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THE TOPOLOGICAL ASYMPTOTIC EXPANSION FOR THE QUASI-STOKES PROBLEM

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Abstract. In this paper, we propose a topological sensitivity analysis for the Quasi-Stokes equations. It consists in an asymptotic expansion of a cost function with respect to the creation of a small hole in the domain. The leading term of this expansion is related to the principal part of the operator. The theoretical part of this work is discussed in both two and three dimensional cases. In the numerical part, we use this approach to optimize the locations of a fixed number of air injectors in an eutrophized lake.

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1. Introduction

The goal of topological optimization is to find an optimal design even with *a priori* poor information on the optimal shape. Unlike the case of classical optimization, the topology of the structure may change during the optimization process, as for example by the inclusion of holes.

Most of the known results in this field concern structural mechanics. In such cases, classical topology optimization involves relaxed formulations or homogenization (see *e.g.* [2, 3, 5, 9, 26, 30]). This method leads to a Neumann condition on the unknown boundary. This boundary condition is quite natural in structural mechanics but this is not the case in fluid dynamics. In that direction, global optimization techniques like genetic algorithms or simulated annealing, have been proposed (see *e.g.* [36]). But these methods are very slow and can hardly be applied to industrial problems.

The recently introduced notion of topological sensitivity gives new perspectives on shape optimization. It provides an asymptotic expansion of a cost function with respect to the creation of a small hole in the domain. To present the basic idea, we consider Ω a domain of $\mathbb{R}^N$, $N = 2, 3$ and $j(\Omega) = J(\Omega, u_\Omega)$ a cost function to be minimized, where u_Ω is the solution to a given PDE problem defined in Ω. For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \backslash \overline{(x_0 + \varepsilon\omega)}$ be the domain obtained by removing a small part $\overline{(x_0 + \varepsilon\omega)}$ from Ω, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^N$ is a fixed bounded domain containing the origin. Then, an asymptotic expansion of the function j is obtained in the following

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form:

$$j(\Omega_\varepsilon) = j(\Omega) + \rho(\varepsilon)\delta j(x_0) + o(\rho(\varepsilon))$$
$$\lim_{\varepsilon\to 0} \rho(\varepsilon) = 0, \quad \rho(\varepsilon) > 0,$$

where $\rho(\varepsilon)$ is a scalar function known explicitly and $\delta j(x_0)$ is easy to compute at each point x_0 of Ω.

The topological sensitivity $\delta j(x_0)$ gives us the information where to create holes, in fact if $\delta j(x_0) < 0$, then $j(\Omega_\varepsilon) < j(\Omega)$ for small ε. The function δj can be used as a descent direction of the domain optimization process.

The optimality condition $\delta j(x_0) \geq 0$ in the domain recalls the one obtained by Buttazzo-Dal Maso [5] for the Laplace equation using the homogenization theory. The notion of topological asymptotic gives an interesting alternative to homogenization methods and genetic algorithms: its applications field is very large and using topological sensitivity information, one can build fast algorithms.

The total energy variation with respect to the creation of a small hole is well known [16]. Schumacher introduced the bubble method that uses this energy variation for topological optimization in [35]. Then Sokolowski extended this idea to more general cost functions using the adjoint approach in [37] but still with Neumann boundary condition. A topological sensitivity framework using an adaptation of the adjoint method [7, 29] and a truncation technique was introduced in [29] in the case of the Laplace equation with a circular hole and a Dirichlet condition on the boundary of the hole. It was generalized in [15] to the elasticity equations in the case of arbitrary shaped holes. Recently, the same technique is adapted, using a Dirichlet boundary condition and non circular holes, to the Poisson equation in [23] and to Stokes equations in [24].

All these contributions concern operators which symbol is an homogeneous polynomial. The goal of this paper is to address the situations where non homogeneous polynomials arise. We will illustrate our approach by the Quasi-Stokes equations case. The basic idea is to say that the leading term of the topological expansion is given by the elementary solution of the principal part of the operator. The theoretical part of this work is discussed in both two and three dimensional cases. Such an expansion is obtained for a large class of cost functions and arbitrary shaped holes.

In the numerical part, we consider the aeration process of eutrophized lakes. Eutrophication leads to a 3-layer situation, the bottom layer being quite poor in oxygen necessary to aquatic life [1]. The aeration process consists in inserting air by the means of injectors located at the bottom of the lake in order to generate a vertical motion mixing up the water of the bottom with that in the top, thus oxygenating the lower part by bringing it in contact with the surface air. A simplified model based on incompressible Quasi-Stokes equations is used, only considering the liquid phase, which is the dominant one. The injected air is taken into account through local boundary conditions for the velocity on the injectors holes. We aim to optimize the injectors location in order to generate the best motion in the fluid with respect to the aeration purpose. The main idea is to compute the asymptotic topological expansion with respect to the insertion of an injector. The injector is modeled as a small hole ω_ε around a point x_0, having an injection velocity $\mathcal{U}_{\rm inj}$. The best locations and orientations are the one for which the cost function decrease most, *i.e.* the sensitivity is as negative as possible. Numerical tests clearly indicate the approach to be quite efficient.

An outline of the paper is as follows. In Section 2, we recall briefly the adaptation of adjoint method to the topological optimization. In Section 3, we derive the Quasi-Stokes equations and we give a description of the shape optimization problem that we consider. Next in Section 4 the truncation technique is applied to the problem. The main results are presented in Section 5. An asymptotic expansion is given in a general form, for a large class of cost functions and arbitrary shaped holes. In Section 6, we present, for the two dimensional case, some numerical experiments validating the above analysis. Finally in Section 7, some background materials related to the Stokes and Quasi-Stokes equations are reviewed.

2. The generalized adjoint method

In this section, we recall the fundamental results introduced in [14, 29] which extends the adjoint method [7] to the topology shape optimization.

Let $\mathcal{V}$ be a fixed Hilbert space. For $\varepsilon \geq 0$, let a_ε be a bilinear, symmetric, continuous and coercive form on $\mathcal{V}$ and l_ε be a linear and continuous form on $\mathcal{V}$, that is, there exist constants $M > 0$, $\gamma > 0$ and $L > 0$, independent of ε such that for all $\varepsilon \geq 0$,

$$\begin{array}{lll} a_\varepsilon(u\,,\,v) & \leq M\,\|u\|\,\|v\|\,, & \forall u,\,v \in \mathcal{V} \\ a_\varepsilon(u\,,\,u) & \geq \gamma\,\|u\|^2\,, & \forall u \in \mathcal{V} \\ l_\varepsilon(v) & \leq L\,\|v\|\,, & \forall v \in \mathcal{V}. \end{array}$$

Assume that there exist a bilinear and continuous form δa, a linear and continuous form δl, and a real function $\rho(\varepsilon) > 0$ defined on $\mathbb{R}_+$ such that

$$\|a_\varepsilon - a_0 - \rho(\varepsilon)\delta a\|_{\mathcal{L}_2(\mathcal{V})} = o(\rho(\varepsilon)), \tag{1}$$

$$\|l_\varepsilon - l_0 - \rho(\varepsilon)\delta l\|_{\mathcal{L}(\mathcal{V})} = o(\rho(\varepsilon)), \tag{2}$$

$$\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0,$$

where $\mathcal{L}(\mathcal{V})$ (respectively $\mathcal{L}_2(\mathcal{V})$) denotes the space of continuous and linear (respectively bilinear) forms on $\mathcal{V}$.
For $\varepsilon \geq 0$, let u_ε be the solution to the problem: find $u_\varepsilon \in \mathcal{V}$ such that

$$a_\varepsilon(u_\varepsilon\,,\,v) = l_\varepsilon(v), \quad \forall v \in \mathcal{V}. \tag{3}$$

Lemma 2.1 [15]**.** *For $\varepsilon \geq 0$, problem (3) has a unique solution u_ε, and*

$$\|u_\varepsilon - u_0\| = O(\rho(\varepsilon)). \tag{4}$$

Next we consider a cost function of the form $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$, where J_ε is defined on $\mathcal{V}$ for $\varepsilon \geq 0$ and J_0 is differentiable with respect to u, its derivative being denoted by $DJ_0(u)$.
Suppose that there exists a function δJ defined on $\mathcal{V}$ such that: for all $\varepsilon > 0$

$$J_\varepsilon(v) - J_0(u) = DJ_0(u)(v-u) + \rho(\varepsilon)\delta J(u) + o(\|v-u\| + \rho(\varepsilon)) \quad \forall u, v \in \mathcal{V}. \tag{5}$$

Theorem 2.1 [15, 29]**.** *Under the hypotheses (1), (2) and (5) the function j has the following asymptotic expansion*

$$j(\varepsilon) = j(0) + \rho(\varepsilon)\,[\delta a(u_0, v_0) - \delta l(v_0) + \delta J(u_0)] + o(\rho(\varepsilon)) \tag{6}$$

where $v_0 \in \mathcal{V}$ is the solution to the adjoint problem: find $v_0 \in \mathcal{V}$ such that

$$a_0(w, v_0) = -DJ_0(u_0)w, \quad \forall w \in \mathcal{V}.$$

3. Position of the problem

3.1. The Quasi-Stokes equations

We consider Ω a bounded domain of $\mathbb{R}^N$, $N = 2{,}3$. We denote by Γ its boundary.
The standard form of the Navier-Stokes equations describing the motion of an incompressible fluid in Ω is given by:

$$\left\{\begin{array}{rl} \dfrac{\partial u_\Omega}{\partial t} + (u_\Omega.\nabla)\,u_\Omega - \nu\Delta u_\Omega + \nabla p_\Omega = F & \text{in } \Omega \\ \nabla.u_\Omega = 0 & \text{in } \Omega \\ u_\Omega = \mathcal{U}_d & \text{on } \Gamma \end{array}\right. \tag{7}$$

where u_Ω and p_Ω denote respectively the velocity and the pressure fields, F is a given body force per unit of mass, ν denotes the kinematic viscosity of the fluid and $\mathcal{U}_d$ is a given boundary velocity.

Because of the divergence-free condition on u_Ω, $\mathcal{U}_d$ must necessary satisfy the compatibility condition

$$\int_\Gamma \mathcal{U}_d.n \ \mathrm{d}s = 0,$$

where n is the unit normal vector along the boundary Γ.

Denoting by Δt the time step and posing $t^n = n\Delta t$ and $t^{n+1} = t^n + \Delta t$.

Using the characteristic method (see, *e.g.*, [12]), an approximation for the convection term is given by

$$\left(\frac{\partial u_\Omega}{\partial t}(t,x) + u_\Omega.\nabla u_\Omega(t,x)\right)\Big|_{(t=t^n)} = \frac{\mathrm{d}u_\Omega}{\mathrm{d}t}(t,x(t))\Big|_{(t=t^n)} \simeq \frac{u_\Omega(t^{n+1},x(t^{n+1})) - u_\Omega(t^n,x(t^n))}{\Delta t}.$$

By an implicit scheme, a time discretization of the system (7) can be written as

$$\begin{cases} \dfrac{1}{\Delta t} u_\Omega^{n+1} - \nu\Delta u_\Omega^{n+1} + \nabla p_\Omega^{n+1} = F^{n+1} + \dfrac{u_\Omega(t^n,x(t^n))}{\Delta t} \\ \nabla.u^{n+1} = 0. \end{cases}$$

Then, at each time step, we have to solve a steady state Quasi-Stokes problem, called also generalized Stokes problem, having the following generic form

$$\begin{cases} \alpha u_\Omega - \nu\Delta u_\Omega + \nabla p_\Omega = f & \text{in } \Omega \\ \nabla.u_\Omega = 0 & \text{in } \Omega \\ u_\Omega = \mathcal{U}_d & \text{on } \Gamma. \end{cases} \tag{8}$$

3.2. The shape optimization problem

For a given $x_0 \in \Omega$, consider the modified domain $\Omega_\varepsilon = \Omega\backslash\overline{\omega}_\varepsilon$, $\omega_\varepsilon = x_0 + \varepsilon\omega$, where ω is a given fixed and bounded domain of $\mathbb{R}^N$, containing the origin, whose boundary $\partial\omega$ is connected and piecewise of class $\mathcal{C}^1$.

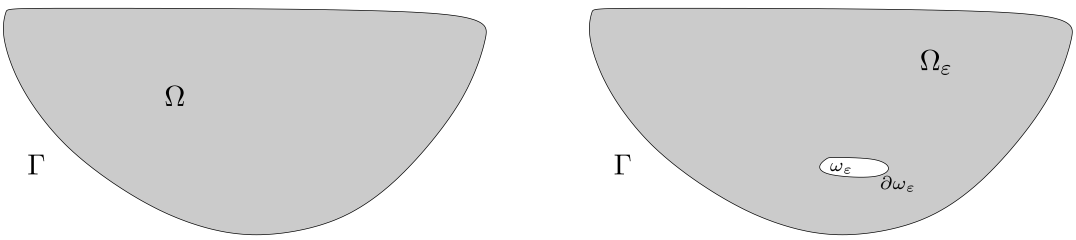


FIGURE 1. The initial domain and the same domain after creation of a small hole ω_ε.

In the modified domain Ω_ε, the velocity and pressure fields are required to satisfy

$$\begin{cases} \alpha u_{\Omega_\varepsilon} - \nu\Delta u_{\Omega_\varepsilon} + \nabla p_{\Omega_\varepsilon} = f & \text{in } \Omega_\varepsilon \\ \nabla.u_{\Omega_\varepsilon} = 0 & \text{in } \Omega_\varepsilon \\ u_{\Omega_\varepsilon} = \mathcal{U}_d & \text{on } \Gamma \\ u_{\Omega_\varepsilon} = \mathcal{U}_{inj} & \text{on } \partial\omega_\varepsilon \end{cases} \tag{9}$$

where $\mathcal{U}_{inj}$ is a given velocity on $\partial\omega_\varepsilon$.

Note that for $\varepsilon = 0$, one has $u_{\Omega_0} = u_\Omega$ and $p_{\Omega_0} = p_\Omega$.

Theorem 3.1 [4,17,38]**.** *Let $\varepsilon \geq 0$, for a given $f \in L^2(\Omega_\varepsilon)^N$, $\mathcal{U}_{inj} \in H^{1/2}(\partial\omega_\varepsilon)^N$ and $\mathcal{U}_d \in H^{1/2}(\Gamma)^N$, the problem (9) has a unique solution $(u_{\Omega_\varepsilon}, p_{\Omega_\varepsilon}) \in H^1(\Omega_\varepsilon)^N \times L^2_0(\Omega_\varepsilon)^N$ where $L^2_0(\Omega_\varepsilon)^N = \left\{\theta \in L^2(\Omega_\varepsilon)^N, \int_{\Omega_\varepsilon} \theta \,\mathrm{d}x = 0\right\}$.*

Let $\overline{u}_{\Omega_\varepsilon}$ an extension of the boundary data $\mathcal{U}_d$ and $\mathcal{U}_{inj}$ in Ω_ε, $(\partial\Omega_\varepsilon = \Gamma \cup \partial\omega_\varepsilon)$, satisfying

$$\begin{cases} \alpha\overline{u}_{\Omega_\varepsilon} - \nu\Delta\overline{u}_{\Omega_\varepsilon} + \nabla\overline{p}_{\Omega_\varepsilon} = 0 & \text{in } \Omega_\varepsilon \\ \nabla.\overline{u}_{\Omega_\varepsilon} = 0 & \text{in } \Omega_\varepsilon \\ \overline{u}_{\Omega_\varepsilon} = \mathcal{U}_d & \text{on } \Gamma \\ \overline{u}_{\Omega_\varepsilon} = \mathcal{U}_{inj} & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{10}$$

The solution u_{Ω_ε} can be recuperated as $u_{\Omega_\varepsilon} = w_{\Omega_\varepsilon} + \overline{u}_{\Omega_\varepsilon}$, with w_{Ω_ε} is the solution of the system (9) with a homogeneous Dirichlet boundary condition on $\partial\Omega_\varepsilon$.

Thanks to the previous variable substitution, in the theoretical part of this work, we will consider only a homogeneous boundary condition. Then, we will assume that $\mathcal{U}_d = 0$ on Γ and $\mathcal{U}_{inj} = 0$ on $\partial\omega_\varepsilon$.

We now consider a cost function $j(\varepsilon)$ of the form

$$j(\varepsilon) = \widetilde{J}_\varepsilon(u_{\Omega_\varepsilon}), \tag{11}$$

with $\widetilde{J}_\varepsilon$ being defined on $H^1(\Omega_\varepsilon)^N$ for $\varepsilon \geq 0$.
Our aim is to obtain an asymptotic expansion of j with respect to ε. The velocity field u_{Ω_ε} is defined in the variable domain Ω_ε, thus it belongs to a functional space which depends on ε. Hence, if we want to derive the asymptotic expansion of j we cannot apply directly the results of Section 2, which require a fixed functional space (*cf.* Th. 2.1).

In classical shape optimization, this condition is satisfied by the mean of a domain parameterization method. This method involves a fixed domain and a bi-Lipshitz map between the initial domain and the modified one. In the topology optimization context, such a map does not exist between Ω and Ω_ε.
However, a functional space independent of ε can be constructed by using a domain truncation technique described in the next paragraph (see also [29] and [15]).

This truncation is needed only for analysis, and will never be used for practical computation. During the optimization process, we have just to solve the system (8) and the adjoint problem associated to the cost function (11).

4. The truncated problem

Let $R > 0$ be such that the closed ball $\overline{B(x_0, R)}$ is included in Ω and $\overline{\omega}_\varepsilon \subset B(x_0, R)$.

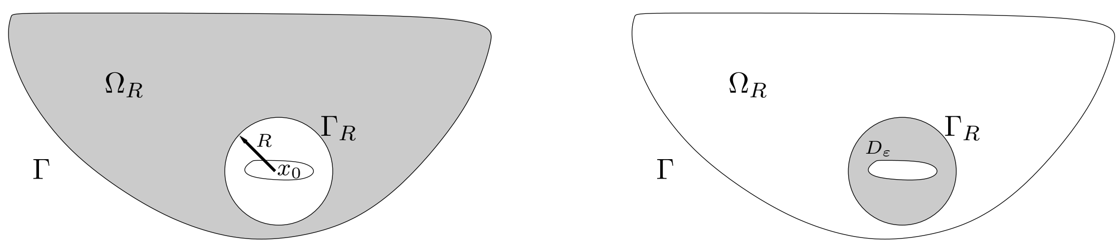


Figure 2. The truncated domain.

We denote by Γ_R the boundary of $B(x_0, R)$ and we consider the fixed domain $\Omega_R = \Omega \backslash \overline{B(x_0, R)}$ and $D_\varepsilon = B(x_0, R) \backslash \overline{\omega_\varepsilon}$. Also we use the following space of traces on Γ_R

$$H_V^{1/2}(\Gamma_R)^N = \left\{ \phi \in H^{1/2}(\Gamma_R)^N; \quad \int_{\Gamma_R} \phi.n \, \mathrm{d}x = 0 \right\} \tag{12}$$

where n is the unit vector normal to Γ_R. Its dual space is denoted by $H_V^{-1/2}(\Gamma_R)^N$.

For a given $f \in L^2(\Omega)^N$, $\varphi \in H_V^{1/2}(\Gamma_R)^N$ and $\varepsilon > 0$, let $u_\varepsilon^{f,\varphi}$, $p_\varepsilon^{f,\varphi}$ be the solution to the problem: find $(u_\varepsilon^{f,\varphi}, p_\varepsilon^{f,\varphi}) \in H^1(D_\varepsilon)^N \times L_0^2(D_\varepsilon)^N$ such that

$$\begin{cases} \alpha u_\varepsilon^{f,\varphi} - \nu \Delta u_\varepsilon^{f,\varphi} + \nabla p_\varepsilon^{f,\varphi} = f & \text{in } D_\varepsilon \\ \text{div } u_\varepsilon^{f,\varphi} = 0 & \text{in } D_\varepsilon \\ u_\varepsilon^{f,\varphi} = \varphi & \text{on } \Gamma_R \\ u_\varepsilon^{f,\varphi} = 0 & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{13}$$

This problem has a unique solution [4, 17, 38].

For $\varepsilon = 0$, $(u_0^{f,\varphi}, p_0^{f,\varphi})$ is the solution to

$$\begin{cases} \alpha u_0^{f,\varphi} - \nu \Delta u_0^{f,\varphi} + \nabla p_0^{f,\varphi} = f & \text{in } B(x_0, R) \\ \text{div } u_0^{f,\varphi} = 0 & \text{in } B(x_0, R) \\ u_0^{f,\varphi} = \varphi & \text{on } \Gamma_R. \end{cases} \tag{14}$$

Clearly we have

$$u_\varepsilon^{f,\varphi} = u_\varepsilon^{f,0} + u_\varepsilon^{0,\varphi}, \quad p_\varepsilon^{f,\varphi} = p_\varepsilon^{f,0} + p_\varepsilon^{0,\varphi}, \quad \forall \varepsilon \geq 0. \tag{15}$$

This decomposition will be used to construct the bilinear form a_ε and the linear form l_ε presented in Section 2. For $\varepsilon \geq 0$, we consider the Dirichlet-to-Neumann operator

$$\begin{aligned} T_\varepsilon \, : \, H_V^{1/2}(\Gamma_R)^N &\longrightarrow H_V^{-1/2}(\Gamma_R)^N \\ \varphi &\longmapsto T_\varepsilon \varphi = \sigma(u_\varepsilon^{0,\varphi}).n \end{aligned} \tag{16}$$

where $\sigma(u_\varepsilon^{0,\varphi}) = (\nu \nabla u_\varepsilon^{0,\varphi} - p_\varepsilon^{0,\varphi} I)$ is the stress tensor.

And the function $f_\varepsilon \in H_V^{-1/2}(\Gamma_R)^N$

$$f_\varepsilon = -\sigma(u_\varepsilon^{f,0}).n = -(\nu \nabla u_\varepsilon^{f,0} - p_\varepsilon^{f,0} I).n \tag{17}$$

with the normal n is chosen outward to D_ε on Γ_R and $\partial\omega_\varepsilon$.

Hence, for all $\varepsilon \geq 0$ and $\varphi \in H_V^{1/2}(\Gamma_R)^N$ we have $\sigma(u_\varepsilon^{f,\varphi}).n = T_\varepsilon \varphi - f_\varepsilon$.

Finally, we define for $\varepsilon \geq 0$ the solution u_ε, p_ε to the truncated problem

$$\begin{cases} \alpha u_\varepsilon - \nu \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_R \\ \text{div } u_\varepsilon = 0 & \text{in } \Omega_R \\ u_\varepsilon = 0 & \text{on } \Gamma \\ \sigma(u_\varepsilon).n + T_\varepsilon u_\varepsilon = f_\varepsilon & \text{on } \Gamma_R. \end{cases} \tag{18}$$

The variational formulation associated to (18) is: find $u_\varepsilon \in \mathcal{V}_R$ such that

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v), \quad \forall v \in \mathcal{V}_R \tag{19}$$

where the functional space $\mathcal{V}_R$, the bilinear form a_ε and the linear form l_ε are defined by

$$\mathcal{V}_R = \left\{u \in H^1(\Omega_R)^N;\ \text{div}\ u = 0,\ u = 0 \text{ on } \Gamma\right\},$$

$$a_\varepsilon(u,v) = \alpha \int_{\Omega_R} u.v\ \mathrm{d}x + \nu \int_{\Omega_R} \nabla u : \nabla v\ \mathrm{d}x + \int_{\Gamma_R} T_\varepsilon uv\ \mathrm{d}\gamma(x), \tag{20}$$

$$l_\varepsilon(v) = \int_{\Omega_R} fv\ \mathrm{d}x + \int_{\Gamma_R} f_\varepsilon v\ \mathrm{d}\gamma(x). \tag{21}$$

Symmetry, continuity and coercivity of a_ε and continuity of l_ε follow directly from

$$\int_{\Gamma_R} T_\varepsilon \varphi\psi\ \mathrm{d}\gamma(x) = \alpha \int_{D_\varepsilon} u_\varepsilon^{0,\varphi}.u_\varepsilon^{0,\psi}\ \mathrm{d}x + \nu \int_{D_\varepsilon} \nabla u_\varepsilon^{0,\varphi} : \nabla u_\varepsilon^{0,\psi}\ \mathrm{d}x, \tag{22}$$

$$\int_{\Gamma_R} f_\varepsilon \psi\ \mathrm{d}\gamma(x) = \int_{D_\varepsilon} f u_\varepsilon^{0,\psi}\ \mathrm{d}x, \tag{23}$$

and the relation

$$\alpha \int_{D_\varepsilon} u_\varepsilon^{f,0}.u_\varepsilon^{0,\psi}\ \mathrm{d}x + \nu \int_{D_\varepsilon} \nabla u_\varepsilon^{f,0} : \nabla u_\varepsilon^{0,\psi}\ \mathrm{d}x = 0. \tag{24}$$

Proposition 4.1. *Let $\varepsilon \geq 0$. Problems (9) and (18) have unique solution.*
Moreover, the restriction to Ω_R of the solution u_{Ω_ε}, p_{Ω_ε} to (9) is the solution u_ε, p_ε to (18), and we have in D_ε

$$(u_{\Omega_\varepsilon})_{|D_\varepsilon} = u_\varepsilon^{f,\varphi}, \qquad (p_{\Omega_\varepsilon})_{|D_\varepsilon} = p_\varepsilon^{f,\varphi} \tag{25}$$

where φ is the trace of u_{Ω_ε} on Γ_R.

Proof. We refer to [4,17,38] for the existence and uniqueness of the solutions to both problems (9) and (18). Recall that we have denoted by $(u_{\Omega_\varepsilon}, p_{\Omega_\varepsilon})$ the solution of (9) and by $(u_\varepsilon, p_\varepsilon)$ the solution of (18).

Let $\varphi = u_{\Omega_\varepsilon|\Gamma_R}$ and $u_R = u_{\Omega_\varepsilon|\Omega_R}$. Clearly (25) holds for this φ, and it remains to prove that $u_R = u_\varepsilon$.

Let $\theta \in \mathcal{V}_R$ and $\psi = \theta_{|\Gamma_R}$. We extend θ on D_ε by $u_\varepsilon^{0,\psi}$. Its extension is still denoted by θ, and it is divergence free on D_ε.

Using (22), (23), (24) and the definition of u_{Ω_ε}, we have

$$\begin{aligned}
\alpha &\int_{\Omega_R} u_R.\theta\ \mathrm{d}x + \nu \int_{\Omega_R} \nabla u_R : \nabla\theta\ \mathrm{d}x + \int_{\Gamma_R} (T_\varepsilon u_R - f_\varepsilon)\theta\ \mathrm{d}\gamma(x) \\
&= \alpha \int_{\Omega_R} u_R.\theta\ \mathrm{d}x + \nu \int_{\Omega_R} \nabla u_R : \nabla\theta\ \mathrm{d}x + \int_{\Gamma_R} T_\varepsilon u_R\theta\ \mathrm{d}\gamma(x) - \int_{\Gamma_R} f_\varepsilon.\theta\ \mathrm{d}\gamma(x) \\
&= \alpha \int_{\Omega_R} u_R.\theta\ \mathrm{d}x + \nu \int_{\Omega_R} \nabla u_R : \nabla\theta\ \mathrm{d}x + \alpha \int_{D_\varepsilon} u_\varepsilon^{0,\varphi}.u_\varepsilon^{0,\psi}\ \mathrm{d}x \\
&\quad + \ \nu \int_{D_\varepsilon} \nabla u_\varepsilon^{0,\varphi} : \nabla u_\varepsilon^{0,\psi}\ \mathrm{d}x - \int_{D_\varepsilon} f.u_\varepsilon^{0,\psi}\ \mathrm{d}x \\
&= \alpha \int_{\Omega_\varepsilon} u_\varepsilon.\theta\ \mathrm{d}x + \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla\theta\ \mathrm{d}x - \int_{\Gamma_R} f.u_\varepsilon^{0,\psi}\ \mathrm{d}\gamma(x) \\
&= \alpha \int_{\Omega_\varepsilon} u_\varepsilon.\theta\ \mathrm{d}x + \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon : \nabla\theta\ \mathrm{d}x - \int_{D_\varepsilon} f.\theta\ \mathrm{d}x = \int_{\Omega_R} f.\theta\ \mathrm{d}x.
\end{aligned}$$

This proves that u_R is the solution to (19). From uniqueness of the solution, we deduce that $u_R = u_\varepsilon$. □

Now we have at our disposal the fixed Hilbert space $\mathcal{V}_R$ required by Section 2. The cost function (11) can be redefined in the following way:

for $u \in \mathcal{V}_R$, let $\varphi = u_{|\Gamma_R}$, one defines $\tilde{u}_\varepsilon \in H^1(\Omega_\varepsilon)^N$ the extension of u on Ω_ε as follow

$$\tilde{u}_\varepsilon = \begin{cases} u & \text{on } \Omega_R \\ u_\varepsilon^{f,\varphi} & \text{on } D_\varepsilon. \end{cases} \tag{26}$$

Then, a function J_ε can be defined on $\mathcal{V}_R$ by

$$J_\varepsilon(u) = \tilde{J}_\varepsilon(\tilde{u}_\varepsilon). \tag{27}$$

Particularly, we have from the previous proposition that

$$j(\varepsilon) = \tilde{J}_\varepsilon(u_{\Omega_\varepsilon}) = J_\varepsilon(u_\varepsilon). \tag{28}$$

Remark that $J_\varepsilon(u_\varepsilon)$ is independent of the choice of R. For example, for a given target function $\mathcal{U}_g$, if

$$\tilde{J}_\varepsilon(u_{\Omega_\varepsilon}) = \int_{\Omega_\varepsilon} |u_{\Omega_\varepsilon} - \mathcal{U}_g|^2 \, \mathrm{d}x \tag{29}$$

then, we have for all $u \in \mathcal{V}_R$

$$J_\varepsilon(u) = \int_{\Omega_R} |u - \mathcal{U}_g|^2 \, \mathrm{d}x + \int_{D_\varepsilon} \left|u_\varepsilon^{f,\varphi} - \mathcal{U}_g\right|^2 \, \mathrm{d}x, \text{ with } \varphi = u_{|\Gamma_R}. \tag{30}$$

5. Main result: the asymptotic expansion

In this section, we present the main results of this paper, which concern the asymptotic analysis with respect to the parameter ε of the functional (27). An asymptotic expansion is obtained for the Quasi-Stokes operator for a large class of cost functions and arbitrary shaped holes.

We begin by the three dimensional case. The principal result is given by Theorem 5.1, it gives the topological sensitivity expression $\delta j(x_0)$ if a hole is created at x_0. The proof of Theorem 5.1 is relegated to Section 7.

5.1. **The three dimensional case**

In order to derive the topological sensitivity of the function j, we introduce two auxiliary problems.

The first problem, which we call the "exterior problem", is formulated in $\mathbb{R}^3 \backslash \overline{\omega}$ and consists to find (U, P) solution to

$$\begin{cases} -\nu \Delta U + \nabla P = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ \operatorname{div} U = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ U = 0 & \text{at infinity} \\ U = u_\Omega(x_0) & \text{on } \partial\omega \end{cases} \tag{31}$$

where u_Ω is the solution to Quasi-Stokes problem (8). Recall that $f \in L^2(\Omega)^3$, so that u_Ω is continuous inside Ω.

Here, one can remark that just the principal part of the Quasi-Stokes operator is used, which is the Stokes equations. A such approach can be justified by the fact that operators $-\Delta$ and $I - \Delta$ have the same behavior near the hole ω_ε and give the same leading terms for topological expansion. Moreover, this approach has a good advantage, it avoids calculations with the complicated expression of the Quasi-Stokes fundamental solution, which involves the convolution product of Green functions for both operators $I - \Delta$ and Δ.

We return now to the system (31). The functions U, P can be expressed by a simple layer potential on $\partial\omega$. Coordinate system can be changed, in what follows one can suppose for convenience that $x_0 = 0$.

Posing $r = \|y\|$ and $e_r = \dfrac{y}{\|y\|}$, the fundamental solution system to the Stokes equations in $\mathbb{R}^3$ can be written as

$$G_U(y) = \frac{1}{8\pi\nu r}(I + e_r e_r^T), \quad G_P(y) = \frac{y}{4\pi r^3} \tag{32}$$

such that

$$-\nu\Delta G_{Uj} + \nabla G_{Pj} = \delta e_j \tag{33}$$

where G_{Uj} denote the jth column of G_U, $\{e_j\}_{j=1,3}$ is the canonical basis of $\mathbb{R}^3$ and δ is the Dirac distribution. Then, the functions U, P read [11]

$$\begin{aligned} U(y) &= \int_{\partial\omega} G_U(y-x)T(x)\ \mathrm{d}\gamma(x), \quad y \in \mathbb{R}^3\backslash\overline{\omega} \\ P(y) &= \int_{\partial\omega} G_P(y-x).T(x)\ \mathrm{d}\gamma(x), \quad y \in \mathbb{R}^3\backslash\overline{\omega} \end{aligned} \tag{34}$$

where $T \in H^{-1/2}(\partial\omega)^3$ is a solution to the boundary integral equation (see *e.g.* [11])

$$\int_{\partial\omega} G_U(y-x)T(x)\ \mathrm{d}\gamma(x) = u_\Omega(x_0), \quad y \in \partial\omega. \tag{35}$$

One can observe that the function T is determined up to a function proportional to the normal, hence it is unique in $H^{-1/2}(\partial\omega)^3/\mathbb{R}n$.

Using the first order Taylor expansion of G_U at the point $y \neq 0$ for x bounded, we have

$$G_U(y-x) = G_U(y) + O\left(\frac{1}{r^2}\right), \quad G_P(y-x) = G_P(y) + O\left(\frac{1}{r^3}\right); \tag{36}$$

from which follows the asymptotic expansion at infinity of U and P:

$$U(y) = S_U(y) + L_U(y), \quad P(y) = S_P(y) + L_P(y) \tag{37}$$

where $S_U(y)$ and $S_P(y)$ are the dominant part respectively of U and P

$$S_U(y) = G_U(y)\, A(u_\Omega(x_0)), \quad S_P(y) = G_P(y).A(u_\Omega(x_0)) \tag{38}$$

with

$$A(u_\Omega(x_0)) = \int_{\partial\omega} T(x)\ \mathrm{d}\gamma(x) \in \mathbb{R}^3. \tag{39}$$

Notice that $\alpha \longrightarrow A(\alpha)$ is linear on $\mathbb{R}^3$ and $S_U \in L^2_{loc}(\mathbb{R}^3)$.

The last parts of U and P are respectively given by

$$L_U(y) = O\left(\frac{1}{r^2}\right), \quad L_P(y) = O\left(\frac{1}{r^3}\right)\cdot \tag{40}$$

The second problem, which we call "interior problem", is formulated in $D_0 = B(x_0, R)$ and consists to find $(R_U^h,\, R_P^h)$ solution to

$$\begin{cases} \alpha R_U^h - \nu\Delta R_U^h + \nabla R_P^h = 0 & \text{in } B(x_0,R) \\ \text{div } R_U^h = 0 & \text{in } B(x_0,R) \\ R_U^h = S_U & \text{on } \Gamma_R. \end{cases} \tag{41}$$

Here, the idea is to consider an interior and exterior problems that gives the asymptotic behavior of $\left(u_\varepsilon^{f,\varphi} - u_0^{f,\varphi}\right)|_{D_\varepsilon}$ with $\varphi = u_\Omega|_{\Gamma_R}$ in a sense which will be stated in Section 7.

It will not be possible to derive the asymptotic behavior of $u_\varepsilon^{f,\varphi} - u_0^{f,\varphi}$ from $R_U^h - S_U$. We have first to take into account the error due to the simplification of the fundamental solution. We propose to cancel this error by adding a correcting term to R_U^h.

In such a case, we consider the correcting term (R_U^c, R_P^c) as solution to

$$\begin{cases} \alpha R_U^c - \nu\Delta R_U^c + \nabla R_P^c = \alpha S_U & \text{in } B(x_0,R) \\ \text{div } R_U^c = 0 & \text{in } B(x_0,R) \\ R_U^c = 0 & \text{on } \Gamma_R. \end{cases} \tag{42}$$

Setting $R_U = R_U^h + R_U^c$, $R_P = R_P^h + R_P^c$, then $(R_U,\, R_P)$ is solution to

$$\begin{cases} \alpha R_U - \nu\Delta R_U + \nabla R_P = \alpha S_U & \text{in } B(x_0,R) \\ \text{div } R_U = 0 & \text{in } B(x_0,R) \\ R_U = S_U & \text{on } \Gamma_R. \end{cases} \tag{43}$$

We will prove in Section 7, using the corrected interior problem (43), it will be possible to derive the asymptotic behavior of $\left(u_\varepsilon^{f,\varphi} - u_0^{f,\varphi}\right)_{|D_\varepsilon}$. The main result is the following. It will be proved in Section 7.

Theorem 5.1. *Let $f \in H^2(\Omega)^3$, and let $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$. Suppose that J_ε satisfy the hypothesis (5): for all $v \in \mathcal{V}_R$ and all $\varepsilon > 0$*

$$J_\varepsilon(v) - J_0(u_0) = DJ_0(u_0)(v-u_0) + \varepsilon\delta J(u_0) + o\left(\varepsilon + \|v-u_0\|_{\mathcal{V}_R}\right), \tag{44}$$

where $DJ_0(u_0)$ is linear and continuous on $\mathcal{V}_R$, and u_ε, $\varepsilon \geq 0$ is the solution to (19).

Let $v_0 \in \mathcal{V}_R$ be the solution to the adjoint equation

$$a_0(w,v_0) = -DJ_0(u_0)w, \quad \forall w \in \mathcal{V}_R. \tag{45}$$

Then, the function j has the following asymptotic expansion

$$j(\varepsilon) = j(0) + \varepsilon\delta j(x_0) + o(\varepsilon) \tag{46}$$

with

$$\delta j(x_0) = \int_{\Gamma_R} (\sigma(R_U - S_U)).nv_0 \,\mathrm{d}\gamma(x) + \delta J(u_0). \tag{47}$$

The functional $\delta j(x_0)$ is called the "topological sensitivity" of the Quasi-Stokes operator. It is also called the "topological gradient".

The cost function j is independent of R and $\delta j(x_0)$ is independent of ε, then $\delta j(x_0)$ is also independent of R. This follows from the uniqueness of an asymptotic expansion. As we will observe in Section 7, this is not necessarily true for the terms $\delta a(u_0,v_0)$, $\delta l(v_0)$ or $\delta J(u_0)$, because a, l and J do depend on R.

Practically, we need just to compute the solution u_Ω to (8) and v_Ω the solution to the associated adjoint problem

$$\alpha\int_\Omega w v_\Omega \,\mathrm{d}x + \nu\int_\Omega \nabla w : \nabla v_\Omega \,\mathrm{d}x = -D\tilde{J}_0(u_\Omega)w, \quad \forall w \in \mathcal{V}_0 \tag{48}$$

with $\mathcal{V}_0 = \left\{v \in H_0^1(\Omega)^3;\ \text{div } v = 0 \text{ in } \Omega\right\}$.

It has been shown in Proposition 4.1 that u_0 is the restriction to Ω_R of u_Ω. Similarly, v_0 is the restriction to Ω_R of v_Ω, this can be proved in the same way. Consequently, the function u_Ω (or u_0) and the adjoint state v_Ω (or v_0) do not depend on x_0. Hence, the basic property of an adjoint technique is here satisfied: only two systems have to be solved in order to compute the topological sensitivity $\delta j(x)$ for all $x \in \Omega$.

Moreover, there exists a unique $q_\Omega \in L_0^2(\Omega)^3$ such that

$$\alpha\int_\Omega w v_\Omega \,\mathrm{d}x + \nu\int_\Omega \nabla w : \nabla v_\Omega \,\mathrm{d}x - \int_\Omega q_\Omega \,\text{div } w \,\mathrm{d}x = -D\tilde{J}_0(u_\Omega)w, \quad \forall w \in H_0^1(\Omega)^3. \tag{49}$$

Corollary 5.1. *Let $x_0 \in \Omega$. Under the hypotheses of Theorem 5.1 and that $\alpha v_\Omega - \nu\Delta v_\Omega + \nabla q_\Omega \in L^2(D_0)^3$, we have*

$$\delta j(x_0) = A(u_\Omega(x_0)).v_\Omega(x_0) + \int_{D_0} (\alpha v_\Omega - \nu\Delta v_\Omega + \nabla q_\Omega)\,(R_U - S_U)\;\mathrm{d}x + \delta J(u_0). \tag{50}$$

Moreover in the particular case; ω is the unit ball $B(0,1)$, $U(y)$, $T(y)$ and $A(u_\Omega(x_0))$ are given explicitly by

$$\begin{aligned} U(y) &= \pi\nu(6G_U + \Delta G_U)(y)\,u_\Omega(x_0)\\ T(y) &= \tfrac{3\nu}{2}u_\Omega(x_0), \quad \forall y \in \partial\omega\\ A(u_\Omega(x_0)) &= 6\pi\nu u_\Omega(x_0). \end{aligned} \tag{51}$$

Proof. Thanks to Green's Formula together with (41) (with $S_U = R_U$ on Γ_R), (47) reads also

$$\begin{aligned} \delta j(x_0) &= \int_{\Gamma_R} \left[(\nu\nabla R_U - R_P I) - (\nu\nabla S_U - S_P I)\right].n\,v_\Omega\;\mathrm{d}\gamma(x) + \delta J(u_0)\\ &= \int_{\Gamma_R} (\nu\nabla v_\Omega - q_\Omega I).n\,S_U\;\mathrm{d}\gamma(x) - \int_{\Gamma_R} (\nu\nabla S_U - S_P I).n\,v_\Omega\;\mathrm{d}\gamma(x)\\ &\quad + \int_{D_0} (\alpha v_\Omega - \nu\Delta v_\Omega + \nabla q_\Omega)\,R_U\;\mathrm{d}x - \alpha\int_{D_0} S_U v_\Omega\;\mathrm{d}x + \delta J(u_0). \end{aligned} \tag{52}$$

Using a regularizing and localization technique, we derive

$$\begin{aligned} &\int_{\Gamma_R} (\nu\nabla v_\Omega - q_\Omega I).n\,S_U\;\mathrm{d}\gamma(x) - \int_{\Gamma_R} (\nu\nabla S_U - S_P I).n\,v_\Omega\;\mathrm{d}\gamma(x)\\ &\qquad = \int_{D_0} (\nu\Delta v_\Omega - \nabla q_\Omega)\,S_U\;\mathrm{d}x - \langle \nu\Delta S_U - \nabla S_P\,,\,\varphi v_\Omega\rangle \end{aligned} \tag{53}$$

where $\varphi \in \mathcal{D}(D_0)$ satisfies $\varphi(x_0) = 1$.

Finally, from (38) and (33) one can check that

$$\begin{aligned} \langle -\nu\Delta S_U + \nabla S_P, \varphi v_\Omega\rangle &= \langle -\nu\Delta(G_U\,A(u_\Omega(x_0))) + \nabla(G_P.A(u_\Omega(x_0)))\,,\,\varphi v_\Omega\rangle\\ &= \sum_j A_j(u_\Omega(x_0))\langle \delta e_j, \varphi v_\Omega\rangle\\ &= A(u_\Omega(x_0)).v_\Omega(x_0), \end{aligned} \tag{54}$$

which proves (50).

For the case $\omega = B(0,1)$, one can derive the explicit expressions of the terms $U(y)$, $T(y)$ and $A(u_\Omega(x_0))$ from (32), (35) and

$$\int_{\partial B(0,1)} G_U(y-x)\;\mathrm{d}\gamma(x) = \frac{2}{3\nu}I, \quad \forall y \in \partial B(0,1). \tag{55}$$

For more details concerning the explicitly calculation of this terms, one may consult [24]. □

Now we discuss briefly the hypothesis (5) used in Theorem 5.1. It concerns the variation of the cost function J_ε. This question has been examined in [23] for the Dirichlet problem and in [24] for the Stokes problem. Here we limit ourselves to cost functions of the form

$$\tilde{J}_\varepsilon(u) = \int_{\Omega_\varepsilon} g(x, u(x))\;\mathrm{d}x, \quad u \in H^1(\Omega_\varepsilon)^3 \tag{56}$$

with g is a given function defined on $\Omega \times \mathbb{R}^3$.

The case where g depends on the pressure p is more complicated and we don't consider it in this paper. Next we suppose that g satisfy the following hypotheses:

- for all $x \in \Omega$, the function $s \longmapsto g(x,s)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^3$, its gradient being denoted by $\nabla_s g(x,s)$;

- for all $x \in \Omega$, the function $s \longmapsto \nabla_s g(x,s)$ is Lipschitz continuous and there exists a constant M such that

$$|\nabla_s g(x,t) - \nabla_s g(x,s)| \leq M|t-s|, \quad \forall (x,s,t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{57}$$

where $|t|$ denotes the usual norm on $\mathbb{R}^n$;

- the function $x \longmapsto \nabla_s g(x,0)$ belongs to $L^2(\Omega)^3$ and $x \longmapsto g(x,0)$ belongs to $L^2(\Omega)^{3/2}$.

These hypotheses imply that for all $(x,s) \in \Omega \times \mathbb{R}^3$

$$\begin{aligned} &|g(x,s)| \leq |g(x,0)| + |\nabla_s g(x,0).s| + \frac{M}{2}|s|^2 \\ &|\nabla_s g(x,s)| \leq |\nabla_s g(x,0)| + M|s|, \end{aligned} \tag{58}$$

and the functions $x \longmapsto g(x,u(x))$ and $x \longmapsto |\nabla_s g(x,u(x))|^2$ are integrable on Ω for all $u \in L^2(\Omega)^3$.

A standard example of this functions is given by

$$g(x,u) = |u(x) - \mathcal{U}_g(x)|^2. \tag{59}$$

The following result is taken from [24].

Proposition 5.1. *Under the previous hypotheses and $f \in L^2(\Omega)^3$, we have*

$$\delta J(u_0) = \int_{D_0} \nabla_s g(x,u_\Omega)(R_U - S_U)\,\mathrm{d}x.$$

The adjoint state $(v_\Omega, q_\Omega) \in \mathcal{V}_0 \times L^2_0(\Omega)$ is the solution to

$$\alpha v_\Omega - \nu\Delta v_\Omega + \nabla q_\Omega = -\nabla_s g(x,u_\Omega). \tag{60}$$

The function j has the asymptotic expansion

$$j(\varepsilon) = j(0) + \varepsilon A(u_\Omega(x_0))v_\Omega(x_0) + o(\varepsilon). \tag{61}$$

If ω is the unit ball $B(1,0)$, then

$$j(\varepsilon) = j(0) + 6\pi\nu\varepsilon\, u_\Omega(x_0).v_\Omega(x_0) + o(\varepsilon). \tag{62}$$

5.2. **The two dimensional case**

In this paragraph, we intend to derive the asymptotic expansion of the function j in the two dimensional case. The technique used is similar to that of the three dimensional case. We use the principal part of the Quasi-Stokes operator to derive the topological sensitivity expression. Next we briefly describe the transposition of the previous results to the two dimensional case. First, let us recall that u_Ω and the adjoint state v_Ω are respectively the solution to (8) and (48).

Let $(U,\, P)$ be the solution to the Stokes exterior problem

$$\begin{cases} -\nu\Delta U + \nabla P = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega} \\ \text{div } U = 0 & \text{in } \mathbb{R}^2\backslash\overline{\omega} \\ U/\log\|y\| = u_\Omega(x_0) & \text{at infinity} \\ U = 0 & \text{on } \partial\omega. \end{cases} \tag{63}$$

In $2D$, the fundamental solution system to the Stokes equations is given by

$$G_U(y) = \frac{1}{4\pi\nu}(-(\log r)I + e_r e_r^T), \quad G_P(y) = \frac{y}{2\pi r^2}\cdot \tag{64}$$

The functions U and P are written

$$U(y) = u_\Omega(x_0)\,\log\|y\| + S_U(y) + L_U(y), \quad P(y) = S_P(y) + L_P(y) \tag{65}$$

where at infinity $S_U(y) = O\,(1)$, $L_U(y) = O\,(1/r)$, $S_P(y) = O\,(1/r)$ and $L_P(y) = O\left(1/r^2\right)$.

The associated interior problem consists in finding $(R_U,\, R_P)$ such that

$$\begin{cases} \alpha R_U - \nu\Delta R_U + \nabla R_P = \alpha S_U & \text{in } D_0 \\ \text{div } R_U = 0 & \text{in } D_0 \\ R_U = S_U & \text{on } \Gamma_R. \end{cases} \tag{66}$$

In this case, a first order expansion of $\left(u_\varepsilon^{f,\varphi} - u_0^{f,\varphi}\right)|_{D_\varepsilon}$ with $\varphi = u_\Omega|_{\Gamma_R}$ is given by

$$\frac{-1}{\log\,\varepsilon}\left(u_\Omega(x_0)\,\log\frac{\|x\|}{R} + R_U - S_U\right)|_{D_\varepsilon}.$$

Theorem 5.2. *Under the same hypotheses of Theorem 5.1, the function j has the following asymptotic expansion*

$$j(\varepsilon) = j(0) - \frac{1}{\log\varepsilon}\delta j(x_0) + o\left(\frac{1}{\log\varepsilon}\right) \tag{67}$$

with

$$\delta j(x_0) = \int_{\Gamma_R} \sigma(R_U - S_U).n\, v_0\, \mathrm{d}\gamma(x) + \delta J(u_0) \tag{68}$$

where $v_0 \in \mathcal{V}_R$ is the solution to the adjoint equation

$$a_0(w, v_0) = -DJ_0(u_0)w, \quad \forall w \in \mathcal{V}_R. \tag{69}$$

And for a cost function of the form (56), we have:

Proposition 5.2. *Let $\tilde{J}_\varepsilon$ a cost function of the form*

$$\tilde{J}_\varepsilon(u) = \int_{\Omega_\varepsilon} g(x, u(x))\,\mathrm{d}x, \quad u \in H^1(\Omega_\varepsilon)^2. \tag{70}$$

Under the same hypotheses of Proposition 5.1, the function j has the following asymptotic expansion

$$j(\varepsilon) = j(0) - \frac{4\pi\nu u_\Omega(x_0).v_\Omega(x_0)}{\log\varepsilon} + o\left(\frac{1}{\log\varepsilon}\right)\cdot \tag{71}$$

6. Numerical results

Here, we limit ourselves to the two dimensional case. As application of the previous theoretical results, we present two examples. The first example concerns the identification of locations and orientations of several injectors in a water reserve. In the second example we treat the water eutrophication phenomena in a lake *via* dynamic aeration process.

In both cases, we deal with a cost function J of the form

$$J(u_\Omega) = \int_{\Omega_m} |u_\Omega - \mathcal{U}_g|^2 \, \mathrm{d}x,$$

with $\Omega_m \subset \Omega$ is the measurement domain and $\mathcal{U}_g$ is a given target flow.

Recall that we consider the Quasi-Stokes equations with a non homogeneous boundary condition on $\partial\omega_\varepsilon$ ($u_{\Omega_\varepsilon} = \mathcal{U}_{inj}$) (see (9)). From (71) we deduce that

$$\delta j(x) = (u_\Omega(x) - \mathcal{U}_{inj})\,.v_\Omega(x), \quad \forall x \in \Omega \tag{72}$$

where u_Ω and v_Ω are, respectively, solution to

$$\begin{cases} \alpha u_\Omega - \nu\Delta u_\Omega + \nabla p_\Omega = 0 & \text{in } \Omega \\ \nabla.u_\Omega = 0 & \text{in } \Omega \end{cases} \tag{73}$$

$$\begin{cases} \alpha v_\Omega - \nu\Delta v_\Omega + \nabla q_\Omega = -2\,(u_\Omega - \mathcal{U}_g)\,\chi_{\Omega_m} & \text{in } \Omega \\ \nabla.v_\Omega = 0 & \text{in } \Omega \end{cases} \tag{74}$$

where χ_{Ω_m} is the characteristic function of the measurement domain.

Our implementation is based on the following optimization algorithm introduced by Céa *et al.* [6] and presented in the topological asymptotic context in [8].

The algorithm:

- initialization: choose $\Omega_0 = \Omega_d$, and set $k = 0$;
- repeat until target is reached:
  - solve (73) and (74) in Ω_k;
  - compute the topological sensitivity δj_k;
  - set $\Omega_{k+1} = \{x \in \Omega_k, \quad \delta j_k(x) \geq c_{k+1}\}$ where c_{k+1} is chosen in such a way that the cost function decreases;
  - $k \longleftarrow k+1$.

This algorithm can be seen as a descent method where the descent direction is determined by the topological sensitivity δj_k and the step length is given by the volume variation.

We propose an adaptation of the previous algorithm to our context. We consider the set $\{x \in \Omega_k; \delta j_k(x) < c_{k+1}\}$. Each connected component of this set is a hole created by the algorithm. Our idea is to replace each hole by an injector located at the local minimum of $\delta j_k(x)$.

In the above algorithm, the systems (73) and (74) are discretized by a finite element method. The computation of the approximate solution is achieved by Uzawa algorithm.

6.1. Test 1: identification of some injectors in a reserve water

In this example, the computational domain Ω is a reserve water. Our purpose is to insert some injectors in Ω in order to reach a given target flow $\mathcal{U}_g$. Each injector Inj_k is supposed as a small hole ω_k around $x_k \in \Omega$, having an injection velocity U^k_{inj}. The velocity field $\mathcal{U}_g$ is chosen as the solution to (73) in $\Omega_l = \Omega\backslash\{\cup_{k=1}^l \omega_k\}$, satisfying the following boundary conditions

$$u = \begin{cases} U_{\text{wind}} & \text{on } \Gamma_s \\ 0 & \text{on } \Gamma_w \\ U^k_{\text{inj}} & \text{on } \partial\omega_k, \quad k = 1, ..., l \end{cases} \tag{75}$$

where Γ_s is the free surface, Γ_w is the wall and U_{wind} is the velocity of wind.

Our aim here is to identify the locations and orientations of injectors from velocity measurement on the upper layer of Ω. The magnitude of the velocity is known. The locations are given by the local minima of the topological sensitivity δj. From the δj expression we deduce that the optimal orientations are given by the adjoint state $\left(\frac{v_\Omega}{\|v_\Omega\|}\right)$.

We consider here three cases, respectively one injector ($l = 1$), two injectors ($l = 2$) and three injectors ($l = 3$). For each case, we give a table summarizing the main parameters used to compute $\mathcal{U}_g$.

6.1.1. *First case: one injector*

Injector	Location	Injection velocity
Injector 1	$x = 0.1338773E+01$, $y = 0.2861623E+00$	$U_x = -0.8$, $U_y = -1.0$

6.1.2. *Second case: two injectors*

Injector	Location	Injection velocity
Injector 1	$x = 0.1338773E+01$, $y = 0.2861623E+00$	$U_x = -0.8$, $U_y = -1.0$
Injector 2	$x = 0.6195151E+00$, $y = 0.2911333E+00$	$U_x = -1.0$, $U_y = 1.3$

6.1.3. *Third case: three injectors*

Injector	Location	Injection velocity
Injector 1	$x = 0.1338773E+01$, $y = 0.2861623E+00$	$U_x = -0.8$, $U_y = -1.0$
Injector 2	$x = 0.6195151E+00$, $y = 0.2911333E+00$	$U_x = -1.0$, $U_y = 1.3$
Injector 3	$x = 0.8965202E+00$, $y = 0.5851877E+00$	$U_x = 0.0$, $U_y = 1.6$

Using the previous algorithm, the numerical results that we present are obtained in one iteration.

In Figure 3, we present the initial flow, which is the same for the three considered cases. The results of this test are given by Figures 4–6. For each case, the injectors locations are given by the local minima of the topological sensitivity δj, see Figures 4c, 5c and 6c. At each local minima, we introduce a pointwise Dirichlet condition (an injector inserted) and new resolution of (73) is performed. The new velocity obtained is shown in Figures 4d, 5d and 6d.

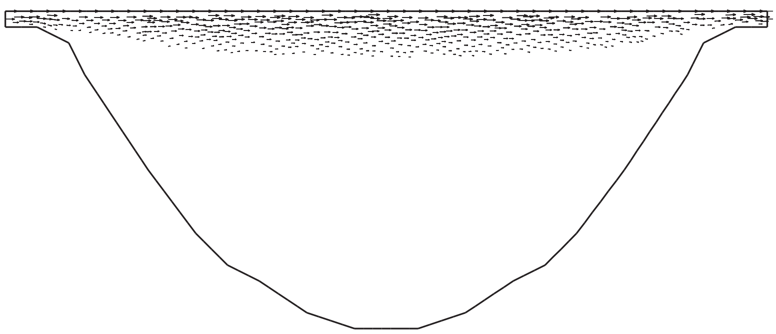


FIGURE 3. The initial flow u_Ω.

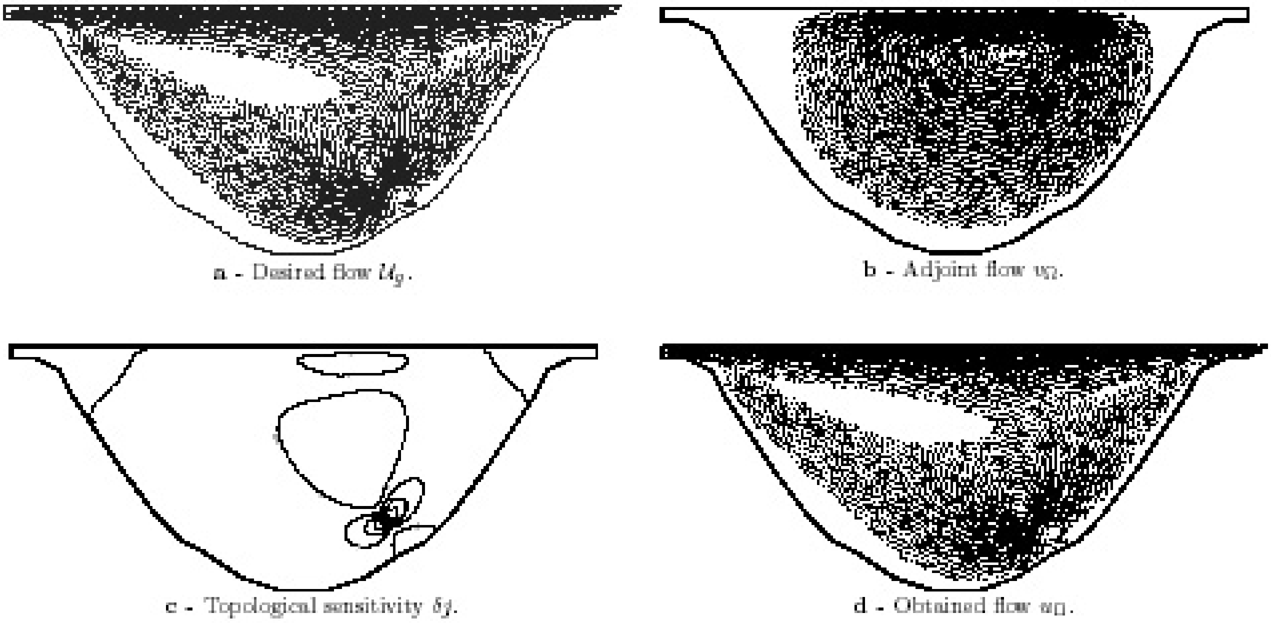


FIGURE 4. One injector case.

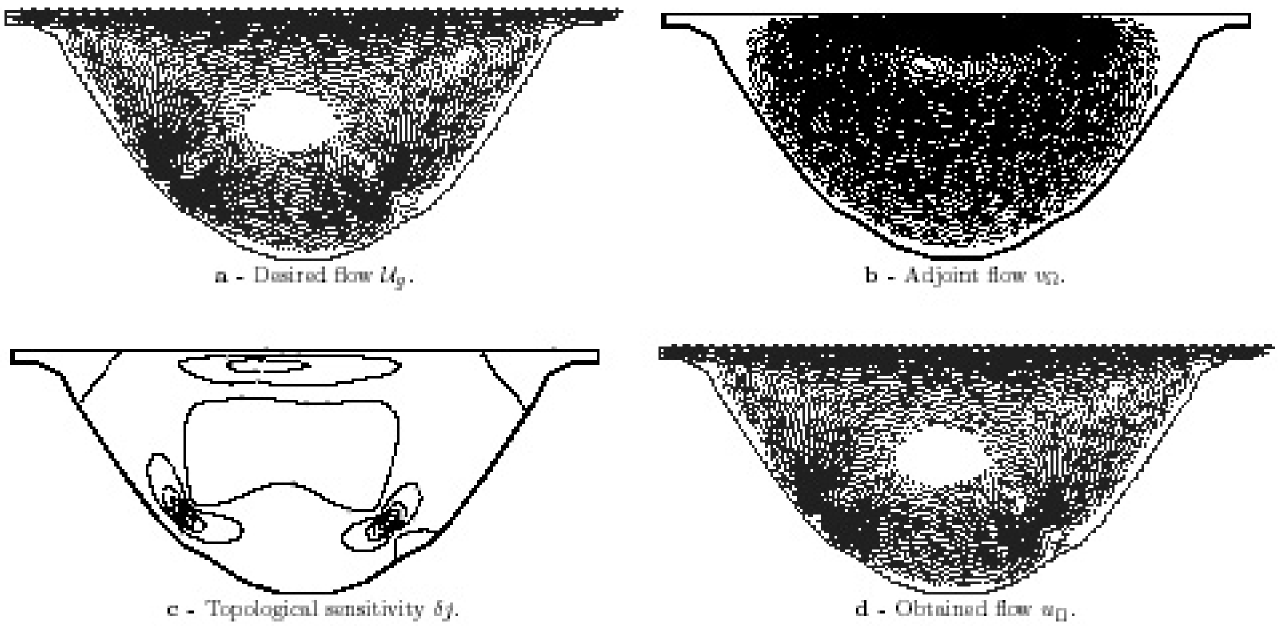


FIGURE 5. Two injectors case.

6.2. **Test 2: dynamic aeration process in an eutrophized lake**

Here, the computational domain Ω is an eutrophized lake. In this example, we treat the water eutrophication phenomena by dynamic aeration process. It consists in inserting some injector holes ω_k at the lower layer of

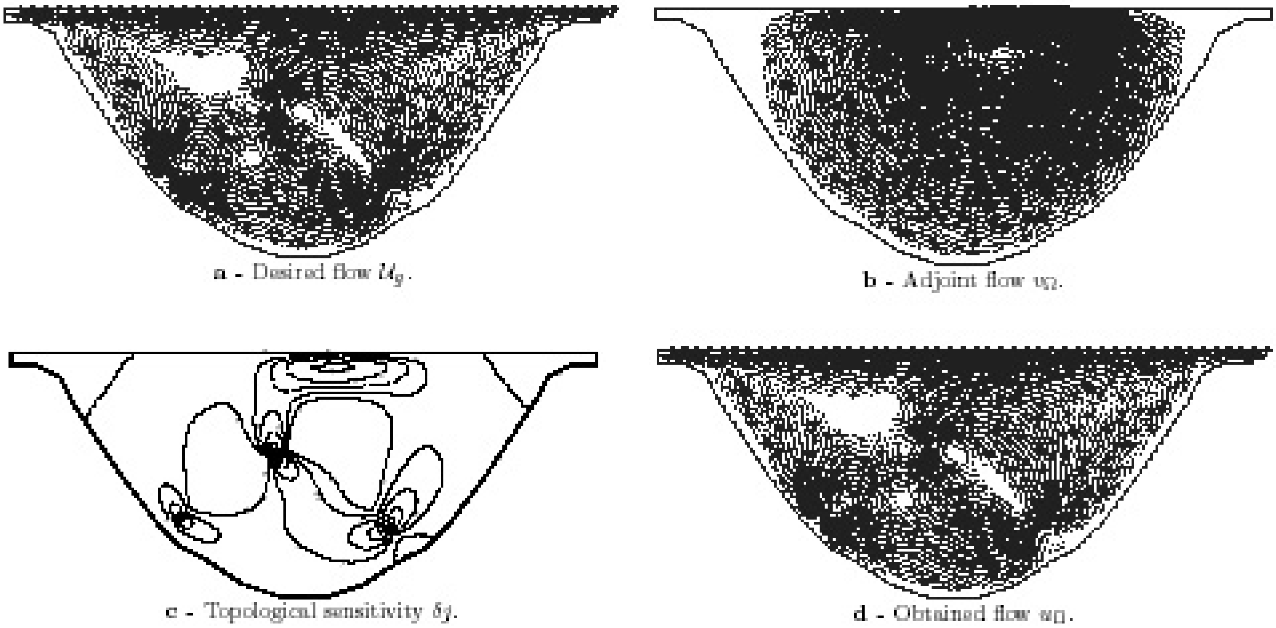


Figure 6. Three injectors case.

the lake in order to create a motion mixing the bottom water with the well oxygenated water from the top. We suppose that a "good" lake oxygenation can be described by a target velocity $\mathcal{U}_g$. Our aim in this test, is to determine the optimal location in Ω of some injector holes ω_k in order to minimize the function $\int_{\Omega_l} |u_\Omega - \mathcal{U}_g|^2 \, \mathrm{d}x$, with $\Omega_l = \Omega \backslash \{\cup_{k=1}^{l} \omega_k\}$.

After only sixth iterations, we obtain a velocity (see Fig. 7) approaching the objective flow $\mathcal{U}_g$. We present for each iteration $l = 1, 6$ the injector location and the obtained flow u_{Ω_l}. Then, Figure 7 shows the initial, desired and the obtained flow. In order to have more idea of what is happening, we represent on Figure 8 several intermediate injectors locations and velocity obtained during the optimization process.

This work can be considered as a preliminary step to study the transient Navier-Stokes problem.

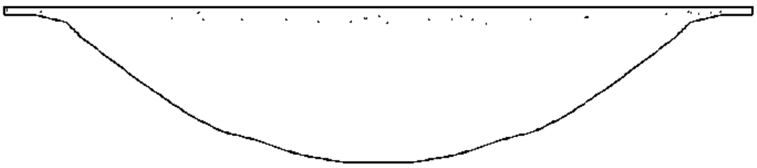
7. Variations of the bilinear and linear form

We now turn to the proof of the main results given by Theorem 5.1. We will use the result given in [15, 29], which is recalled in Section 2. More precisely, we will prove in Sections 7.3 and 7.4 that there exist a bilinear form $\delta a \in \mathcal{L}_2(\mathcal{V}_R)$ and a linear form $\delta l \in \mathcal{L}(\mathcal{V}_R)$ such that

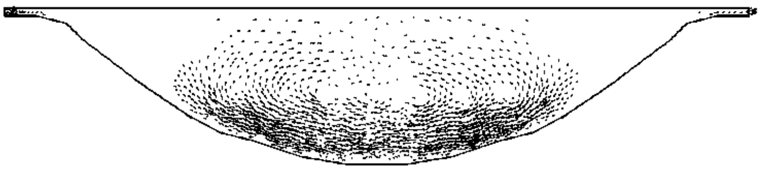
$$\|a_\varepsilon - a_0 - \varepsilon \delta a\|_{\mathcal{L}_2(\mathcal{V}_R)} = O\left(\varepsilon^{3/2}\right), \tag{76}$$

$$\|l_\varepsilon - l_0 - \varepsilon \delta l\|_{\mathcal{L}(\mathcal{V}_R)} = O\left(\varepsilon^{3/2}\right). \tag{77}$$

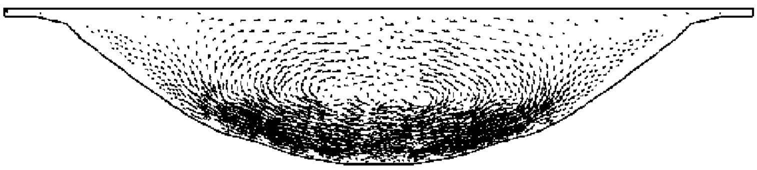
First we need some definitions and preliminary lemmas.



a - The initial flow u_Ω.



b - The desired flow $\mathcal{U}_g$.



c - The obtained flow u_Ω (after sixth iterations).

FIGURE 7. The initial, desired and obtained flow.

7.1. **Definitions**

Let $\mathcal{O}$ be a bounded open domain of $\mathbb{R}^3$ and $\partial\mathcal{O}$ its boundary, assumed polygonal and simply connected.

• $H^m(\mathcal{O})^3$ stands for the Hilbert Sobolev space of order m, where m is a positive integer. It is provided with the norm

$$\|u\|^2_{m,\mathcal{O}} = \sum_{k=0}^{m} |u|^2_{k,\mathcal{O}} \tag{78}$$

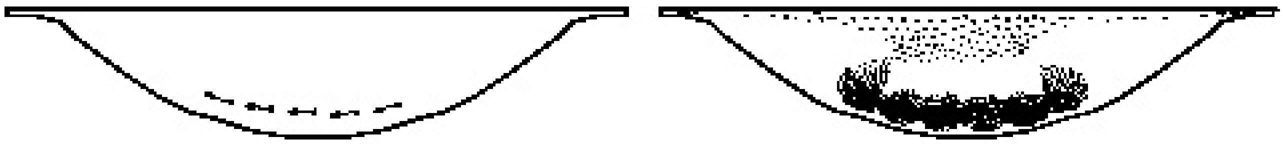
where the semi-norms $|.|_{k,\mathcal{O}}$ are defined

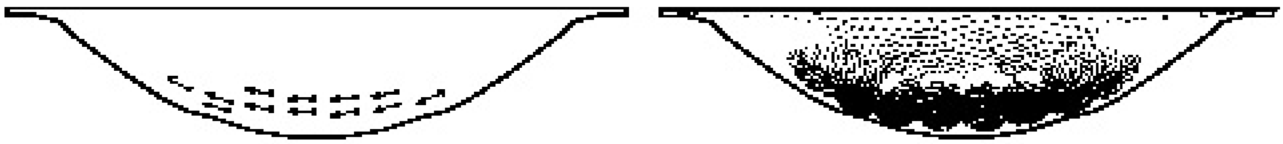
$$|u|^2_{k,\mathcal{O}} = \sum_{|\alpha|=k} \int_{\mathcal{O}} |\partial_\alpha u|^2 \,\mathrm{d}x. \tag{79}$$

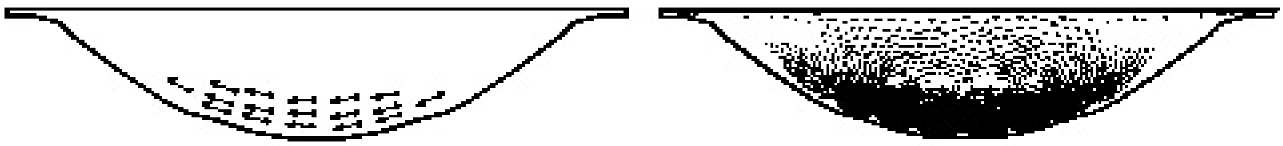
• The usual space of traces (of $H^1(\mathcal{O})$ elements) on the boundary of $\mathcal{O}$ is denoted $H^{1/2}(\partial\mathcal{O})$, and its norm is denoted by $\|.\|_{1/2,\partial\mathcal{O}}$. The subspace

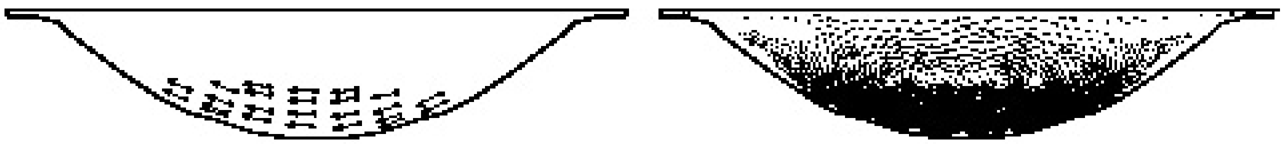
$$H_V^{1/2}(\partial\mathcal{O})^3 = \left\{\varphi \in H^{1/2}(\partial\mathcal{O})^3; \quad \int_{\partial\mathcal{O}} \varphi.n\mathrm{d}\gamma(x) = 0\right\} \tag{80}$$

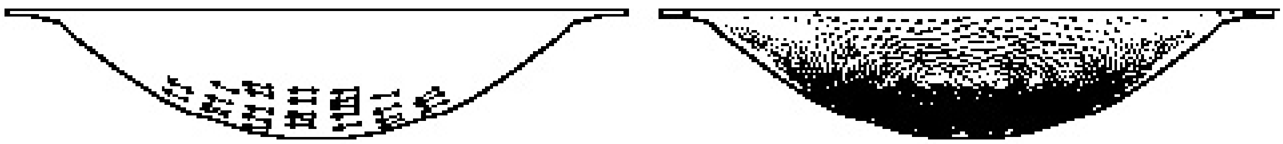
is equipped with the norm induced by $H^{1/2}(\partial\mathcal{O})^3$ and $H_V^{-1/2}(\partial\mathcal{O})^3$ denotes its dual space.


a - First iteration.


b - Second iteration.


c - Third iteration.


d - Fourth iteration.


e - Fifth iteration.

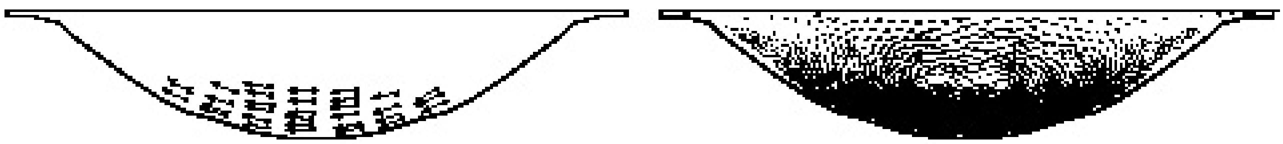

f - Sixth iteration.

FIGURE 8. Injectors locations and velocity obtained during optimization process.

• Let $\varepsilon > 0$; for a given function u defined on $\mathcal{O}$, we define the function $\tilde{u}$ on $\tilde{\mathcal{O}} := \mathcal{O}/\varepsilon$ by

$$\tilde{u}(y) = u(x), \quad y = x/\varepsilon.$$

Using that $Du(x) = D\tilde{u}(y)/\varepsilon$ and the definition (79), we obtain

$$|u|_{1,\mathcal{O}}^2 = \int_{\mathcal{O}} |Du(x)|^2 \, \mathrm{d}x = 1/\varepsilon \int_{\tilde{\mathcal{O}}} |D\tilde{u}(y)|^2 \, \mathrm{d}y,$$

hence

$$|u|_{1,\mathcal{O}} = \varepsilon^{1/2} \, |\tilde{u}|_{1,\tilde{\mathcal{O}}} \, . \tag{81}$$

Similarly, we have

$$\|u\|_{0,\mathcal{O}} = \varepsilon^{3/2} \, \|\tilde{u}\|_{0,\tilde{\mathcal{O}}} \, . \tag{82}$$

7.2. Preliminary lemmas

The aim of this section is to give some technical results which will be used in Sections 7.3 and 7.4. Let us begin by recalling some estimates describing the behavior of the two parts S_U and L_U of the fundamental solution U, see (37).

Lemma 7.1 [15]**.** *For $\phi \in H_V^{1/2}(\partial\omega)^3$; let U, P be the solution to the Stokes exterior problem*

$$\begin{cases} -\nu\Delta U + \nabla Q = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ \operatorname{div} U = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ U = 0 & \text{at infinity} \\ U = \phi & \text{on } \partial\omega. \end{cases} \tag{83}$$

The function U is splited into

$$U(y) = S_U(y) + L_U(y) \quad \forall y \in \mathbb{R}^3 \backslash \overline{\omega},$$
$$\text{with } \; S_U(y) = G_U(y) \int_{\partial\omega} T(x) \, \mathrm{d}\gamma(x)$$

where $G_U(y)$ is defined in (32) and $T \in H_V^{-1/2}(\partial\omega)^3$ is the unique solution [11] *to*

$$\int_{\partial\omega} G_U(y-x) T(x) \, \mathrm{d}\gamma(x) = \phi(y), \quad \forall y \in \partial\omega. \tag{84}$$

There exists a constant $c > 0$, independent of ϕ and ε, such that

$$\begin{aligned} \|S_U\|_{0,C(R/(2\varepsilon),R/\varepsilon)} &\le c\varepsilon^{-1/2} \|\phi\|_{1/2,\partial\omega} \\ \|S_U\|_{0,D_\varepsilon/\varepsilon} &\le c\varepsilon^{-1/2} \|\phi\|_{1/2,\partial\omega} \\ |S_U|_{1,C(R/(2\varepsilon),R/\varepsilon)} &\le c\varepsilon^{1/2} \|\phi\|_{1/2,\partial\omega} \\ |S_U|_{1,D_\varepsilon/\varepsilon} &\le c \|\phi\|_{1/2,\partial\omega} \\ \|L_U\|_{0,C(R/(2\varepsilon),R/\varepsilon)} &\le c\varepsilon^{1/2} \|\phi\|_{1/2,\partial\omega} \\ |L_U|_{1,C(R/(2\varepsilon),R/\varepsilon)} &\le c\varepsilon^{3/2} \|\phi\|_{1/2,\partial\omega} \\ \|L_U\|_{1,D_\varepsilon/\varepsilon} &\le c \|\phi\|_{1/2,\partial\omega} \, . \end{aligned} \tag{85}$$

Lemma 7.2. *For a given $\varepsilon > 0$ and $\varphi \in H_V^{1/2}(\Gamma_R)^3$, let v_ε, q_ε be the solution to the problem*

$$\begin{cases} \alpha v_\varepsilon - \nu\Delta v_\varepsilon + \nabla q_\varepsilon = 0 & \text{in } D_\varepsilon \\ \operatorname{div} v_\varepsilon = 0 & \text{in } D_\varepsilon \\ v_\varepsilon = \varphi & \text{on } \Gamma_R \\ v_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{86}$$

Then, there exists two constants $c > 0$ (independent of φ and ε) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$, we have

$$\|v_\varepsilon\|_{1,D_\varepsilon} \leq c\,\|\varphi\|_{1/2,\Gamma_R}\,. \tag{87}$$

Proof. For all $\varepsilon > 0$, problem (86) is well posed and the solution $v_\varepsilon \in H^1(D_\varepsilon)^3$. Particularly for a given $\varepsilon_0 > 0$, $v_{\varepsilon_0} \in H^1(D_{\varepsilon_0})^3$ and there exists a constant $c > 0$ such that

$$\|v_{\varepsilon_0}\|_{1,D_{\varepsilon_0}} \leq c\,\|\varphi\|_{1/2,\Gamma_R}\,.$$

Let $\varepsilon_1 < \varepsilon_0$ be such that $D_{\varepsilon_0} \subset D_\varepsilon$ for all $\varepsilon < \varepsilon_1$, and let $\tilde{v}_{\varepsilon_0}$ be the extension of v_{ε_0} to D_ε by 0. The function v_ε is solution to the Quasi-Stokes problem (86), it can be shown also as a solution to the following minimization problem

$$\min_{v\in\mathcal{U}} \left\{\alpha\,\|v\|_{0,D_\varepsilon} + \nu\,|v|_{1,D_\varepsilon}\right\}$$

where $\mathcal{U} = \left\{v \in H^1(D_\varepsilon)^3;\ v = \varphi \text{ on } \Gamma_R,\ \operatorname{div} v = 0 \text{ in } D_\varepsilon \text{ and } v = 0 \text{ on } \partial\omega_\varepsilon\right\}$. Hence, for all $\varepsilon < \varepsilon_1$ we have

$$\|v_\varepsilon\|_{1,D_\varepsilon} \leq \|\tilde{v}_{\varepsilon_0}\|_{1,D_\varepsilon} = \|v_{\varepsilon_0}\|_{1,D_{\varepsilon_0}} \leq c\,\|\varphi\|_{1/2,\Gamma_R}\,. \tag{88}$$

□

Lemma 7.3. *For $\varepsilon > 0$ and $\psi \in H^1(D_0)^3$ such that,* $\operatorname{div}\psi = 0$*, let u_ε, p_ε be the solution to the Quasi-Stokes problem*

$$\begin{cases} \alpha u_\varepsilon - \nu\Delta u_\varepsilon + \nabla p_\varepsilon = 0 & \text{in } D_\varepsilon \\ \operatorname{div} u_\varepsilon = 0 & \text{in } D_\varepsilon \\ u_\varepsilon = 0 & \text{on } \Gamma_R \\ u_\varepsilon = \psi & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{89}$$

Then, there exists two constants $c > 0$ (independent of ψ and ε) and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} |u_\varepsilon|_{1,C(R/2,R)} &\leq c\varepsilon\,\|\psi(\varepsilon y)\|_{1/2,\partial\omega} \\ \|u_\varepsilon\|_{0,D_\varepsilon} &\leq c\varepsilon\,\|\psi(\varepsilon y)\|_{1/2,\partial\omega} \\ |u_\varepsilon|_{1,D_\varepsilon} &\leq c\varepsilon^{1/2}\,\|\psi(\varepsilon y)\|_{1/2,\partial\omega}\,. \end{aligned} \tag{90}$$

Proof. First, we denote by $(V_\varepsilon, Q_\varepsilon)$ the solution to the exterior problem (83) obtained using $\phi(y) = \psi(\varepsilon y)$ on the boundary $\partial\omega$.

Posing $v_\varepsilon = V_\varepsilon(x/\varepsilon)$, then the function $w_\varepsilon = v_\varepsilon - u_\varepsilon$ itself is the solution to

$$\begin{cases} \alpha w_\varepsilon - \nu\Delta w_\varepsilon + \nabla s_\varepsilon = \alpha v_\varepsilon & \text{in } D_\varepsilon \\ \operatorname{div} w_\varepsilon = 0 & \text{in } D_\varepsilon \\ w_\varepsilon = v_\varepsilon & \text{on } \Gamma_R \\ w_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{91}$$

Thanks to Lemma 7.2 and elliptic regularity it can be shown that there exists $c > 0$ and $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$

$$\|w_\varepsilon\|_{1,D_\varepsilon} \leq c\left(\|v_\varepsilon\|_{1/2,\Gamma_R} + \alpha\,\|v_\varepsilon\|_{0,D_\varepsilon}\right). \tag{92}$$

In the other hand, we have

$$\|v_\varepsilon\|_{1/2,\Gamma_R} \leq \|v_\varepsilon\|_{1,C(R/2,R)} \leq \|v_\varepsilon\|_{0,C(R/2,R)} + |v_\varepsilon|_{1,C(R/2,R)}\,.$$

From (81), (82) and Lemma 7.1, we obtain

$$\begin{aligned} \|v_\varepsilon\|_{0,C(R/2,R)} &= \varepsilon^{3/2}\|V_\varepsilon\|_{0,C(R/2\varepsilon,R/\varepsilon)} \le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega}, \\ |v_\varepsilon|_{1,C(R/2,R)} &= \varepsilon^{1/2}|V_\varepsilon|_{1,C(R/2\varepsilon,R/\varepsilon)} \le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega}. \end{aligned}$$

Hence,

$$\|v_\varepsilon\|_{1/2,\Gamma_R} \le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega}. \tag{93}$$

Similarly we get

$$\begin{aligned} \|v_\varepsilon\|_{0,D_\varepsilon} &= \varepsilon^{3/2}\|V_\varepsilon\|_{0,D_\varepsilon/\varepsilon} \le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega}, \\ |v_\varepsilon|_{1,D_\varepsilon} &= \varepsilon^{1/2}|V_\varepsilon|_{1,D_\varepsilon/\varepsilon} \le c\varepsilon^{1/2}\|\psi(\varepsilon y)\|_{1/2,\partial\omega}. \end{aligned} \tag{94}$$

Relation (92), (93) and (94) implies that

$$\|w_\varepsilon\|_{1,D_\varepsilon} \le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega}.$$

The desired results follows from the following inequalities

$$\begin{aligned} |u_\varepsilon|_{1,C(R/2,R)} = |v_\varepsilon - w_\varepsilon|_{1,C(R/2,R)} &\le |v_\varepsilon|_{1,C(R/2,R)} + |w_\varepsilon|_{1,C(R/2,R)} \\ &\le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega} \\ \|u_\varepsilon\|_{0,D_\varepsilon} \le \|v_\varepsilon\|_{0,D_\varepsilon} + \|w_\varepsilon\|_{0,D_\varepsilon} &\le c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega} \\ |u_\varepsilon|_{1,D_\varepsilon} \le |v_\varepsilon|_{1,D_\varepsilon} + |w_\varepsilon|_{1,D_\varepsilon} &\le c\varepsilon^{1/2}\|\psi(\varepsilon y)\|_{1/2,\partial\omega} + c\varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega} \\ &\le c\varepsilon^{1/2}\|\psi(\varepsilon y)\|_{1/2,\partial\omega}. \end{aligned}$$

□

The following lemma summarize the results shown in Lemmas 7.2 and 7.3.

Lemma 7.4. *For a given* $\varepsilon > 0$, $h_\varepsilon \in L^2(D_\varepsilon)^3$, $\varphi \in H_V^{1/2}(\Gamma)^3$ *and* $\psi \in H^1(D_0)^3$ *such that ,* $\operatorname{div}\psi = 0$, *let* $v_\varepsilon, q_\varepsilon$ *be the solution to the Quasi-Stokes problem*

$$\begin{cases} \alpha v_\varepsilon - \nu\Delta v_\varepsilon + \nabla q_\varepsilon = h_\varepsilon & \text{in } D_\varepsilon \\ \operatorname{div} v_\varepsilon = 0 & \text{in } D_\varepsilon \\ v_\varepsilon = \varphi & \text{on } \Gamma_R \\ v_\varepsilon = \psi & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{95}$$

Then, there exists a constant $c > 0$ *(independent of* φ, ψ *and* ε*), and* $\varepsilon_1 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} |v_\varepsilon|_{1,C(R/2,R)} &\le c\left(\|\varphi\|_{1/2,\Gamma_R} + \varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega} + \|h_\varepsilon\|_{0,D_\varepsilon}\right) \\ \|v_\varepsilon\|_{0,D_\varepsilon} &\le c\left(\|\varphi\|_{1/2,\Gamma_R} + \varepsilon\|\psi(\varepsilon y)\|_{1/2,\partial\omega} + \|h_\varepsilon\|_{0,D_\varepsilon}\right) \\ |v_\varepsilon|_{1,D_\varepsilon} &\le c\left(\|\varphi\|_{1/2,\Gamma_R} + \varepsilon^{1/2}\|\psi(\varepsilon y)\|_{1/2,\partial\omega} + \|h_\varepsilon\|_{0,D_\varepsilon}\right). \end{aligned} \tag{96}$$

Proof. The proof follows easily by combining Lemmas 7.2 and 7.3 and using the linearity of the Quasi-Stokes operator. □

7.3. Variation of the bilinear form

Let us now compute the variation of the bilinear form a_ε with respect to the creation of a small hole ω_ε in the domain Ω. Then, according to (20), we have

$$a_\varepsilon(u,v) - a_0(u,v) = \int_{\Gamma_R} (T_\varepsilon - T_0)u.v \ \mathrm{d}\gamma(x). \tag{97}$$

So, we first need to compute the operator T_ε variation. To this aim, we introduce some notations.

For $\varphi \in H_V^{1/2}(\Gamma_R)$, recall that $u_\varepsilon^{0,\varphi}$ is the solution to (13) or (14) if $\varepsilon = 0$. We denote by $(U^{0,\varphi}, P^{0,\varphi})$ the solution to the exterior problem (83) with $\phi = u_0^{0,\varphi}(x_0)$ on the boundary $\partial\omega$.

As in (37), $(U^{0,\varphi}, P^{0,\varphi})$ can be decomposed into two parts:

$$U^{0,\varphi} = S_U^{0,\varphi} + L_U^{0,\varphi}, \quad P^{0,\varphi} = S_P^{0,\varphi} + L_P^{0,\varphi} \tag{98}$$

with $S_U^{0,\varphi} = G_U(y)A(u_0^{0,\varphi}(x_0))$, $S_P^{0,\varphi} = G_P(y)A(u_0^{0,\varphi}(x_0))$ are the dominant parts respectively of $U^{0,\varphi}$ and $P^{0,\varphi}$.

Now, let $R_U^{0,\varphi}$ be the solution to the associated interior problem

$$\begin{cases} \alpha R_U^{0,\varphi} - \nu\Delta R_U^{0,\varphi} + \nabla R_P^{0,\varphi} = \alpha S_U^{0,\varphi} & \text{in } D_0 \\ \operatorname{div} R_U^{0,\varphi} = 0 & \text{in } D_0 \\ R_U^{0,\varphi} = S_U^{0,\varphi} & \text{on } \Gamma_R. \end{cases} \tag{99}$$

Then, the linear operator δT (independent of ε) is defined as follows:

$$\begin{aligned} \delta T : H_V^{1/2}(\Gamma_R)^3 &\longrightarrow H_V^{-1/2}(\Gamma_R)^3 \\ \varphi &\to \delta T\varphi = \sigma(R_U^{0,\varphi} - S_U^{0,\varphi}).n. \end{aligned} \tag{100}$$

Proposition 7.1. *The operator T_ε has the following asymptotic expansion*

$$\|T_\varepsilon - T_0 - \varepsilon\delta T\|_{\mathcal{L}(H_V^{1/2}(\Gamma_R)^3; H_V^{-1/2}(\Gamma_R)^3)} = O\left(\varepsilon^{3/2}\right). \tag{101}$$

Proof. As we have shown in Section 5 (see (37), (38), (40)); for $y = x/\varepsilon$, we have

$$S_U^{0,\varphi}(x/\varepsilon) = \varepsilon S_U^{0,\varphi}(x),\ L_U^{0,\varphi}(y) = O\left(1/\|y\|^2\right),\ S_P^{0,\varphi}(x/\varepsilon) = \varepsilon^2 S_P^{0,\varphi}(x),\ \text{and } L_P^{0,\varphi}(y) = O\left(1/\|y\|^3\right).$$

Next, for simplicity, we may drop the subscripts $(.)^{0,\varphi}$. Let

$$\psi_\varepsilon(x) = (T_\varepsilon - T_0 - \varepsilon\delta T)\,\varphi(x). \tag{102}$$

We have

$$\begin{aligned} \psi_\varepsilon &= \sigma(u_\varepsilon - u_0).n - \varepsilon\sigma(R_U - S_U).n \\ &= (\nu\nabla u_\varepsilon - p_\varepsilon I).n - (\nu\nabla u_0 - p_0 I).n - \varepsilon\left[(\nu\nabla R_U - R_P I).n - (\nu\nabla S_U - S_P I).n\right]. \end{aligned}$$

Posing

$$\begin{aligned} w_\varepsilon &= u_\varepsilon - u_0 + U(x/\varepsilon) - \varepsilon R_U \\ s_\varepsilon &= p_\varepsilon - p_0 + 1/\varepsilon P(x/\varepsilon) - \varepsilon R_P(x). \end{aligned} \tag{103}$$

Then, ψ_ε is written as

$$\psi_\varepsilon = \nu\nabla(w_\varepsilon(x) - L_U(x/\varepsilon)).n - (s_\varepsilon(x) - 1/\varepsilon\, L_P(x/\varepsilon))I.n = \sigma(w_\varepsilon(x) - L_U(x/\varepsilon)).n.$$

The functions w_ε, s_ε are solution to

$$\begin{cases} \alpha w_\varepsilon - \nu\Delta w_\varepsilon + \nabla s_\varepsilon = \alpha L_U(x/\varepsilon) & \text{in } D_\varepsilon \\ \operatorname{div} w_\varepsilon = 0 & \text{in } D_\varepsilon \\ w_\varepsilon = U(x/\varepsilon) - \varepsilon R_U(x) & \text{on } \Gamma_R \\ w_\varepsilon = -u_0(x) + u_0(x_0) - \varepsilon R_U(x) & \text{on } \partial\omega_\varepsilon. \end{cases} \tag{104}$$

In order to use Lemma 7.4, we need to estimate the right hand side terms. For the boundary terms, we will proceed in the same way as in [24] for the Stokes problem.

• On Γ_R, due to $R_U(x) = S_U(x)$, we have

$$U(x/\varepsilon) - \varepsilon R_U(x) = L_U(x/\varepsilon)$$

and from the definitions of the functions U and R_U, we get

$$\int_{\Gamma_R} (U(x/\varepsilon) - \varepsilon R_U(x)) .n \,\mathrm{d}\gamma(x) = 0.$$

Using (81), (82) and Lemma 7.1, we check that

$$\begin{aligned} \|U(x/\varepsilon) - \varepsilon R_U(x)\|_{1/2,\Gamma_R} &= \|L_U(x/\varepsilon)\|_{1/2,\Gamma_R} \\ &\le c\,\|L_U(x/\varepsilon)\|_{1,C(R/2,R)} \\ &\le c\left(\|L_U(x/\varepsilon)\|_{0,C(R/2,R)} + |L_U(x/\varepsilon)|_{1,C(R/2,R)}\right) \\ &\le c\left(\varepsilon^{3/2}\,\|L_U(y)\|_{0,C(R/2\varepsilon,R/\varepsilon)} + \varepsilon^{1/2}\,|L_U(y)|_{1,C(R/2\varepsilon,R/\varepsilon)}\right) \\ &\le c\varepsilon^2\,\|u_0(x_0)\|_{1/2,\partial\omega} \\ &\le c\varepsilon^2\,\|\varphi\|_{1/2,\Gamma_R}. \end{aligned} \tag{105}$$

• On $\partial\omega_\varepsilon$, let $\theta_\varepsilon(x) = (-u_0(x) + u_0(x_0) - \varepsilon R_U(x))/\varepsilon$, we have $\operatorname{div}\,\theta_\varepsilon = 0$ in D_0 and for small ε

$$\begin{aligned} \|\theta_\varepsilon(\varepsilon y)\|_{1/2,\partial\omega} &\le c\,\|\theta_\varepsilon(\varepsilon y)\|_{1,\omega} \\ &\le c\left\|\frac{u_0(\varepsilon y) - u_0(x_0}{\varepsilon} + R_U(\varepsilon y)\right\|_{1,\omega} \\ &\le c\left(\|u_0\|_{\mathcal{C}^2(B(0,R/2))} + \|R_U\|_{\mathcal{C}^1(B(0,R/2))}\right) \\ &\le c\left(\|\varphi\|_{1/2,\Gamma_R} + \|S_U\|_{1/2,\Gamma_R}\right) \\ &\le c\,\|\varphi\|_{1/2,\Gamma_R}. \end{aligned}$$

Then, using Lemmas 7.4 and 7.1 we get

$$\begin{aligned} |w_\varepsilon|_{1,C(R/2,R)} &\le c\left(\varepsilon^2\,\|\varphi\|_{1/2,\Gamma_R} + \varepsilon\,\|\theta_\varepsilon(\varepsilon y)\|_{1/2,\partial\omega} + \alpha\,\|L_U(x/\varepsilon)\|_{0,D_\varepsilon}\right) \\ &\le c\varepsilon^{3/2}\,\|\varphi\|_{1/2,\Gamma_R}. \end{aligned} \tag{106}$$

Now, noting that $\operatorname{div}\,(w_\varepsilon(x) - L_U(x/\varepsilon)) = 0$ in $C(R/2,R)$ and $w_\varepsilon(x) = L_U(x/\varepsilon)$ on Γ_R.

Then, from the fact that

$$\alpha(w_\varepsilon(x) - L_U(x/\varepsilon)) - \nu\Delta(w_\varepsilon(x) - L_U(x/\varepsilon)) + \nabla(s_\varepsilon(x) - 1/\varepsilon\,L_P(x/\varepsilon)) = \alpha L_U(x/\varepsilon) \text{ in } C(R/2,R),$$

we deduce

$$\begin{aligned} \|\psi\|_{-1/2,\Gamma_R} &= \|\sigma(w_\varepsilon(x) - L_U(x/\varepsilon)).n\|_{-1/2,\Gamma_R} \\ &\le c\left(|w_\varepsilon(x) - L_U(x/\varepsilon)|_{1,C(R/2R)} + \|L_U(x/\varepsilon)\|_{0,C(R/2R)}\right). \end{aligned}$$

Finally, due to (106), (81), (82) and Lemma 7.1, we obtain

$$\|\psi\|_{-1/2,\Gamma_R} \leq c\varepsilon^{3/2}\|\varphi\|_{1/2,\Gamma_R}.$$

Hence

$$\|(T_\varepsilon - T_0 - \varepsilon\delta T)\varphi\|_{1/2,\Gamma_R} = O\left(\varepsilon^{3/2}\right). \qquad \square$$

Proposition 7.2. *Let*

$$\delta a(u,v) = \int_{\Gamma_R} \delta T u.v \, \mathrm{d}\gamma(x) \quad \forall u, v \in \mathcal{V}_R. \tag{107}$$

The asymptotic expansion of the linear form a_ε is given by

$$\|a_\varepsilon - a_0 - \varepsilon\delta a\|_{\mathcal{L}_2(\mathcal{V}_R)} = O\left(\varepsilon^{3/2}\right). \tag{108}$$

7.4. Variation of the linear form

We search now to compute the variation of the linear form l_ε. For that purpose, we will use the same technique as in the preceding section. The unique difference comes from the boundary condition imposed on $\partial\omega$ to the solution of the exterior problem. Indeed, for the study of the bilinear form we have used $U^{0,\varphi} = u_0^{0,\varphi}(x_0)$ but for the study of the linear form we will use $U^{f,0} = u_0^{f,0}(x_0)$. As consequence, estimations involving $\|\varphi\|_{1/2,\Gamma_R}$ will be replaced by estimations involving $\|f\|_{2,D_0}$.

The variation of the linear form l_ε reads

$$l_\varepsilon(v) - l_0(v) = \int_{\Gamma_R} (f_\varepsilon - f_0).v \, \mathrm{d}\gamma(x). \tag{109}$$

First, we denote by $(U^{f,0}, P^{f,0})$ the solution to the exterior problem (83) corresponding to the boundary condition $\phi = u_0^{f,0}(x_0)$ on $\partial\omega$, with $u_0^{f,0}(x_0)$ is the solution to (13) or (14) if $\varepsilon = 0$.

Using the same decomposition like in (38), we have

$$U^{f,0} = S_U^{f,0} + L_U^{f,0}, \quad P^{f,0} = S_P^{f,0} + L_P^{f,0} \tag{110}$$

with $S_U^{f,0}(y) = G_U(y)A(u_0^{f,0}(x_0))$, $S_P^{f,0}(y) = G_P(y)A(u_0^{f,0}(x_0))$.

Now, let $(R_U^{f,0}, R_P^{f,0})$ be the associated solution to (41) with $R_U^{f,0} = S_U^{f,0}$ on Γ_R.

Then the linear form δf (independent of ε) is given by

$$\delta f = \sigma(R_U^{f,0} - S_U^{f,0}).n. \tag{111}$$

Proposition 7.3. *Let $f \in H^2(\Omega)^3$. The asymptotic expansion of f_ε is given by*

$$\|f_\varepsilon - f_0 - \varepsilon\delta f\|_{-1/2,\Gamma_R} = O\left(\varepsilon^{3/2}\right). \tag{112}$$

Proof. We use the same proof as in Proposition 7.1 with

$$\begin{aligned} w_\varepsilon &= u_\varepsilon^{f,0} - u_0^{f,0} + U^{f,0}(x/\varepsilon) - \varepsilon R_U^{f,0}, \\ \theta_\varepsilon(x) &= \left(-u_0^{f,0}(x) + u_0^{f,0}(x_0) - \varepsilon R_U^{f,0}(x)\right)/\varepsilon. \end{aligned}$$

In this case, elliptic regularity implies

$$\left|u_0^{f,0}(x_0)\right| \leq \left\|u_0^{f,0}\right\|_{\mathcal{C}^0(\mathcal{D}_0)} \leq \left\|u_0^{f,0}\right\|_{2,D_0} \leq c\,\|f\|_{0,D_0}$$

and for small ε, we have

$$\begin{aligned}\|\theta_\varepsilon(\varepsilon y)\|_{1/2,\partial\omega} &\leq c\,\|\theta_\varepsilon(\varepsilon y)\|_{1,\omega}\\ &\leq c\left\|\frac{u_0^{f,0}(\varepsilon y)-u_0^{f,0}(x_0}{\varepsilon}+R_U^{f,0}(\varepsilon y)\right\|_{1,\omega}\\ &\leq c\left(\left\|u_0^{f,0}\right\|_{\mathcal{C}^2(B(0,R/2))}+\left\|R_U^{f,0}\right\|_{\mathcal{C}^1(B(0,R/2))}\right)\\ &\leq c\,\|f\|_{2,D_0}\,. \end{aligned}$$ □

Proposition 7.4. *Let* $\delta l(v)=\displaystyle\int_{\Gamma_R}\delta f.v\,\mathrm{d}\gamma(x),\quad v\in\mathcal{V}_R.$

The asymptotic expansion of l_ε *is given by*

$$\|l_\varepsilon-l_0-\varepsilon\delta l\|_{-1/2,\Gamma_R}=O\left(\varepsilon^{3/2}\right). \tag{113}$$

7.5. Proof of Theorem 5.1

Thanks to the previous results given in Propositions 7.2 and 7.4, we deduce that the hypotheses (1) and (2) hold. Then, we are now ready to apply the tools of Section 2. So that, from Theorem 2.1 we have

$$j(\varepsilon)=j(0)+\varepsilon\left[\delta a(u_0,v_0)-\delta l(v_0)+\delta J(u_0)\right]+o(\varepsilon).$$

Due to (31), (98) and (110) we derive

$$U=U^{f,0}+U^{0,\varphi},$$

and

$$\begin{aligned}S_U&=S_U^{f,0}+S_U^{0,\varphi},\\ R_U&=R_U^{f,0}+R_U^{0,\varphi}.\end{aligned}$$

Then, using (100), (111), Propositions 7.2 and 7.4, we obtain

$$\begin{aligned}\delta a(u_0,v_0)-\delta l(v_0)&=\int_{\Gamma_R}\sigma(R_U^{0,\varphi}-S_U^{0,\varphi}).nv_0\ \mathrm{d}\gamma(x)+\int_{\Gamma_R}\sigma(R_U^{f,0}-S_U^{f,0}).nv_0\ \mathrm{d}\gamma(x)\\ &=\int_{\Gamma_R}[(\nu\nabla R_U^{f,\varphi}-R_P^{f,\varphi}I)-(\nu\nabla S_U^{f,\varphi}-S_P^{f,\varphi}I)].nv_0\ \mathrm{d}\gamma(x)\\ &=\int_{\Gamma_R}\sigma(R_U^{f,\varphi}-S_U^{f,\varphi}).nv_0\ \mathrm{d}\gamma(x).\end{aligned}$$

This ends the proof of the theorem. □

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