SHARP SUMMABILITY FOR MONGE TRANSPORT DENSITY
VIA INTERPOLATION

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This paper is concerned with the transport problem, which consists in minimizing

$$\int_\Omega |x - t(x)| \, df^+(x)$$

among the transports, which are the measurable functions $t : \text{spt}(f^+) \rightarrow \text{spt}(f^-)$ such that $t \# f^+ = f^-$, i.e. for any Borel set $B$ it is $f^+(t^{-1}(B)) = f^-(B)$; here $f = f^+ - f^-$ is a $L^1$ function on $\Omega$ with $\int f = 0$, while $\Omega$ is a convex and bounded subset of $\mathbb{R}^N$ (to find more general descriptions of the transport problem, see [1,9]). To each optimal – i.e. minimizing (1) – transport $t$ it is possible to associate a positive measure $\sigma$ on $\Omega$ defined by

$$\langle \sigma, \varphi \rangle := \int_\Omega \left( \int_\Omega \varphi(z) \, d\mathcal{H}^1_{x,t(x)}(z) \right) \, df^+(x)$$

where $\varphi$ is any function in $C_0(\Omega)$ and $\mathcal{H}^1_{x,y}$ is the one-dimensional Hausdorff measure on the segment $xy$. It has been proved (see [1,8]) that there always exist (in this setting) optimal transports and in particular there are invertible optimal transports whose inverse is also an optimal transport for $-f$. A fundamental result, due to [1,8], is that, even if there can be many different optimal transports, all define \textit{via} (2) the same measure $\sigma$, which is then called transport density relative to $f$. This measure is very interesting for the transport problem (for example it plays an important role in [7]), and moreover it represents the connection between this problem and some shape optimization problem (see [3,4]), which can be reduced to the research of a positive measure $\sigma$.

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and a 1-Lipschitz function $u$ solving

$$\begin{cases}
-\text{div } (\sigma Du) = f & \text{on } \Omega \\
|Du| = 1 & \sigma - \text{a.e.}
\end{cases} \tag{3}$$

The relationship between the two problems relies on the fact that the (unique) transport density is also the unique solution of (3) (see [1,3,6]); the functions $u$ solving (3) together with $\sigma$ are also meaningful in the context of the transport problem, they are referred to as Kantorovich potentials. Equation (3) is often referred to as Monge-Kantorovich equation. Thus the study of the regularity of $\sigma$ is useful both for the transport problem and for the shape optimization problem. It was proved (see [1,6,8]) that the fact that $f \in L^1$ implies also that $\sigma \in L^1$.

In this paper we will show some sharp relationship between the summability of $f$ and that of $\sigma$. The problem to derive regularity of $\sigma$ from that of $f$ has already been studied in [5,6] following two different methods: in [6] we used a geometric construction starting from the definition (2), while in [5] the proofs used PDE tools starting from the equivalent definition (3). In the first work it was proved that

$$f \in L^1 \implies \sigma \in L^1, \quad f \in L^\infty \implies \sigma \in L^\infty,$$

$$f \in L^p \implies \sigma \in L^{p-\epsilon} \text{ for any } \epsilon > 0,$$  \tag{4}

and some examples were given in which $f \in L^p$ and $\sigma \notin L^q$ for any $q > p$. Thus it was left open the problem whether or not it is true that $f \in L^p$ implies $\sigma \in L^p$ for $p \neq 1, +\infty$. In the second work this problem was partially solved, since it was proved that

$$f \in L^p \implies \sigma \in L^p \text{ for any } 2 \leq p < +\infty.$$  \tag{5}

Since the cases $p = 1, \infty$ had already been solved in the first work, it was left open only the case with $1 < p < 2$. In this work we will show how the classic Marcinkieicz interpolation result can be used to infer from the results already mentioned the general property for any $p$. Note that this is not trivial since the map associating the transport density $\sigma$ to any function $f$ with $\int f = 0$ is far from being linear or sublinear, as easy examples show; however, this map is 1-homogeneous, as one can hope in view of (6).

The result of this paper is the following

**Theorem A.** For any $1 \leq p \leq +\infty$, if $f \in L^p$ is a function with $\int f = 0$, then the associated transport density $\sigma$ is also in $L^p$. More precisely, there exist a constant $C_p$, depending only on $\Omega$, such that

$$\|\sigma\|_{L^p} \leq C_p \|f\|_{L^p}.$$  \tag{6}

To prove the theorem, we first of all recall the known results we use, the regularity results proved in [5,6] and the Marcinkieicz interpolation result, which can be found for example in [10].

**Theorem 1.** For $p = 1$ and for any $p \geq 2$ there exists a constant $C_p$ depending on $\Omega$ such that, for any $f \in L^p$ with $\int f = 0$,

$$\|\sigma\|_{L^p} \leq C_p \|f\|_{L^p}$$  \tag{7}

where $\sigma$ is the transport density associated to $f$.

**Theorem 2** (Marcinkieicz). If $T : L^1 \to L^1$ is a linear mapping such that, for two suitable constants $M_p$ and $M_q$ with $1 \leq p < q \leq +\infty$,

$$\|T(g)\|_{L^p} \leq M_p \|g\|_{L^p} \quad \text{and} \quad \|T(g)\|_{L^q} \leq M_q \|g\|_{L^q},$$  \tag{8}
then it is also true that for any $s \in (p,q)$

$$\|T(g)\|_{L^s} \leq C M_p^{\frac{s(q-p)}{p(q-1)}} M_q^{\frac{s(q-p)}{p(q-1)}} \|g\|_{L^p},$$

(9)

where $C$ is a geometric constant depending only on $\Omega$.

To prove our result, let us fix now a function $f \in L^p$ with $\int f = 0$ and $1 \leq p \leq +\infty$. We can assume $p \neq 1, +\infty$, since otherwise we already know that $\sigma \in L^p$. Fix also an invertible optimal transport $t$ for $f$ (as we said, this always exists when $f \in L^1$, even though it is not unique). For any function $g \in L^1$, there are of course two uniquely determined measurable functions $\lambda$ and $\nu$ supported respectively on $\text{spt}(f^+)$ and on $\Omega \setminus \text{spt}(f^+)$ such that

$$g = \lambda f^+ + \nu.$$  
(10)

Let us finally define the operator $T : L^1 \to \mathcal{M}(\Omega)$ to which we will apply later the Marcinkiewicz theorem: given any $g \in L^1$ and following the notations of (10), we define

$$T(g) := \int_{\Omega} \lambda(x) f^+(x) \mathcal{H}^1_{xt(x)}(z) \, dx,$$

(11)

which can also be rewritten as

$$(T(g), \varphi) = \int_{x \in \Omega} \left( \int_{z \in \Omega} \varphi(z) \, d\mathcal{H}^1_{xt(x)}(z) \right) \lambda(x) f^+(x) \, dx$$

for any $\varphi \in C_0(\Omega)$. Notice that $T(g)$ is a priori a measure, and that the definition of $T$ depends on the function $f$ we fixed; moreover, we point out that of course $T(f)$ is the transport density $\sigma$ associated to $f$ (just recall (2) and (11)).

We define now $\sigma_1, \sigma_2$ and $f_1, f_2$ (depending on $f$ and $g$) as follows:

$$\begin{align*}
\sigma_1 & := \int_{\Omega} \lambda^+(x) f^+(x) \mathcal{H}^1_{xt(x)}(z) \, dx & f_1 & := \lambda f^+ - (\lambda^+ \circ t^{-1}) f^- \\
\sigma_2 & := \int_{\Omega} \lambda^-(x) f^+(x) \mathcal{H}^1_{xt(x)}(z) \, dx & f_2 & := \lambda f^+ - (\lambda^- \circ t^{-1}) f^-;
\end{align*}$$

(12)

note that also these definitions depend on $f$ and $g$, and that $T(g) = \sigma_1 + \sigma_2$. First we prove the

**Lemma 3.** The function $t : \text{spt}(f^+) \to \text{spt}(f^-)$ is defined $f_i^+ - a.e.$ and it is an optimal transport for the functions $f_i, i = 1, 2$ defined in (12); moreover, each $\sigma_i$ is the transport density associated to $f_i$.

**Proof.** The optimality of a transport is equivalent to the cyclical monotonicity of its graph (see [2,9] to find the definition of the cyclical monotonicity and the proof of this assert). Then the fact that $t$ is optimal for $f$ assures that its graph is monotonically cyclic; thus, given any function $h$ on $\Omega$ with 0 mean and such that $h^+ \ll f^+$, $t$ is defined $h^+ - a.e.$ and it is an optimal transport for $h$ if and only if it is a transport. Then to prove the first part of the assert it is enough to check that $t\#f_i^+ = f_i^-$ for $i = 1, 2$, which is a straightforward consequence of the fact that $t\#f^+ = f^-$ and of the properties of the push-forward. Finally, the fact that each $\sigma_i$ is the transport density associated to $f_i$ follows comparing (12) with the definition (2) of the transport density (replace $f$ and $\sigma$ in (2) by $f_i$ and $\sigma_i$). \hfill $\square$

We can then prove the following

**Lemma 4.** $T : L^1 \to L^1$ is a linear operator.

**Proof.** The fact that $T$ is linear follows immediately from the definition (11); moreover $T(g)$ is a $L^1$ function (recall that a priori we knew it only to be a measure) since it is the sum of the two transport densities $\sigma_i$, thanks to the preceding Lemma, and thanks to Theorem 1 each of these densities is in $L^1$ since so is each $f_i$ – recall (12) and that $g \in L^1$. \hfill $\square$
We prove now the validity of (8) with $p = 1$ and $q = +\infty$ in order to apply the Marcinkiezcz Theorem.

**Lemma 5.** The inequalities (8) hold for $T$ with $p = 1$ and $q = +\infty$; in particular, $M_1 = 2C_1$ and $M_\infty = 2C_\infty$, where the $C_i$’s are the constants of (7).

**Proof.** In view of Lemma 3 and Theorem 1, $\sigma_1$ is the transport density relative to $f_1$ and then $\Vert \sigma_1 \Vert_{L^1} \leq C_1 \Vert f_1 \Vert_{L^1}$; but since $t_\mu f_1^+ = f_1^-$, then $\Vert f_1^+ \Vert_{L^1} = \Vert f_1^- \Vert_{L^1}$ and we infer

$$\Vert \sigma_1 \Vert_{L^1} \leq 2C_1 \Vert f_1^- \Vert_{L^1}.$$ 

In the same way we deduce also $\Vert \sigma_2 \Vert_{L^1} \leq 2C_1 \Vert f_2^- \Vert_{L^1}$. Using now the fact that the supports of the $f_i$’s are essentially disjoint – that is clear from (12) –, we have

$$\Vert T(g) \Vert_{L^1} = \Vert \sigma_1 + \sigma_2 \Vert_{L^1} \leq \Vert \sigma_1 \Vert_{L^1} + \Vert \sigma_2 \Vert_{L^1} \leq 2C_1 \left( \Vert f_1^+ \Vert_{L^1} + \Vert f_2^+ \Vert_{L^1} \right) \leq 2C_1 \Vert g \Vert_{L^1},$$

which gives the first estimate.

On the other hand, to show the $L^\infty$ inequality we note that, thanks to (10) and (12), it is $\Vert f_i \Vert_{L^\infty} \leq \Vert g \Vert_{L^\infty}$ for each $i$. Since $T(g) = \sigma_1 + \sigma_2$, from Lemma 3 and Theorem 1 we infer

$$\Vert T(g) \Vert_{L^\infty} \leq 2C_\infty \Vert g \Vert_{L^\infty},$$

and then also the $L^\infty$ inequality follows. \qed

Thanks to Lemmas 4 and 5, we can apply Theorem 2 to prove (6), recalling that $T(f)$ is the transport density $\sigma$ associated to $f$. Recall now that the function $f \in L^p$ was fixed at the beginning, but the constants $C_p$ we obtained do not depend on $f$, but only on $p$ and $\Omega$. Then the estimate (6) is true, with the same constants, for any function $f \in L^p$.

**References**


