm{Re}\,(s+\mu) = \frac{s}{2} > 0$, $\mathrm{Re}\,(2\mu) = \mu + \bar{\mu} = -s < 0$. Straightforward computations give

$$\int_{-L}^{0} v_0^2(x)\,\mathrm{d}x = \frac{\mathrm{e}^{sL}}{s}\left[2 - \cos\left(\frac{2\pi}{3} - \sqrt{3s^2+4}\right) + O(s^{-2})\right]. \tag{109}$$

Then it follows from (108) and (109) for $s = s_n$ that (105) fails to be true. $\square$

*Acknowledgements*. The author gratefully acknowledges several discussions with Jerry Bona, Amy Cohen, Jean-Michel Coron and Jean-Claude Saut.

## References

[1] S.N. Antontsev, A.V. Kazhikov and V.N. Monakhov, *Boundary values problems in mechanics of nonhomogeneous fluids*. North-Holland, Amsterdam (1990).
[2] P. Benilan and R. Gariepy, Strong solutions in $L^1$ of degenerate parabolic equations. *J. Differ. Equations* **119** (1995) 473-502.
[3] J.L. Bona, M. Chen and J.-C. Saut, Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory. *J. Nonlinear Sci.* **12** (2002) 283-318.
[4] J.L. Bona, S. Sun and B.-Y. Zhang, A Non-homogeneous Boundary-Value Problem for the Korteweg-de Vries Equation Posed on a Finite Domain. *Commun. Partial Differ. Equations* **28** (2003) 1391-1436.
[5] J.L. Bona and R. Winther, The Korteweg-de Vries equation, posed in a quarter-plane. *SIAM J. Math. Anal.* **14** (1983) 1056-1106.
[6] J.-M. Coron, On the controllability of the 2-D incompressible perfect fluids. *J. Math. Pures Appl.* **75** (1996) 155-188.
[7] J.-M. Coron, Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations, A tribute to J.L. Lions. *ESAIM: COCV* **8** (2002) 513-554.
[8] E. Crépeau, Exact boundary controllability of the Korteweg-de Vries equation around a non-trivial stationary solution. *Int. J. Control* **74** (2001) 1096-1106.
[9] E. Fernández-Cara, Null controllability of the semilinear heat equation. *ESAIM: COCV* **2** (1997) 87-103.
[10] A.V. Fursikov and O.Y. Imanuvilov, On controllability of certain systems simulating a fluid flow, in *Flow Control,* M.D. Gunzburger Ed., Springer-Verlag, New York, *IMA Vol. Math. Appl.* **68** (1995) 149-184.
[11] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equations. *Stud. App. Math.* **8** (1983) 93-128.
[12] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes*, Vol. 1. Dunod, Paris (1968).
[13] G. Mathieu-Girard, *Étude et contrôle des équations de la théorie "Shallow water" en dimension un.* Ph.D. thesis, Université Paul Sabatier, Toulouse III (1998).
[14] S. Micu, On the controllability of the linearized Benjamin-Bona-Mahony equation. *SIAM J. Control Optim.* **39** (2001) 1677-1696.
[15] S. Micu and J.H. Ortega, *On the controllability of a linear coupled system of Korteweg-de Vries equations. Mathematical and numerical aspects of wave propagation* (Santiago de Compostela, 2000). Philadelphia, PA *SIAM* (2000) 1020-1024.
[16] S. Mottelet, Controllability and stabilization of a canal with wave generators. *SIAM J. Control Optim.* **38** (2000) 711-735.
[17] S. Mottelet, Controllability and stabilization of liquid vibration in a container during transportation. (Preprint.)
[18] N. Petit and P. Rouchon, Dynamics and solutions to some control problems for water-tank systems. *IEEE Trans. Automat. Control* **47** (2002) 594-609.
[19] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM: COCV* **2** (1997) 33-55, `http://www.edpsciences.org/cocv`
[20] L. Rosier, Exact boundary controllability for the linear Korteweg-de Vries equation – a numerical study. *ESAIM Proc.* **4** (1998) 255-267, `http://www.edpsciences.org/proc`

[21] L. Rosier, Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line. *SIAM J. Control Optim.* **39** (2000) 331-351.

[22] D.L. Russell and B.-Y. Zhang, Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain. *SIAM J. Control Optim.* **31** (1993) 659-673.

[23] D.L. Russell and B.-Y. Zhang, Exact controllability and stabilizability of the Korteweg-de Vries equation. *Trans. Amer. Math. Soc.* **348** (1996) 3643-3672.

[24] J. Simon, Compact Sets in the Space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (IV)* **CXLVI** (1987) 65-96.

[25] G.B. Whitham, Linear and nonlinear waves. A Wiley-Interscience publication, Wiley, New York (1999) reprint of the 1974 original.

[26] E. Zeidler, Nonlinear functional analysis and its applications, Part 1. Springer-Verlag, New York (1986).

[27] B.-Y. Zhang, Exact boundary controllability of the Korteweg-de Vries equation. *SIAM J. Control Optim.* **37** (1999) 543-565.