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ANISOTROPIC FUNCTIONS: A GENERICITY RESULT WITH CRYSTALLOGRAPHIC IMPLICATIONS

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Abstract. In the 1950's and 1960's surface physicists/metallurgists such as Herring and Mullins applied ingenious thermodynamic arguments to explain a number of experimentally observed surface phenomena in crystals. These insights permitted the successful engineering of a large number of alloys, where the major mathematical novelty was that the surface response to external stress was **anisotropic**. By examining step/terrace (**vicinal**) surface defects it was discovered through lengthy and tedious experiments that the stored energy density (**surface tension**) along a step edge was a smooth symmetric function β of the azimuthal angle θ to the step, and that the positive function β attains its minimum value at $\theta = \pi/2$ and its maximum value at $\theta = 0$. The function β provided the crucial thermodynamic parameters needed for the engineering of these materials. Moreover the minimal energy configuration of the step is determined by the values of the **stiffness function** $\beta'' + \beta$ which ultimately leads to the magnitude and direction of surface mass flow for these materials. In the 1990's there was a dramatic improvement in electron microscopy which permitted real time observation of the meanderings of a step edge under Brownian heat oscillations. These observations provided much more rapid determination of the relevant thermodynamic parameters for the step edge, even for crystals at temperatures below their **roughening** temperature. Use of these tools led J. Hannon and his coexperimenters to discover that some crystals behave in a highly anti-intuitive manner as their temperature is varied. The present article is devoted to a model described by a class of variational problems. The main result of the paper describes the solutions of the corresponding problem for a generic integrand.

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1. Introduction

It is well-known in surface physics that when a crystalline substance is maintained at a temperature T above its *roughening temperature* T_R then the surface stored energy integrand, usually referred to as *surface tension*, is a smooth function β of the azimuthal angle of orientation θ. Furthermore, β obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta),\ 0 < \beta(\pi/2) \le \beta(\theta) \le \beta(0)$$

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(*cf.* for example [1, 2, 5]). We note that given any point P on an arc in the x, y plane, the convention here is that $\theta(P) \in [-\pi, \pi]$ denotes the angle (measured counterclockwise) that the tangent line at P makes with the positive x-axis. The issue we proceed to discuss is the relative availability of smooth surface tension functions β for which the quantity $\beta'' + \beta$, usually referred to as the *stiffness*, is negative at the angle $\theta = 0$ where β attains its maximum, *i.e.*, $\beta''(0) + \beta(0) < 0$. That condition is present in a type of Silicon crystal which exhibited very unorthodox behavior during experimental observations involving mass transport along the surface of the crystal (*cf.* [3, 4]). This phenomenon led us to undertake the present investigation.

For each function $f : X \to R$ set $\inf(f) = \inf\{f(x) :\ x \in X\}\cdot$

Denote by $\mathcal{M}$ the set of all functions $\beta \in C^2(R)$ which satisfy the following assumption:

(A)

$$\beta(t) \geq 0 \text{ for all } t \in R, \tag{1.1}$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for all } t \in R, \tag{1.2}$$

$$\beta(t) = \beta(-t) \text{ for all } t \in R, \tag{1.3}$$

$$\beta(t + \pi) = \beta(t) \text{ for all } t \in R, \tag{1.4}$$

$$\beta(0) + \beta''(0) \leq 0. \tag{1.5}$$

For each $\beta_1, \beta_2 \in \mathcal{M}$ set

$$\rho(\beta_1, \beta_2) = \sup\left\{|\beta_1^{(i)}(t) - \beta_2^{(i)}(t)| :\ t \in R,\ i = 0, 1, 2\right\}\cdot \tag{1.6}$$

It is not difficult to see that the metric space $(\mathcal{M}, \rho)$ is complete.

Denote by $\mathcal{M}_r$ the set of all $\beta \in \mathcal{M}$ such that

$$\beta(t) > 0 \text{ for all } t \in R, \tag{1.7}$$

$$\beta(0) + \beta''(0) < 0. \tag{1.8}$$

It is obvious that $\mathcal{M}_r$ is nonempty.

Proposition 1.1. *$\mathcal{M}_r$ is an open everywhere dense subset of $(\mathcal{M}, \rho)$.*

Proof. Evidently $\mathcal{M}_r$ is an open subset of $(\mathcal{M}, \rho)$. Let us show that $\mathcal{M}_r$ is an everywhere dense subset of $(\mathcal{M}, \rho)$.

Let $\beta \in \mathcal{M}$ and $\tilde{\beta} \in \mathcal{M}_r$. Then for each natural number n the function $\beta + (n)^{-1}\tilde{\beta} \in \mathcal{M}_r$,

$$(\beta + (n)^{-1}\tilde{\beta})(t) \geq (n)^{-1}\tilde{\beta}(t) > 0$$

for all $t \in R$ and

$$(\beta + (n)^{-1}\tilde{\beta})(0) + (\beta + (n)^{-1}\tilde{\beta})''(0) = \beta(0) + \beta''(0) + [\tilde{\beta}(0) + \tilde{\beta}''(0)]/n < 0.$$

Thus $\beta + n^{-1}\tilde{\beta} \in \mathcal{M}_r$ for all natural numbers n. It is easy to see that $\beta + n^{-1}\tilde{\beta} \to \beta$ as $n \to \infty$ in $(\mathcal{M}, \rho)$. Therefore $\mathcal{M}_r$ is an everywhere dense subset of $(\mathcal{M}, \rho)$. Proposition 1.1 is proved. □

Let $\beta \in \mathcal{M}_r$. Define

$$G_\beta(z) = \beta(\arctan(z))(1 + z^2)^{1/2},\ z \in R. \tag{1.9}$$

Clearly G_β is a continuous function and

$$G_\beta(z) \to \infty \text{ as } z \to \pm\infty. \tag{1.10}$$

Remark 1.1. Note that $\inf(G_\beta) < \beta(0)$ (see [4]).

The classical model for the free energy of certain crystals is given by

$$J(y) = \int_0^S \beta(\theta)\mathrm{d}s$$

where s is arclength and y is a function defined on a fixed interval $[0, L]$ whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L),\ \theta = \arctan(y') \in [-\pi/2, \pi/2].$$

We can rewrite J in the form

$$J(y) = \int_0^L G_\beta(y')\mathrm{d}x.$$

It was shown in [4] that $y \in W^{1,1}(0, L)$ is a minimizer of J if and only if

$$|y'| \in \{z \in R :\ G_\beta(z) = \inf(G_\beta)\} \text{ a.e.}$$

In this paper we show that for a generic function β the set

$$\{z \in R :\ G_\beta(z) = \inf(G_\beta)\} = \{z_\beta, -z_\beta\}$$

where z_β is a unique positive number depending only on β.

Denote by $\mathcal{F}$ the set of all $\beta \in \mathcal{M}_r$ which satisfy the following condition:

There is $z_\beta \in R$ such that

$$G_\beta(z) > G_\beta(z_\beta) \text{ for all } z \in R \setminus \{z_\beta, -z_\beta\}. \tag{1.11}$$

We will establish the following result.

Theorem 1.1. *$\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, \rho)$.*

2. Preliminary results

Proposition 2.1. *Let $\beta \in \mathcal{M}_r$. Then there exist $M_0 > 0$ and a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{M}_r$ and the following assertion holds:*

if $\phi \in \mathcal{U}$, $z \in R$ and $G_\phi(z) \le \inf(G_\phi) + 1$, then $|z| \le M_0$.

Proof. There is $c_0 > 0$ such that

$$\beta(t) \ge c_0 \text{ for all } t \in R. \tag{2.1}$$

There is a neighborhood $\mathcal{U}$ of β in $(\mathcal{M}, \rho)$ such that

$$\mathcal{U} \subset \mathcal{M}_r \tag{2.2}$$

and for each $\phi \in \mathcal{U}$

$$\phi(t) \ge c_0/2 \text{ for all } t \in R, \tag{2.3}$$

$$\phi(0) \le 2\beta(0). \tag{2.4}$$

Assume that $\phi \in \mathcal{U}$, $z \in R$,

$$G_\phi(z) \le \inf(G_\phi) + 1. \tag{2.5}$$

Then (2.3), (2.4) hold. By (1.9), (2.3), (2.5), (2.4)

$$(1 + z^2)^{1/2}c_0/2 \le \phi(\arctan(z))(1 + z^2)^{1/2} = G_\phi(z) \le G_\phi(0) + 1 = \phi(0) + 1 \le 2\beta(0) + 1$$

and

$$|z| \le 2(2\beta(0) + 1)c_0^{-1}.$$

Thus the assertion of Proposition 2.1 holds with $M_0 = 2(2\beta(0) + 1)c_0^{-1}$. □

It is not difficult to see that the next proposition is true.

Proposition 2.2. *Let $\beta \in \mathcal{M}_r$, $\epsilon, M > 0$. Then there exists a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{M}_r$ and the following assertion holds:*

if $\phi \in \mathcal{U}$, $z \in R$, $|z| \leq M$, then

$$|G_\phi(z) - G_\beta(z)| \leq \epsilon.$$

Proposition 2.3. *Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{M}_r$ and the following assertion holds:*

if $\phi_1, \phi_2 \in \mathcal{U}$, $z \in R$,

$$G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1, \tag{2.6}$$

then

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon.$$

Proof. By Proposition 2.1 there exist $M > 0$ and a neighborhood $\mathcal{U}_1$ of β in $(\mathcal{M}, \rho)$ such that

$$\mathcal{U}_1 \subset \mathcal{M}_r \tag{2.7}$$

and the following property holds:

(P1) if $\phi \in \mathcal{U}_1$, $z \in R$, $G_\phi(z) \leq \inf(G_\phi) + 1$, then $|z| \leq M$.

By Proposition 2.2 there is a neighborhood $\mathcal{U}$ of β in $(\mathcal{M}, \rho)$ such that

$$\mathcal{U} \subset \mathcal{U}_1$$

and the following property holds:

(P2) If $\phi_1, \phi_2 \in \mathcal{U}$, $z \in R$, $|z| \leq M$, then

$$|G_{\phi_1}(z) - G_{\phi_2}(z)| \leq \epsilon. \tag{2.8}$$

Now assume that $\phi_1, \phi_2 \in \mathcal{U}$, $z \in R$ and (2.6) holds. By (2.6) and the property (P1),

$$|z| \leq M. \tag{2.9}$$

By (2.9) and property (P2), the inequality (2.8) is true. Proposition 2.3 is proved. □

Proposition 2.4. *Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ such that $\mathcal{U} \subset \mathcal{M}_r$ and for each $\phi \in \mathcal{U}$,*

$$|\inf(G_\phi) - \inf(G_\beta)| \leq \epsilon.$$

Proof. Let a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ be as guaranteed by Proposition 2.3. Let $\phi_1, \phi_2 \in \mathcal{U}$. It is enough to show that

$$\inf(G_{\phi_2}) \leq \inf(G_{\phi_1}) + \epsilon.$$

By the choice of $\mathcal{U}$ and Proposition 2.3 the inequality (2.8) holds for any $z \in R$ satisfying (2.6). This implies that

$$\begin{aligned} \inf(G_{\phi_2}) &\leq \inf\{G_{\phi_2}(z) :\ z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &\leq \inf\{G_{\phi_1}(z) + \epsilon :\ z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &= \epsilon + \inf\{G_{\phi_1}(z) :\ z \in R \text{ and } G_{\phi_1}(z) \leq \inf(G_{\phi_1}) + 1\} \\ &= \epsilon + \inf(G_{\phi_1}) \end{aligned}$$

and

$$\inf(G_{\phi_2}) \le \inf(G_{\phi_1}) + \epsilon.$$

This completes the proof of Proposition 2.4. □

Proposition 2.5. *Let $\beta \in \mathcal{M}_r$, $\bar{z} \in R$,*

$$G_\beta(z) > G_\beta(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}\cdot \tag{2.10}$$

Let $\epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of β in $\mathcal{M}$ and $\delta > 0$ such that $\mathcal{U} \subset \mathcal{M}_r$ and that for each $\phi \in \mathcal{U}$ and each $z \in R$ satisfying

$$G_\phi(z) \le \inf(G_\phi) + \delta \tag{2.11}$$

the inequality

$$\min\{|z - \bar{z}|,\ |z + \bar{z}|\} \le \epsilon$$

is true.

Proof. Let us assume the converse. Then for each natural number n there exist $\phi_n \in \mathcal{M}_r$ and $z_n \in R$ such that

$$\rho(\beta, \phi_n) \le 1/n, \tag{2.12}$$

$$G_{\phi_n}(z_n) \le \inf(G_{\phi_n}) + 1/n \tag{2.13}$$

and

$$\min\{|z_n - \bar{z}|,\ |z_n + \bar{z}|\} > \epsilon. \tag{2.14}$$

By (2.12), (2.13) and Proposition 2.1 the sequence $\{z_n\}_{n=1}^\infty$ is bounded. Extracting a subsequence and reindexing, if necessary, we may assume without loss of generality that there exists

$$z_* = \lim_{n\to\infty} z_n. \tag{2.15}$$

By (2.12) and Proposition 2.4

$$\lim_{n\to\infty} \inf(G_{\phi_n}) = \inf(G_\beta). \tag{2.16}$$

Since the sequence $\{z_n\}_{n=1}^\infty$ is bounded it follows from (2.12) and Proposition 2.2 that

$$\lim_{n\to\infty} [G_{\phi_n}(z_n) - G_\beta(z_n)] = 0. \tag{2.17}$$

It follows from (2.15), (2.17), (2.13), (2.16) that

$$G_\beta(z_*) = \lim_{n\to\infty} G_\beta(z_n) = \lim_{n\to\infty} G_{\phi_n}(z_n) = \lim_{n\to\infty} \inf(G_{\phi_n}) = \inf(G_\beta).$$

Thus

$$G_\beta(z^*) = \inf(G_\beta).$$

By (2.10) either $z_* = \bar{z}$ or $z_* = -\bar{z}$ and

$$\text{either } z_n \to \bar{z} \text{ or } z_n \to -\bar{z} \text{ as } n \to \infty.$$

This contradicts (2.14). The contradiction we have reached proves Proposition 2.5. □

3. A basic lemma

For $\psi \in C^2(R)$ set

$$||\psi||_{C^2(R)} = \sup\left\{|\psi^{(i)}(t)|:\ t \in R,\ i = 0, 1, 2\right\}\cdot$$

Lemma 3.1. *Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$,*

$$\beta(t) > \beta(\pi/2) \text{ for all } t \in [0, \pi/2). \tag{3.1}$$

Then there exist $\phi \in \mathcal{M}_r$, $\bar{z} \in R$ such that

$$\rho(\phi, \beta) \le \epsilon,$$

$$G_\phi(z) > G_\phi(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}\cdot$$

Proof. There is $\bar{z} \in R$ such that

$$G_\beta(\bar{z}) = \inf(G_\beta). \tag{3.2}$$

By Remark 1.1 $\bar{z} \neq 0$. We may assume that

$$\bar{z} > 0. \tag{3.3}$$

Set

$$\bar{\theta} = \arctan(\bar{z}) \in (0, \pi/2). \tag{3.4}$$

There exists $\psi \in C^\infty(R)$ such that

$$0 \le \psi(t) \le 1 \text{ for all } t \in R,\ \psi(t) = 0 \text{ if } |t| \ge 1,\ \psi(t) = 1, \text{ if } |t| \le 1/2. \tag{3.5}$$

Set

$$\psi_1(t) = \psi(t)(1 - t^2),\ t \in R. \tag{3.6}$$

Clearly $\psi_1 \in C^\infty(R)$,

$$0 \le \psi_1(t) \le 1 \text{ for all } t \in R,\ \psi_1(t) = 0 \text{ if } |t| \ge 1,\ \psi_1(t) = 1 - t^2, \text{ if } |t| \le 1/2, \tag{3.7}$$

$$1 = \psi_1(0) > \psi_1(t) \text{ for all } t \in R \setminus \{0\}\cdot \tag{3.8}$$

By (3.4), (3.1), and the relation $\beta \in \mathcal{M}_r$, we can choose positive constants c_0, c_1 such that $c_0 > 1$, $c_1 < 1$,

$$\left[\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}\right] \subset (0, \pi/2), \tag{3.9}$$

$$\inf\left\{\beta(t):\ t \in \left[\bar{\theta} - c_0^{-1}, \bar{\theta} + c_0^{-1}\right]\right\} - \beta(\pi/2) > 4c_1,\ c_1 < -\left[\beta(0) + \beta''(0)\right], \tag{3.10}$$

$$||\psi_1||_{C^2(R)} c_1 c_0^2 < \epsilon. \tag{3.11}$$

Consider the function

$$\psi_2(t) = c_1 - c_1\psi_1(c_0(t - \bar{\theta})),\ t \in R. \tag{3.12}$$

Clearly $\psi_2 \in C^\infty(R)$. By (3.12), (3.7)

$$0 \le \psi_2(t) \le c_1,\ t \in R, \tag{3.13}$$

$$\psi_2(t) = c_1 \text{ for each } t \in R \text{ satisfying } |t - \bar{\theta}| \ge c_0^{-1}. \tag{3.14}$$

By (3.12), (3.8)

$$\psi_2(\bar{\theta}) = 0, \tag{3.15}$$

$$\psi_2(t) > 0 \text{ for each } t \in R \setminus \{\bar{\theta}\}\cdot \tag{3.16}$$

It is not difficult to see that there exists a function $\psi_3 : R \to R$ such that

$$\psi_3(t) = \psi_2(t),\ t \in [0, \pi/2], \tag{3.17}$$

$$\psi_3(-t) = \psi_3(t),\ t \in R,$$

$$\psi_3(t + \pi) = \psi_3(t),\ t \in R.$$

It is not difficult to see that $\psi_3 \in C^\infty(R)$,

$$0 \le \psi_3(t) \le c_1,\ t \in R, \tag{3.18}$$

$$\begin{aligned} &\psi_3(\bar\theta) = 0,\\ &\psi_3(t) > 0 \text{ for all } t \in [0, \pi/2] \setminus \{\bar\theta\},\\ &\psi_3(t) > 0 \text{ for all } t \in [-\pi/2, 0] \setminus \{-\bar\theta\}. \end{aligned} \tag{3.19}$$

Define

$$\phi(t) = \beta(t) + \psi_3(t),\ t \in R. \tag{3.20}$$

Clearly $\phi \in C^2(R)$. By (3.20), (1.1), (3.18), (1.3), (3.17), (1.4) we have

$$\phi(t) \ge 0 \text{ for all } t \in R \text{ and } \phi(t) = \phi(-t) = \phi(t+\pi) \text{ for all } t \in R. \tag{3.21}$$

By (3.20), the relation $\beta \in \mathcal{M}_r$, (3.18)

$$\phi(t) > 0 \text{ for all } t \in R. \tag{3.22}$$

We show that

$$\phi(0) + \phi''(0) < 0. \tag{3.23}$$

By (3.20), (3.17)

$$\begin{aligned} \phi(0) + \phi''(0) &= \beta(0) + \beta''(0) + \psi_3(0) + \psi_3''(0)\\ &= \beta(0) + \beta''(0) + \psi_2(0) + \psi_2''(0). \end{aligned} \tag{3.24}$$

By (3.14), (3.9),

$$\psi_2(0) = c_1,\ \psi_2''(0) = 0. \tag{3.25}$$

Combined with (3.24), (3.10) this implies that

$$\phi(0) + \phi''(0) = \beta(0) + \beta''(0) + c_1 < 0.$$

Thus (3.23) is true. We show that

$$\phi(\pi/2) \le \phi(t) \le \phi(0) \tag{3.26}$$

for all $t \in R$. Clearly it is enough to show that this inequality holds for all $t \in [0, \pi/2]$.

Let $t \in [0, \pi/2]$. Then by (3.20), (3.17)

$$\phi(t) = \beta(t) + \psi_2(t). \tag{3.27}$$

By (3.27), (3.13), (3.25), (1.2), (3.17), (3.20)

$$\begin{aligned} \phi(t) \le \beta(t) + c_1 = \beta(t) + \psi_2(0) &\le \beta(0) + \psi_2(0)\\ &= \beta(0) + \psi_3(0) = \phi(0). \end{aligned}$$

Thus

$$\phi(t) \le \phi(0). \tag{3.28}$$

By (3.20), (3.17), (3.14), (3.9)

$$\phi(\pi/2) = \beta(\pi/2) + \psi_2(\pi/2) = \beta(\pi/2) + c_1. \tag{3.29}$$

There are two cases:

(a) $|t - \bar\theta| \ge c_0^{-1}$; (b) $|t - \bar\theta| < c_0^{-1}$.

Consider the case (a). Then it follows from (3.17), (3.14) that

$$\psi_3(t) = \psi_2(t) = c_1.$$

Combined with (3.20), (1.2), (3.29) this implies that

$$\phi(t) = \beta(t) + c_1 \geq \beta(\pi/2) + c_1 = \phi(\pi/2)$$

and

$$\phi(t) \geq \phi(\pi/2). \tag{3.30}$$

Consider the case (b) with

$$|t - \bar{\theta}| < c_0^{-1}. \tag{3.31}$$

By (3.20), (3.17), (3.31), (3.13), (3.10), (3.29)

$$\phi(t) = \beta(t) + \psi_2(t) \geq \beta(t) > \beta(\pi/2) + 4c_1 > \phi(\pi/2)$$

and (3.30) is true. Thus (3.30) is true in both cases. (3.30), (3.28) imply (3.26).

We have shown that $\phi \in \mathcal{M}_r$. By (1.6), (3.20), (3.17), (3.13), (3.11), (3.12)

$$\begin{aligned}\rho(\beta, \phi) &= \sup\{\psi_3(t),\ |\psi_3'(t)|,\ |\psi_3''(t)| :\ t \in [-\pi/2, \pi/2]\} \\ &= \sup\{\psi_2(t),\ |\psi_2'(t)|,\ |\psi_2''(t)| :\ t \in [0, \pi/2]\} \\ &\leq \max\left\{c_1, ||\psi_1||_{C^2(R)} c_1 c_0,\ ||\psi_1||_{C^2(R)} c_1 c_0^2\right\} = c_1 c_0^2 ||\psi_1||_{C^2} < \epsilon.\end{aligned}$$

Thus

$$\rho(\phi, \beta) < \epsilon. \tag{3.32}$$

We have

$$\phi(t) \geq \beta(t) \text{ for all } t \in R. \tag{3.33}$$

This implies that

$$G_\phi(t) \geq G_\beta(t) \text{ for all } t \in R. \tag{3.34}$$

By (1.9), (3.4), (3.20), (3.17), (3.15), (3.2)

$$\begin{aligned}G_\phi(\bar{z}) &= \phi(\arctan(\bar{z}))(1 + \bar{z}^2)^{1/2} = \phi(\bar{\theta})(1 + \bar{z}^2)^{1/2} \\ &= (\beta + \psi_3)(\bar{\theta})(1 + \bar{z}^2)^{1/2} = \beta(\bar{\theta})(1 + \bar{z}^2)^{1/2} = G_\beta(\bar{z}) = \inf(G_\beta).\end{aligned} \tag{3.35}$$

(3.35), (3.34) imply that

$$\inf(G_\phi) = \inf(G_\beta) = G_\phi(\bar{z}) = G_\beta(\bar{z}). \tag{3.36}$$

Let

$$z \in R \setminus \{\bar{z}, -\bar{z}\}. \tag{3.37}$$

It follows from (1.9), (3.20) that

$$\begin{aligned}G_\phi(z) = \phi(\arctan(z))(1 + z)^{1/2} = \beta(\arctan(z))(1 + z^2)^{1/2} \\ + \psi_3(\arctan(z))(1 + z^2)^{1/2} = G_\beta(z) + \psi(\arctan(z))(1 + z^2)^{1/2}.\end{aligned} \tag{3.38}$$

By (3.37), (3.19)

$$\psi_3(\arctan(z)) > 0.$$

Combined with (3.38), (3.36) this inequality implies that

$$G_\phi(z) > G_\beta(z) \geq G_\beta(\bar{z}) = G_\phi(\bar{z}).$$

Lemma 3.1 is proved. □

Lemma 3.2. *Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists $\tilde{\beta} \in \mathcal{M}_r$ such that $\rho(\beta, \tilde{\beta}) < \epsilon$, $\tilde{\beta}(t) > \tilde{\beta}(\pi/2)$ for all $t \in [0, \pi/2)$.*

Proof. Consider the function

$$\beta_0(t) = \cos(2t) + 3/2, \ t \in R.$$

Clearly $\beta_0 \in \mathcal{M}_r$. For all $t \in [0, \pi/2)$

$$\beta_0(t) = \cos(2t) + 3/2 > (\cos \pi) + 3/2 = \beta_0(\pi/2).$$

For each natural number n set

$$\beta_n(t) = \beta(t) + n^{-1}\beta_0(t), \ t \in R.$$

Clearly for all natural n, $\beta_n \in \mathcal{M}_r$, $\beta_n(t) > \beta_n(\pi/2)$ for all $t \in [0, \pi/2)$,

$$\beta_n \to \beta \text{ as } n \to \infty \text{ in } (\mathcal{M}, \rho).$$

Lemma 3.2 is proved. □

Lemmas 3.1 and 3.2 imply

Lemma 3.3 (basic lemma)**.** *Let $\beta \in \mathcal{M}_r$, $\epsilon > 0$. Then there exists $\phi \in \mathcal{M}_r$, $\bar{z} \in R$ such that $\rho(\phi, \beta) < \epsilon$,*

$$G_\phi(z) > G_\phi(\bar{z}) \text{ for all } z \in R \setminus \{\bar{z}, -\bar{z}\}.$$

4. Proof of Theorem 1.1

By Proposition 1.1 and Lemma 3.3, $\mathcal{F}$ is an everywhere dense subset of $(\mathcal{M}, \rho)$. Let $\beta \in \mathcal{F}$, $z_\beta > 0$ satisfy (1.11), n be a natural number.

By Proposition 2.5 there are an open neighborhood $\mathcal{U}(\beta, n)$ of β in $(\mathcal{M}, \rho)$ and a number $\delta(\beta, n) > 0$ such that

$$\mathcal{U}(\beta, n) \subset \mathcal{M}_r$$

and the following property holds:

(P3) if $\phi \in \mathcal{U}(\beta, n)$, $z \in R$, $G_\phi(z) \leq \inf(G_\phi) + \delta(\beta, n)$, then

$$\min\{|z - z_\beta|, \ |z + z_\beta|\} \leq 1/n.$$

Set

$$\mathcal{F}_0 = \cap_{n=1}^\infty \cup \{\mathcal{U}(\beta, n) : \ \beta \in \mathcal{F}, \ n \text{ is a natural number}\}\cdot \tag{4.1}$$

Clearly $\mathcal{F} \subset \mathcal{F}_0$ and $\mathcal{F}_0$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, \rho)$. Let $\phi \in \mathcal{F}_0$, $z_1, z_2 \in [0, \infty)$,

$$G_\phi(z_1) = G_\phi(z_2) = \inf(G_\beta). \tag{4.2}$$

Let $n \geq 1$ be an integer. By (4.1) there is $\beta \in \mathcal{F}$ such that

$$\phi \in \mathcal{U}(\beta, n). \tag{4.3}$$

By (4.3), (P3), (4.2),

$$|z_i - z_\beta| \le 1/n,\ i = 1, 2,$$
$$|z_1 - z_2| \le 2/n.$$

Since n is any natural number we conclude that $z_1 = z_2$, $\phi \in \mathcal{F}$ and $\mathcal{F}_0 = \mathcal{F}$. This completes the proof of Theorem 1.1.

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