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ENTIRE SOLUTIONS IN $\mathbb{R}^2$ FOR A CLASS OF ALLEN-CAHN EQUATIONS *

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Abstract. We consider a class of semilinear elliptic equations of the form

$$-\varepsilon^2\Delta u(x,y)+a(x)W'(u(x,y))=0,\quad (x,y)\in\mathbb{R}^2 \tag{0.1}$$

where $\varepsilon>0$, $a:\mathbb{R}\to\mathbb{R}$ is a periodic, positive function and $W:\mathbb{R}\to\mathbb{R}$ is modeled on the classical two well Ginzburg-Landau potential $W(s)=(s^2-1)^2$. We look for solutions to (0.1) which verify the asymptotic conditions $u(x,y)\to\pm1$ as $x\to\pm\infty$ uniformly with respect to $y\in\mathbb{R}$. We show *via* variational methods that if ε is sufficiently small and a is not constant, then (0.1) admits infinitely many of such solutions, distinct up to translations, which do not exhibit one dimensional symmetries.

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1. Introduction

In paper we deal with a class of semilinear elliptic equations of the form

$$-\varepsilon^2\Delta v(x,y)+a(x)W'(v(x,y))=0, \tag{1.1}$$

$((x,y)\in\mathbb{R}^2)$ or equivalently (setting $u(x,y)=v(\varepsilon x,\varepsilon y)$)

$$-\Delta u(x,y)+a(\varepsilon x)W'(u(x,y))=0, \tag{1.2}$$

where we assume $\varepsilon>0$ and

(H_1) $a:\mathbb{R}\to\mathbb{R}$ is not constant, 1-periodic, positive and Hölder continuous,
(H_2) $W\in\mathcal{C}^2(\mathbb{R})$ satisfies $W(s)\geq0$ for any $s\in\mathbb{R}$, $W(s)>0$ for any $s\in(-1,1)$, $W(\pm1)=0$ and $W''(\pm1)>0$.

This kind of equation arises in various fields of Mathematical Physics. As an example, when W is the classical two well Ginzburg-Landau potential, $W(s)=(s^2-1)^2$, (1.2) can be viewed as a generalization of the stationary Allen-Cahn equation introduced as a model for phase transitions in binary metallic alloys. Another kind of equation of the Mathematical Physics that fits in our assumption is the stationary version of the so called Sine-Gordon equation, corresponding to taking $W(s)=1+\cos(\pi s)$, potential which has been applied to several problems in condensed state Physics like for instance the propagation of dislocations in crystals. The function v,

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in these models, is considered as an order parameter describing pointwise the state of the material. The global minima of W represent energetically favorite *pure phases* and different values of v depict mixed configurations.

We look for existence and multiplicity of two phases solutions of (1.2), *i.e.*, solutions of the boundary value problem

$$\begin{cases} -\Delta u(x,y) + a(\varepsilon x)W'(u(x,y)) = 0, & (x,y) \in \mathbb{R}^2 \\ \lim_{x\to\pm\infty} u(x,y) = \pm 1, & \text{uniformly w.r.t. } y \in \mathbb{R}. \end{cases} \tag{1.3}$$

That kind of problem has been extensively studied under various points of view. In [11], N. Ghoussoub and C. Gui partially proved a De Giorgi's conjecture (see [9]) regarding (1.3). The following result is a particular consequence of their study.

Theorem 1.1. *If $a(x) = a_0 > 0$ for any $x \in \mathbb{R}$ and if $u \in \mathcal{C}^2(\mathbb{R}^2)$ is a solution of (1.3), then $u(x,y) = q(x)$ for all $(x,y) \in \mathbb{R}^2$, where $q \in \mathcal{C}^2(\mathbb{R})$ is a solution of the problem*

$$\begin{cases} -\ddot{q}(x) + a_0 W'(q(x)) = 0, & x \in \mathbb{R} \\ \lim_{x\to\pm\infty} q(x) = \pm 1. \end{cases}$$

By Theorem 1.1, when equation (1.2) is autonomous, any solution of (1.3) depends only on the x variable being in fact a solution of the corresponding ordinary differential equation. We have to remark that the result in [11], as the De Giorgi conjecture, deals with an asymptotic condition weaker than the one in (1.3), asking that the limits ± 1 are realized only pointwise with respect to $y \in \mathbb{R}$ and that u is increasing in the variable x. In such a case in [11] it is proved that the conclusion in Theorem 1.1 is still true modulo a space roto-translation. We mention that in this form the De Giorgi conjecture is still open in $\mathbb{R}^n$ for $n \geq 4$ while it was recently proved in [5] for $n = 3$ (see also [2]). The assumption, as in the problem (1.3), that the limits are uniform with respect to $y \in \mathbb{R}^{n-1}$, simplifies in fact the matter and the question of De Giorgi, known in this setting as Gibbons conjecture, is nowadays completely solved for any $n \geq 2$, see [7, 8, 10].

All these results show that, in the autonomous case, the problem (1.3) is in fact one dimensional and the set of its solutions can be considered in this sense trivial. This is not the case, in general, for systems of autonomous Allen Cahn equations as shown in [1] and this is not the case for non autonomous x-dependent Allen Cahn type equation as shown in [3]. In fact, in [3] it is proved that introducing in the potential a non trivial periodic dependence on the single variable x, as in (1.2), the one dimensional symmetry of the problem disappears. The existence of at least two solutions of problem (1.3), distinct up to translations, depending on both the planar variables x and y is displayed when ε is sufficiently small. This reveals that for the x-dependent Allen Cahn type equations (1.2) even the weaker Gibbons conjecture decades.

Pursuing the study started in [3], aim of the present paper is to show that the introduction of a space x-dependence leads to a complicated structure of the set of solutions of problem (1.3). In particular we show that, if ε is sufficiently small, it always admits infinitely many solutions depending on both the planar variables and distinct up to space translations.

To state precisely our result it is better to recall some of the properties of the one dimensional problem associated to (1.3),

$$\begin{cases} -\ddot{q}(x) + a(\varepsilon x)W'(q(x)) = 0, & x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} q(x) = \pm 1. \end{cases} \tag{1.4}$$

As it is nowadays well known, when a is not constant and ε is sufficiently small, the problem (1.4) admits the so called multibump dynamics. To be precise, let $z_0 \in C^\infty(\mathbb{R})$ be an increasing function such that $z_0(x) \to \pm 1$ as $x \to \pm\infty$ and $|z_0(x)| = 1$ for any $|x| \geq 1$, and define the action functional

$$F(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}(x)|^2 + a(\varepsilon x)W(q(x))\,\mathrm{d}x$$

on the class

$$\Gamma = \{q \in H^1_{loc}(\mathbb{R}) \,/\, \|q\|_{L^\infty(\mathbb{R})} = 1 \text{ and } q - z_0 \in H^1(\mathbb{R})\}.$$

Then (see Sect. 2) there exists $\delta_0 \in (0, 1/4)$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ there is a family of open intervals $\{(t_j^-, t_j^+) \,/\, j \in \mathbb{Z}\}$ verifying

$$t_j^+ = t_{j+1}^- \text{ and } t_j^+ - t_j^- = \frac{1}{\varepsilon}, \text{ for any } j \in \mathbb{Z},$$

for which, for any odd integer number k, for any $p = (p_1, \dots, p_k)$ with $p_1 < p_2 < \cdots < p_k \in \mathbb{Z}$, setting

$$c_{k,p} = \inf\{F(q) \,/\, q \in \Gamma,\ |q(t_{p_i}^-) - (-1)^i| \le \delta_0 \text{ and } |q(t_{p_i}^+) - (-1)^{i+1}| \le \delta_0 \text{ for } i = 1, \dots, k\}$$

and

$$\mathcal{K}_{k,p} = \{q \in \Gamma \,/\, F(q) = c_{k,p},\ |q(t_{p_i}^-) - (-1)^i| \le \delta_0 \text{ and } |q(t_{p_i}^+) - (-1)^{i+1}| \le \delta_0 \text{ for } i = 1, \dots, k\}$$

we have that $\mathcal{K}_{k,p}$ is not empty, and constituted by k-bump solutions of (1.4).

In particular the 1-bump solutions are global minima of F on Γ at the level $c = c_{1,p} = \min_\Gamma F(q)$. Moreover it can be proved that, since a is not constant, when ε is sufficiently small the following non-degeneracy condition holds:

$$\mathcal{K} = \{q \in \Gamma \,/\, F(q) = c\} = \cup_{p \in \mathbb{Z}} \mathcal{K}_{1,p},$$

where the sets $\mathcal{K}_{1,p} \subset \Gamma$ are compact and uniformly separated in Γ (with respect to the $H^1(\mathbb{R})$ metric).

In [3] the existence of solutions depending on both the planar variables is proved looking for solutions to problem (1.3) which are asymptotic as $y \to \pm\infty$ to different minimal sets $\mathcal{K}_{1,p_-}$, $\mathcal{K}_{1,p_+}$. More precisely, letting

$$\mathcal{H} = \{u \in H^1_{loc}(\mathbb{R}^2) \,/\, \|u\|_{L^\infty(\mathbb{R}^2)} \le 1 \text{ and } u - z_0 \in \cap_{(\zeta_1, \zeta_2) \subset \mathbb{R}} H^1(\mathbb{R} \times (\zeta_1, \zeta_2))\},$$

in [3] it is shown that fixed $p_- = 0$ there exists at least two different values of $p_+ \in \mathbb{Z} \setminus \{0\}$ for which there exists a solution $u \in \mathcal{H}$ of (1.3) such that

$$\mathsf{d}(u(\cdot, y), \mathcal{K}_{1,p_\pm}) \to 0, \qquad \text{as } y \to \pm\infty,$$

where, if $q \in \Gamma$ and $A \subset \Gamma$, we denote $\mathsf{d}(q(\cdot), A) = \inf\{\|q - \bar{q}\|_{L^2(\mathbb{R})} \,/\, \bar{q} \in A\}$ (in fact, as shown in [13], there results $p_+ = \pm 1$),

In the present paper we strengthen that result showing that (1.3) admits infinitely many solutions $u \in \mathcal{H} \cap C^2(\mathbb{R}^2)$ with $\partial_y u \ne 0$, which emanate as $y \to -\infty$ from different 3-bump solutions of (1.4). In fact we prove

Theorem 1.2. *There exists $\varepsilon_0 > 0$ for which for any $\varepsilon \in (0, \varepsilon_0)$ there exists $\mathsf{p}_0 \in \mathbb{N}$ such that if $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$ verifies $\min\{p_2 - p_1, p_3 - p_2\} \ge \mathsf{p}_0$ then there exists a solution $u \in \mathcal{H} \cap C^2(\mathbb{R}^2)$ of (1.3) such that $\partial_y u \ne 0$ and $\lim_{y \to -\infty} \mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) = 0$.*

We find these solutions using a variational argument which generalizes the one introduced in [3].

In [3], following a renormalization procedure inspired by the one introduced by P.H. Rabinowitz in [15, 16], the solutions are found as minima of the renormalized action functional

$$\varphi(u) = \int_{\mathbb{R}} \frac{1}{2} \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbb{R})} + (F(u(\cdot, y)) - c)\, \mathrm{d}y$$

on the set

$$\mathcal{M}_{p_-, p_+} = \{u \in \mathcal{H} \,|\, \lim_{y \to \pm\infty} \mathsf{d}(u(\cdot, y), \mathcal{K}_{1,p_\pm}) = 0\}.$$

The fact that c is the minimal level of F on Γ implies that for any $u \in \mathcal{M}_{p_-, p_+}$, the function $y \to F(u(\cdot, y)) - c$ is non negative and so the functional φ is well defined with values in $[0, +\infty]$. Moreover, the discreteness of the minimal set $\mathcal{K}$ allows us to show that if $u \in \mathcal{M}_{p_-, p_+}$ and $\varphi(u) < +\infty$ then $\sup_{y \in \mathbb{R}} \mathsf{d}(u(\cdot, y), \mathcal{K}_{1,p_-}) < +\infty$.

This excludes "sliding" phenomena for the minimizing sequences (see [1]) possibly due to the non compactness of the domain in the x-direction. The lack of compactness in the y-direction is then overcame *via* concentration compactness techniques.

Following that argument, to find solutions to (1.3) asymptotic to 3-bump solutions, fixed $p \in \mathcal{P} = \{(p_1, p_2, p_3) \in \mathbb{Z}^3 \,/\, p_1 < p_2 < p_3\}$, we may consider the set

$$\mathcal{M}_p^* = \{u \in \mathcal{H} \,/\, \lim_{y \to -\infty} \mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) = 0,\ \liminf_{y \to +\infty} \mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) > 0\}.$$

A difficulty arises when we try to define on $\mathcal{M}_p^*$ a suitable renormalized functional. Indeed, differently from the 1-bump case, the set $\mathcal{K}_{3,p}$ is not minimal for F on Γ and if $u \in \mathcal{M}_p^*$, the function $y \to F(u(\cdot, y)) - c_{3,p}$ is indefinite in sign. In other word the natural renormalized functional

$$\varphi_p(u) = \int_{\mathbb{R}} \frac{1}{2} \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbb{R})} + (F(u(\cdot, y)) - c_{3,p}) \, \mathrm{d}y$$

is not well defined on $\mathcal{M}_p^*$. To overcome that difficulty we make use of a natural constraint of the problem. Indeed, we observe that any solution $u \in \mathcal{H}$ of (1.2) on the half plane $\mathbb{R} \times (-\infty, y_0)$, which satisfies $\mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) \to 0$ as $y \to -\infty$, and $\int_{\mathbb{R} \times (-\infty, y_0)} |\partial_y u(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y < +\infty$, verifies the property

$$F(u(\cdot, y)) = c_{3,p} + \frac{1}{2} \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbb{R})}, \qquad \forall y \in (-\infty, y_0), \tag{1.5}$$

a sort of conservation of *Energy*. In particular (1.5) implies that $F(u(\cdot, y)) \geq c_{3,p}$ for any $y \in (-\infty, y_0)$ and that suggests us to define, given $p \in \mathcal{P}$, the set

$$\mathcal{M}_p = \left\{ u \in \mathcal{M}_p^* / \inf_{y \in \mathbb{R}} F(u(\cdot, y)) \geq c_{3,p} \right\}$$

on which we look for minima of the natural renormalized functional φ_p which is well-defined there.

As in the one bump case, to avoid sliding phenomena, we have to show that if $u \in \mathcal{M}_p$ and $\varphi_p(u) < +\infty$ then $\sup_{y \in \mathbb{R}} \mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) < +\infty$. This is done studying the discreteness properties of the level set $\{F = c_{3,p}\}$ showing that we have sufficient compactness in the problem whenever $\min\{p_2 - p_1, p_3 - p_2\}$ is sufficiently large, but not in general for any $p \in \mathcal{P}$.

That allows us to prove that for such kind of p, the minimizing sequences of φ_p on $\mathcal{M}_p$ converges up to subsequences, weakly in $H^1_{loc}(\mathbb{R}^2)$. Unfortunately we can not say that the limit functions u_p are minima of φ_p on $\mathcal{M}_p$ since the constraint $\inf_{y \in \mathbb{R}} F(u_p(\cdot, y)) \geq c_{3,p}$ is not necessarily satisfied. However we recover that $u_p \in \mathcal{H}$ and that $\lim_{y \to -\infty} \mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) = 0$. Moreover, setting

$$y_{0,u} = \inf\{y \in \mathbb{R} \,/\, \mathsf{d}(u_p(\cdot, y), \mathcal{K}_{3,p}) > 0 \text{ and } F(u_p(\cdot, y)) \leq c_{3,p}\}$$

we prove that $\liminf_{y \to y_{0,u}^-} \mathsf{d}(u_p(\cdot, y), \mathcal{K}_{3,p}) \geq d_0 > 0$, that $\liminf_{y \to y_{0,u}^-} F(u_p(\cdot, y)) = c_{3,p}$ and that u_p is a classical solution of (1.2) on the set $R \times (-\infty, y_{0,u})$.

If $y_{0,u} = +\infty$, we conclude that $u_p \in \mathcal{M}_p \cap C^2(\mathbb{R}^2)$ is a solution to (1.2). Differently from the one bump case we are not able to precisely characterize the asymptotic behaviour of this kind of solutions as $y \to +\infty$. Anyway we can say that in this case there exists a $p' \neq p \in \mathcal{P}$ such that $u_p(\cdot, y)$ remains for large values of y nearby the set $\mathcal{K}_{3,p'}$.

If otherwise $y_{0,u} \in \mathbb{R}$, we have that u_p solves (1.2) only on the half plane $\mathbb{R} \times (-\infty, y_{0,u})$. Using the conservation of *Energy* (1.5), we show that in such case u_p satisfies the Neumann boundary condition $\partial_y u_p(\cdot, y_{0,u}) \equiv 0$. This will allow us to recover, by reflection, an entire solution to (1.2) even in this case.

More precisely, setting

$$v_p \equiv u_p, \quad \text{if } y_{0,u} = +\infty, \qquad \text{and} \qquad v_p(x,y) = \begin{cases} u_p(x,y), & \text{if } y \le y_{0,u}, \\ u_p(x, 2y_{0,u} - y), & \text{if } y > y_{0,u}, \end{cases} \quad \text{if } y_{0,u} \in \mathbb{R},$$

we get that $v_p \in \mathcal{H} \cap C^2(\mathbb{R})$ is a classical solution (of homoclinic type if $y_{0,u} \in \mathbb{R}$ and of heteroclinic type if $y_{0,u} = +\infty$) of (1.2) verifying $\partial_y v_p \neq 0$. The fact that v_p is actually a solution of (1.3) follows now in a standard way since $\sup_{y\in\mathbb{R}} \mathsf{d}(v_p(\cdot, y), \mathcal{K}_{3,p}) < +\infty$ and $\varphi_p(v_p) < +\infty$.

We finally want to point out some comments on our result.

First, we note that the proof described above can be adapted to find solutions asymptotic as $y \to -\infty$ to k-bump solutions for any $k \in \mathbb{N}$. Moreover that *Energy* constraint can be used even in other contests and to find other kind of solutions. We think in particular to the possibility of finding periodic solutions of the *brake orbits* type (in the y-variable), and to study the case in which the function a has more general recurrence properties (*e.g.* a almost periodic).

Another remark regards the connection of our result with the ones obtained for the "fully" non autonomous case, *i.e.*,

$$\begin{cases} -\varepsilon^2 \Delta u + a(x,y) W'(u(x,y)) = 0, & (x,y) \in \mathbb{R}^2 \\ \lim_{x\to\pm\infty} u(x,y) = \pm 1, & \text{uniformly w.r.t. } y \in \mathbb{R}, \end{cases} \tag{1.6}$$

where a is periodic in both variables. That kind of problem, and even in a more general setting, has been already considered for example in the papers [4, 6, 12–14]. In these papers the existence of a wide variety of solutions has been shown. However, in this setting, the existence of solutions asymptotic to periodic solutions in the variable y and of the k-bump type in the variable x is still an open problem and the present work gives a partial positive answer in that direction.

Some constants and notation. Before starting in our study we fix here some constants and notation which will be used in the rest of the paper.

By (H_1) there exist $\underline{x}, \overline{x} \in [0,1)$ such that

$$a(\underline{x}) = \underline{a} \equiv \min_{t\in\mathbb{R}} a(t), \quad a(\overline{x}) = \overline{a} \equiv \max_{t\in\mathbb{R}} a(t). \tag{1.7}$$

Considering if necessary a translation of the function a, we can assume that $\underline{x} < \overline{x}$.

By (H_2) there exists $\overline{\delta} \in (0, \frac{1}{4})$ and $\overline{w} > \underline{w} > 0$ such that

$$\overline{w} \ge W''(s) \ge \underline{w} \text{ for any } |s| \in [1 - 2\overline{\delta}, 1 + 2\overline{\delta}]. \tag{1.8}$$

In particular, setting $\chi(s) = \min\{|1-s|, |1+s|\}$, we have that

$$\text{if } |s| \in [1 - 2\overline{\delta}, 1], \text{ then } \frac{\underline{w}}{2}\chi(s)^2 \le W(s) \le \frac{\overline{w}}{2}\chi(s)^2 \text{ and } |W'(s)| \le \overline{w}\chi(s). \tag{1.9}$$

Therefore there exist $\underline{b}$, $\overline{b} > 0$ such that

$$\underline{b}\,\chi(s)^2 \le W(s) \text{ and } |W'(s)| \le \overline{b}\,\chi(s), \ \forall |s| \le 1. \tag{1.10}$$

For any $\delta \in (0,1)$ we denote

$$\omega_\delta = \min_{|s|\le 1-\delta} W(s) \tag{1.11}$$

and we note that $\omega_\delta > 0$ for any $\delta \in (0,1)$. Moreover we define the constants

$$m = \sqrt{2\underline{a}\omega_{\overline{\delta}}}\,\overline{\delta} \quad \text{and} \quad m_0 = \min\left\{\frac{1}{2}, \frac{\sqrt{\overline{a}}}{\sqrt{\underline{a}}} - 1\right\} m. \tag{1.12}$$

Note that, since a is not constant, $\overline{a} > \underline{a}$ and so $m_0 > 0$.

Finally, for a given $q \in L^2(\mathbb{R})$ we denote $\|q\| \equiv \|q\|_{L^2(\mathbb{R})}$.

2. The one dimensional problem

In this section, letting $a_\varepsilon(x) = a(\varepsilon x)$, we focalize our study to the ODE problem associated to (1.3), namely,

$$\begin{cases} -\ddot{q}(t) + a_\varepsilon(t) W'(q(t)) = 0, & t \in \mathbb{R}, \\ \lim_{t\to\pm\infty} q(t) = \pm 1. \end{cases} \tag{2.1}$$

In particular we are interested in some variational aspects of the problem.

Let $z_0 \in C^\infty(\mathbb{R})$ be an increasing function such that $z_0(t) \to \pm 1$ as $t \to \pm\infty$ and $|z_0(t)| = 1$ for any $|t| \geq 1$. We define the class

$$\Gamma = \left\{ q \in H^1_{loc}(\mathbb{R}) \,/\, \|q\|_{L^\infty(\mathbb{R})} = 1 \text{ and } q - z_0 \in H^1(\mathbb{R}) \right\},$$

on which we consider the action functional

$$F_{a_\varepsilon}(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}(t)|^2 + a_\varepsilon(t) W(q(t))\,\mathrm{d}t.$$

Note that the functional F_{a_ε} is a continuous and in fact a Lipschitz continuous positive functional on Γ endowed with the $H^1(\mathbb{R})$ metric (see Lem. 2.13 in the appendix). Note also that in fact $F_{a_\varepsilon}(q)$ is well defined with value in $[0, +\infty]$ whenever $q \in H^1_{loc}(\mathbb{R})$.

For future references we introduce also the set

$$\overline{\Gamma} = \{ q \in L^\infty(\mathbb{R}) \,/\, \|q\|_{L^\infty(\mathbb{R})} = 1 \text{ and } q - z_0 \in L^2(\mathbb{R}) \}$$

which is in fact the completion of Γ with respect to the metric

$$\mathsf{d}(q_1, q_2) = \|q_1 - q_2\|_{L^2(\mathbb{R})}.$$

The main objective of this section will be to study some discreteness properties of a particular sublevel of F_{a_ε} in $\overline{\Gamma}$ for ε small enough.

First we remark that, due to the unboundedness of $\mathbb{R}$, the sublevels of F_{a_ε} in Γ are not precompact in any sense. In fact, it is sufficient to note that given a function $q \in \Gamma$ we have that the sequence $q_n(\cdot) = q(\cdot - \frac{n}{\varepsilon})$ is such that $F_{a_\varepsilon}(q_n) = F_{a_\varepsilon}(q)$ for any $n \in \mathbb{N}$ and $q_n(t) \to -1 \notin \Gamma$ for any $t \in \mathbb{R}$ as $n \to \infty$. Anyway, it is simple to recognize that the sublevels of F_{a_ε} are (sequentially) weakly precompact in $H^1_{loc}(\mathbb{R})$. In fact, denoting by $\{F_{a_\varepsilon} \leq c\}$ the set $\{q \in \Gamma \,|\, F_{a_\varepsilon}(q) \leq c\}$ for every $c > 0$, the following result holds

Lemma 2.1. *Let $(q_n) \subset \{F_{a_\varepsilon} \leq c\}$ for some $c > 0$. Then, there exists $q \in H^1_{loc}(\mathbb{R})$ with $\|q\|_{L^\infty(\mathbb{R})} \leq 1$ such that, along a subsequence, $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$, $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$ and moreover $F_{a_\varepsilon}(q) \leq \liminf_{n\to\infty} F_{a_\varepsilon}(q_n)$*

Proof. Let $\ell = \liminf_{n\to\infty} F_{a_\varepsilon}(q_n)$. Up to a subsequence, we can assume that $F_{a_\varepsilon}(q_n) \to \ell$ as $n \to \infty$. Since $\|q_n\|_{L^\infty(\mathbb{R})} \leq 1$ and $\|\dot{q}_n\| \leq 2c$, there exists $q \in H^1_{loc}(\mathbb{R})$ and a subsequence of (q_n), denoted again (q_n), such that $q_n \to q$ weakly in $H^1_{loc}(\mathbb{R})$ (and so strongly in $L^\infty_{loc}(\mathbb{R})$) and such that $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$. Then, $\|q\|_{L^\infty(\mathbb{R})} \leq 1$. By weak semicontinuity of the norm, we obtain $\|\dot{q}\| \leq \liminf_{n\to\infty} \|\dot{q}_n\|$ and by the pointwise convergence and the Fatou Lemma, we get $\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t$. $\square$

The autonomous problem

Given a positive continuous function β, we denote $F_\beta(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot q|^2 + \beta(t)W(q)\,\mathrm{d}t$. Moreover, if I is an interval in $\mathbb{R}$ we set $F_{\beta,I}(q) = \int_I \frac{1}{2}|\dot q|^2 + \beta(t)W(q)\,\mathrm{d}t$.

It will be useful to recall some properties of the functional F_β when β is a given positive constant. First, setting

$$c_b = \inf_{\Gamma} F_b,$$

using standard argument (see *e.g.* [3]), it can be proved that if $Q \in \Gamma$ is such that $F_b(Q) = c_b$, then it is a classical solution to the autonomous problem

$$\begin{cases} -\ddot q(t) + bW'(q(t)) = 0, & t \in \mathbb{R} \\ \lim_{t\to\pm\infty} q(t) = \pm 1. \end{cases} \tag{P_b}$$

Moreover we have

Lemma 2.2. *For every $b > 0$ the problem (P_b) admits a unique solution in Γ, modulo time translation. Such solution is increasing, is a minimum of F_b on Γ and $c_b = \sqrt{b}c_1$.*

Proof. It is standard to show that (P_b) admits a solution $Q \in \Gamma$ which is a minimum on Γ of the functional F_b. To show that Q is increasing we argue by contradiction assuming that there exist $\sigma < \tau \in \mathbb{R}$ such that $Q(\sigma) = Q(\tau)$. Then the function

$$\hat q(t) = \begin{cases} Q(t) & \text{if } t \le \sigma, \\ Q(t + \tau - \sigma) & \text{if } t > \sigma, \end{cases}$$

belongs to Γ and moreover $c_b \le F_b(\hat q) = F_b(Q) - \int_\sigma^\tau \frac{1}{2}|\dot Q|^2 + bW(Q)\,\mathrm{d}t$. Then $\int_\sigma^\tau \frac{1}{2}|\dot Q|^2 + bW(Q)\,\mathrm{d}t = 0$ and we deduce that Q is constantly equal to 1 or -1 on the interval (σ,τ), a contradiction since Q solves (P_b). Note now that since the problem (P_b) is invariant by time translations, all the functions $Q(\cdot - \tau)$, $\tau \in \mathbb{R}$, are classical solutions to (P_b) and in fact, it is a simple consequence of the maximum principle that all the solutions to (P_b) which are in Γ belong to this family. Indeed, let $\bar q \in \Gamma$ be a solution of (P_b) and $t_0 \in \mathbb{R}$ be such that $\bar q(t) < -1 + 2\bar\delta$ for any $t < t_0$ and $\bar q(t_0) = -1 + 2\bar\delta$. Let moreover $\tau_0 \in \mathbb{R}$ be such that $Q(t_0 - \tau_0) + 1 = 2\bar\delta$ and set $h(t) = (\bar q(t) - Q(t - \tau_0))^2$. We have

$$\ddot h(t) = 2(\ddot{\bar q}(t) - \ddot Q(t - \tau_0))(\bar q(t) - Q(t - \tau_0)) + 2|\dot h(t)|^2 \ge 2b(W'(\bar q(t)) - W'(Q(t - \tau_0)))(\bar q(t) - Q(t - \tau_0)),$$

and since, by (1.8), there results $(W'(s_1) - W'(s_2))(s_1 - s_2) \ge \underline{w}(s_1 - s_2)^2$ for any $s_1, s_2 \in [-1 - 2\bar\delta, -1 + 2\bar\delta]$, we conclude that

$$\ddot h(t) \ge 2b\underline{w}\, h(t) \qquad \forall t \le \tau.$$

Since $h(t_0) = 0$ and $\lim_{t\to-\infty} h(t) = 0$, by the maximum principle we conclude that $h(t) = 0$ for any $t \le t_0$ and so, by uniqueness of the solution of the Cauchy problem, that $\bar q(t) \equiv Q(t - \tau_0)$ for any $t \in \mathbb{R}$.

Let now $b \ne d > 0$. Setting $q(t) = Q(\sqrt{\frac{d}{b}}t)$ we have

$$\ddot q(t) = \frac{d}{b}\ddot Q\left(\sqrt{\frac{d}{b}}t\right) = \mathrm{d}W'\left(Q\left(\sqrt{\frac{d}{b}}t\right)\right) = \mathrm{d}W'(q(t)),$$

i.e., q is a solution of problem (P_d). Since, as we have proved, all the solutions of (P_d) in Γ are minima of F_d on Γ, we have $F_d(q) = c_d$. Therefore

$$\begin{aligned} c_d &= \int_{\mathbb{R}} \frac{1}{2}|\dot q(t)|^2 + \mathrm{d}W(q(t))\,\mathrm{d}t \\ &= \frac{1}{2}\frac{d}{b}\int_{\mathbb{R}}\left|\dot Q\left(\sqrt{\frac{d}{b}}t\right)\right|^2\mathrm{d}t + d\int_{\mathbb{R}} W\left(Q\left(\sqrt{\frac{d}{b}}t\right)\right)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{d}{b}}\int_{\mathbb{R}}|\dot Q(s)|^2\,\mathrm{d}s + \sqrt{bd}\int_{\mathbb{R}} W(Q(s))\,\mathrm{d}s \\ &= \sqrt{\frac{d}{b}}\left(\frac{1}{2}\int_{\mathbb{R}}|\dot Q(s)|^2\,\mathrm{d}s + b\int_{\mathbb{R}} W(Q(s))\,\mathrm{d}s\right) = \sqrt{\frac{d}{b}}\,F_b(Q) = \sqrt{\frac{d}{b}}\,c_b. \end{aligned}$$

In particular we obtain $c_b = \sqrt{b}\,c_1$ for any $b > 0$. □

In the following, we will denote by q_b the unique solution to (P_b) in Γ which verifies $q_b(0) = 0$.

Remark 2.1. Since the equation $-\ddot q(t) + bW'(q(t)) = 0$ is reversible, the results concerning (P_b) reflect on the symmetric problem

$$\begin{cases} -\ddot q(t) + bW'(q(t)) = 0, & x \in \mathbb{R} \\ \lim_{t\to\pm\infty} q(t) = \mp 1. \end{cases} \tag{$\tilde P_b$}$$

In fact, the problem $(\tilde P_b)$ has only one solution in $\tilde\Gamma = \{q \in H^1_{loc}(\mathbb{R})\,|\,q(-t) \in \Gamma\}$, modulo time translations, and if we denote by $\tilde q_b$ the solution which verifies $\tilde q_b(0) = 0$, we have $\tilde q_b(t) = q_b(-t)$ for all $t \in \mathbb{R}$. Moreover we have $F_b(\tilde q_b) = \inf_{\tilde\Gamma} F_b = c_b$.

The constants ρ_0, δ_0, ε_0 and c^*

Here below we display some estimates concerning the functionals F_b with $b \in [\underline{a}, \overline{a}]$, the range of the function a_ε, and fix some constants which will remain unchanged along the paper. In particular, note that the functionals $F_{\underline{a}}$ and $F_{\overline{a}}$ bound respectively from below and above the functional F_{a_ε} for any $\varepsilon > 0$.

The basic remark is that for any $\delta \in (0,1)$, if $q \in \Gamma$ is such that $|q(t)| \le 1-\delta$ for any $t \in (\sigma,\tau) \subset \mathbb{R}$, then by (1.11)

$$F_{\underline{a},(\sigma,\tau)}(q) \ge \frac{1}{2(\tau-\sigma)}|q(\tau)-q(\sigma)|^2 + \underline{a}\omega_\delta(\tau-\sigma) \ge \sqrt{2\underline{a}\omega_\delta}\;|q(\tau)-q(\sigma)|. \tag{2.2}$$

As direct consequence, recalling the definition of $\overline{\delta}$ in (1.8) and the ones of m and m_0 given in (1.12), we recover that if $q \in \Gamma$ is such that $|q(t)| \le 1-\overline{\delta}$ for any $t \in (\sigma,\tau)$ and $|q(\tau)-q(\sigma)| = \overline{\delta}$ then, by (2.2), $F_{\underline{a},(\sigma,\tau)}(q) \ge m \ge 2m_0$. Hence, since $\overline{\delta} \le \frac{1}{4}$, we recognize that $F_{\underline{a}}(q) \ge 6m$ for any $q \in \Gamma$. Then $c_{\underline{a}} > 6m$, and, by Lemma 2.2 and the definition of m_0 in (1.12), we obtain

$$c_{\overline{a}} - c_{\underline{a}} = \left(\frac{\sqrt{\overline{a}}}{\sqrt{\underline{a}}} - 1\right)c_{\underline{a}} \ge 6\left(\frac{\sqrt{\overline{a}}}{\sqrt{\underline{a}}} - 1\right)m \ge 6m_0.$$

By the previous estimate, since by Lemma 2.2 the function $b \to c_b$ is continuous on $[\underline{a},\overline{a}]$, we can fix $\underline{\alpha} < \overline{\alpha} \in (\underline{a},\overline{a})$ and $\rho_0 > 0$ such that

$$c_{\underline{\alpha}} - c_{\underline{a}} \le \frac{m_0}{6} \quad\text{and}\quad c_{\overline{\alpha}} - c_{\underline{a}} \ge 3m_0, \tag{2.3}$$

$$a(\underline{x}+t) \le \underline{\alpha} \quad\text{and}\quad a(\overline{x}+t) \ge \overline{\alpha}, \qquad \forall\,|t| \le 2\rho_0. \tag{2.4}$$

Given $\delta \in [0,1)$, assume that a function $q \in H^1_{loc}(\mathbb{R})$ verifies $q(\sigma) = (-1)^l(1-\delta)$ and $q(\tau) = (-1)^{l+1}(1-\delta)$ for certain $\sigma < \tau \in \mathbb{R}$, $l \in \mathbb{N}$. If $\delta > 0$, one easily guess that $F_{b,(\sigma,\tau)}(q) \ge c_b - o_\delta$ with $o_\delta \to 0$ as $\delta \to 0$. The following lemma fix a $\delta_0 > 0$ in such a way the o_{δ_0} is comparable with the constant m_0 fixed in (1.12) for any $b \in [\underline{a},\overline{a}]$.

Lemma 2.3. *There exists $\delta_0 \in (0, \overline{\delta})$ such that if $q \in \Gamma$ verifies $q(\sigma) = (-1)^l(1-\delta_0)$ and $q(\tau) = (-1)^{l+1}(1-\delta_0)$ for some $\sigma < \tau \in \mathbb{R}$, $l \in \mathbb{N}$, then*

$$F_{b,(\sigma,\tau)}(q) \geq c_b - \frac{m_0}{8}, \qquad \forall b \in [\underline{a}, \overline{a}].$$

Proof. Let $\delta_0 \in (0, \overline{\delta})$ such that $\lambda_{\delta_0} \equiv (1 + \frac{\overline{a}\,\overline{w}}{3})\frac{{\delta_0}^2}{2} < \frac{m_0}{16}$. We define

$$\hat{q}(t) = \begin{cases} (-1)^l & \text{if } t \leq \sigma - 1, \\ q(\sigma)(t - \sigma + 1) + (-1)^l(t - \sigma) & \text{if } \sigma - 1 \leq t \leq \sigma, \\ q(t) & \text{if } \sigma < t < \tau, \\ q(\tau)(\tau + 1 - t) + (-1)^{l+1}(t - \tau) & \text{if } \tau \leq t \leq \tau + 1, \\ (-1)^{l+1} & \text{if } t \geq \tau + 1, \end{cases}$$

observing that $\hat{q} \in \Gamma \cup \tilde{\Gamma}$ and so that

$$c_b \leq F_b(\hat{q}) = F_{b,(\sigma-1,\sigma)}(\hat{q}) + F_{b,(\sigma,\tau)}(q) + F_{b,(\tau,\tau+1)}(\hat{q}). \tag{2.5}$$

Now, let us note that since $q(\tau) = (-1)^{l+1}(1 - \delta_0)$, then $|(-1)^{l+1} - \hat{q}(t)| = \delta_0(1 - (t - \tau)) \leq \delta_0 \leq \overline{\delta}$ for any $t \in (\tau, \tau + 1)$ and by (1.8) we obtain

$$\begin{aligned} F_{b,(\tau,\tau+1)}(\hat{q}) &= \frac{{\delta_0}^2}{2} + \int_\tau^{\tau+1} bW(\hat{q})\,\mathrm{d}t \leq \frac{{\delta_0}^2}{2} + \frac{b\,\overline{w}{\delta_0}^2}{2}\int_\tau^{\tau+1}(1 - (t - \tau))^2\,\mathrm{d}t \\ &= \frac{{\delta_0}^2}{2} + \frac{b\,\overline{w}{\delta_0}^2}{6} \leq \lambda_{\delta_0}. \end{aligned} \tag{2.6}$$

Similar estimates allow us to conclude that also $F_{b,(\sigma-1,\sigma)}(\hat{q}) \leq \lambda_{\delta_0}$ and by (2.5) the lemma follows. □

In relation with ρ_0 and δ_0 we set

$$\varepsilon_0 = \frac{1}{2}\rho_0 \min\left\{1, \frac{\underline{a}}{2c_{\underline{a}}}\omega_{\delta_0}\right\}. \tag{2.7}$$

Remark 2.2. Note that, by (2.2) and (2.7), if $q \in \Gamma$ is such that $|q(t)| < 1 - \delta_0$ for every $t \in I$, where I is an interval with length $|I| \geq \frac{\rho_0}{\varepsilon_0}$, then

$$F_{\underline{a},I}(q) = \int_I \frac{1}{2}|\dot{q}|^2 + \underline{a}W(q)\,\mathrm{d}t \geq \underline{a}\omega_{\delta_0}|I| \geq 4c_{\underline{a}}.$$

The properties and the constants fixed above exhaust the preliminaries we need to tackle the principal object of the study of this section. In the sequel we will denote

$$c^* = 3c_{\underline{a}} + m_0. \tag{2.8}$$

The set $\{F_{a_\varepsilon} \leq c^*\}$: concentration and local compactness properties

Our goal now is to characterize some discreteness properties of the sublevel $\{F_{a_\varepsilon} \leq c^*\}$ which will be basic in the proof of the existence of two dimensional solutions of (1.3) in the next section.

As useful tool to study this problem we first introduce the function $\mathsf{nt} : \Gamma \to \mathbb{N}$, which counts the number of transitions of a function $q \in \Gamma$ between the values $-1 + \delta_0$ and $1 - \delta_0$.

Given $q \in \Gamma$, let us consider the set

$$\mathcal{D}_{\delta_0,q} = \{t \in \mathbb{R} \,|\, |q(t)| < 1 - \delta_0\},$$

the set of times in which $q(t)$ has distance from the equilibria ± 1 greater than δ_0. The set $\mathcal{D}_{\delta_0,q}$ is an open subset of $\mathbb{R}$ and so it is the disjoint union of open intervals which we denote by $(s_{i,q}, t_{i,q})$, $i \in \mathcal{I}$. We note that, by (2.2), $F_{\underline{a}}(q) \geq \sum_{i\in\mathcal{I}} \int_{s_{i,q}}^{t_{i,q}} \frac{1}{2}|\dot q|^2 + \underline{a}W(q)\,\mathrm{d}x \geq \sum_{i\in\mathcal{I}} \underline{a}\omega_{\delta_0}(t_{i,q} - s_{i,q}) = \underline{a}\omega_{\delta_0}|\mathcal{D}_{\delta_0,q}|$, and so

$$|\mathcal{D}_{\delta_0,q}| \leq \frac{1}{\underline{a}\omega_{\delta_0}} F_{\underline{a}}(q) \qquad \forall q \in \Gamma. \tag{2.9}$$

Now, for any $i \in \mathcal{I}$, we have $|q(s_{i,q})| = |q(t_{i,q})| = 1 - \delta_0$ and then we define

$$\mathsf{nt}(q, (s_{i,q}, t_{i,q})) = \begin{cases} 1 & \text{if } q(s_{i,q}) \neq q(t_{i,q}), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, given any interval $A \subset \mathbb{R}$ we set

$$\mathsf{nt}(q, A) = \sum_{i\in\mathcal{I}\,|\,(s_{i,q},t_{i,q})\subset A} \mathsf{nt}(q, (s_{i,q}, t_{i,q})).$$

and finally $\mathsf{nt}(q) = \mathsf{nt}(q, \mathbb{R})$.

The function nt counts the number of transitions of the function q between the values $-1 + \delta_0$ and $1 - \delta_0$. If $q \in \Gamma$ we always have that $\mathsf{nt}(q)$ is an odd number and the space Γ splits in the countable union of the disjoint classes:

$$\Gamma_k = \{q \in \Gamma \,|\, \mathsf{nt}(q) = k\}, \qquad k = 2n + 1, \quad n \in \mathbb{N}.$$

If $q \in \Gamma_k$, by definition, there exist $\{i_1, \ldots, i_k\} \subset \mathcal{I}$ such that

$$\mathsf{nt}(q, (s_{i,q}, t_{i,q})) = \begin{cases} 1 & \text{if } i \in \{i_1, \ldots, i_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

We can assume that for any $l \in \{1, \ldots, k-1\}$ the interval $(s_{i_l,q}, t_{i_l,q})$ is on the left of the interval $(s_{i_{l+1},q}, t_{i_{l+1},q})$ and we set

$$(\sigma_{l,q}, \tau_{l,q}) = (s_{i_l,q}, t_{i_l,q}), \qquad l = 1, \ldots, k.$$

With this position we have that for any $l \in \{1, \ldots, k\}$,

$$q(\sigma_{l,q}) = (-1)^l(1 - \delta_0) \quad \text{and} \quad q(\tau_{l,q}) = (-1)^{l+1}(1 - \delta_0).$$

Fixed any $\varepsilon \in (0, \varepsilon_0)$ and given $j \in \mathbb{Z}$, we define the intervals

$$A_j = \left(\frac{j + \overline{x}}{\varepsilon}, \frac{j + 1 + \overline{x}}{\varepsilon}\right), \qquad O_j = \left(\frac{j + \overline{x} - \rho_0}{\varepsilon}, \frac{j + \overline{x} + \rho_0}{\varepsilon}\right) \tag{2.10}$$

where $\overline{x}$ is a maximum for a as defined in (1.7) and ρ_0 is defined by (2.4). Note that, if $i \neq j$ then $A_i \cap A_j = \emptyset$ and $\mathbb{R} = \cup_{i\in\mathbb{Z}} \bar{A}_j$, where $\bar{A}_j$ denotes the closure of A_j. Moreover for any $j \in \mathbb{Z}$, the intervals O_j and O_{j+1} are centered respectively on the left and on the right extreme of A_j and, by (2.4), we have

$$\text{if} \quad \sup_{s\in O_j} |t-s| \le \frac{\rho_0}{\varepsilon} \quad \text{then} \quad a_\varepsilon(t) \ge \overline{\alpha}. \tag{2.11}$$

Note finally that $|A_j| = \frac{1}{\varepsilon}$ and $|O_j| = \frac{2\rho_0}{\varepsilon}$ for any $j \in Z$.

We can now describe simple concentration properties of the functions in $\{F_{a_\varepsilon} \le c^*\}$. Firstly we show that if $q \in \{F_{a_\varepsilon} \le c^*\}$ then $\mathsf{nt}(q) \le 3$, in other words q makes at most three transitions between the values $-1+\delta_0$ and $1-\delta_0$.

Lemma 2.4. *If $q \in \Gamma$ and $F_{a_\varepsilon}(q) \le c^*$ then $\mathsf{nt}(q) \le 3$.*

Proof. We simply observe that if $\mathsf{nt}(q) = k > 3$ then by Lemma 2.3 we have

$$F_{a_\varepsilon}(q) > \sum_{l=1}^{4} F_{\underline{a},(\sigma_{l,q},\tau_{l,q})}(q) \ge 4c_{\underline{a}} - \frac{m_0}{2},$$

which contradicts the assumption $F_{a_\varepsilon}(q) \le c^*$ since by definition $c^* = 3c_{\underline{a}} + m_0$ and, as we know, $c_{\underline{a}} > 6m \ge 12m_0$. □

Now, given $q \in \{F_{a_\varepsilon} \le c^*\}$ with $\mathsf{nt}(q) = 3$, we show that the intervals of transition $(\sigma_{l,q}, \tau_{l,q})$ can not intersect the set $\cup_{j\in\mathbb{Z}} O_j$ and outside of these intervals q is nearby ± 1 for less than $2\overline{\delta}$.

Lemma 2.5. *If $q \in \Gamma$ is such that $F_{a_\varepsilon}(q) \le c^*$ and $\mathsf{nt}(q) = 3$ then*

(i) $|q(t)| > 1 - 2\overline{\delta}$ *for all* $t \in \mathbb{R} \setminus (\cup_{l=1}^3 (\sigma_{l,q}, \tau_{l,q}))$.

(ii) $(\sigma_{l,q}, \tau_{l,q}) \cap (\cup_{j\in\mathbb{Z}} O_j) = \emptyset$, *for all* $l \in \{1,2,3\}$.

Proof. To prove (i), let $q \in \Gamma$ be such that $\mathsf{nt}(q) = 3$ and assume by contradiction that there exists $t_0 \in \mathbb{R} \setminus (\cup_{l=1}^3 (\sigma_{l,q}, \tau_{l,q}))$ for which $|q(t_0)| \le 1 - 2\overline{\delta}$. Since $\delta_0 < \overline{\delta}$ we have $|q(\sigma_{l,q})| > 1 - \overline{\delta}$ and $|q(\tau_{l,q})| > 1 - \overline{\delta}$ for $l = 1,2,3$. Then, by the intermediate values Theorem, there exists $(\sigma,\tau) \subset \mathbb{R} \setminus (\cup_{l=1}^3 (\sigma_{l,q}, \tau_{l,q}))$ such that $|q(t)| \le 1 - \overline{\delta}$ for any $t \in (\sigma,\tau)$ and $|q(\tau) - q(\sigma)| = \overline{\delta}$. Then by (2.2) we have $F_{\underline{a},(\sigma,\tau)}(q) \ge 2m_0$ and so using Lemma 2.3 and (2.8) we obtain

$$F_{a_\varepsilon}(q) \ge \sum_{l=1}^{3} F_{\underline{a},(\sigma_{l,q},\tau_{l,q})}(q) + F_{\underline{a},(\sigma,\tau)}(q) \ge 3c_{\underline{a}} - \frac{3m_0}{8} + 2m_0 > c^*,$$

a contradiction which proves (i).

To prove (ii), first we note that

$$\tau_{l,q} - \sigma_{l,q} \le \frac{\rho_0}{\varepsilon_0}, \qquad \forall l \in \{1,2,3\}. \tag{2.12}$$

Otherwise, by Remark 2.2 there exists $\overline{l} \in \{1,2,3\}$ such that $F_{a_\varepsilon}(q) \ge F_{\underline{a},(\sigma_{\overline{l},q},\tau_{\overline{l},q})}(q) \ge 4c_{\underline{a}} > c^*$. By (2.12) and (2.11), if there exists $\overline{l} \in \{1,2,3\}$ such that $(\sigma_{\overline{l},q}, \tau_{\overline{l},q}) \cap (\cup_{j\in\mathbb{Z}} O_j) \ne \emptyset$, we then have that $a_\varepsilon(t) \ge \overline{\alpha}$ for any $t \in (\sigma_{\overline{l},q}, \tau_{\overline{l},q})$. Therefore, by Lemma 2.3

$$F_{a_\varepsilon,(\sigma_{\overline{l},q},\tau_{\overline{l},q})}(q) \ge F_{\overline{\alpha},(\sigma_{\overline{l},q},\tau_{\overline{l},q})}(q) \ge c_{\overline{\alpha}} - \frac{m_0}{8}$$

and so, again by Lemma 2.3,

$$F_{a_\varepsilon}(q) \geq F_{\overline{\alpha},(\sigma_{\bar{l},q},\tau_{\bar{l},q})}(q) + \sum_{l\neq\bar{l}} F_{\underline{a},(\sigma_{l,q},\tau_{l,q})}(q) \geq c_{\overline{\alpha}} - \frac{m_0}{8} + \Big(2c_{\underline{a}} - \frac{m_0}{4}\Big)\cdot$$

By (2.8), since by (2.3) we have $c_{\overline{\alpha}} - c_{\underline{a}} \geq 3m_0$, this contradicts the assumption $F_{a_\varepsilon}(q) \leq c^*$. □

These concentration properties allow us to start in discretizing the set $\{F_{a_\varepsilon} \leq c^*\} \cap \{\mathsf{nt} = 3\}$. We let $\mathcal{P} = \{p = (p_1, p_2, p_3) \in \mathbb{Z}^3 \,|\, p_1 < p_2 < p_3\}$ and for $p \in \mathcal{P}$ we define

$$\Gamma_{3,p} = \{q \in \{F_{a_\varepsilon} \leq c^*\} \,|\, \mathsf{nt}(q) = 3,\ (\sigma_{l,q}, \tau_{l,q}) \subset A_{p_l},\ l = 1,2,3\}.$$

We study here below the existence, for all $p \in \mathcal{P}$, of solutions to the problem (2.1) belonging to the set $\Gamma_{3,p}$. In fact, setting $c_{3,p} = \inf\{F_{a_\varepsilon}(q) \,|\, q \in \Gamma_{3,p}\}$, we will prove that for any $p \in \mathcal{P}$ the set

$$\mathcal{K}_{3,p} = \{q \in \Gamma_{3,p} \,|\, F_{a_\varepsilon}(q) = c_{3,p}\}$$

is not empty, compact, with respect to the $H^1(\mathbb{R})$ metric, and consists of solutions of (2.1). The following preliminary result shows in particular that $\Gamma_{3,p}$ is not empty for any $p \in \mathcal{P}$.

Lemma 2.6. *$c_{3,p} \leq c^* - \frac{m_0}{8}$ for any $p \in \mathcal{P}$.*

Proof. Let us consider the above defined function $q_{\underline{\alpha}}$, solution to the problems $(P_{\underline{\alpha}})$ with $q_{\underline{\alpha}}(0) = 0$. Since $q_{\underline{\alpha}}$ is increasing and $q_{\underline{\alpha}}(0) = 0$ we have that there exist $\sigma < 0 < \tau$ such that $\mathcal{D}_{\delta_0,q_{\underline{\alpha}}} = (\sigma, \tau)$. Moreover, since by (2.3) we have $F_{\underline{a}}(q_{\underline{\alpha}}) \leq F_{\underline{\alpha}}(q_{\underline{\alpha}}) = c_{\underline{\alpha}} < 2c_{\underline{a}}$, by Remark 2.2 we obtain

$$\tau - \sigma < \frac{\rho_0}{\varepsilon_0}\cdot \tag{2.13}$$

We define the function

$$Q(t) = \begin{cases} -1 & \text{if } t \leq \sigma - 1, \\ q_{\underline{\alpha}}(\sigma)(t - \sigma + 1) + (t - \sigma) & \text{if } \sigma - 1 < t \leq \sigma, \\ q_{\underline{\alpha}}(t) & \text{if } \sigma < t < \tau, \\ q_{\underline{\alpha}}(\tau)(\tau + 1 - t) + (t - \tau) & \text{if } \tau \leq t < \tau + 1, \\ 1 & \text{if } t \geq \tau + 1, \end{cases}$$

noting that, arguing as in the proof of Lemma 2.3,

$$F_{\underline{\alpha}}(Q) = F_{\underline{\alpha},(\sigma-1,\sigma)}(Q) + F_{\underline{\alpha},(\sigma,\tau)}(q_{\underline{\alpha}}) + F_{\underline{\alpha},(\tau,\tau+1)}(Q) \leq c_{\underline{\alpha}} + \frac{m_0}{8}\cdot$$

Letting $p = (p_1, p_2, p_3) \in \mathcal{P}$ we set

$$Q_{p_1}(t) = Q\left(t - \frac{p_1 + \underline{x}}{\varepsilon}\right),\ Q_{p_2}(t) = Q\left(-t + \frac{p_2 + \underline{x}}{\varepsilon}\right) \text{ and } Q_{p_3}(t) = Q\left(t - \frac{p_3 + \underline{x}}{\varepsilon}\right)\cdot$$

Since $F_{\underline{\alpha}}$ is invariant by time translation and reflection, we have

$$F_{\underline{\alpha}}(Q_{p_j}) = F_{\underline{\alpha}}(Q) \leq c_{\underline{\alpha}} + \frac{m_0}{8} \qquad j = 1,2,3. \tag{2.14}$$

We note also that, by (2.13) and (2.4), if $|Q_{p_j}(t)| \neq 1$ then $t \in A_{p_j}$ and $a_\varepsilon(t) \leq \underline{\alpha}$ $(j = 1,2,3)$. Therefore, by (2.14), we obtain

$$F_{a_\varepsilon,A_{p_j}}(Q_{p_j}) = F_{a_\varepsilon}(Q_{p_j}) \leq F_{\underline{\alpha}}(Q_{p_j}) \leq c_{\underline{\alpha}} + \frac{m_0}{8}, \qquad j = 1,2,3. \tag{2.15}$$

We finally consider the function $Q_{3,p} \in \Gamma$ defined as follows: $|Q_{3,p}(t)| = 1$ if $t \in \mathbb{R} \setminus (\cup_{j=1}^3 A_{p_j})$ and

$$Q_{3,p}(t) = \begin{cases} Q_{p_1}(t) & \text{if } t \in A_{p_1}, \\ Q_{p_2}(t) & \text{if } t \in A_{p_2}, \\ Q_{p_3}(t) & \text{if } t \in A_{p_3}. \end{cases}$$

Then by (2.8), (2.15) and (2.3) we have

$$F_{a_\varepsilon}(Q_{3,p}) = \sum_{j=1}^{3} F_{a_\varepsilon, A_{p_j}}(Q_{p_j}) \le 3c_{\underline{\alpha}} + \frac{3m_0}{8} = c^* + (3(c_{\underline{\alpha}} - c_{\underline{a}}) + \frac{3m_0}{8} - m_0) < c^* - \frac{m_0}{8},$$

and the lemma follows. □

We now show that the sets $\Gamma_{3,p}$ are sequentially compact with respect to the weak topology in $H^1_{loc}(\mathbb{R})$.

Lemma 2.7. *If $p \in \mathcal{P}$ and $(q_n) \subset \Gamma_{3,p}$ then there exists $q \in \Gamma_{3,p}$ such that, along a subsequence, $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot q_n \to \dot q$ weakly in $L^2(\mathbb{R})$.*

Proof. Let $p = (p_1, p_2, p_3) \in \mathcal{P}$ and $(q_n) \subset \Gamma_{3,p}$. Since $F_{a_\varepsilon}(q_n) \le c^*$ for any $n \in \mathbb{N}$, by Lemma 2.1 there exists $q \in H^1_{loc}(\mathbb{R})$ with $\|q\|_{L^\infty} \le 1$ and $F_{a_\varepsilon}(q) \le c^*$ such that, along a subsequence, still denoted (q_n), we have $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot q_n \to \dot q$ weakly in $L^2(\mathbb{R})$.

Since $F_{a_\varepsilon}(q) < +\infty$, by (2.2), one plainly obtains that $|q(t)| \to 1$ as $t \to \pm\infty$. To show that $q(t) \to \pm 1$ as $t \to \pm\infty$ and so that $q \in \Gamma$, note that by Lemma 2.5 we have that for any $n \in \mathbb{N}$, if $t \in \mathbb{R} \setminus (\cup_{l=1}^3 A_{p_l})$ then $|q_n(t)| \ge 1 - 2\overline{\delta}$. Hence by pointwise convergence we obtain that

$$q(t) \le -1 + 2\delta \ \text{ if } \ t \le \frac{p_1 - 1 + \overline{x}}{\varepsilon} \quad \text{and} \quad q(t) \ge 1 - 2\overline{\delta} \ \text{ if } \ t \ge \frac{p_3 + 1 + \overline{x}}{\varepsilon},$$

and so that $q(t) \to \pm 1$ as $t \to \pm\infty$.

To show that $q \in \Gamma_{3,p}$ note that, by Lemma 2.5 we have $(\sigma_{l,q_n}, \tau_{l,q_n}) \subset A_{p_l} \setminus (O_{p_l} \cup O_{p_l+1})$ for any $l \in \{1,2,3\}$ and $n \in \mathbb{N}$. Then, for any $l \in \{1,2,3\}$ there exists $\sigma_l < \tau_l \in A_{p_l} \setminus (O_{p_l} \cup O_{p_l+1})$ such that, up to a subsequence, $\tau_{l,q_n} \to \tau_l$, $\sigma_{l,q_n} \to \sigma_l$ as $n \to +\infty$ and, by L^∞_{loc} convergence,

$$q(\sigma_l) = (-1)^l(1 - \delta_0) \quad \text{and} \quad q(\tau_l) = (-1)^{l+1}(1 - \delta_0).$$

Hence $\mathsf{nt}(q, A_{p_l}) \ge 1$ for any $l \in \{1,2,3\}$ and since $F_{a_\varepsilon}(q) \le c^*$, by Lemma 2.4, we can conclude that $\mathsf{nt}(q) = 3$. Then, $q \in \Gamma_{3,p}$ and the lemma is proved. □

Thanks to Lemmas 2.1 and 2.7 it is now possible to apply the direct method of the Calculus of Variations to show that the set $\mathcal{K}_{3,p}$ is not empty for any $p \in \mathcal{P}$.

Proposition 2.1. *For every $\varepsilon \in (0, \varepsilon_0)$ and $p \in \mathcal{P}$ we have $\mathcal{K}_{3,p} \neq \emptyset$. Moreover, if $q \in \mathcal{K}_{3,p}$ then $q \in \mathcal{C}^2(\mathbb{R})$ and it is a classical solution to* (2.1).

Proof. Let $p = (p_1, p_2, p_3) \in \mathcal{P}$ and $(q_n) \subset \Gamma_{3,p}$ be such that $F_{a_\varepsilon}(q_n) \to c_{3,p}$ as $n \to +\infty$. By Lemmas 2.1 and 2.7 we obtain that (q_n) converges along a subsequence, in the specified way, to a function $q \in \Gamma_{3,p}$ with $F_{a_\varepsilon}(q) \le c_{3,p}$. Then $q \in \Gamma_{3,p}$ and $F_{a_\varepsilon}(q) = c_{3,p}$, *i.e.*, $q \in \mathcal{K}_{3,p}$.

To complete the proof we have to show that if $q \in \mathcal{K}_{3,p}$ then $q \in \mathcal{C}^2(\mathbb{R})$ and it is a classical solution to (2.1). To this aim we firstly note that for any $l \in \{1,2,3\}$, there exist $s_l \in A_{p_l} \cap O_{p_l}$ and $t_l \in A_{p_l} \cap O_{p_l+1}$ such that $|q(s_l)| > 1 - \delta_0$ and $|q(t_l)| > 1 - \delta_0$.

Indeed, otherwise, there exists $\bar{l} \in \{1,2,3\}$ for which $|q(t)| \le 1-\delta_0$ for any $t \in A_{p_{\bar{l}}} \cap O_{p_{\bar{l}}}$ or for any $t \in A_{p_{\bar{l}}} \cap O_{p_{\bar{l}}+1}$. Then, since $|A_{p_{\bar{l}}} \cap O_{p_{\bar{l}}}| = |A_{p_{\bar{l}}} \cap O_{p_{\bar{l}}+1}| = \frac{\rho_0}{\varepsilon_0}$, by Remark 2.2 we have in both the cases that $F_{a_\varepsilon, A_{p_{\bar{l}}}}(q) \ge 2c_{\underline{a}}$ and so, by Lemma 2.3 and (2.8),

$$F_{a_\varepsilon}(q) \ge F_{a_\varepsilon, A_{p_{\bar{l}}}}(q) + \sum_{l \ne \bar{l}} F_{a_\varepsilon, A_{p_l}}(q) \ge 2c_{\underline{a}} + \left(2c_{\underline{a}} - \frac{m_0}{4}\right) > c^*.$$

Now, let $\tilde{\delta} = \max_{l=1,2,3}\{1-|q(s_l)|, 1-|q(t_l)|\}$ and note that $\delta_0 > \tilde{\delta}$. Given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\|\varphi\|_{L^\infty(\mathbb{R})} < \delta_0 - \tilde{\delta}$, let $\psi = q + \varphi$ and consider $\hat{\psi}(t) = \min\{\max\{\psi(t), -1\}, 1\}$ noting that $F_{a_\varepsilon}(\psi) \ge F_{a_\varepsilon}(\hat{\psi})$. If $F_{a_\varepsilon}(\hat{\psi}) > c^*$ we have

$$F_{a_\varepsilon}(\psi) \ge F_{a_\varepsilon}(\hat{\psi}) > c^* \ge c_{3,p} = F_{a_\varepsilon}(q). \tag{2.16}$$

If otherwise $F_{a_\varepsilon}(\hat{\psi}) \le c^*$ we claim that $\hat{\psi} \in \Gamma_{3,p}$ and so that

$$F_{a_\varepsilon}(\psi) \ge F_{a_\varepsilon}(\hat{\psi}) \ge c_{3,p} = F_{a_\varepsilon}(q). \tag{2.17}$$

To show that $\hat{\psi} \in \Gamma_{3,p}$, first note that since $\hat{\psi} \in \Gamma$ and $F_{a_\varepsilon}(\hat{\psi}) \le c^*$, by Lemma 2.4 we have that $\mathsf{nt}(\hat{\psi}) \le 3$. Moreover, note that

$$|q(s_l) - (-1)^l| < \tilde{\delta} \quad \text{and} \quad |q(t_l) - (-1)^{l+1}| < \tilde{\delta}, \quad \forall\, l \in \{1,2,3\}$$

and since $\|q - \hat{\psi}\|_{L^\infty(\mathbb{R})} \le \|\varphi\|_{L^\infty(\mathbb{R})} < \delta_0 - \tilde{\delta}$, we obtain that

$$|\hat{\psi}(s_l) - (-1)^l| < \delta_0 \quad \text{and} \quad |\hat{\psi}(t_l) - (-1)^{l+1}| < \delta_0, \quad \forall\, l \in \{1,2,3\}$$

from which we deduce that $\mathsf{nt}(\hat{\psi}, A_{p_l}) \ge 1$ for any $l \in \{1,2,3\}$. Therefore $\hat{\psi} \in \Gamma_{3,p}$ as we claimed.

By (2.16) and (2.17) we conclude that

$$F_{a_\varepsilon}(q) \le F_{a_\varepsilon}(q+\varphi), \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}) \text{ with } \|\varphi\|_{L^\infty(\mathbb{R})} < \delta_0 - \tilde{\delta}.$$

Then q is a weak solution to $-\ddot{q} + a_\varepsilon(t) W'(q) = 0$ on $\mathbb{R}$ and, by standard bootstrap arguments, a classical $\mathcal{C}^2(\mathbb{R})$ solution to (2.1). □

The next lemma shows in particular that the sets $\mathcal{K}_{3,p}$, $p \in \mathcal{P}$, are sequentially compact with respect to the $H^1(\mathbb{R})$ metric.

Lemma 2.8. *If $(q_n) \subset \Gamma_{3,p}$ is such that $F_{a_\varepsilon}(q_n) \to c_{3,p}$ then there exists $q \in \mathcal{K}_{3,p}$ such that along a subsequence $\|q_n - q\|_{H^1(\mathbb{R})} \to 0$ as $n \to \infty$.*

Proof. As in the proof of Proposition 2.1 we obtain that there exists $q \in \mathcal{K}_{3,p}$ and a subsequence of (q_n), still denoted (q_n), such that $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$. Note that since $\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t \le \liminf_{n\to\infty} \int_{\mathbb{R}} a_\varepsilon W(q_n)\,\mathrm{d}t$ and since $F_{a_\varepsilon}(q_n) \to F_{a_\varepsilon}(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \|\dot{q}_n\|^2 &= 2\lim_{n\to\infty} F_{a_\varepsilon}(q_n) - 2\liminf_{n\to\infty} \int_{\mathbb{R}} a_\varepsilon W(q_n)\,\mathrm{d}t \\ &\le 2F_{a_\varepsilon}(q) - 2\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t = \|\dot{q}\|^2 \le \liminf_{n\to\infty} \|\dot{q}_n\|^2.\end{aligned}$$

Therefore $\|\dot{q}_n\| \to \|\dot{q}\|$ as $n \to \infty$ and so $\|\dot{q}_n - \dot{q}\| \to 0$.

To show that $\|q_n - q\| \to 0$ as $n \to \infty$, note that since $\|\dot{q}_n\| \to \|\dot{q}\|$ and $F_{a_\varepsilon}(q_n) \to F_{a_\varepsilon}(q)$ as $n \to \infty$ we have $\int_{\mathbb{R}} a_\varepsilon W(q_n)\,\mathrm{d}t \to \int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t$ as $n \to \infty$. Moreover, since by L^∞_{loc} convergence we have that $\int_{-T}^{T} a_\varepsilon W(q_n)\,\mathrm{d}t \to$

$\int_{-T}^{T} a_\varepsilon W(q)\,\mathrm{d}t$ for any $T > 0$ we obtain that $\int_{|t|>T} a_\varepsilon W(q_n)\,\mathrm{d}t \to \int_{|t|>T} a_\varepsilon W(q)\,\mathrm{d}t$ for any $T > 0$. We deduce that for any $\eta > 0$ there exists $T > 0$ and $\bar{n} \in \mathbb{N}$ such that $\int_{|t|>T} a_\varepsilon W(q)\,\mathrm{d}t < \eta$ and $\int_{|t|>T} a_\varepsilon W(q_n)\,\mathrm{d}t < \eta$ for any $n \ge \bar{n}$. Hence, since by Lemma 2.5, there exists $T_0 > 0$ such that $|q(t)| \ge 1 - 2\overline{\delta}$ and $|q_n(t)| \ge 1 - 2\overline{\delta}$ for any $|t| \ge T_0$ and $n \in \mathbb{N}$, using (1.9) we conclude that for any $\eta > 0$ there exists $T \ge T_0$ and $\bar{n} \in \mathbb{N}$ such that for any $n \ge \bar{n}$ we have

$$\begin{aligned}\int_{|t|>T} |q_n - q|^2 &\le 2\left(\int_{t<-T} (q_n+1)^2 + (q+1)^2\,\mathrm{d}t + \int_{t>T} (q_n-1)^2 + (q-1)^2\,\mathrm{d}t\right)\\ &= 2\left(\int_{t<-T} \chi(q_n)^2 + \chi(q)^2\,\mathrm{d}t + \int_{t>T} \chi(q_n)^2 + \chi(q)^2\,\mathrm{d}t\right)\\ &= 2\int_{|t|>T} \chi(q_n)^2 + \chi(q)^2\,\mathrm{d}t \le \frac{2}{\underline{w}\,\underline{a}} \int_{|t|>T} a_\varepsilon W(q_n) + a_\varepsilon W(q)\,\mathrm{d}t \le \frac{4\eta}{\underline{w}\,\underline{a}}.\end{aligned}$$

Therefore $\|q_n - q\| \to 0$ as $n \to \infty$ follows since, by L^∞_{loc} convergence, we have $\int_{|t|\le T} |q_n - q|^2\,\mathrm{d}t \to 0$ as $n \to \infty$. □

Note that by Lemma 2.8, using a direct contradiction argument, we obtain that for any $r > 0$ there exists $\nu_r > 0$ such that

$$\text{if } q \in \Gamma_{3,p} \text{ and } \inf_{\bar{q}\in\mathcal{K}_{3,p}} \|q - \bar{q}\|_{H^1(\mathbb{R})} \ge r \text{ then } F(q) \ge c_{3,p} + \nu_r. \tag{2.18}$$

The set $\{F_{a_\varepsilon} \le c^*\}$: discreteness properties

We end the section characterizing some metric properties of the sets $\Gamma_{3,p}$ as subsets of the metric space $\overline{\Gamma}$. All these properties will be used in the next section to prove the existence of two dimensional solutions to (1.3).

Given two subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\overline{\Gamma}$, with abuse of notation we set

$$\mathsf{d}(\mathcal{U}_1,\mathcal{U}_2) = \inf_{q_1\in\mathcal{U}_1,\, q_2\in\mathcal{U}_2} \|q_1 - q_2\| \quad\text{and}\quad \mathsf{diam}(\mathcal{U}_1) = \sup_{q_1,q_2\in\mathcal{U}_1} \|q_1 - q_2\|.$$

For $p \in \mathcal{P}$ and $i \in \{1, ..., 4\}$ we set

$$B_i(p) = \left(\frac{p_{i-1} + 1 + \overline{x}}{\varepsilon}, \frac{p_i + \overline{x}}{\varepsilon}\right)$$

with the agreement that $p_0 = -\infty$ and $p_4 = +\infty$. Note that $\mathbb{R} = (\cup_{i=1}^4 B_i(p)) \cup (\cup_{l=1}^3 \bar{A}_{p_l})$ and that $B_i(p) \cap (\cup_{l=1}^3 \bar{A}_{p_l}) = \emptyset$ for all $i \in \{1, ..., 4\}$. Moreover, it is a direct consequence of Lemma 2.5 that for any $q \in \Gamma_{3,p}$ we have

$$|q(t) - (-1)^i| \le 2\overline{\delta} \text{ for any } t \in B_i(p),\ i \in \{1, \ldots, 4\}. \tag{2.19}$$

Next Lemma shows that $\mathsf{diam}(\Gamma_{3,p})$ is uniformly bounded with respect to $p \in \mathcal{P}$.

Lemma 2.9. *There exists* $\mathsf{D} \in \mathbb{R}$ *such that* $\mathsf{diam}(\Gamma_{3,p}) \le \mathsf{D}$ *for all* $p \in \mathcal{P}$.

Proof. Let $p \in \mathcal{P}$ and $q_1, q_2 \in \Gamma_{3,p}$. Then

$$\|q_1 - q_2\|^2 = \sum_{i=1}^{3} \int_{A_{p_i}} |q_1 - q_2|^2\,\mathrm{d}x + \sum_{\iota=1}^{4} \int_{B_\iota(p)} |q_1 - q_2|^2\,\mathrm{d}x.$$

We clearly have $\int_{A_{p_i}} |q_1 - q_2|^2\,\mathrm{d}x \le 4|A_0|$. Moreover, by (2.19) and (1.9) we have that for any $t \in B_\iota(p)$ and $1 \le \iota \le 4$

$$|q_1 - q_2|^2 \le 2(|q_1 - (-1)^\iota|^2 + |q_2 - (-1)^\iota|^2) = 2(\chi(q_1)^2 + \chi(q_2)^2) \le \frac{4a_\varepsilon}{\underline{w}\,\underline{a}}(W(q_1) + W(q_2)).$$

Therefore

$$\sum_{\iota=1}^{4}\int_{B_\iota(p)}|q_1-q_2|^2\,\mathrm{d}t\le\frac{4}{\underline{w}\,\underline{a}}\sum_{\iota=1}^{4}\int_{B_\iota(p)}a_\varepsilon(W(q_1)+W(q_2))\,\mathrm{d}t\le\frac{8c^*}{\underline{w}\,\underline{a}}$$

and the lemma follows. □

In the next lemma we show that the sets $\Gamma_{3,p}$, $p\in\mathcal{P}$, are well separated in $\overline{\Gamma}$.

Lemma 2.10. *If $p,\bar p\in\mathcal{P}$ and $p\ne\bar p$ then $\mathsf{d}(\Gamma_{3,p},\Gamma_{3,\bar p})\ge(\frac{2\rho_0}{\varepsilon})^{1/2}$.*

Proof. Let $p\ne\bar p\in\mathcal{P}$, $q\in\Gamma_{3,p}$ and $\bar q\in\Gamma_{3,\bar p}$. We set

$$k=\min\{l\in\{1,2,3\}\,/\,p_l\ne\bar p_l\}.$$

Assume that $p_k<\bar p_k$ (the other case can be handled in the same way). Then, by Lemma 2.5, we have $|q(t)-(-1)^{k+1}|\le 2\overline{\delta}$ and $|\bar q(t)-(-1)^k|\le 2\overline{\delta}$ for any $t\in O_{p_k+1}$. Since $\overline{\delta}<\frac14$, we have $|q(t)-\bar q(t)|\ge 1$ for all $t\in O_{p_k+1}$. Then $\|q-\bar q\|^2_{L^2(O_{p_k+1})}\ge|O_{p_k+1}|=\frac{2\rho_0}{\varepsilon}$ from which, since q and $\bar q$ are arbitrary, the lemma follows. □

We remark that for any $p\in\mathcal{P}$ we have

$$\min\{|B_2(p)|,|B_3(p)|\}=\frac{1}{\varepsilon}\max\{([p]-2),0\}\tag{2.20}$$

where $[p]\equiv\min\{p_2-p_1,p_3-p_2\}$. Next lemma shows that any bounded set in $\overline{\Gamma}$ can intersect at most a finite number of $\Gamma_{3,p}$, $p\in\mathcal{P}$.

Lemma 2.11. *For any $\bar p\in\mathcal{P}$ we have $\mathsf{d}(\Gamma_{3,p},\Gamma_{3,\bar p})\to+\infty$ whenever $[p]\to\infty$ or $p_3\to-\infty$ or $p_1\to+\infty$.*

Proof. Let p, $\bar p\in\mathcal{P}$, $q\in\Gamma_{3,p}$, $\bar q\in\Gamma_{3,\bar p}$ and, for $1\le\iota\le 4$ denote $B_\iota\equiv B_\iota(p)$, $\bar B_\iota\equiv B_\iota(\bar p)$. Clearly

$$\|q-\bar q\|^2\ge\int_{B_2(p)}|q-\bar q|^2\,\mathrm{d}t+\int_{B_3(p)}|q-\bar q|^2\,\mathrm{d}t.$$

Since $\overline{\delta}\le\frac14$, by (2.19) we have $|q(t)-\bar q(t)|\ge 1$ whenever $t\in B_2\cap(\bar B_1\cup\bar B_3)$ or $t\in B_3\cap(\bar B_2\cup\bar B_4)$. Then

$$\|q-\bar q\|^2\ge|B_2\cap\bar B_1|+|B_2\cap\bar B_3|+|B_3\cap\bar B_2|+|B_2\cap\bar B_4|.$$

Since $B_2=(\cup_{i=1}^3 B_2\cap A_{\bar p_i})\cup(\cup_{i=1}^4 B_2\cap\bar B_i)$ we have $|B_2|\le\frac{3}{\varepsilon}+\sum_{\iota=1}^4|B_2\cap\bar B_\iota|$ and then

$$|B_2\cap\bar B_1|+|B_2\cap\bar B_3|\ge|B_2|-|\bar B_2|-|B_2\cap\bar B_4|-\frac{3}{\varepsilon}\cdot$$

Analogously

$$|B_3\cap\bar B_2|+|B_3\cap\bar B_4|\ge|B_3|-|\bar B_3|-|B_3\cap\bar B_1|-\frac{3}{\varepsilon}\cdot$$

Then, noting that $\min\{|B_2\cap\bar B_4|,|B_3\cap\bar B_1|\}=0$, we obtain

$$\begin{aligned}\|q-\bar q\|^2&\ge\max\{|B_2|-|\bar B_2|-|B_2\cap\bar B_4|,|B_3|-|\bar B_3|-|B_3\cap\bar B_1|\}-\frac{3}{\varepsilon}\\&\ge\min\{|B_2|-|\bar B_2|,|B_3|-|\bar B_3|\}-\frac{3}{\varepsilon}\\&\ge\min\{|B_2|,|B_3|\}-\max\{|\bar B_2|,|\bar B_3|\}-\frac{3}{\varepsilon}\end{aligned}$$

and by (2.20), we conclude

$$\|q-\bar{q}\|^2 \geq \frac{1}{\varepsilon}([p]-2) - \max\{|\bar{B}_2|, |\bar{B}_3|\} - \frac{3}{\varepsilon},$$

from which we obtain that $\lim_{[p]\to\infty} \mathsf{d}(\Gamma_{3,p}, \Gamma_{3,\bar{p}}) = +\infty$.

One analogously argues in the cases $p_3 \to -\infty$ and $p_1 \to +\infty$ and the lemma follows. □

Finally we let

$$\Lambda = \{F_{a_\varepsilon} \leq c^*\} \setminus (\cup_{p\in\mathcal{P}}\Gamma_{3,p})$$

and we show that Λ has positive distance in $\overline{\Gamma}$ from the set $\cup_{p\in\mathcal{P}}\Gamma_{3,p}$.

Lemma 2.12. *We have* $\mathsf{d}(\Gamma_{3,p}, \Lambda) \geq (\frac{5m}{\underline{a}\omega_{\delta_0}})^{1/2}$ *for any* $p \in \mathcal{P}$ *and moreover*

$$\mathsf{d}(\Gamma_{3,p}, \Lambda) \to +\infty \quad \textit{as } [p] \to +\infty.$$

Proof. Let $p \in \mathcal{P}$, $q \in \Gamma_{3,p}$ and $\bar{q} \in \Lambda$. For $1 \leq \iota \leq 4$, we denote $B_\iota = B_\iota(p)$.

We note that the set Λ can be written as the disjoint union of the two subset

$$\Lambda_1 = \{q \in \Gamma \,/\, F_{a_\varepsilon}(q) \leq c^* \text{ and } \mathsf{nt}(q) = 1\},$$

$$\Lambda_2 = \{q \in \Gamma \,/\, F_{a_\varepsilon}(q) \leq c^*,\ \mathsf{nt}(q) = 3 \text{ and } \exists j \in \mathbb{Z} \text{ such that } \mathsf{nt}(q, A_j) \geq 2\}$$

and in the following we will separately consider the two cases $\bar{q} \in \Lambda_1$ and $\bar{q} \in \Lambda_2$.

If $\bar{q} \in \Lambda_1$ then there exists $I_- \prec (\sigma, \tau) \prec I_+$ ($E_1 \prec E_2$ if $t \in E_1$ and $s \in E_2$ implies $t < s$) such that $\bar{q}(\sigma) = -1 + \delta_0$, $\bar{q}(\tau) = 1 - \delta_0$ and $\mathcal{D}_{\delta_0,\bar{q}} = I_- \cup (\sigma, \tau) \cup I_+$. Note that, by (2.9) and since $c_{\underline{a}} \geq 6m$, we have

$$|\mathcal{D}_{\delta_0,\bar{q}}| \leq \frac{c^*}{\underline{a}\omega_{\delta_0}} = \frac{3c_{\underline{a}} + m_0}{\underline{a}\omega_{\delta_0}} \leq \frac{4c_{\underline{a}}}{\underline{a}\omega_{\delta_0}} - \frac{5m}{\underline{a}\omega_{\delta_0}} \leq \frac{\rho_0}{\varepsilon_0} - \frac{5m}{\underline{a}\omega_{\delta_0}}. \tag{2.21}$$

Setting $\bar{B}_- = (-\infty, \sigma) \setminus I_-$, $\bar{B}_+ = (\tau, +\infty) \setminus I_+$ we have $\bar{q}(t) \leq -1 + \delta_0$ for any $t \in \bar{B}_-$, $\bar{q}(t) \geq 1 - \delta_0$ for any $t \in \bar{B}_+$.

Since $q \in \Gamma_{3,p}$ by Lemma 2.5 we have that $q(t) \leq -1 + 2\overline{\delta}$ for any $t \in A_{p_1} \cap O_{p_1}$ and $q(t) \geq 1 - 2\overline{\delta}$ for any $t \in A_{p_1} \cap O_{p_1+1}$. Therefore $|q(t) - \bar{q}(t)| \geq 1$ for any $t \in A_{p_1} \cap O_{p_1} \cap \bar{B}_+$ and for any $t \in A_{p_1} \cap O_{p_1+1} \cap \bar{B}_-$. Hence

$$\begin{aligned}\|q-\bar{q}\|^2 &\geq |A_{p_1} \cap O_{p_1} \cap \bar{B}_+| + |A_{p_1} \cap O_{p_1+1} \cap \bar{B}_-| \\ &= |(A_{p_1} \cap O_{p_1}) \setminus (I_+ \cup (-\infty, \tau))| + |(A_{p_1} \cap O_{p_1+1}) \setminus (I_- \cup (\sigma, +\infty))|\end{aligned}$$

and since

$$\begin{aligned}&\min\{|(A_{p_1} \cap O_{p_1}) \setminus (I_+ \cup (-\infty, \tau))|, |(A_{p_1} \cap O_{p_1+1}) \setminus (I_- \cup (\sigma, +\infty))|\} \\ &\quad\geq \min\{|(A_{p_1} \cap O_{p_1}) \setminus I_+|, |(A_{p_1} \cap O_{p_1+1}) \setminus I_-|\} - |(\sigma, \tau)| \\ &\quad\geq \min\{|A_{p_1} \cap O_{p_1}|, |A_{p_1} \cap O_{p_1+1})|\} - \max\{|I_+|, |I_-|\} - |(\sigma, \tau)| \geq \frac{\rho_0}{\varepsilon} - |\mathcal{D}_{\delta_0,\bar{q}}|\end{aligned}$$

by (2.21) we conclude that $\|q-\bar{q}\|^2 \geq \frac{5m}{\underline{a}\omega_{\delta_0}}$, from which, since q and $\bar{q}$ are arbitrary, we deduce that $\mathsf{d}(\Gamma_{3,p}, \Lambda_1)^2 \geq \frac{5m}{\underline{a}\omega_{\delta_0}}$.

Let us show now that if $[p] \to \infty$ then $\mathsf{d}(\Gamma_{3,p}, \Lambda_1) \to +\infty$. Since $q \in \Gamma_{3,p}$ we have $q(t) \geq 1 - 2\overline{\delta}$ for any $t \in B_2$, $q(t) \leq -1 + 2\overline{\delta}$ for any $t \in B_3$ and since $\delta_0 \leq \overline{\delta} \leq \frac{1}{4}$ there results $|q(t) - \bar{q}(t)| \geq 1$ for any $t \in B_3 \cap \bar{B}_+$

and for any $t \in B_2 \cap \bar{B}_-$. Hence

$$\begin{aligned}\|q-\bar{q}\|^2 &\geq \int_{B_2\cap\bar{B}_-} |q-\bar{q}|^2\,\mathrm{d}x + \int_{B_3\cap\bar{B}_+} |q-\bar{q}|^2\,\mathrm{d}x \geq |B_2\cap\bar{B}_-| + |B_3\cap\bar{B}_+| \\ &= |B_2 \setminus (I_- \cup (\overline{\sigma}, +\infty))| + |B_3 \setminus (I_+ \cup (-\infty, \overline{\tau}))| \\ &\geq |B_2 \setminus (\overline{\sigma}, +\infty)| - |I_-| + |B_3 \setminus (-\infty, \overline{\tau})| - |I_+|.\end{aligned}$$

Then, since

$$\max\{|B_2 \setminus (\sigma, +\infty)|, |B_3 \setminus (-\infty, \tau)|\} \geq \min\{|B_2|, |B_3|\} - |(\sigma, \tau)|$$

by (2.21), and (2.20) we conclude

$$\|q-\bar{q}\|^2 \geq ([p]-2)\frac{1}{\varepsilon} - |\mathcal{D}_{\delta_0,\bar{q}}| \geq ([p]-2)\frac{1}{\varepsilon} - \frac{c^*}{\underline{a}\omega_{\delta_0}},$$

from which we derive that $\mathsf{d}(\Gamma_{3,p}, \Lambda_1) \to +\infty$ as $[p] \to +\infty$.

Let us now consider the case $\bar{q} \in \Lambda_2$. Then there exist two (not necessarily different) indices $\bar{l}, \bar{m} \in \mathbb{Z}$, such that $\mathsf{nt}(\bar{q}, A_{\bar{l}}) \geq 2$, $\mathsf{nt}(\bar{q}, A_{\bar{m}}) \geq 1$ and $\mathsf{nt}(\bar{q}, A_j) = 0$ for any $j \notin \{\bar{l}, \bar{m}\}$.

If $\bar{l} \leq \bar{m}$ we set

$$\bar{B}_- = (\cup_{j<\bar{m}} A_j) \setminus A_{\bar{l}}, \quad \bar{B}_+ = (\cup_{j>\bar{m}} A_j)$$

while if $\bar{l} > \bar{m}$ we set

$$\bar{B}_- = \cup_{j<\bar{m}} A_j, \quad \bar{B}_+ = (\cup_{j>\bar{m}} A_j) \setminus A_{\bar{l}}.$$

In any case, by Lemma 2.5, we have that $\bar{q}(t) \leq -1 + 2\overline{\delta}$ for any $t \in \bar{B}_-$, $\bar{q}(t) \geq 1 - 2\overline{\delta}$ for any $t \in \bar{B}_+$.

Since $q \in \Gamma_{3,p}$ there exists $k \in \{1, 2, 3\}$ such that $p_k \notin \{\bar{l}, \bar{m}\}$ and so there results either $A_{p_k} \subset \bar{B}_-$ or $A_{p_k} \subset \bar{B}_+$. Since $\mathsf{nt}(q, A_{p_k}) = 1$ and since by Lemma 2.5 we have $|q(t)| \geq 1 - 2\overline{\delta}$ for any $t \in A_{p_k} \cap (O_{p_k} \cup O_{p_k+1})$, it is simple to recognize that in both the cases we have either $|q(t) - \bar{q}(t)| \geq 1$ for any $t \in A_{p_k} \cap O_{p_k}$ or $|q(t) - \bar{q}(t)| \geq 1$ for any $t \in A_{p_k} \cap O_{p_k+1}$. Then

$$\|q(t) - \bar{q}(t)\|^2 \geq \min\{|A_{p_k} \cap O_{p_k}|, |A_{p_k} \cap O_{p_k+1}|\} \geq \frac{\rho_0}{\varepsilon_0}$$

from which we derive that $\mathsf{d}(\Gamma_{3,p}, \Lambda_3)^2 \geq \frac{\rho_0}{\varepsilon_0} \geq \frac{5m}{\underline{a}\omega_{\delta_0}}$ and so the first part of the Lemma. To end the proof we show now that if $[p] \to \infty$ then $\mathsf{d}(\Gamma_{3,p}, \Lambda_3) \to +\infty$.

Since $q \in \Gamma_{3,p}$ we have $q(t) \geq 1 - 2\overline{\delta}$ for any $t \in B_2$, $q(t) \leq -1 + 2\overline{\delta}$ for any $t \in B_3$ and since $\overline{\delta} \leq \frac{1}{4}$ we obtain $|q(t) - \bar{q}(t)| \geq 1$ for any $t \in B_2 \cap \bar{B}_-$ and for any $t \in B_3 \cap \bar{B}_+$. Hence

$$\|q-\bar{q}\|^2 \geq \int_{B_2\cap\bar{B}_-} |q-\bar{q}|^2\,\mathrm{d}x + \int_{B_3\cap\bar{B}_+} |q-\bar{q}|^2\,\mathrm{d}x \geq |B_2\cap\bar{B}_-| + |B_3\cap\bar{B}_+|.$$

We observe that if $\bar{m} \geq p_2$ then $|B_2 \cap B_-| \geq |B_2| - |A_{\bar{l}}|$ while if $\bar{m} < p_2$ then $|B_3 \cap B_+| \geq |B_3| - |A_{\bar{l}}|$. Then in any case

$$\max\{|B_2 \cap B_-|, |B_3 \cap B_+|\} \geq \min\{|B_2|, |B_3|\} - \frac{1}{\varepsilon}$$

and by (2.20) we conclude that

$$\|q-\bar{q}\|^2 \geq ([p]-3)\frac{1}{\varepsilon}$$

and the lemma follows. □

Remark 2.3. Since $\frac{2\rho_0}{\varepsilon_0} \geq \frac{5m}{\underline{a}\omega_{\delta_0}}$, by Lemmas 2.10 and 2.12 we recover that setting

$$3d_0 = \left(\frac{5m}{\underline{a}\omega_{\delta_0}}\right)^{1/2},$$

then $\mathsf{d}(\Gamma_{3,p}, \Gamma_{3,\bar{p}}) \geq 3d_0$ and $\mathsf{d}(\Gamma_{3,p}, \Lambda) \geq 3d_0$ for any $p, \bar{p} \in \mathcal{P}$, $p \neq \bar{p}$. In particular, if $q \in \Gamma$ is such that and $0 < \mathsf{d}(q, \Gamma_{3,p}) < 3d_0$ for a $p \in \mathcal{P}$, then $F(q) \geq c^* > c_{3,p}$, by Lemma 2.6. Moreover, if $q \in \Gamma$ is such that $0 < \mathsf{d}(q, \mathcal{K}_{3,p}) < 3d_0$ for a $p \in \mathcal{P}$, then $F(q) > c_{3,p}$.

The following technical result will be used in the next section

Lemma 2.13. *For any $q \in \overline{\Gamma}$ there results $\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t < +\infty$. Moreover for any $q_1, q_2 \in \overline{\Gamma}$ we have*

$$\int_{\mathbb{R}} a_\varepsilon |W(q_1) - W(q_2)|\,\mathrm{d}t \leq \bar{b}\bar{a}\|q_1 - q_2\| \left(\left(\frac{1}{\underline{b}\,\underline{a}} \int_{\mathbb{R}} a_\varepsilon W(q_1)\,\mathrm{d}t \right)^{\frac{1}{2}} + \|q_1 - q_2\| \right).$$

Proof. For any $q \in \overline{\Gamma}$ we have $\int_{\mathbb{R}} |W(q) - W(z_0)|\,\mathrm{d}t = \int_{\mathbb{R}} |\int_0^1 W'(z_0 + s(q - z_0))(q - z_0)\,\mathrm{d}s|\,\mathrm{d}t$. Then, since $|z_0(t) + s(q(t) - z_0(t))| \leq 1$ for any $(s,t) \in [0,1] \times \mathbb{R}$, by (1.10) we obtain

$$\int_{\mathbb{R}} |W(q) - W(z_0)|\,\mathrm{d}t \leq \bar{b} \int_{\mathbb{R}} \int_0^1 \chi(z_0 + s(q - z_0))\,\mathrm{d}s\,|q - z_0|\,\mathrm{d}t.$$

Since $\chi(s_1 + s_2) \leq \chi(s_1) + |s_2|$ for any $s_1, s_2 \in \mathbb{R}$, we conclude

$$\int_{\mathbb{R}} |W(q) - W(z_0)|\,\mathrm{d}t \leq \bar{b}\,\|q - z_0\|(\|\chi(z_0)\| + \|q - z_0\|),$$

and so

$$\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t \leq \bar{a}(\int_{\mathbb{R}} W(z_0)\,\mathrm{d}t + \bar{b}\,\|q - z_0\|(\|\chi(z_0)\| + \|q - z_0\|) < +\infty.$$

Note now that by (1.10) we have $\|\chi(q(\cdot))\| \leq (\frac{1}{\underline{b}\,\underline{a}} \int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}t)^{\frac{1}{2}} < +\infty$ for any $q \in \overline{\Gamma}$. Therefore, given q_1, $q_2 \in \overline{\Gamma}$, to complete the proof of the Lemma it is sufficient to exactly repeat the argument above with q_1 and q_2 which play respectively the role of z_0 and q. ☐

3. Two dimensional solutions

In this section we will show that (1.3) admits infinitely many two dimensional solutions for any $\varepsilon \in (0, \varepsilon_0)$. In fact, we will prove that for every $p \in \mathcal{P}$, with $[p]$ large enough, there exists a solution $u_p \in \mathcal{C}^2(\mathbb{R}^2)$ of (1.3) such that $\partial_y u_p \not\equiv 0$ and

$$\mathsf{d}(u_p(\cdot, y), \mathcal{K}_{3,p}) \to 0 \qquad \text{as } y \to -\infty.$$

In the following, for $(y_1, y_2) \subset \mathbb{R}$ we set $S_{(y_1,y_2)} = \mathbb{R} \times (y_1, y_2)$. Let us consider the set

$$\mathcal{H} = \{u \in H^1_{loc}(\mathbb{R}^2) \,/\, \|u\|_{L^\infty(\mathbb{R}^2)} \leq 1 \text{ and } u - z_0 \in \cap_{(y_1,y_2) \subset \mathbb{R}} H^1(S_{(y_1,y_2)})\}.$$

Note that, by Fubini Theorem, we have that if $u \in \mathcal{H}$ then $u(\cdot, y) \in \Gamma$ for *a.e.* $y \in \mathbb{R}$.

Moreover we have also $u(x, \cdot) \in H^1_{loc}(\mathbb{R})$ for *a.e.* $x \in \mathbb{R}$. Therefore, if $(\zeta_1, \zeta_2) \subset \mathbb{R}$ then $u(x, \zeta_2) - u(x, \zeta_1) = \int_{\zeta_1}^{\zeta_2} \partial_y u(x, y)\,\mathrm{d}y$ holds for *a.e.* $x \in \mathbb{R}$ and so

$$\int_{\mathbb{R}} |u(x, \zeta_2) - u(x, \zeta_1)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \left| \int_{\zeta_1}^{\zeta_2} \partial_y u(x, y)\,\mathrm{d}y \right|^2 \mathrm{d}x \leq |\zeta_2 - \zeta_1| \int_{\mathbb{R}} \int_{\zeta_1}^{\zeta_2} |\partial_y u(x, y)|^2\,\mathrm{d}y\mathrm{d}x.$$

According to that, if $u \in \mathcal{H}$, then the function $y \in \mathbb{R} \to u(\cdot, y) \in \overline{\Gamma}$, defines a continuous trajectory in $\overline{\Gamma}$ verifying

$$\|u(\cdot,\zeta_2) - u(\cdot,\zeta_1)\|^2 \le \|\partial_y u\|^2_{L^2(S(\zeta_1,\zeta_2))}|\zeta_2 - \zeta_1|, \quad \forall\,(\zeta_1,\zeta_2) \subset \mathbb{R}. \tag{3.1}$$

In the sequel, we fix $\varepsilon \in (0, \varepsilon_0)$ and we denote

$$F(q) = \begin{cases} F_{a_\varepsilon}(q), & \text{if } q \in \Gamma, \\ +\infty, & \text{if } q \in \overline{\Gamma} \setminus \Gamma. \end{cases}$$

As we will see below (see Lem. 3.10), any solution $u \in \mathcal{H}$ of (1.2) which satisfies the further conditions $\mathsf{d}(u(\cdot,y), \mathcal{K}_{3,p}) \to 0$ as $y \to -\infty$, for some $p \in \mathcal{P}$, and $\int_{\mathbb{R}^2} |\partial_y u(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y < +\infty$, verifies the property

$$F(u(\cdot,y)) = c_{3,p} + \frac{1}{2}\|\partial_y u(\cdot,y)\|^2, \qquad \forall y \in \mathbb{R},$$

and so in particular that $F(u(\cdot,y)) \ge c_{3,p}$ for any $y \in \mathbb{R}$. Such consideration suggest us to define, given $p \in \mathcal{P}$, the set

$$\mathcal{M}_p = \{u \in \mathcal{H}\,/\, \lim_{y\to-\infty} \mathsf{d}(u(\cdot,y), \mathcal{K}_{3,p}) = 0, \liminf_{y\to+\infty} \mathsf{d}(u(\cdot,y), \Gamma_{3,p}) \ge d_0 \text{ and } \inf_{y\in\mathbb{R}} F(u(\cdot,y)) \ge c_{3,p}\}$$

on which we look for a minima of the functional

$$\varphi_p(u) = \int_{\mathbb{R}} \frac{1}{2}\|\partial_y u(\cdot,y)\|^2 + (F(u(\cdot,y)) - c_{3,p})\,\mathrm{d}y.$$

Remark 3.1. The problem of finding a minimum of φ_p on $\mathcal{M}_p$ is well posed. In fact, if $u \in \mathcal{M}_p$ then $F(u(\cdot,y)) \ge c_{3,p}$ for every $y \in \mathbb{R}$ and so the functional φ_p is well defined and non negative on $\mathcal{M}_p$. Moreover, as we will prove in Lemma 4.2 in the appendix, for any $p \in \mathcal{P}$ there results $\mathcal{M}_p \neq \emptyset$ and setting

$$m_p \equiv \inf_{\mathcal{M}_p} \varphi_p, \quad p \in \mathcal{P},$$

we have $\inf_{p\in\mathcal{P}} m_p > d_0\frac{\sqrt{m_0}}{8}$ and $\sup_{p\in\mathcal{P}} m_p < +\infty$.

Remark 3.2. In general the functional φ_p is not well defined on $\mathcal{H}$. Indeed, if $u \in \mathcal{H}$, the function $y \to F(u(\cdot,y)) - c_{3,p}$ is indefinite in sign and we cannot say, in general, that it is Lebesgue integrable on $\mathbb{R}$. However, if $u \in \mathcal{H}$ then $u(\cdot,y) \in \Gamma$ for *a.e.* $y \in \mathbb{R}$ and so $F(u(\cdot,y)) - c_{3,p} \ge c_{a_\varepsilon} - c_{3,p} > -\infty$ for any $y \in \mathbb{R}$. Therefore, given an interval $I \subset \mathbb{R}$ the functional

$$\varphi_{p,I}(u) = \int_I \frac{1}{2}\|\partial_y u(\cdot,y)\|^2 + (F(u(\cdot,y)) - c_{3,p})\,\mathrm{d}y$$

is well defined for any $u \in \mathcal{H}$ such that the set $\{y \in I\,/\,F(u(\cdot,y)) < c_{3,p}\}$ has bounded measure.

It is standard to show (see *e.g.* [3], Lem. 3.1, for a similar argument) that the following semicontinuity property holds: letting $I \subset \mathbb{R}$ and $u \in \mathcal{H}$, if $\varphi_{p,I}(u)$ is well defined and $(u_n) \subset \mathcal{M}_p$ is such that $u_n \to u$ weakly in $H^1_{loc}(\mathbb{R}^2)$, then $\varphi_{p,I}(u) \le \liminf_{n\to\infty} \varphi_{p,I}(u_n)$.

Finally we point out an important inequality concerning the functional $\varphi_{p,I}$ which constitutes the analogous of (2.2) in the one dimensional problem and, as there, has many useful consequences. Given $u \in \mathcal{H}$, if $(y_1, y_2) \subset \mathbb{R}$

is such that $F(u(\cdot,y)) \geq c_{3,p} + \nu$ for any $y \in (y_1, y_2)$, then

$$\varphi_{p,(y_1,y_2)}(u) \geq \frac{1}{2}\int_{y_1}^{y_2} \|\partial_y u(\cdot,y)\|^2 \,\mathrm{d}y + \nu(y_2 - y_1) \geq \frac{1}{2(y_2-y_1)}\int_{\mathbb{R}}\left(\int_{y_1}^{y_2} |\partial_y u(x,y)|\,\mathrm{d}y\right)^2 \mathrm{d}x + \nu(y_2-y_1)$$
$$\geq \frac{1}{2(y_2-y_1)}\|u(\cdot,y_1) - u(\cdot,y_2)\|^2 + \nu(y_2-y_1) \geq \sqrt{2\nu}\,\|u(\cdot,y_1)-u(\cdot,y_2)\|. \tag{3.2}$$

Concentration and compactness properties of the minimizing sequences in $\mathcal{M}_p$

As first step in studying the minimum problem of φ_p in $\mathcal{M}_p$, we characterize here below some properties of the minimizing sequences in $\mathcal{M}_p$.

The following Lemma, obtained combining (3.2) with Lemmas 2.6, 2.9 and 2.10, tells us in particular that if $u \in \mathcal{M}_p$, $\varphi_p(u) < +\infty$ and $u(\cdot,y) \notin \Lambda$ for any $y \in \mathbb{R}$, then the trajectory $y \in \mathbb{R} \to u(\cdot,y) \in \overline{\Gamma}$ is bounded.

Lemma 3.1. *There exists $\mathsf{C} > 0$ such that given any $p \in \mathcal{P}$, if $u \in \mathcal{M}_p$ satisfies $\mathsf{d}(u(\cdot,y),\Lambda) > 0$ for any $y \in (y_1,y_2)$ then*

$$\|u(\cdot,y_1) - u(\cdot,y_2)\| \leq \mathsf{C}\,\varphi_p(u).$$

Proof. Let $\bar{y}_1 = \inf\{y \in [y_1,y_2] \mid F(u(\cdot,y)) \leq c^*\}$ and $\bar{y}_2 = \sup\{y \in [y_1,y_2] \mid F(u(\cdot,y)) \leq c^*\}$. We have

$$\|u(\cdot,y_1)-u(\cdot,y_2)\| \leq \|u(\cdot,y_1)-u(\cdot,\bar{y}_1)\| + \|u(\cdot,\bar{y}_1)-u(\cdot,\bar{y}_2)\| + \|u(\cdot,\bar{y}_2)-u(\cdot,y_2)\|$$

and since $F(u(\cdot,y)) > c^*$ for any $y \in (y_1,\bar{y}_1)\cup(\bar{y}_2,y_2)$ and, by Lemma 2.6, $c^* \geq c_{3,p} + \frac{m_0}{8}$, using (3.2) we obtain

$$\|u(\cdot,y_1)-u(\cdot,y_2)\| \leq \|u(\cdot,\bar{y}_1)-u(\cdot,\bar{y}_2)\| + \frac{4}{\sqrt{m_0}}\varphi_p(u).$$

To estimate $\|u(\cdot,\bar{y}_1)-u(\cdot,\bar{y}_2)\|$, note that we can write $\{y \in (\bar{y}_1,\bar{y}_2) \mid F(u(\cdot,y)) > c^*\} = \cup_{i\in\mathcal{I}}(y_{1,i},y_{2,i})$, disjoint union. Then, since $u(\cdot,y) \notin \Lambda$ for all $y \in (y_1,y_2)$, we obtain that for every $i \in \mathcal{I}$, there exist $p_{1,i},\, p_{2,i} \in \mathcal{P}$ such that $u(\cdot,y_{1,i}) \in \Gamma_{3,p_{1,i}}$ and $u(\cdot,y_{2,i}) \in \Gamma_{3,p_{2,i}}$. Let $\mathcal{I}_1 = \{i \in \mathcal{I} \,/\, p_{1,i} \neq p_{2,i}\}$ and $\#\mathcal{I}_1$ its cardinality. By Lemmas 2.10, 2.6 and by (3.2) we have that for any $i \in \mathcal{I}_1$

$$3d_0 \leq \|u(\cdot,y_{1,i})-u(\cdot,y_{2,i})\| \leq \frac{2}{\sqrt{m_0}}\varphi_{p,(y_{1,i},y_{2,i})}(u)$$

and so, summing on $i \in \mathcal{I}_1$, we obtain

$$\#\mathcal{I}_1 \leq \frac{2}{3d_0\sqrt{m_0}}\varphi_p(u).$$

Since as one easily recognizes

$$\|u(\cdot,\bar{y}_1)-u(\cdot,\bar{y}_2)\| \leq (\#\mathcal{I}_1+1)\sup_{p\in\mathcal{P}}\mathsf{diam}(\Gamma_{3,p}) + \sum_{i\in\mathcal{I}_1}\|u(\cdot,y_{1,i})-u(\cdot,y_{2,i})\|,$$

by Lemma 2.9 and (3.2) we conclude

$$\|u(\cdot,\bar{y}_1)-u(\cdot,\bar{y}_2)\| \leq \left(\frac{2}{3d_0\sqrt{m_0}}\varphi_p(u)+1\right)\mathsf{D} + \frac{2}{\sqrt{m_0}}\varphi_p(u)$$

then, by Remark 3.1, the Lemma follows with $\mathsf{C} = \frac{9}{\sqrt{m_0}}(1+\frac{\mathsf{D}}{d_0})$. □

Since by Lemma 2.12, $\mathsf{d}(\Gamma_{3,p},\Lambda)\to\infty$ as $[p]\to\infty$, setting $\mathsf{C}_0=\mathsf{C}(\sup_{p\in\mathcal{P}}m_p+1)$ we have that

$$\exists\,\mathsf{p}_0\in\mathbb{N}\text{ such that if }[p]\ge\mathsf{p}_0\text{ then }\mathsf{d}(\Gamma_{3,p},\Lambda)>\mathsf{C}_0. \tag{3.3}$$

Then, using Lemma 3.1 we obtain

Lemma 3.2. *Let* $[p]\ge\mathsf{p}_0$*, if* $u\in\mathcal{M}_p$ *is such that* $\varphi_p(u)\le m_p+1$*, then*

$$\mathsf{d}(u(\cdot,y),\Gamma_{3,p})\le\mathsf{C}_0,\quad\forall y\in\mathbb{R}.$$

Proof. The lemma follows by Lemma 3.1 once we prove that if $[p]\ge\mathsf{p}_0$, and $u\in\mathcal{M}_p$ is such that $\varphi_p(u)\le m_p+1$ then $\mathsf{d}(u(\cdot,y),\Lambda)>0$ for any $y\in\mathbb{R}$.

Assume by contradiction that there exist $p\in\mathcal{P}$, $u\in\mathcal{M}_p$ and $y_0\in\mathbb{R}$ such that $[p]\ge\mathsf{p}_0$, $\varphi_p(u)\le m_p+1$, $\mathsf{d}(u(\cdot,y_0),\Lambda)=0$ and $\mathsf{d}(u(\cdot,y),\Lambda)>0$ for any $y<y_0$. By (3.1) and (3.3) there exists $y_1<y_0$ such that

$$\|u(\cdot,y_1)-u(\cdot,y_0)\|<\mathsf{d}(\Gamma_{3,p},\Lambda)-\mathsf{C}_0.$$

By Lemma 3.1 we have moreover that $\|u(\cdot,y)-u(\cdot,y_1)\|\le\mathsf{C}_0$ for any $y\le y_1$. Therefore, since $u\in\mathcal{M}_p$, we obtain also

$$\mathsf{d}(\Gamma_{3,p},u(\cdot,y_1))\le\limsup_{y\to-\infty}\|u(\cdot,y)-u(\cdot,y_1)\|\le\mathsf{C}_0$$

and so

$$\mathsf{d}(\Gamma_{3,p},\Lambda)\le\mathsf{d}(\Gamma_{3,p},u(\cdot,y_1))+\|u(\cdot,y_1)-u(\cdot,y_0)\|<\mathsf{d}(\Gamma_{3,p},\Lambda),$$

a contradiction. □

Remark 3.3. Given $p\in\mathcal{P}$ we define

$$\Omega(p)=\{\bar p\in\mathcal{P}\,/\,\mathsf{d}(\Gamma_{3,p},\Gamma_{3,\bar p})\le\mathsf{C}_0\}.$$

Note that, by Lemma 2.11, the set $\Omega(p)$ is finite. Moreover, by Lemma 3.2, if $[p]\ge\mathsf{p}_0$, $u\in\mathcal{M}_p$ and $y\in\mathbb{R}$ are such that $\varphi_p(u)\le m_p+1$ and $F(u(\cdot,y))\le c^*$ then $u(\cdot,y)\in\cup_{p\in\Omega(p)}\Gamma_{3,p}$.

Lemma 3.3. *If* $[p]\ge p_0$ *and* $u\in\mathcal{M}_p$ *is such that* $\varphi_p(u)\le m_p+1$ *then there exists* $\bar p\in\Omega(p)\setminus\{p\}$ *such that*

$$\lim_{y\to+\infty}\mathsf{d}(u(\cdot,y),\Gamma_{3,\bar p})=0.$$

Proof. Since $u\in\mathcal{M}_p$ and $\varphi_p(u)\le m_p+1$ we have $\liminf_{y\to+\infty}F(u(\cdot,y))=c_{3,p}$. Then, since $[p]\ge\mathsf{p}_0$, by Remark 3.3 and Lemma 2.6, one plainly obtains that there exists $\bar p\in\Omega(p)$ such that $\liminf_{y\to+\infty}\mathsf{d}(u(\cdot,y),\Gamma_{3,\bar p})=0$. Moreover, since $u\in\mathcal{M}_p$ we have $\lim_{y\to+\infty}\mathsf{d}(u(\cdot,y),\Gamma_{3,p})\ge d_0$ and so $\bar p\ne p$.

Assume by contradiction that $\limsup_{y\to+\infty}\mathsf{d}(u(\cdot,y),\Gamma_{3,\bar p})=3d>0$ and set $\bar d=\min\{d,d_0\}$. Then by (3.1) we have that there exist two sequences $(y_{1,i})$, $(y_{2,i})\subset\mathbb{R}$ such that $y_{1,i}<y_{2,i}$ for any $i\in\mathbb{N}$, $y_{1,i}\to+\infty$ as $i\to+\infty$, $\mathsf{d}(u(\cdot,y),\Gamma_{3,\bar p})\in(\bar d,2\bar d)$ for any $y\in(y_{1,i},y_{2,i})$, $i\in\mathbb{N}$, and finally $\|u(\cdot,y_{1,i})-u(\cdot,y_{2,i})\|=\bar d$. Then, by Remark 2.3 we have $F(u(\cdot,y))\ge c^*$ for any $y\in(y_{1,i},y_{2,i})$, $i\in\mathbb{N}$ and by (3.2) we obtain that $\varphi_p(u)\ge\sum_{i=1}^{\infty}\frac{\sqrt{m_0}}{2}\bar d=+\infty$, a contradiction. □

Remark 3.4. By Lemma 3.3 we obtain that for any $p\in\mathcal{P}$ such that $[p]\ge\mathsf{p}_0$, there exists $p^*\in\Omega(p)\setminus\{p\}$ such that setting

$$\mathcal{M}_{p,p^*}=\{u\in\mathcal{M}_p\,/\,\lim_{y\to+\infty}\mathsf{d}(u(\cdot,y),\Gamma_{3,p^*})=0\}$$

we have

$$m_p=\inf_{u\in\mathcal{M}_{p,p^*}}\varphi_p(u).$$

Indeed if $(u_n) \subset \mathcal{M}_p$ is such that $\varphi_p(u_n) \to m_p$ then by Lemma 3.3 there exist $\bar{n} \in \mathbb{N}$ and $\bar{p}_n \in \Omega(p) \setminus \{p\}$ such that $\lim_{y\to+\infty} \mathsf{d}(u(\cdot, y), \Gamma_{3,\bar{p}_n}) = 0$ for any $n \geq \bar{n}$. Then the property follows since, by Remark 3.3, $\Omega(p)$ is finite and so there exists $p^* \in \Omega(p) \setminus \{p\}$ such that, along a subsequence, $p_n = p^*$.

In the sequel we will denote

$$\lambda_0 = \min\left\{1, \frac{\sqrt{m_0}}{4} d_0\right\}.$$

In the proofs of the following lemmas we make use of a technical result whose statement and proof is postponed in the appendix (see Lem. 4.3).

Lemma 3.4. *For every $p \in \mathcal{P}$ with $[p] \geq \mathsf{p}_0$ there exists $\hat{\nu} \in (0, \frac{m_0}{8})$ such that if $u \in \mathcal{M}_{p,p^*}$, $\varphi_p(u) \leq m_p + \lambda_0$ and $u(\cdot, y) \in \cup_{\bar{p}\in\mathcal{P}\setminus\{p,p^*\}} \Gamma_{3,\bar{p}}$, then $F(u(\cdot, y)) \geq c_{3,p} + \hat{\nu}$.*

Proof. Let $p \in \mathcal{P}$ such that $[p] \geq \mathsf{p}_0$ and assume by contradiction that there exists a sequence $(u_n) \subset \mathcal{M}_{p,p^*}$, a sequence $(\bar{p}_n) \subset \mathcal{P} \setminus \{p, p^*\}$ and a sequence $(y_n) \subset \mathbb{R}$ such that for any $n \in \mathbb{N}$ there results

$$\varphi_p(u_n) \leq m_p + \lambda_0,\ u(\cdot, y_n) \in \Gamma_{3,\bar{p}_n} \text{ and } \lim_{n\to\infty} F(u_n(\cdot, y_n)) = c_{3,p}.$$

By Lemmas 3.2 and 3.3 we have that $(\bar{p}_n) \subset \Omega(p) \setminus \{p, p^*\}$ which is a finite set. Therefore, extracting a subsequence if necessary, we can assume that there exists $\bar{p} \in \Omega(p) \setminus \{p, p^*\}$ such that $\bar{p}_n = \bar{p}$ for any $n \in \mathbb{N}$.

By translating the function u_n, we can furthermore assume that $y_n = 0$ for any $n \in \mathbb{N}$ and, by Lemma 4.3, we obtain $\liminf_{n\to\infty} \varphi_{p,(-\infty,0)}(u_n) \geq m_p$.

Since $u_n(\cdot, 0) \in \Gamma_{3,\bar{p}}$ and $\lim_{y\to\infty} \mathsf{d}(u_n(\cdot, y), \Gamma_{3,p^*}) = 0$, by Lemma 2.10 and (3.1) we derive that there exists $(\zeta_1, \zeta_2) \subset (0, +\infty)$ such that $\|u(\cdot, \zeta_1) - u(\cdot, \zeta_2)\| \geq d_0$ and $F(u(\cdot, y)) \geq c^*$ for any $y \in (\zeta_1, \zeta_2)$. Then, by (3.2), $\varphi_{p,(0,+\infty)}(u_n) \geq \varphi_{p,(\zeta_1,\zeta_2)}(u_n) \geq \frac{\sqrt{m_0}}{2} d_0$ for any $n \in \mathbb{N}$ and we conclude that

$$\frac{\sqrt{m_0}}{2} d_0 \leq \liminf_{n\to\infty}(\varphi_p(u_n) - \varphi_{p,(-\infty,0)}(u_n)) \leq m_p + \lambda_0 - \limsup_{n\to\infty} \varphi_{p,(-\infty,0)}(u_n) \leq \frac{\sqrt{m_0}}{4} d_0,$$

a contradiction. □

Lemma 3.4 shows that if $u \in \mathcal{M}_{p,p^*}$ and $\varphi_p(u) \leq m_p + \lambda_0$ then $u(\cdot, y)$ is forced to be in $\Gamma_{3,p} \cup \Gamma_{3,p^*}$ whenever $F(u(\cdot, y)) < c_{3,p} + \hat{\nu}$. Next Lemma strengthens that result describing how the set $\Gamma_{3,p} \cup \Gamma_{3,p^*}$ "*absorbs*" the trajectories $u(\cdot, y) \in \mathcal{M}_{p,p^*} \cap \{\varphi_p < m_p + \lambda_0\}$.

Lemma 3.5. *For any $p \in \mathcal{P}$ with $[p] \geq \mathsf{p}_0$ there exists $\bar{\nu} \in (0, \hat{\nu}]$ such that if $u \in \mathcal{M}_{p,p^*}$, $\varphi_p(u) \leq m_p + \lambda_0$ and $F(u(\cdot, \bar{y})) < c_{3,p} + \bar{\nu}$ for some $\bar{y} \in \mathbb{R}$, then, either*

(i) $u(\cdot, \bar{y}) \in \Gamma_{3,p}$ and $\mathsf{d}(u(\cdot, y), \Gamma_{3,p}) \leq d_0$ for all $y \leq \bar{y}$; or
(ii) $u(\cdot, \bar{y}) \in \Gamma_{3,p^*}$ and $\mathsf{d}(u(\cdot, y), \Gamma_{3,p^*}) \leq d_0$ for all $y \geq \bar{y}$.

Proof. Let us prove (ii), being the proof of (i) analogous.

Assume by contradiction that there exists a sequence $(u_n) \subset \mathcal{M}_{p,p^*}$ such that $\varphi_p(u_n) \leq m_p + \lambda_0$, and two sequences $(y_{n,1}), (y_{n,2}) \subset \mathbb{R}$ such that for any $n \in \mathbb{N}$ there results

$$y_{n,1} \leq y_{n,2},\ \lim_{n\to\infty} F(u_n(\cdot, y_{n,1})) = c_{3,p},\ u_n(\cdot, y_{n,1}) \in \Gamma_{3,p^*} \text{ and } \mathsf{d}(u_n(\cdot, y_{n,2}), \Gamma_{3,p^*}) > d_0.$$

By Lemma 4.3 we obtain $\liminf_{n\to\infty} \varphi_{p,(-\infty,y_{n,1})}(u_n) \geq m_p$.

Moreover, since $u_n(\cdot, y_{n,1}) \in \Gamma_{3,p^*}$ and $\mathsf{d}(u_n(\cdot, y_{n,2}), \Gamma_{3,p^*}) > d_0$, by (3.1) we obtain that there exists $\bar{y}_{n,1}, \bar{y}_{n,2} \in [y_{n,1}, y_{n,2}]$ such that $F(u(\cdot, y)) \geq c^*$ for any $y \in (\bar{y}_n, \bar{y}_{n,2})$ and $\mathsf{d}(u(\cdot, \bar{y}_{n,1}), u(\cdot, \bar{y}_{n,2})) = d_0$. By (3.2) this implies $\varphi_{p,(y_{n,1},+\infty)}(u_n) \geq \varphi_{p,(\bar{y}_{n,1},\bar{y}_{n,2})}(u_n) \geq \frac{\sqrt{m_0}}{2} d_0$ for any $n \in \mathbb{N}$. As in the proof of Lemma 3.4, that gives rise to the contradiction

$$\frac{\sqrt{m_0}}{2} d_0 \leq \liminf_{n\to\infty} (\varphi_p(u_n) - \varphi_{p,(-\infty,y_{n,1})}(u_n)) \leq m_p + \lambda_0 - \limsup_{n\to\infty} \varphi_{p,(-\infty,,y_{n,1})}(u_n) \leq \frac{\sqrt{m_0}}{4} d_0,$$

and the lemma follows. □

Note that Lemma 3.5 holds true also for minimizing sequences of φ_p on $\mathcal{M}_{p,p^*}$. This fact will be used in the next lemma to derive analogous asymptotic properties of the their limits points.

Lemma 3.6. *Let $p \in \mathcal{P}$ with $[p] \geq \mathsf{p}_0$ and let $(u_n) \subset \mathcal{M}_{p,p^*}$ be such that $\varphi_p(u_n) \to m_p$ as $n \to \infty$ and $\mathsf{d}(u_n(\cdot,0), \Gamma_{3,p}) = \frac{3}{2} d_0$ for all $n \in \mathbb{N}$. Then, there exists $u_p \in \mathcal{H}$ and a subsequence of (u_n), still denoted (u_n), such that*

(*i*) $u_n \to u_p$ *as* $n \to \infty$ *weakly in* $H^1_{loc}(\mathbb{R}^2)$*;*
(*ii*) $\mathsf{d}(u_p(\cdot,y), \Gamma_{3,p}) \leq \mathsf{C}_0$ *for any* $y \in \mathbb{R}$*;*
(*iii*) $\lim_{y\to-\infty} \mathsf{d}(u_p(\cdot,y), \mathcal{K}_{3,p}) = 0$ *and* $\limsup_{y\to+\infty} \mathsf{d}(u_p(\cdot,y), \Gamma_{3,p^*}) \leq d_0$.

Proof. Pick any function $q \in \Gamma_{3,p}$ and consider a sequence of bounded intervals $(y_{1,j}, y_{2,j}) \subset \mathbb{R}$ such that $y_{1,j} \to -\infty$ and $y_{2,j} \to +\infty$. Since $\varphi_p(u_n) \to m_p$ as $n \to \infty$ we can assume that $\varphi_p(u_n) \leq m_p + \lambda_0$ for any $n \in \mathbb{N}$ and so, by Lemmas 2.9 and 3.2 we have that for any $y \in \mathbb{R}$

$$\|u_n(\cdot,y) - q\| \leq \mathsf{d}(u_n(\cdot,y), \Gamma_{3,p}) + \mathsf{D} \leq \mathsf{C}_0 + \mathsf{D}.$$

Then, $\|u_n - q\|^2_{L^2(S_{(y_{1,j},y_{2,j})})} \leq (y_{2,j} - y_{1,j})(\bar{C} + \mathsf{D})^2$ for any $n \in \mathbb{N}$ and $j \in \mathbb{N}$. Since moreover $\|\nabla u_n\|^2_{L^2(S_{(y_{1,j},y_{2,j})})} \leq 2(\varphi_p(u_n) + (y_{2,j} - y_{1,j}) c_{3,p})$ we conclude that the sequence $(u_n - q)$, and so the sequence $(u_n - z_0)$, is bounded in $H^1(S_{(y_{1,j},y_{2,j})})$ for any $j \in \mathbb{N}$. Then, with a diagonal argument, we derive that there exists $u_p \in H^1_{loc}(\mathbb{R}^2)$ such that along a subsequence $u_n - z_0 \to u_p - z_0$ weakly in $H^1(S_{(y_{1,j},y_{2,j})})$ for any $j \in \mathbb{N}$ (and *a.e.* in $\mathbb{R}^2$). Then $u_p \in \mathcal{H}$ and $u_n - u_p \to 0$ weakly in $H^1(S_{(y_1,y_2)})$ for any $(y_1, y_2) \subset \mathbb{R}$ and (*i*) follows.

Since by Lemma 3.2 we have $\mathsf{d}(u_n(\cdot,y), \Gamma_{3,p}) \leq \mathsf{C}_0$ for any $n \in \mathbb{N}$, there exists $q_n \in \Gamma_{3,p}$ such that $\limsup_{n\to\infty} \|u_n(\cdot,y) - q_n(\cdot)\| \leq \mathsf{C}_0$. By Lemma 2.7 we have that along a subsequence, still denoted (q_n), $q_n \to q \in \Gamma_{3,p}$ as $n \to \infty$ in $L^2_{loc}(\mathbb{R})$ and so, by the Fatou Lemma, we obtain that for *a.e.* $y \in \mathbb{R}$ there results

$$\mathsf{d}(u_p(\cdot,y), \Gamma_{3,p}) \leq \|u_p(\cdot,y) - q\| \leq \liminf_{n\to\infty} \|u_n(\cdot,y) - q_n(\cdot)\| \leq \mathsf{C}_0.$$

Then $\mathsf{d}(u_p(\cdot,y), \Gamma_{3,p}) \leq \mathsf{C}_0$ for *a.e.* $y \in \mathbb{R}$ and since $u \in \mathcal{H}$, by (3.1) we obtain in fact that $\mathsf{d}(u_p(\cdot,y), \Gamma_{3,p}) \leq \mathsf{C}_0$ for any $y \in \mathbb{R}$ and (*ii*) follows.

Let us finally prove (*iii*). By (3.2) there exists $L > 0$ such that, for any $n \in \mathbb{N}$ there exist $y_{n,1} \in (-L, 0)$ and $y_{n,2} \in (0, L)$ for which $F(u_n(\cdot,y_{n,1}))$, $F(u_n(\cdot,y_{n,2})) \leq c_{3,p} + \bar{\nu}$ and so, by Lemma 3.4 and Remark 3.3, $u_n(\cdot,y_{n,1})$, $u_n(\cdot,y_{n,2}) \in \Gamma_{3,p} \cup \Gamma_{3,p^*}$. By Lemma 3.5 it is simple to show that in fact

$$u_n(\cdot,y_{n,1}) \in \Gamma_{3,p} \text{ and } u_n(\cdot,y_{n,2}) \in \Gamma_{3,p^*}. \tag{3.4}$$

Indeed if $u_n(\cdot,y_{n,1}) \in \Gamma_{3,p^*}$ then Lemma 3.5 implies that $\mathsf{d}(u_n(\cdot,y), \Gamma_{3,p^*}) \leq d_0$ for any $y \geq y_{n,1}$ and so in particular $\mathsf{d}(u_n(\cdot,0), \Gamma_{3,p^*}) \leq d_0$ in contradiction with the assumption $\mathsf{d}(u_n(\cdot,0), \Gamma_{3,p}) = \frac{3}{2} d_0$ since as we know $\mathsf{d}(\Gamma_{3,p}, \Gamma_{3,p^*}) \geq 3d_0$. Analogously one obtains a contradiction assuming $u_n(\cdot,y_{n,2}) \in \Gamma_{3,p}$.

By (3.4) and Lemma 3.5 we conclude that for any $n \in \mathbb{N}$ there results

$$\mathsf{d}(u_n(\cdot,y), \Gamma_{3,p}) \leq d_0 \text{ for any } y \leq -L \text{ and } \mathsf{d}(u_n(\cdot,y), \Gamma_{3,p^*}) \leq d_0 \text{ for any } y \geq L$$

and so, as in the proof of (*ii*), in the limit we obtain

$$\mathsf{d}(u_p(\cdot,y),\Gamma_{3,p}) \le d_0 \text{ for any } y \le -L \text{ and } \mathsf{d}(u_p(\cdot,y),\Gamma_{3,p^*}) \le d_0 \text{ for any } y \ge L.$$

This proves in particular that, as stated in (*iii*), $\limsup_{y\to+\infty} \mathsf{d}(u_p(\cdot,y),\Gamma_{3,p^*}) \le d_0$.

To complete the proof let us show now that $\lim_{y\to-\infty} \mathsf{d}(u_p(\cdot,y),\mathcal{K}_{3,p}) = 0$.

We first observe that, by Remark 2.3, $F(u_p(\cdot,y)) \ge c_{3,p}$ for any $y \le -L$, and, by Remark 3.2, we deduce that $\varphi_{p,(-\infty,-L)}(u_p)$ is well defined and $\varphi_{p,(-\infty,-L)}(u_p) \le \liminf_{n\to\infty} \varphi_p(u_n) \le m_p$.

By (2.18) we have that for any $r > 0$, if $y \le -L$ and $\mathsf{d}(u_p(\cdot,y),\mathcal{K}_{3,p}) \ge r$ then $F(u_p(\cdot,y)) \ge \min\{c_{3,p}+\nu_r, c^*\}$. Then, since $\varphi_{p,(-\infty,-L)}(u_p) \le m_p$, we deduce that $\liminf_{y\to-\infty} \mathsf{d}(u_p(\cdot,y),\mathcal{K}_{3,p}) = 0$. Finally, if we assume by contradiction that

$$\limsup_{y\to-\infty} \mathsf{d}(u_p(\cdot,y),\mathcal{K}_{3,p}) = r > 0$$

we obtain the existence of a sequence of intervals $(y_{1,j}, y_{2,j})$ with $y_{2,j+1} < y_{1,j} < y_{2,j} < -L$ for any $j \in \mathbb{N}$, $y_{2,j} \to -\infty$ as $j \to \infty$, $\|u_p(\cdot,y_{1,j}) - u_p(\cdot,y_{2,j})\| = \frac{r}{2}$ and $F(u_p(\cdot,y)) \ge \min\{c_{3,p}+\nu_{r/4}, c^*\}$ for any $y \in (y_{1,j}, y_{2,j})$. Then, by (3.2) we obtain

$$\varphi_{p,(-\infty,-L)}(u_p) \ge \sum_{j=1}^{\infty} \varphi_{p,(y_{1,j},y_{2,j})}(u_p) \ge \sqrt{2\min\{\nu_{r/4}, c^* - c_{3,p}\}} \sum_{j=1}^{\infty} \frac{r}{2} = +\infty,$$

a contradiction. □

The conservation of “Energy” and the existence of two dimensional solutions

Note that the function u_p given by Lemma 3.6 does not necessarily satisfies the condition $F(u_p(\cdot,y)) \ge c_{3,p}$ for any $y \in \mathbb{R}$. Hence, we cannot say that u_p belongs to $\mathcal{M}_p$ and so that it is a minimum for φ_p on $\mathcal{M}_p$. Anyway, as we will show below, as limit of a minimizing sequence, u_p inherits suitable minimality properties which allow us to construct from it a two dimensional solution of (1.3).

First of all, we introduce the set of limit points of the minimizing sequences of φ_p in $\mathcal{M}_p$. More precisely, for $p \in \mathcal{P}$ such that $[p] \ge \mathsf{p}_0$, we set

$$\mathcal{L}_p = \{u \in \mathcal{H} \,/\, \exists (u_n) \in \mathcal{M}_{p,p^*} \text{ such that } \mathsf{d}(u_n(\cdot,0),\Gamma_{3,p}) = \frac{3}{2}d_0 \text{ for any } n \in \mathbb{N},$$
$$\varphi_p(u_n) \to m_p \text{ and } u_n \to u \text{ weakly in } H^1_{loc}(\mathbb{R}^2) \text{ as } n \to \infty\}.$$

Remark 3.5. Note that, using the invariance with respect to the y-translation of φ_p, there always exists a sequence $(u_n) \subset \mathcal{M}_{p,p^*}$ such that $\varphi_p(u_n) \to m_p$ and $\mathsf{d}(u_n(\cdot,0),\Gamma_{3,p}) = \frac{3}{2}d_0$ for any $n \in \mathbb{N}$. Then, by Lemma 3.6, $\mathcal{L}_p$ is not empty and constituted by functions u verifying the properties

$$\sup_{y\in\mathbb{R}} \mathsf{d}(u(\cdot,y),\Gamma_{3,p}) \le \mathsf{C}_0, \quad \lim_{y\to-\infty} \mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p}) = 0 \text{ and } \limsup_{y\to+\infty} \mathsf{d}(u(\cdot,y),\Gamma_{3,p^*}) \le d_0.$$

For any $u \in \mathcal{L}_p$ we set

$$D_u = \{y \in \mathbb{R} \,/\, \mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p}) \ge d_0\}.$$

Note that, by Remark 3.5 and (3.1), D_u is not empty and, by Remark 2.3, we recover that if $y \notin D_u$ then $F(u(\cdot,y)) \ge c_{3,p}$. We define

$$y_{0,u} = \begin{cases} +\infty & \text{if } F(u(\cdot,y)) > c_{3,p} \text{ for any } y \in D_u, \\ \inf\{y \in D_u \,/\, F(u(\cdot,y)) \le c_{3,p}\} & \text{otherwise.} \end{cases}$$

Note that, since $\lim_{y\to-\infty} \mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p}) = 0$ we have in fact $y_{0,u} > -\infty$.

Remark 3.6. We remark that, by definition, for any $u \in \mathcal{L}_p$

$$\text{if } y < y_{0,u} \text{ then, } \mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p}) < d_0 \text{ or } F(u(\cdot,y)) > c_{3,p}. \tag{3.5}$$

Then, by (3.5) and Remark 2.3, we always have that

$$F(u(\cdot,y)) \geq c_{3,p} \quad \text{for any} \quad y < y_{0,u}$$

and so, by Remark 3.2, we obtain that $\varphi_{p,(-\infty,y_{0,u})}(u)$ is well defined for any $u \in \mathcal{L}_p$ and $\varphi_{p,(-\infty,y_{0,u})}(u) \leq m_p$.

Remark 3.7. Note that if $y_{0,u} = +\infty$, then $F(u(\cdot,y)) \geq c_{3,p}$ for any $y \in \mathbb{R}$ and so, by Remark 3.2, $\varphi_p(u) \leq m_p$. By Remark 3.5 we have in fact that in this case u is a minimum for φ_p on $\mathcal{M}_p$, *i.e.*,

$$\text{if } y_{0,u} = +\infty \text{ then } u \in \mathcal{M}_p \text{ and } \varphi_p(u) = m_p.$$

Let us consider the case $y_{0,u} \in \mathbb{R}$. We point out that, by definition, there exists a sequence $(y_n) \subset [y_{0,u},+\infty)$ such that $y_n \to y_{0,u}$ as $n \to \infty$, $F(u(\cdot,y_n)) \leq c_{3,p}$ and $\mathsf{d}(u(\cdot,y_n),\mathcal{K}_{3,p}) \geq d_0$ for any $n \in \mathbb{N}$. Since by Remark 3.5 we have $\mathsf{d}(u(\cdot,y),\Gamma_{3,p}) \leq \mathsf{C}_0$ for any $y \in \mathbb{R}$, by Remark 3.3, there exists $\bar{p} \in \Omega(p) \setminus \{p\}$ for which, along a subsequence, $u(\cdot,y_n) \in \Gamma_{3,\bar{p}}$. Therefore, since the function $y \in \mathbb{R} \to F(u(\cdot,y)) \in [0,+\infty]$ is lower semicontinuous (see Lemmas 4.1 in the appendix), by Lemma 2.7 and (3.1) we conclude that

$$\text{if } y_{0,u} \in \mathbb{R} \text{ then } \exists \bar{p} \in \Omega(p) \setminus \{p\} \text{ such that } u(\cdot,y_{0,u}) \in \Gamma_{3,\bar{p}} \text{ and } F(u(\cdot,y_{0,u})) \leq c_{3,p}. \tag{3.6}$$

As stated in the following lemma, in the case $y_{0,u} \in \mathbb{R}$, we can say more than (3.6). In the proof we make use of a technical result whose statement and proof is postponed in the appendix (see Lem. 4.4).

Lemma 3.7. *Let $u \in \mathcal{L}_p$ with $y_{0,u} \in \mathbb{R}$. Then we have*

$$\liminf_{y \to y_{0,u}^-} F(u(\cdot,y)) = c_{3,p} \quad \textit{and} \quad \varphi_{p,(-\infty,y_{0,u})}(u) = m_p.$$

Proof. Let us assume, by translating u if necessary, that $y_{0,u} = 0$.

To show that $\liminf_{y \to 0^-} F(u(\cdot,y)) = c_{3,p}$ assume by contradiction that there exists $y_0 < 0$ and $\mu > 0$ such that $F(u(\cdot,y)) \geq c_{3,p} + \mu$ for any $y \in (y_0,0)$. We set $v(x,y) = u(x,y) - u(x,0)$ and note that by (3.1) we have

$$\|v(\cdot,y)\|^2 \leq -y \int_y^0 \|\partial_y u(\cdot,s)\|^2 \,\mathrm{d}s, \qquad \forall y \in (y_0,0). \tag{3.7}$$

Then, in particular, $\|v(\cdot,y)\| \to 0$ as $y \to 0^-$ and taking y_0 bigger if necessary, we can assume that $\|v(\cdot,y)\| \leq d_0$ for any $y \in (y_0,0)$.

For $f,\, g \in L^2(\mathbb{R})$ in the sequel we will denote $(f,g) = \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x$.

Since $\|v(\cdot,y)\| \leq d_0$ for any $y \in (y_0,0)$, by Lemma 2.13 we obtain that there exists $C > 0$ depending on d_0, $u(\cdot,0)$ and W such that $\forall t \in [0,1]$ and $\forall y \in (y_0,0)$ there results

$$\left| \int_{\mathbb{R}} a_\varepsilon (W(u(\cdot,0) + v(\cdot,y)) - W(u(\cdot,0) + tv(\cdot,y)))\,\mathrm{d}x \right| \leq (1-t)C\|v(\cdot,y)\|. \tag{3.8}$$

By (3.7) and (3.8), since by assumption $\mu \leq F(u(\cdot,0) + v(\cdot,y)) - F(u(\cdot,0))$, we obtain

$$\mu \leq \frac{1}{2}\|\partial_x v(\cdot,y)\|^2 - (\partial_x u(\cdot,0),\, \partial_x v(\cdot,y)) - \int_{\mathbb{R}} a_\varepsilon (W(u(\cdot,0) + v(\cdot,y)) - W(u(\cdot,0)))\,\mathrm{d}x \tag{3.9}$$

for any $y \in (y_0, 0)$. Then, we obtain

$$\liminf_{y\to 0^-} \|\partial_x v(\cdot, y)\|^2 \geq 2\mu. \tag{3.10}$$

Indeed, given any sequence $y_n \to 0^-$ as $n \to +\infty$, if $(\partial_x v(\cdot, y_n)))$ is unbounded in $L^2(\mathbb{R})$ we have nothing to prove. If otherwise, the sequence $(\partial_x v(\cdot, y_n)))$ is bounded in $L^2(\mathbb{R})$ by (3.7) we deduce that $\partial_x v(\cdot, y_n) \to 0$ weakly in $L^2(\mathbb{R})$ and then $(\partial_x u(\cdot, 0),\, \partial_x v(\cdot, y_n)) \to 0$. Hence, by (3.7) and (3.9), we obtain $\lim_{n\to+\infty} \|\partial_x v(\cdot, y_n)\|^2 \geq 2\mu$.

Now, by (3.10) we can assume, taking y_0 bigger if necessary, that $\|\partial_x v(\cdot, y)\|^2 \geq \mu$ for any $y \in (y_0, 0)$. Then, let $(y_j) \subset (y_0, 0)$ be such that $y_j \to 0$ as $j \to +\infty$. Then, by (3.7) we obtain $\frac{v(\cdot,y_j)}{\|\partial_x v(\cdot,y_j)\|} \to 0$ in $L^2(\mathbb{R})$ as $j \to +\infty$ and since the sequence $(\frac{\partial_x v(\cdot,y_j)}{\|\partial_x v(\cdot,y_j)\|})$ is bounded in $L^2(\mathbb{R})$, we deduce that $\frac{\partial_x v(\cdot,y_j)}{\|\partial_x v(\cdot,y_j)\|} \to 0$ weakly in $L^2(\mathbb{R})$ and so $\frac{|(\partial_x u(\cdot,0),\partial_x v(\cdot,y_j))|}{\|\partial_x v(\cdot,y)\|^2} \leq \frac{1}{\sqrt{\mu}} |(\partial_x u(\cdot, 0), \frac{\partial_x v(\cdot,y_j)}{\|\partial_x v(\cdot,y)\|})| \to 0$ as $j \to \infty$. That shows that

$$\lim_{y\to 0^-} \frac{(\partial_x u(\cdot, 0), \partial_x v(\cdot, y))}{\|\partial_x v(\cdot, y)\|^2} = 0. \tag{3.11}$$

Note now that, thanks to (3.8), for any $y \in (y_0, 0)$ and any $t \in [0, 1]$ we have

$$\begin{aligned} F(u(\cdot, 0) + v(\cdot, y)) - F(u(\cdot, 0) + tv(\cdot, y)) &= \frac{\|\partial_x v(\cdot, y)\|^2}{2}(1 - t^2) + (1 - t)(\partial_x u(\cdot, 0), \partial_x v(\cdot, y)) \\ &\quad + \int_{\mathbb{R}} a_\varepsilon (W(u(\cdot, 0) + v(\cdot, y)) - W(u(\cdot, 0) + tv(\cdot, y)))\, \mathrm{d}x \\ &\geq \|\partial_x v(\cdot, y)\|^2(1 - t)\left(\frac{(1 + t)}{2} - \frac{|(\partial_x u(\cdot, 0), \partial_x v(\cdot, y))|}{\|\partial_x v(\cdot, y)\|^2} - C\frac{\|v(\cdot, y)\|}{\|\partial_x v(\cdot, y)\|^2}\right)\cdot \end{aligned}$$

Then, by (3.7), (3.10) and (3.11) there exists $y_1 \in (y_0, 0)$ such that

$$F(u(\cdot, 0) + v(\cdot, y)) - F(u(\cdot, 0) + tv(\cdot, y)) \geq \frac{\|\partial_x v(\cdot, y)\|^2}{4}(1 - t), \qquad \forall y \in [y_1, 0),\ \forall t \in [0, 1]. \tag{3.12}$$

Let $\ell = \liminf_{y\to 0^-} F(u(\cdot, y))$, since $F(u(\cdot, 0) + v(\cdot, y)) = F(u(\cdot, y))$, there exists $y_2 \in [y_1, 0)$ such that

$$F(u(\cdot, 0) + v(\cdot, y_2)) \leq \ell \left(1 + \frac{\mu}{32\ell}\right) \text{ and } F(u(\cdot, 0) + v(\cdot, y)) \geq \ell \left(1 - \frac{\mu}{32\ell}\right) \text{ for any } y \in [y_2, 0)$$

and so for any $y \in [y_2, 0)$ there results

$$F(u(\cdot, 0) + v(\cdot, y)) - F(u(\cdot, 0) + v(\cdot, y_2)) \geq -\frac{\mu}{16}\cdot \tag{3.13}$$

By definition of $y_{0,u}$, we have $F(u(\cdot, 0)) \leq c_{3,p}$, then we can choose $\bar{y} \in (y_2, 0]$ such that $F(u(\cdot, 0) + \frac{\bar{y}}{y_2} v(\cdot, y_2)) = c_{3,p}$ and $F(u(\cdot, 0) + \frac{y}{y_2} v(\cdot, y_2)) > c_{3,p}$ for any $y \in [y_2, \bar{y})$. We define

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{if } y < y_2 \\ u(\cdot, 0) + \frac{y}{y_2} v(\cdot, y_2) & \text{if } y_2 \leq y < \bar{y} \\ u(\cdot, 0) + \frac{\bar{y}}{y_2} v(\cdot, y_2) & \text{if } y \geq \bar{y} \end{cases}$$

and note that $\tilde{u} \in \mathcal{M}_p$.

Now we show that $\varphi_p(\tilde u) < \varphi_{p,(-\infty,0)}(u)$ obtaining a contradiction since, by Remark 3.6, $\varphi_{p,(-\infty,0)}(u) < m_p$. Note that

$$\varphi_{p,(-\infty,0)}(u) - \varphi_p(\tilde u) = \frac{1}{2}\int_{y_2}^{0} \|\partial_y u\|^2 - \|\partial_y \tilde u\|^2 \,\mathrm{d}y + \int_{y_2}^{0} F(u(\cdot,y)) - F(\tilde u(\cdot,y))\,\mathrm{d}y.$$

Since by (3.7) we have

$$\int_{y_2}^{0} \|\partial_y \tilde u\|^2 \,\mathrm{d}y = \frac{1}{y_2^2}\int_{y_2}^{\bar y} \|v(\cdot,y_2)\|^2 \,\mathrm{d}y \le \frac{-(\bar y - y_2)}{y_2}\int_{y_2}^{0} \|\partial_y u\|^2 \,\mathrm{d}y \le \int_{y_2}^{0} \|\partial_y u\|^2 \,\mathrm{d}y,$$

to show that $\varphi_{p,(-\infty,0)}(u) > \varphi_p(\tilde u)$ it is sufficient to prove that $\int_{y_2}^{0} F(u(\cdot,y)) - F(\tilde u(\cdot,y))\,\mathrm{d}y > 0$.

Indeed, if $\bar y \le \frac{y_2}{2}$, since $F(\tilde u(\cdot,y)) = c_{3,p}$ for any $y \ge \bar y$ and by assumption $F(u(\cdot,y)) > c_{3,p} + \mu$ for any $y \in (y_0, 0)$, by (3.13) and (3.12) we obtain

$$\begin{aligned}\int_{y_2}^{0} F(u(\cdot,y)) - F(\tilde u(\cdot,y))\,\mathrm{d}y &\ge \int_{y_2}^{\bar y} F(u(\cdot,y)) - F(u(\cdot,0) + v(\cdot,y_2))\,\mathrm{d}y + \int_{\bar y}^{0} F(u(\cdot,y)) - c_{3,p}\,\mathrm{d}y \\ &\ge -(\bar y - y_2)\frac{\mu}{16} - \bar y\mu \ge -\frac{y_2}{2}\left(-\frac{\mu}{16} + \mu\right) > 0.\end{aligned}$$

If otherwise $\bar y > \frac{y_2}{2}$, we have $\frac{\bar y + y_2}{y_2} < \frac{3}{2}$ and so by (3.13), (3.12) and (3.10) we obtain

$$\begin{aligned}\int_{y_2}^{0} F(u(\cdot,y)) - F(\tilde u(\cdot,y))\,\mathrm{d}y &\ge \int_{y_2}^{\bar y} F(u(\cdot,y)) - F(u(\cdot,0) + v(\cdot,y_2))\,\mathrm{d}y \\ &\quad + \int_{y_2}^{\bar y} F(u(\cdot,0) + v(\cdot,y_2)) - F\left(u(\cdot,0) + \frac{y}{y_2}v(\cdot,y_2)\right)\mathrm{d}y \\ &\ge -(\bar y - y_2)\frac{\mu}{16} + \frac{\mu}{4}\int_{y_2}^{\bar y}\left(1 - \frac{y}{y_2}\right)\mathrm{d}y = (\bar y - y_2)\left(-\frac{\mu}{16} + \frac{\mu}{4} - \frac{\mu}{8}\frac{\bar y + y_2}{y_2}\right) > 0.\end{aligned}$$

This proves that $\liminf_{y\to 0^-} F(u(\cdot,y)) = c_{3,p}$.

To conclude, note that, by Remark 3.5, $\mathsf{d}(u(\cdot,y), \mathcal{K}_{3,p}) \to 0$ as $y \to -\infty$. Moreover, by Remark 3.6, we know that $F(u(\cdot,y)) \ge c_{3,p}$ for any $y < 0$ and $\varphi_{p,(-\infty,0)}(u) \le m_p$. Finally, by (3.6), we have $u(\cdot,0) \in \Gamma_{3,\bar p}$ for some $\bar p \ne p$. Then, we can directly apply Lemma 4.4 to obtain $\varphi_{p,(-\infty,0)}(u) = m_p$. □

We are now able to prove that any function $u \in \mathcal{L}_p$ is a *weak* solution to (1.2) in $\mathbb{R} \times (-\infty, y_{0,u})$.

Lemma 3.8. *Let $u \in \mathcal{L}_p$, then*

$$\int_{\mathbb{R}^2} \nabla u \nabla\psi + a_\varepsilon(x)W'(u)\psi \,\mathrm{d}x\,\mathrm{d}y = 0 \qquad \forall \psi \in C_0^\infty(\mathbb{R} \times (-\infty, y_{0,u})).$$

Proof. Given any $\psi \in C_0^\infty(\mathbb{R} \times (-\infty, y_{0,u}))$ we set, for $t \in (0,1)$,

$$v_t(x,y) = \begin{cases} 1 - (u + t\psi - 1) & \text{if } u + t\psi > 1, \\ u + t\psi & \text{if } 1 \ge u + t\psi \ge -1, \\ -1 - (u + t\psi + 1) & \text{if } u + t\psi < -1. \end{cases}$$

First of all note that, assuming without loss of generality, that $\|\psi\|_{L^\infty(R^2)} \le 1$, $\|\psi\|_{H^1(R^2)} \le 1$ and $\|\psi(\cdot,y)\|_{H^1(R)} \le 1$ for any $y \in \mathbb{R}$, we have $|v_t(x,y)| \le 1$ for almost every $(x,y) \in \mathbb{R}^2$ and since $v_t(x,y) = u(x,y)$ on $\mathbb{R}^2 \setminus \operatorname{supp}\psi$ we have, by Remark 3.5, that $v_t \in \mathcal{H}$.

We claim that there exists $t_\psi \in (0,1)$ such that $\varphi_{p,(-\infty,y_{0,u})}(v_t) \geq m_p$ for all $t \in (0,t_\psi)$ and then, by Remark 3.7 and Lemma 3.7,

$$\varphi_{p,(-\infty,y_{0,u})}(v_t) \geq \varphi_{p,(-\infty,y_{0,u})}(u), \quad \forall t \in (0,t_\psi) \tag{3.14}$$

from which we can conclude the proof.

First, let us show that there exists $t_\psi \in (0,d_0)$ such that

$$F(v_t(\cdot,y)) \geq c_{3,p} \text{ for every } y < y_{0,u},\ t \in (0,t_\psi). \tag{3.15}$$

If $y \in (-\infty,y_{0,u}) \setminus D_u$, note that since $|v_t(\cdot,y) - u(\cdot,y)| \leq t|\psi(\cdot,y)|$, in particular we have $\|v_t(\cdot,y) - u(\cdot,y)\|^2 \leq t^2\|\psi(\cdot,y)\|^2 \leq t^2$, and for $t < d_0$ we have $\|v_t(\cdot,y)-u(\cdot,y)\| \leq d_0$ for any $y \in \mathbb{R}$. Then, we derive $\mathsf{d}(v_t,\mathcal{K}_{3,p}) < 2d_0$ for all $t \in (0,d_0)$. Hence, by Remark 2.3, we have $F(v_t(\cdot,y)) \geq c_{3,p}$ for any $t \in (0,d_0)$ and $y \in (-\infty,y_{0,u}) \setminus D_u$.

Let $x_1 < x_2$, $\zeta_1 < \zeta_2 < y_{0,u}$ be such that $\operatorname{supp}\psi \subset (x_1,x_2) \times (\zeta_1,\zeta_2)$ and let $y \in D = (-\infty,\zeta_2] \cap D_u$. Since by (3.1) and Lemma 2.8 we have that $y \to \mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p})$ is continuous, we deduce that D is closed in $\mathbb{R}$. Since by Lemma 4.1 $y \to F(u(\cdot,y))$ is lower semicontinuous and since by (3.5) $F(u(\cdot,y)) > c_{3,p}$ for any $y \in D$, we have that there exists $\mu > 0$ such that $F(u(\cdot,y)) \geq c_{3,p} + \mu$ for any $y \in D \cap [\zeta_1,\zeta_2]$. If $u(\cdot,y) \in \overline{\Gamma} \setminus \Gamma$ then also $v_t(\cdot,y) \in \overline{\Gamma} \setminus \Gamma$ and so $F(v_t(\cdot,y)) = +\infty$. If $u(\cdot,y) \in \Gamma$, setting $C_1 = \bar{a}\max_{|s|\leq 1}|W'(s)|$ and $C_2 = C_1(x_2-x_1)^{\frac{1}{2}}$ we obtain

$$\begin{aligned}
|F(u(\cdot,y)) - F(v_t(\cdot,y))| &\leq \frac{1}{2}\left|\int_{\mathbb{R}} |\partial_x u(x,y)|^2 - |\partial_x v_t(x,y)|^2\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}} a_\varepsilon(W(u(x,y)) - W(v_t(x,y)))\,\mathrm{d}x\right| \\
&\leq \frac{1}{2}\left|\int_{\mathbb{R}} |\partial_x u(x,y)|^2 - |\partial_x(u(x,y) + t\psi(x,y))|^2\mathrm{d}x\right| + C_1\int_{\mathbb{R}} t|\psi(x,y)|\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}} \frac{t^2}{2}|\partial_x\psi(x,y)|^2 + t|\partial_x\psi(x,y)||\partial_x u(x,y)|\,\mathrm{d}x + C_1\int_{\mathbb{R}} t|\psi(x,y)|\mathrm{d}x \\
&\leq \frac{t^2}{2}\|\psi(\cdot,y)\|^2_{H^1(\mathbb{R})} + 2t\|\psi(\cdot,y)\|_{H^1(\mathbb{R})}\left(\frac{F(u(\cdot,y))}{2}\right)^{\frac{1}{2}} + tC_2\|\psi(\cdot,y)\| \\
&\leq \frac{t^2}{2} + 2t\left(\frac{F(u(\cdot,y))}{2}\right)^{\frac{1}{2}} + tC_2.
\end{aligned}$$

Then for any $y \in D$ we have

$$F(v_t(\cdot,y)) \geq F(u(\cdot,y))(1-t) - \frac{1}{2}(t^2+t) - tC_2 \geq (c_{3,p}+\mu)(1-t) - \frac{1}{2}(t^2+t) - tC_2$$

from which we plainly derive that there exists $t_\psi \in (0,d_0)$ such that $F(v_t(\cdot,y)) \geq c_{3,p}$ for any $y \in D$ and $t \in (0,t_\psi)$.

Finally, if $y \in (\zeta_2,y_{0,u}) \cap D_u$, we have $v_t(x,y) = u(x,y)$ and we know, by Remark 3.5, that $F(u(\cdot,y)) \geq c_{3,p}$. Gathering the estimates above (3.15) follows.

Now, note that if $y_{0,u} = +\infty$, since by Remark 3.7 we have $u \in \mathcal{M}_p$, by (3.15) we obtain $v_t \in \mathcal{M}_p$, and then $\varphi_p(v_t) \geq m_p$, for all $t \in (0,t_\psi)$ and (3.14) follows in this case.

If otherwise $y_{0,u} \in \mathbb{R}$, note that by (3.6) and Lemma 3.7, for all $t \in (0,1)$, v_t verifies

$$\lim_{y\to-\infty} \mathsf{d}(v_t(\cdot,y),\mathcal{K}_{3,p}) = 0,\ v_t(x,y_{0,u}) = u(x,y_{0,u}) \in \cup_{\bar{p}\neq p}\Gamma_{3,\bar{p}} \text{ and } \liminf_{y\to y_{0,u}^-} F(v_t(x,y)) = c_{3,p}. \tag{3.16}$$

Hence, by (3.15), we obtain that, for all $t \in (0,t_\psi)$, v_t verifies the conditions of Lemma 4.4 with $y_0 = y_{0,u}$ and we can conclude that $\varphi_{p,(-\infty,y_{0,u})}(v_t) \geq m_p$, for all $t \in (0,t_\psi)$ and so (3.14) is completely proved.

Finally, let us define

$$\tilde W(s)=\begin{cases}W(1-(s-1)) & \text{if } 1<s\le 2,\\ W(s) & \text{if } |s|\le 1,\\ W(-1-(s+1)) & \text{if } -2\le s<-1\end{cases}$$

observing that $\tilde W\in C^1([-2,2])$ and $\tilde W(s)=W(s)$ for any $s\in[-1,1]$. There results $W(v_t)=\tilde W(u+t\psi)$, $|\partial_x v_t|=|\partial_x(u+t\psi)|$ and $|\partial_y v_t|=|\partial_y(u+t\psi)|$, *a.e.* on $\mathbb{R}^2$. Therefore

$$\int_{-\infty}^{y_{0,u}}\int_{\mathbb{R}}|\nabla(u+t\psi)|^2+a_\varepsilon\tilde W(u+t\psi)\,\mathrm{d}x-c_{3,p}\,\mathrm{d}y=\varphi_{p,(-\infty,y_{0,u})}(v_t). \tag{3.17}$$

Then, by (3.17) and (3.14), we conclude that for any $t\in(0,t_\psi)$ there results

$$\int_{-\infty}^{y_{0,u}}\int_{\mathbb{R}}\frac{1}{2}|\nabla(u+t\psi)|^2+a_\varepsilon\tilde W(u+t\psi)\,\mathrm{d}x-c_{3,p}\,\mathrm{d}y-\varphi_{p,(-\infty,y_{0,u})}(u)\ge 0.$$

Hence, since $|u(x,y)|\le 1$ for *a.e.* $(x,y)\in\mathbb{R}^2$, and since $\frac{1}{t}|\tilde W(u+t\psi)-W(u)|\le\psi\max_{|s|<2}|\tilde W'(s)|$ for *a.e.* $(x,y)\in\mathbb{R}^2$ and for any $t\in(0,1)$, by using the Fubini and the dominated convergence Theorems we obtain

$$\begin{aligned}0&\le\lim_{t\to0^+}\frac{1}{t}\left(\int_{-\infty}^{y_{0,u}}\int_{\mathbb{R}}|\nabla(u+t\psi)|^2+a_\varepsilon\tilde W(u+t\psi)\,\mathrm{d}x-c_{3,p}\,\mathrm{d}y-\varphi_{p,(-\infty,y_{0,u})}(u)\right)\\&=\lim_{t\to0^+}\frac{1}{t}\int_{\operatorname{supp}\psi}\frac{1}{2}(|\nabla(u+t\psi)|^2-|\nabla u|^2)+a_\varepsilon(x)(\tilde W(u+t\psi)-W(u))\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{\mathbb{R}^2}\nabla u\nabla\psi+a_\varepsilon(x)W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Considering $-\psi$ as test function we deduce that $\int_{\mathbb{R}^2}\nabla u\nabla\psi+a_\varepsilon(x)W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y\le 0$ from which in fact $\int_{\mathbb{R}^2}\nabla u\nabla\psi+a_\varepsilon(x)W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y=0$ and the lemma follows. □

By the following lemma we obtain that in fact any function $u\in\mathcal{L}_p$ is a a *classical* solution to (1.2) on $\mathbb{R}\times(-\infty,y_{0,u})$.

Lemma 3.9. *Let $y_1<y_2\in\mathbb{R}$. If $u\in\mathcal{H}$ verifies*

$$\int_{S_{(y_1,y_2)}}\nabla u\nabla\psi+a_\varepsilon(x)W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y=0,\qquad\forall\psi\in H_0^1(S_{(y_1,y_2)}). \tag{3.18}$$

Then $u\in C^2(S_{(y_1,y_2)})$ and verifies $-\Delta u+a_\varepsilon W'(u)=0$ on $S_{(y_1,y_2)}$. Moreover, for any $[\zeta_1,\zeta_2]\subset(y_1,y_2)$ there results $u-z_0\in H^2(S_{(\zeta_1,\zeta_2)})$ and

$$\lim_{x\to\pm\infty}u(x,y)=\pm1\qquad\textit{unif. w.r. to } y\in[\zeta_1,\zeta_2]. \tag{3.19}$$

Proof. Let $[\zeta_1,\zeta_2]\subset(\bar\zeta_1,\bar\zeta_2)\subset[\bar\zeta_1,\bar\zeta_2]\subset(y_1,y_2)$ and $\theta\in C^2(\mathbb{R})$ be such that $\theta(y)=0$ if $y\notin(\bar\zeta_1,\bar\zeta_2)$ and $\theta(y)=1$ for any $y\in[\zeta_1,\zeta_2]$. Defining $v(x,y)=\theta(y)(u(x,y)-z_0(x))$ we have $v\in H_0^1(S_{(\zeta_1,\zeta_2)})$ and moreover

$$\int_{S_{(\bar\zeta_1,\bar\zeta_2)}}\nabla v\nabla\psi\,\mathrm{d}x\,\mathrm{d}y=\int_{S_{(\bar\zeta_1,\bar\zeta_2)}}(-\theta a_\varepsilon W'(u)+\theta\partial_x^2z_0-\partial_y^2\theta(u-z_0)-\partial_y\theta\partial_y u)\psi \tag{3.20}$$

for any $\psi\in H_0^1(S_{(\bar\zeta_1,\bar\zeta_2)})$. Then one plainly recognizes that $f=-a_\varepsilon W'(u)\theta+\theta\partial_x^2z_0-\partial_y^2\theta(u-z_0)-\partial_y u\partial_y\theta\in L^2(S_{(\bar\zeta_1,\bar\zeta_1)})$ and by classical elliptic argument recovers that $v\in H^2(S_{(\bar\zeta_1,\bar\zeta_2)})$ and so that $u-z_0\in H^2(S_{(\zeta_1,\zeta_2)})$.

Then $-\Delta u + a_\varepsilon W'(u) = 0$ as element of $\cap_{[\zeta_1,\zeta_2]\subset(y_1,y_2)} L^2(S_{(y_1,y_2)})$ and since $\|u\|_{L^\infty(\mathbb{R}^2)} = 1$, by a bootstrap argument we obtain that u verifies the equation in a classical sense with $\|u\|_{C^2(S_{(\zeta_1,\zeta_2)})} < +\infty$ for any $[\zeta_1,\zeta_2] \subset (y_1, y_2)$.

To show that (3.19) holds true observe that since $u - z_0 \in \cap_{[\zeta_1,\zeta_2]\subset(y_1,y_2)} H^2(S_{(\zeta_1,\zeta_2)})$ we have that $u(\cdot, y) - z_0(\cdot) \in H^1(\mathbb{R})$ (in the sense of traces) for any $y \in (\zeta_1, \zeta_2)$ and so that $u(x, y) \to \pm 1$ as $x \to \pm\infty$ for any $y \in (\zeta_1, \zeta_2)$. Then, assume by contradiction that (3.19) does not hold and so that there exist $[\zeta_1, \zeta_2] \subset (y_1, y_2)$, $\mu > 0$, a sequence $(y_n) \subset [\zeta_1, \zeta_2]$, $y_n \to \bar{y} \in [\zeta_1, \zeta_2]$, and a sequence $(x_n) \subset \mathbb{R}$, $|x_n| \to \infty$ such that $1 - |u(x_n, y_n)| \geq \mu$. Since $\|u\|_{C^2(S_{(\zeta_1,\zeta_2)})} < +\infty$, one obtains that there exists $\rho > 0$ such that $1 - |u(x_n, y)| \geq \frac{\mu}{2}$ for any $y \in [\zeta_1, \zeta_2]$ such that $|y - \bar{y}| \leq \rho$ whenever n is sufficiently large, a contradiction since we already know that $1 - |u(x, y)| \to 0$ as $|x| \to +\infty$ for any $y \in (\zeta_1, \zeta_2)$. □

By Lemma 3.9 we obtain that if $u \in \mathcal{L}_p$ and $y_{0,u} = +\infty$, then $u \in C^2(\mathbb{R}^2)$ and $-\Delta u + a_\varepsilon W'(u) = 0$ on $\mathbb{R}^2$, *i.e.*, u is a solution to (1.2). If otherwise $u \in \mathcal{L}_p$ is such that $y_{0,u} \in \mathbb{R}$, by Lemma 3.9 we have that u solves (1.2) only on the half plane $\mathbb{R} \times (-\infty, y_{0,u})$. We will prove, by the following Lemma, that in such case u satisfies the Neumann boundary condition $\partial_y u(\cdot, y_{0,u}) \equiv 0$. This will allow us to recover, by reflection, an entire solution to (1.2) even in this case.

In fact, in the next lemma, noting that in the equation (1.2) the variable y is cyclic, we prove that a sort of *Energy* has to be conserved for the functions $u \in \mathcal{L}_p$.

Lemma 3.10. *If $u \in \mathcal{L}_p$, then the energy function*

$$y \to E_u(y) = -\frac{1}{2}\|\partial_y u(\cdot, y)\|^2 + F(u(\cdot, y))$$

is constant on $(-\infty, y_{0,u})$. In particular

$$E_u(y) = c_{3,p} \ \textit{ for all } y \in (-\infty, y_{0,u}) \quad \textit{ and } \quad \liminf_{y\to y_{0,u}^-} \|\partial_y u(\cdot, y)\| = 0. \tag{3.21}$$

Proof. Let $u \in \mathcal{L}_p$ and $(\zeta_1, \zeta_2) \subset (-\infty, y_{0,u})$. By Lemma 3.9 we know that $u \in C^2(S_{(\zeta_1,\zeta_2)})$ verifies $-\Delta u + a_\varepsilon W'(u) = 0$ on $S_{(\zeta_1,\zeta_2)}$. Multiplying both the terms of the equation by $\partial_y u(x, y)$ we get that

$$\begin{aligned} 0 &= -\partial_{x,x} u\, \partial_y u - \partial_{y,y} u\, \partial_y u + a_\varepsilon(x) W'(u) \partial_y u = -\partial_{x,x} u\, \partial_y u + \partial_y \left(-\frac{1}{2}|\partial_y u|^2 + a_\varepsilon(x) W(u) \right) \\ &= -\partial_x(\partial_x u\, \partial_y u) + \partial_y \left(\frac{1}{2}|\partial_x u|^2 - \frac{1}{2}|\partial_y u|^2 + a_\varepsilon(x) W(u) \right). \end{aligned}$$

Given $[\bar{\zeta}_1, \bar{\zeta}_2] \subset (\zeta_1, \zeta_2)$, by Lemma 3.9 we know that $u - z_0 \in H^2(S_{(\bar{\zeta}_1,\bar{\zeta}_2)})$ and hence $\nabla u \in H^1(S_{(\bar{\zeta}_1,\bar{\zeta}_2)})$. Then, integrating on $S_{(\bar{\zeta}_1,\bar{\zeta}_2)}$ and using Fubini Theorem we obtain

$$\begin{aligned} 0 &= -\int_{S_{(\bar{\zeta}_1,\bar{\zeta}_2)}} \partial_x(\partial_x u\, \partial_y u)\, \mathrm{d}x\, \mathrm{d}y + \int_{S_{(\bar{\zeta}_1,\bar{\zeta}_2)}} \partial_y \left(\frac{1}{2}|\partial_x u|^2 - \frac{1}{2}|\partial_y u|^2 + a_\varepsilon(x) W(u) \right) \mathrm{d}x\, \mathrm{d}y \\ &= -\int_{\bar{\zeta}_1}^{\bar{\zeta}_2} \int_{\mathbb{R}} \partial_x(\partial_x u\, \partial_y u)\, \mathrm{d}x\, \mathrm{d}y + \int_{\mathbb{R}} \int_{\bar{\zeta}_1}^{\bar{\zeta}_2} \partial_y \left(\frac{1}{2}|\partial_x u|^2 - \frac{1}{2}|\partial_y u|^2 + a_\varepsilon(x) W(u) \right) \mathrm{d}y\, \mathrm{d}x \\ &= -\int_{\bar{\zeta}_1}^{\bar{\zeta}_2} \int_{\mathbb{R}} \partial_x(\partial_x u\, \partial_y u)\, \mathrm{d}x\, \mathrm{d}y + E_u(\bar{\zeta}_2) - E_u(\bar{\zeta}_1). \end{aligned}$$

By Lemma 3.9 $u - z_0 \in H^2(S_{(\bar{\zeta}_1,\bar{\zeta}_2)})$. Then $\partial_x u(\cdot, y)$, $\partial_y u(\cdot, y) \in H^1(\mathbb{R})$ for *a.e.* $y \in (\bar{\zeta}_1, \bar{\zeta}_2)$ and so $\partial_x u(x, y)$, $\partial_y u(x, y) \to 0$ as $|x| \to +\infty$ for *a.e.* $y \in (\bar{\zeta}_1, \bar{\zeta}_2)$. Therefore $\int_{\mathbb{R}} \partial_x(\partial_x u\, \partial_y u)\, \mathrm{d}x = 0$ for *a.e.* $y \in [\bar{\zeta}_1, \bar{\zeta}_2]$ and $E_u(\bar{\zeta}_2) = E_u(\bar{\zeta}_1)$ follows.

That proves that the function $E_u(y)$ is constant on $(-\infty, y_{0,u})$. It is not difficult to recognize that $E_u(y) = c_{3,p}$. Indeed, By Lemma 3.7 we have $\int_{-\infty}^{y_{0,u}} \frac{1}{2}\|\partial_y u(\cdot,y)\|^2 + (F(u(\cdot,y)) - c_{3,p})\,\mathrm{d}y = m_p < +\infty$ and so there exists a sequence $(y_j) \subset (-\infty, y_{0,u})$ such that $y_j \to -\infty$ and $\frac{1}{2}\|\partial_y u(\cdot,y_j)\|^2 + (F(u(\cdot,y_j)) - c_{3,p}) \to 0$. Since, by Remarks 2.3 and 3.6, $F(u(\cdot,y_j)) \geq c_{3,p}$ for any $j \in \mathbb{N}$ we have $\|\partial_y u(\cdot,y_j)\|^2 \to 0$ and $F(u(\cdot,y_j)) \to c_{3,p}$. Then $E_u(y) = c_{3,p}$ for any $y \in (-\infty, y_{0,u})$ follows. Since by Lemma 3.7, $\liminf_{y\to y_{0,u}^-} F(u(\cdot,y)) = c_{3,p}$, we can conclude that $\liminf_{y\to y_{0,u}^-} \|\partial_y u(\cdot,y)\| = 2\liminf_{y\to y_{0,u}^-}(F(u(\cdot,y)) - E_u(y)) = 0$ and (3.21) follows. □

We are now able to prove the existence of solutions to (1.2).

Proposition 3.1. *Let $u \in \mathcal{L}_p$. Then, setting*

$$v_p \equiv u, \quad \text{if } y_{0,u} = +\infty, \qquad \text{or} \qquad v_p(x,y) = \begin{cases} u(x,y), & \text{if } y \leq y_{0,u}, \\ u(x, 2y_{0,u} - y), & \text{if } y > y_{0,u}, \end{cases} \quad \text{if } y_{0,u} \in \mathbb{R},$$

we have that $v_p \in C^2(\mathbb{R})$ is a classical solution to (1.2) on $\mathbb{R}^2$. Moreover, $\|v_p\|_{C^2(\mathbb{R}^2)} < +\infty$ and $v_p(x,y) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $|y| \leq T$, for any $T > 0$.

Proof. If $y_{0,u} = +\infty$, the statement follows by Lemmas 3.8 and 3.9 noting that $\|u\|_{C^2(\mathbb{R}^2)} < +\infty$ derive from $\|u\|_{L^\infty(\mathbb{R}^2)} \leq 1$ using local Schauder estimates.

Let $y_{0,u} \in \mathbb{R}$ and since the functional is invariant with respect to the y-translations, it is non restrictive to assume that $y_{0,u} = 0$. By (3.21) we already know that $E_u = c_{3,p} = -\frac{1}{2}\|u(\cdot,y)\|^2 + F(u(\cdot,y))$ for any $y < 0$.

By Lemma 3.7 we know that

$$\exists\,(y_j) \subset \mathbb{R}_- \text{ such that } y_j \to 0 \text{ and } F(u(\cdot,y_j)) \to c_{3,p} \tag{3.22}$$

and hence $\|\partial_y u(\cdot,y_j)\| \to 0$. By Lemma 3.9, using the Green formula, we have that for any $\psi \in C_0^\infty(\mathbb{R}^2)$ and $j \in \mathbb{N}$

$$0 = \int_{\mathbb{R}\times(-\infty,y_j)} -\Delta u\,\psi + a_\varepsilon W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}\times(-\infty,y_j)} \nabla u \nabla\psi + a_\varepsilon W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}} \partial_y u(x,y_j)\psi(x,y_j)\,\mathrm{d}x,$$

and so

$$\begin{aligned}\int_{\mathbb{R}\times(-\infty,0)} \nabla u \nabla\psi + a_\varepsilon W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y &= \lim_{j\to\infty} \int_{\mathbb{R}\times(-\infty,y_j)} \nabla u \nabla\psi + a_\varepsilon W'(u)\psi\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{j\to\infty} \int_{\mathbb{R}} \partial_y u(x,y_j)\psi(x,y_j)\,\mathrm{d}x = 0.\end{aligned}$$

With a simple change of coordinates we obtain also that

$$\int_{\mathbb{R}\times(0,+\infty)} \nabla u(x,-y)\nabla\psi + a_\varepsilon W'(u(x,-y))\psi\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}^2).$$

Then, v_p satisfies

$$\int_{\mathbb{R}^2} \nabla v_p \nabla\psi + a_\varepsilon(x) W'(v_p)\psi\,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}^2)$$

and using Lemma 3.9 the proposition follows as in the case $y_{0,u} = +\infty$. □

Thanks to Proposition 3.1 and Lemma 3.9, we can say that (1.2) always admits a two dimensional solution verifying $\mathsf{d}(u(\cdot,y),\mathcal{K}_{3,p}) \to 0$ as $y \to -\infty$ whenever $p \in \mathcal{P}$ is such that $[p] \geq p_0$. Moreover,

- if $y_{0,u} = +\infty$, such solution is of the *heteroclinic type*, *i.e.* $v_p \in C^2(\mathbb{R}^2)$ verifies (1.2) and $v_p \in \mathcal{M}_p$;
- if $y_{0,u} \in \mathbb{R}$, then the solution is of the *homoclinic type*, *i.e.* $v_p \in C^2(\mathbb{R}^2)$ verifies (1.2), $\mathsf{d}(v_p,\mathcal{K}_{3,p}) \to 0$ as $y \to \pm\infty$ and, by Remark 3.5 and Lemma 2.10, $\mathsf{d}(v_p(\cdot,y_{0,u}),\mathcal{K}_{3,p}) \geq 3d_0$.

To complete the proof of the main theorem we have to show that in any case $v_p(x,y) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $y \in \mathbb{R}$.

Lemma 3.11. *Let $p \in \mathcal{P}$ with $[p] \geq \mathsf{p}_0$, for every $u \in \mathcal{L}_p$, let v_p be given by Proposition 3.1. Then, $v_p(x,y) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $y \in \mathbb{R}$.*

Proof. By Proposition 3.1 we know that $v_p \in C^2(\mathbb{R})$, $\|v_p\|_{C^2(\mathbb{R}^2)} < +\infty$ and that for any $T > 0$ there results $v_p(x,y) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $|y| \leq T$.

As first step in the proof we claim now that for any $(\zeta_1,\zeta_2) \subset \mathbb{R}$ there exists a constant $C > 0$ depending only on $\zeta_2 - \zeta_1$ such that
$$\|v_p\|_{H^2(S_{(\zeta_1,\zeta_2)})} \leq C.$$
To this aim note firstly that, by Lemma 3.6, $\mathsf{d}(v_p(\cdot,y),\Gamma_{3,p}) \leq \mathsf{C}_0$ for any $y \in \mathbb{R}$.

In particular we obtain that $\sup_{y\in\mathbb{R}} \|v_p(\cdot,y) - z_0(\cdot)\|^2 = C_1 < +\infty$ and so by Lemma 2.13 we recover that $\int_{\mathbb{R}} W(v_p(x,y))\,\mathrm{d}x \leq C_2 < +\infty$ for any $y \in \mathbb{R}$. Then, since by (1.10), we have $|W'(s)|^2 \leq \frac{\bar{b}^2}{\underline{b}} W(s)$ for any $|s| \leq 1$ we derive that there exists a constant $C_3 > 0$ such that $\|W'(v_p)\|^2_{L^2(S_{(\zeta_1,\zeta_2)})} \leq C_3(\zeta_2 - \zeta_1)$ for any $(\zeta_1,\zeta_2) \subset \mathbb{R}$.

Secondly we observe that since $\varphi_p(v_p) = m_p$ if $y_{0,u} = +\infty$ and $\varphi_p(v_p) = 2m_p$ if $y_{0,u} < +\infty$ we always have that $\|\partial_y v_p\|^2_{L^2(\mathbb{R}^2)} \leq 4m_p$.

Then, for any $(\zeta_1,\zeta_2) \subset \mathbb{R}$ we let $\theta \in C^2(\mathbb{R})$ be such that $\theta(y) = 0$ if $y \notin (\zeta_1 - 1, \zeta_2 + 1)$, $\theta(y) = 1$ for any $y \in (\zeta_1,\zeta_2)$ and $\|\theta\|_{C^2(\mathbb{R})} \leq 2$. Defining $v(x,y) = \theta(y)(v_p(x,y) - z_0(x))$ we have $v \in H^1_0(S_{(\zeta_1-1,\zeta_2+1)})$,
$$\|v\|^2_{L^2(S_{(\zeta_1-1,\zeta_2+1)})} \leq 4\|v_p - z_0\|^2_{L^2(S_{(\zeta_1-1,\zeta_2+1)})} \leq 4C_1(\zeta_2 - \zeta_1 + 2)$$
and moreover, since v_p is a classical solution to (1.2),
$$\Delta v = \theta a_\varepsilon W'(v_p) - \theta\partial_x^2 z_0 + \partial_y^2\theta(v_p - z_0) + \partial_y\theta\partial_y v_p.$$
Then, since $\Delta v = \theta a_\varepsilon W'(v_p) - \theta\partial_x^2 z_0 + \partial_y^2\theta(v_p - z_0) + \partial_y\theta\partial_y v_p \in L^2(S_{(\zeta_1-1,\zeta_2+1)})$ and
$$\|\Delta v\|_{L^2(S_{(\zeta_1-1,\zeta_2+1)})} \leq 2(\overline{a}C_3^{1/2} + C_1^{1/2} + \|\partial_x^2 z_0\|)(\zeta_2 - \zeta_1 + 2)^{1/2} + 4m_p^{1/2}$$
by classical elliptic argument we recover that $v \in H^2(S_{(\zeta_1-1,\zeta_2+1)})$ and that there exists a constant C depending only on $\zeta_2 - \zeta_1$ such that $\|v\|_{H^2(S_{(\zeta_1-1,\zeta_2+1)})} \leq C$. Then, since $\|v_p - z_0\|_{H^2(S_{(\zeta_1,\zeta_2)})} = \|v\|_{H^2(S_{(\zeta_1,\zeta_2)})}$, our claim follows.

In particular we obtain that the function $y \in \mathbb{R} \to \partial_y v_p(\cdot,y) \in L^2(\mathbb{R})$ is uniformly continuous. Indeed, as in (3.1), for any $(\zeta_1,\zeta_2) \subset \mathbb{R}$ we have
$$\|\partial_y v_p(\cdot,\zeta_1) - \partial_y v_p(\cdot,\zeta_2)\|^2 \leq (\zeta_2 - \zeta_1)\|\partial_y^2 v_p\|_{L^2(S_{(\zeta_1,\zeta_2)})} \leq (\zeta_2 - \zeta_1)\|v_p - z_0\|_{H^2(S_{(\zeta_1,\zeta_2)})}$$
from which we derive that
$$\lim_{y\to\pm\infty} \|\partial_y v_p(\cdot,y)\| = 0.$$
Indeed, if there exist a sequence $|y_j| \to \infty$ as $j \to \infty$ and $r > 0$ such that $\|\partial_y v_p(\cdot,y_j)\| \geq 2r$ for any $j \in \mathbb{N}$, then by uniform continuity there exists $\rho > 0$ such that $\|\partial_y v_p(\cdot,y)\| \geq r$ for any $y \in \cup_{j\in\mathbb{N}}(y_j - \rho, y_j + \rho)$ and so $\varphi_p(v_p) \geq \sum_{j=1}^{\infty} \frac{r^2}{2}\rho = +\infty$, a contradiction since $\varphi_p(v_p) < +\infty$.

By Lemma 3.10 we then obtain that

$$\lim_{y\to\pm\infty} F(v_p(\cdot,y)) = c_{3,p}$$

and so that there exists $L > 0$ such that

$$v_p(\cdot,y) \in \Gamma_{3,p} \text{ for any } y < -L \text{ and } v_p(\cdot,y) \in \Gamma_{3,\bar p} \text{ for any } y > L$$

where $\bar p = p$ if $y_{0,u} < +\infty$ and $\bar p = p^*$ if $y_{0,u} = +\infty$. By Lemma 2.5 we deduce that there exists $T > 0$ such that that if $|y| > L$ then

$$v_p(x,y) \geq 1 - 2\bar\delta \text{ for any } x > T \text{ and } v_p(x,y) \leq -1 + 2\bar\delta \text{ for any } x < -T. \tag{3.23}$$

We will assume that T is such that also $1 - |z_0(x)| \geq 2\bar\delta$ for any $|x| \geq T$.

Assume now by contradiction that $v_p(x,y)$ does not converge to ± 1 as $x \to \pm\infty$ uniformly with respect to $y \in \mathbb{R}$. Then there exists a sequence $(x_j, y_j) \subset \mathbb{R}^2$ such that $|x_j| \to \infty$, $|y_j| \to \infty$ as $j \to \infty$ and $1 - |v_p(x_j,y_j)| \geq 2r > 0$ for any $j \in \mathbb{N}$. Since $\|v_p\|_{C^2(\mathbb{R}^2)} < +\infty$ we obtain that there exists $\rho \in (0,1)$ such that $1 - |v_p(x,y)| \geq r$ for any $(x,y) \in \cup_{j\in\mathbb{N}} B_\rho((x_j,y_j))$ (as usual we denote $B_\rho((x_j,y_j)) = \{(x,y) \in \mathbb{R}^2 \,/\, (x-x_j)^2 + (y-y_j)^2 < \rho^2\}$). Since $|x_j| \to \infty$ as $j \to \infty$ and $z_0(x) \to \pm 1$ as $x \to \pm\infty$ we deduce that

$$\liminf_{j\to\infty} \|v_p - z_0\|_{L^2(B_\rho((x_j,y_j)))} \geq \pi^{1/2} r\rho. \tag{3.24}$$

For any $j, n \in \mathbb{N}$ we set $Q_{j,n} = \{(x,y) \in \mathbb{R}^2 \,/\, n-1 < |x| < n,\ |y-y_j| < \rho\}$. Since, as we know there exists a constant $C > 0$ depending only on ρ such that for any $j \in \mathbb{N}$

$$\sum_{n\in\mathbb{N}} \|v_p - z_0\|^2_{H^2(Q_{j,n})} = \|v_p - z_0\|^2_{H^2(S_{(y_j-\rho,y_j+\rho)})} \leq C^2$$

we obtain that for any $j \in \mathbb{N}$ big enough

$$\frac{[|x_j|]}{2} \min_{n\in\{[\frac{|x_j|}{2}]+1,[|x_j|]-1\}} \|v_p - z_0\|^2_{H^2(Q_{j,n})} \leq \sum_{n=[\frac{|x_j|}{2}]+1}^{[|x_j|]-1} \|v_p - z_0\|^2_{H^2(Q_{j,n})} \leq C^2,$$

where we denote with $[x]$ the entire part of $x \in \mathbb{R}$.

Therefore for any $j \in \mathbb{N}$ there exists $\bar n_j \in \{[\frac{|x_j|}{2}]+1, [|x_j|]-1\}$ such that

$$\|v_p - z_0\|^2_{H^2(Q_{j,\bar n_j})} \leq \frac{2C^2}{[|x_j|]}.$$

Now, for any $j \in \mathbb{N}$ we set

$$\mathcal{A}_j = \{(x,y) \in \mathbb{R}^2 \,/\, |x| \geq \bar n_j,\ |y - y_j| < \rho\}$$

and we let $\theta_j \in C^2(\mathbb{R}^2)$ to be a function which verifies, $\|\theta_j\|_{C^2(\mathbb{R}^2)} \leq 2$, $\theta_j(x,y) = 1$ on $\mathcal{A}_j$ and $\theta_j(x,y) = 0$ if $|x| \leq \bar n_j - 1$ or $|y - y_j| \geq \rho + 1$.

Note that $\theta_j(v_p - z_0) \in H_0^1(\mathbb{R}^2)$ and so integrating the equation $-\theta_j(v_p - z_0)\Delta v_p + \theta_j(v_p - z_0)a_\varepsilon W'(v_p) = 0$ on the strip $S_{(y_j-\rho,y_j+\rho)}$, and applying the Green Formula, we obtain

$$\int_{S_{(y_j-\rho,y_j+\rho)}} \nabla v_p \,\nabla(\theta_j(v_p - z_0)) + \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y - \int_{y=y_j+\rho} \partial_y v_p(\theta_j(v_p - z_0))\,\mathrm{d}x$$
$$+ \int_{y=y_j-\rho} \partial_y v_p(\theta_j(v_p - z_0))\,\mathrm{d}x = 0.$$

Then, since $\|\partial_y v_p(\cdot, y)\| \to 0$ as $y \to \pm\infty$, we conclude that

$$\int_{S_{(y_j-\rho,y_j+\rho)}} \nabla v_p \,\nabla(\theta_j(v_p - z_0)) + \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y \to 0 \quad \text{as } j \to \infty. \tag{3.25}$$

On the other hand we note that

$$\begin{aligned}
\int_{S_{(y_j-\rho,y_j+\rho)}} \nabla v_p \,\nabla(\theta_j(v_p - z_0))\,\mathrm{d}x\mathrm{d}y &= \int_{Q_{j,\bar{n}_j}} \nabla v_p \,\nabla(\theta_j(v_p - z_0))\,\mathrm{d}x\mathrm{d}y + \int_{\mathcal{A}_j} \nabla v_p \,\nabla(v_p - z_0)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{Q_{j,\bar{n}_j}} \nabla\theta_j \nabla v_p \,(v_p - z_0) + \theta_j \nabla v_p \,\nabla(v_p - z_0)\,\mathrm{d}x\mathrm{d}y + \int_{\mathcal{A}_j} (\nabla(v_p - z_0))^2 + \partial_x z_0 \partial_x(v_p - z_0)\,\mathrm{d}x\mathrm{d}y \\
&\geq \|\nabla(v_p - z_0)\|^2_{L^2(\mathcal{A}_j)} - \|\partial_x z_0\|_{L^2(\mathcal{A}_j)}\|\partial_x(v_p - z_0)\|_{L^2(\mathcal{A}_j)} - 2\|\nabla v_p\|_{L^2(Q_{j,\bar{n}_j})}\|\nabla(v_p - z_0)\|_{L^2(Q_{j,\bar{n}_j})} \\
&\quad -2\|\nabla v_p\|_{L^2(Q_{j,\bar{n}_j})}\|v_p - z_0\|_{L^2(Q_{j,\bar{n}_j})}.
\end{aligned}$$

Then, since $\|\partial_x z_0\|_{L^2(\mathcal{A}_j)} \to 0$, $\|\nabla(v_p - z_0)\|_{L^2(Q_{j,\bar{n}_j})} \to 0$ and $\|v_p - z_0\|_{L^2(Q_{j,\bar{n}_j})} \to 0$ as $j \to \infty$, we conclude that

$$\liminf_{j\to\infty} \int_{S_{(y_j-\rho,y_j+\rho)}} \nabla v_p \,\nabla(\theta_j(v_p - z_0))\,\mathrm{d}x\mathrm{d}y \geq 0.$$

Note finally that for any $j \in \mathbb{N}$ such that $\bar{n}_j > T$, by (3.23) and (1.8), we have that $W''(v_p(x, y)) \geq \underline{w}$ for all $(x, y) \in \mathcal{A}_j$ and so we deduce that

$$\begin{aligned}
\int_{S_{(y_j-\rho,y_j+\rho)}} \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y &= \int_{Q_{j,\bar{n}_j}} \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathcal{A}_j} (v_p - z_0)a_\varepsilon(W'(v_p) - W'(z_0))\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathcal{A}_j} (v_p - z_0)a_\varepsilon W'(z_0)\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \underline{a}\,\underline{w}\|v_p - z_0\|^2_{L^2(\mathcal{A}_j)} - 2\overline{a}\|v_p - z_0\|_{L^2(\mathcal{A}_j)}\|W'(z_0)\|_{L^2(\mathcal{A}_j)} - 2\overline{a}\|v_p - z_0\|_{L^2(Q_{j,\bar{n}_j})}\|W'(v_p)\|_{L^2(Q_{j,\bar{n}_j})}.
\end{aligned}$$

Then, since by (3.24) we have

$$\liminf_{j\to\infty} \|v_p - z_0\|^2_{L^2(\mathcal{A}_j)} \geq \liminf_{j\to\infty} \|v_p - z_0\|^2_{L^2(B_\rho((x_j,y_j)))} \geq \pi r^2 \rho^2,$$

and since $\|v_p - z_0\|_{L^2(Q_{j,\bar{n}_j})} \to 0$ and $\|W'(z_0)\|_{L^2(\mathcal{A}_j)} \to 0$ as $j \to \infty$, we conclude that

$$\liminf_{j\to\infty} \int_{S_{(y_j-\rho,y_j+\rho)}} \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y \geq \underline{a}\,\underline{w}\pi r^2 \rho^2.$$

Gathering the estimates above, we deduce that

$$\liminf_{j\to\infty} \int_{S_{(y_j-\rho,y_j+\rho)}} \nabla v_p \,\nabla(\theta_j(v_p - z_0)) + \theta_j(v_p - z_0)a_\varepsilon W'(v_p)\,\mathrm{d}x\,\mathrm{d}y \geq \underline{a}\,\underline{w}\pi r^2 \rho^2$$

a contradiction with (3.25). □

4. Appendix

In this section we will display the details of some technical result used in the previous section.

Lemma 4.1. *Given $u \in \mathcal{H}$, the function $y \in \mathbb{R} \to F(u(\cdot, y)) \in [0, +\infty]$ is lower semicontinuous.*

Proof. If $y_n \to y_0$ and $\liminf F(u(\cdot, y_n)) = +\infty$ there is nothing to prove. If otherwise there exists a subsequence $(y_{n_k}) \subset (y_n)$ such that $\lim F(u(\cdot, y_{n_k})) = \liminf F(u(\cdot, y_n)) < +\infty$, then, by Lemma 2.1, there exists $q \in H^1_{loc}(\mathbb{R})$ such that, along a subsequence, $u(\cdot, y_{n_k}) \to q(\cdot)$ weakly in $H^1_{loc}(\mathbb{R})$ and $F(q) \le \lim F(u(\cdot, y_{n_k}))$. Since, by (3.1), $u(\cdot, y_{n_k}) - u(\cdot, y_0) \to 0$ strongly in $L^2(\mathbb{R})$ we conclude that $q(\cdot) = u(\cdot, y_0)$ and the lemma follows. □

Lemma 4.2. *There results $\mathcal{M}_p \neq \emptyset$. Moreover*

$$\inf_{p \in \mathcal{P}} m_p \ge d_0 \frac{\sqrt{m_0}}{8} \quad \textit{and} \quad \sup_{p \in \mathcal{P}} m_p < +\infty.$$

Proof. Let $q \in \mathcal{K}_{3,p}$. We isolate the transitions of q defining q_{p_1}, q_{p_2}, q_{p_3} in such a way that $q_{p_l}(x) = q(x)$ if $x \in (\sigma_{l,q}, \tau_{l,q})$ and $1 - |q_{p_l}(x)| = 0$ if $x \in \mathbb{R} \setminus (\sigma_{l,q} - 1, \tau_{l,q} + 1)$, $l = 1, 2, 3$. In fact given $l = \{1, 2, 3\}$, we set

$$q_{p_l}(x) = \begin{cases} (-1)^l & \text{if } x \le \sigma_{l,q} - 1, \\ (-1)^l(\sigma_{l,q} - x) + (-1)^l(1 - \delta_0)(x - \sigma_{l,q} + 1) & \text{if } \sigma_{l,q} - 1 < x \le \sigma_{l,q}, \\ q(x) & \text{if } \sigma_{l,q} < x < \tau_{l,q}, \\ (-1)^{l+1}(1 - \delta_0)(\tau_{l,q} + 1 - x) + (-1)^{l+1}(x - \tau_{l,q}) & \text{if } \tau_{l,q} \le x \le \tau_{l,q} + 1, \\ (1)^{l+1} & \text{if } x \ge \tau_{l,q} + 1. \end{cases}$$

We define now $q_0(x) = q_{p_1}(x) + q_{p_2}(x) + q_{p_3}(x)$ and we observe that $\mathsf{nt}(q_0) = 3$ and $(\sigma_{l,q_0}, \tau_{l,q_0}) = (\sigma_{l,q}, \tau_{l,q}) \subset A_{p_l} \setminus \cup_{j \in \mathbb{Z}} O_j$ for any $l \in \{1, 2, 3\}$.

Let t_1, t_2, t_3 be such that $t_l \in (\sigma_{l,q_0}, \tau_{l,q_0})$ and $q_0(t_l) = 0$. Set moreover $T_l = \frac{p_l + \overline{x} + 1}{\varepsilon} - t_l$, $\tilde{T}_l = t_l - \frac{p_l + \overline{x}}{\varepsilon}$ and note that $T_l + \tilde{T}_l = \frac{1}{\varepsilon}$ for any $l \in \{1, 2, 3\}$.

Define

$$u(x, y) = \begin{cases} q(x) & \text{if } y \le -1, \\ -q(x)y + q_0(x)(y + 1) & \text{if } -1 < y \le 0, \\ \sum_{l=1}^3 q_{p_l}(x - yT_l) & \text{if } 0 < y \le 1, \\ \sum_{l=1}^3 q_{p_l}(x - T_l - (y - 1)\tilde{T}_l) & \text{if } 1 < y \le 2, \\ q_0(x - \frac{1}{\varepsilon})(3 - y) + q(x - \frac{1}{\varepsilon})(y - 2) & \text{if } 2 < y \le 3, \\ q(x - \frac{1}{\varepsilon}) & \text{if } y > 3, \end{cases}$$

and note that $u \in \mathcal{H}$, $\lim_{y \to -\infty} \mathsf{d}(u(x, y), \mathcal{K}_{3,p}) = 0$ and letting $p' = p + (1, 1, 1)$ we have $p' \in \mathcal{P}$, $q(x - \frac{1}{\varepsilon}) \in \mathcal{K}_{3,p'}$ and so $\lim_{y \to +\infty} \mathsf{d}(u(x, y), \mathcal{K}_{3,p'}) = 0$. Since, by Lemma 2.10, $\mathsf{d}(\mathcal{K}_{3,p}, \mathcal{K}_{3,p'}) \ge 3d_0$, we obtain $\liminf_{y \to +\infty} \mathsf{d}(u(x, y), \mathcal{K}_{3,p}) \ge 3d_0$.

We show now that $F(u(\cdot, y)) \ge c_{3,p}$ for any $y \in \mathbb{R}$ and so $u \in \mathcal{M}_p$.

First of all note that $F(u(\cdot, y)) = c_{3,p}$ for any $y \in \mathbb{R} \setminus (-1, 3)$. If $y \in (-1, 0]$ then $u(\cdot, y)$ is a convex combination of the two functions q and q_0, therefore $\mathsf{nt}(u(\cdot, y)) = 3$ and $(\sigma_{l,u(\cdot,y)}, \tau_{l,u(\cdot,y)}) = (\sigma_{l,q}, \tau_{l,q}) \subset A_{p_l}$ for any $l \in \{1, 2, 3\}$. We have either $F(u(\cdot, y)) \ge c^*$ or $F(u(\cdot, y)) < c^*$ and in both the cases we have $F(u(\cdot, y)) \ge c_{3,p}$. Indeed, if $F(u(\cdot, y)) < c^*$ then $u \in \Gamma_{3,p}$ and so $F(u(\cdot, y)) \ge c_{3,p}$. The same reasoning can be used to show that if $y \in (2, 3]$ then $F(u(\cdot, y)) \ge c_{3,p}$. Let us now consider the case $y \in (0, 1]$. We have $u(\cdot, y) = \sum_{l=1}^3 q_{p_l}(x - yT_l)$ and so $\mathsf{nt}(u(\cdot, y)) = 3$ and

$$(\sigma_{l,u(\cdot,y)}, \tau_{l,u(\cdot,y)}) = (\sigma_{l,q} + yT_l, \tau_{l,q} + yT_l) \text{ for any } l \in \{1, 2, 3\}.$$

If $(\cup_{l=1}^3(\sigma_{l,q}+yT_l,\tau_{l,q}+yT_l))\cap(\cup_{j\in\mathbb{Z}}J_j)\neq\emptyset$ then, by Lemma 2.5 we have $F(u(\cdot,y))>c^*$. If $(\cup_{l=1}^3(\sigma_{l,q}+yT_l,\tau_{l,q}+yT_l))\cap(\cup_{j\in\mathbb{Z}}J_j)=\emptyset$, since $\sigma_{l,q}+yT_l\in A_{p_l}$, we have $(\sigma_{l,q}+yT_l,\tau_{l,q}+yT_l)\subset A_{p_l}$ for any $l\in\{1,2,3\}$. Then, as above, if $F(u(\cdot,y))\le c^*$, we have $u(\cdot,y)\in\Gamma_{3,p}$ and so $F(u(\cdot,y))\ge c_{3,p}$. Analogous is the case $y\in(1,2]$ and, as claimed, $u\in\mathcal{M}_p$ follows.

Now, we will find a constant $C>0$, independent on $p\in\mathcal{P}$, such that $\varphi_p(u)\le C$, proving in this way that $\sup_{p\in\mathcal{P}}m_p<+\infty$.

To this aim note that since $T_l+\tilde{T}_l\le\frac{1}{\varepsilon}$ for any $l\in\{1,2,3\}$, we have

$$\begin{aligned}\varphi_p(u)\le&\int_{-1}^0\frac{1}{2}\|q-q_0\|^2+F(-yq+(y+1)q_0)-c_{3,p}\,\mathrm{d}y+\int_0^2\frac{1}{2\varepsilon^2}\sum_{l=1}^3\|\dot{q}_{p_l}\|^2+\sum_{l=1}^3F(q_{p_l})-c_{3,p}\,\mathrm{d}y\\&+\int_2^3\frac{1}{2}\|q_0-q\|^2+F((3-y)q_0+(y-2)q)-c_{3,p}\,\mathrm{d}y.\end{aligned}$$

It is simple to recognize that there exists $C>0$ such that

$$\int_0^2\frac{1}{2\varepsilon^2}\sum_{l=1}^3\|\dot{q}_{p_l}\|^2+\sum_{l=1}^3F(q_{p_l})-c_{3,p}\,\mathrm{d}y\le C$$

for any $p\in\mathcal{P}$. Indeed $\sum_{l=1}^3\|\dot{q}_{p_l}\|^2\le\|\dot{q}\|^2+6{\delta_0}^2\le 2c^*+6{\delta_0}^2$ and, arguing as in the proof of Lemma 2.3 and by Lemma 2.6, (2.14), $\sum_{l=1}^3F(q_{p_l})\le F(q)+6\frac{m_0}{16}<c^*+m_0$ for any $p\in\mathcal{P}$.

Let us now estimate the term

$$\int_{-1}^0\frac{1}{2}\|q-q_0\|^2+F(-yq+(y+1)q_0)-c_{3,p}\,\mathrm{d}y.$$

Since $F(q)=c_{3,p}$ and since, by Lemma 2.5, $1-|q(x)|\le 2\overline{\delta}$ for any $x\in\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))$, then, by (1.9) we obtain that

$$\int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))}\chi(q)^2\,\mathrm{d}x\le\frac{2c^*}{\underline{a}\,\underline{w}}.$$

Therefore

$$\|q-q_0\|^2\le 2\int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))}\chi(q)^2+\chi(q_0)^2\,\mathrm{d}x\le\frac{4c^*}{\underline{a}\,\underline{w}}+12{\delta_0}^2.$$

To evaluate $\int_{-1}^0F(-yq+(y+1)q_0)\,\mathrm{d}y$ note firstly that since $\|\dot{q}_0\|^2\le\|\dot{q}\|^2+6{\delta_0}^2$ and since $\|\dot{q}\|^2\le 2c^*$, we have

$$\|-y\dot{q}+(y+1)\dot{q}_0\|^2\le 2(\|\dot{q}\|^2+\|\dot{q}_0\|^2)\le 8c^*+12{\delta_0}^2\text{ for any }y\in(-1,0).$$

Hence

$$\int_{-1}^0F(-yq+(y+1)q_0)\,\mathrm{d}y\le 4c^*+6{\delta_0}^2+\int_{-1}^0\int_{\mathbb{R}}a_\varepsilon W(-yq+(y+1)q_0)\,\mathrm{d}x\,\mathrm{d}y.$$

Observe now that for any $y\in(-1,0)$ we plainly have

$$\int_{\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q})}W(-yq+(y+1)q_0)\,\mathrm{d}x\le\frac{3}{\varepsilon}\max_{|s|\le 1}W(s).$$

Note moreover that for any $y\in(-1,0)$ there results

$$|-yq(x)+(y+1)q_0(x)|\ge 1-2\overline{\delta}\text{ for any }x\in\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))$$

and so by (1.9) we obtain that for any $y \in (-1,0)$

$$\int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))} W(-yq+(y+1)q_0)\,\mathrm{d}x \le \frac{\overline{w}}{2}\int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))} \chi(-yq+(y+1)q_0)^2\,\mathrm{d}x.$$

Then observe that

$$\chi(-yq(x)+(y+1)q_0(x)) \le \chi(q_0(x)) + |y(q_0(x)-q(x))| \le \chi(q_0(x)) + |q(x)-q_0(x)|,$$

for any $x \in \mathbb{R}$ and $y \in (-1,0]$. Therefore since as one plainly recognizes

$$\chi(q_0(x)) = 0 \text{ and } \chi(q(x)) = |q(x)-q_0(x)| \text{ for any } x \in \mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q}-1,\tau_{l,q}+1))$$

and since

$$\chi(-yq(x)+(y+1)q_0(x)) \le 2\overline{\delta} \text{ for any } x \in \cup_{l=1}^3((\sigma_{l,q}-1,\sigma_{l,q})\cup(\tau_{l,q},\tau_{l,q}+1)),$$

we obtain that,

$$\begin{aligned}\int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q},\tau_{l,q}))} \chi(-yq+(y+1)q_0)^2\,\mathrm{d}x &\le 24\overline{\delta}^2 + \int_{\mathbb{R}\setminus(\cup_{l=1}^3(\sigma_{l,q}-1,\tau_{l,q}+1))} \chi(q)^2\,\mathrm{d}x \\ &\le 24\overline{\delta}^2 + \frac{c^*}{\underline{a}\,\underline{w}}\cdot\end{aligned}$$

This proves that there exists a constant $C>0$ independent from $p \in \mathcal{P}$ such that $\int_{-1}^0 \frac{1}{2}\|q-q_0\|^2 + F_{\overline{a}}(-yq+(y+1)q_0) - c_{3,p}\,\mathrm{d}y \le C$. Similarly one shows that $\int_2^3 \frac{1}{2}\|q_0-q\|^2 + F_{\overline{a}}((3-y)q_0+(y-2)q) - c_{3,p}\,\mathrm{d}y \le C$ for any $p \in \mathcal{P}$.

Finally observe that if $u \in \mathcal{M}_p$ then by (3.1) there exists $(a,b) \subset \mathbb{R}$ such that $\mathsf{d}(u(\cdot,y),\Gamma_{3,p}) \in (\frac{d_0}{4},\frac{d_0}{2})$ for any $y \in (a,b)$ and $\|u(\cdot,a)-u(\cdot,b)\| = \frac{d_0}{4}\cdot$ Then by Remark 2.3 we recover that $F(u(\cdot,y)) \ge c^*$ for any $y \in (a,b)$ and by (3.2) and Lemma 2.6 we obtain

$$m_p \ge \sqrt{2(c^*-c_{3,p})}\frac{d_0}{4} \ge d_0\frac{\sqrt{m_0}}{8} \quad \text{for any } p \in \mathcal{P}, \tag{4.1}$$

concluding the proof of the lemma. □

Lemma 4.3. *If* $(u_n) \subset \mathcal{M}_p$, $F(u_n(\cdot,0)) \to c_{3,p}$ *and* $u_n(\cdot,0) \in \Gamma_{3,\bar{p}}$ *for a* $\bar{p} \ne p$ *then* $\liminf \varphi_{p,(-\infty,0)}(u_n) \ge m_p$.

Proof. By Lemmas 2.1 and 2.7 there exists $q \in \Gamma_{3,\bar{p}}$ such that $F(q) \le c_{3,p}$ and a subsequence of (u_n), still denoted (u_n), such that, setting $v_n(\cdot) = u_n(\cdot,0) - q(\cdot)$, there results $v_n(\cdot,0) \to 0$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot{v}_n \to 0$ weakly in $L^2(\mathbb{R})$.

We set $t_n = \sup\{t \in [0,1]\,/\,F(q+tv_n) \le c_{3,p}\}$, and we note that by continuity $F(q+t_nv_n) = c_{3,p}$ for any $n \in \mathbb{N}$.

We define the new sequence

$$\tilde{u}_n(x,y) = \begin{cases} u_n(x,y) & \text{if } y \le 0, \\ q_n(x) + (1-t_n-y)v_n(x) & \text{if } 0 \le y \le 1-t_n, \\ q_n(x) & i\text{if } y \ge 1-t_n. \end{cases}$$

Let us observe that $\tilde{u}_n \in \mathcal{M}_p$ and so that $\varphi_p(\tilde{u}_n) \ge m_p$. Then, since

$$\varphi_{p,(-\infty,0)}(u_n) = \varphi_p(\tilde{u}_n) - \int_0^{1-t_n} \frac{1}{2}\|v_n\|^2 + F(q_n(\cdot) + (1-t_n-y)v_n(\cdot)) - c_{3,p}\,\mathrm{d}y,$$

the Lemma follows once we prove that as $n \to \infty$ we have

$$\int_0^{1-t_n} \frac{1}{2}\|v_n\|^2 + F(q_n(\cdot) + (1 - t_n - y)v_n(\cdot)) - c_{3,p}\,\mathrm{d}y \to 0. \tag{4.2}$$

To this aim we observe that since $F(q+v_n) - F(q+t_nv_n) \to 0$, $v_n \to 0$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot v_n \to 0$ weakly in $L^2(\mathbb{R})$, we have that for any $T > 0$ there results

$$\frac{(1-t_n^2)}{2}\|\dot v_n\|^2 + \int_{|x|>T} a_\varepsilon(W(q+v_n) - W(q+t_nv_n))\,\mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Since q and $q+v_n$ belong to $\Gamma_{3,\bar p}$, by Lemma 2.5 there exists $T_0 > 0$ such that if $|x| > T_0$ then $|q(x)| \in [1-2\overline{\delta}, 1]$, $|q(x) + v_n(x)| \in [1 - 2\overline{\delta}, 1]$ and so also $|q(x) + t_nv_n(x)| \in [1 - 2\overline{\delta}, 1]$. By convexity of W around the points -1 and 1, we recover that for any $|x| \geq T_0$ we have $W(q(x) + t_nv_n(x))) \leq (1 - t_n)W(q) + t_nW(q + v_n)$ and so that for any $T > T_0$

$$\frac{1-t_n^2}{2}\|\dot v_n\|^2 + (1-t_n)\int_{|x|>T} a_\varepsilon W(q+v_n)\,\mathrm{d}x \leq (1-t_n)\int_{|x|\geq T} a_\varepsilon W(q)\,\mathrm{d}x + o(1) \text{ as } n \to \infty.$$

Moreover, by (1.9), we obtain that for any $|x| \geq T_0$ there results $a_\varepsilon W(q + v_n) \geq \underline{a}\,\underline{b}\chi(q + v_n)^2 = \underline{a}\,\underline{b}(\chi(q) - \mathrm{sgn}(x)v_n)^2$ and so we recover that for any $T > T_0$

$$\frac{1-t_n^2}{2}\|\dot v_n\|^2 + (1-t_n)\underline{a}\,\underline{b}\int_{|x|>T} (\chi(q) - \mathrm{sgn}(x)v_n)^2\,\mathrm{d}x \leq (1-t_n)\int_{|x|\geq T} a_\varepsilon W(q)\,\mathrm{d}x + o(1) \text{ as } n \to \infty.$$

Then, since $\int_{\mathbb{R}} a_\varepsilon W(q)\,\mathrm{d}x < +\infty$, $\int_{\mathbb{R}} \chi(q)^2\,\mathrm{d}x < +\infty$ and since $\|v_n\| \leq \mathsf{diam}(\Gamma_{3,\bar p}) \leq \mathsf{D}$ for any $n \in \mathbb{N}$, it is immediate to verify that for any $\eta > 0$ there exists $T_\eta > 0$ such that

$$\frac{(1-t_n^2)}{2}\|\dot v_n\|^2 + (1-t_n)\underline{a}\,\underline{b}\int_{|x|>T_\eta} v_n^2\,\mathrm{d}x \leq \eta + o(1) \text{ as } n \to \infty.$$

This last inequality, since $v_n \to 0$ in $L^\infty_{loc}(\mathbb{R})$, implies

$$(1-t_n)\|v_n\|^2_{H^1(\mathbb{R})} \to 0 \text{ as } n \to \infty. \tag{4.3}$$

By (4.3) it is simple to derive (4.2). Indeed $\int_0^{1-t_n} \frac{1}{2}\|v_n\|^2\,\mathrm{d}y = \frac{1}{2}(1-t_n)\|v_n\| \to 0$ directly by (4.3). Moreover, using Lemma 2.13, since $F(q + t_nv_n) = c_{3,p}$, it is not difficult to obtain that there exists $C > 0$ such that

$$F(q_n + (1-t_n-y)v_n) - c_{3,p} = F(q + (1-y)v_n) - F(q + t_nv_n) \leq C(1-t_n-y)\|v_n\|_{H^1(\mathbb{R})}$$

for any $n \in \mathbb{N}$ and $y \in [0, 1-t_n]$. Then, by (4.3), $\int_0^{1-t_n} F(q_n + (1-t_n-y)v_n) - c_{3,p}\,\mathrm{d}y \to 0$ and the lemma follows. $\square$

Lemma 4.4. *If $u \in \mathcal{H}$, $p \in \mathcal{P}$ and $y_0 \in \mathbb{R}$ are such that:*

i) $\mathsf{d}(u(\cdot, y), \mathcal{K}_{3,p}) \to 0$ *as* $y \to -\infty$ *and* $u(\cdot, y_0) \in \Gamma_{3,\bar{p}}$ *for a* $\bar{p} \neq p$*;*

ii) $F(u(\cdot, y)) \geq c_{3,p}$ *for all* $y < y_0$ *and* $\liminf_{y \to y_0^-} F(u(\cdot, y)) = c_{3,p}$,

then $\varphi_{p,(-\infty,y_0)}(u) \geq m_p$.

Proof. Let $y_n \to y_0^-$ be such that $u(\cdot, y_n) \in \Gamma_{3,\bar{p}}$ for any $n \in \mathbb{N}$ and $F(u(\cdot, y_n)) \to c_{3,p}$. Setting $v_n(\cdot) = u(\cdot, y_0) - u(\cdot, y_n)$ let moreover $t_n \in [0, 1]$ be such that $F(u(\cdot, y_n) + t_n v_n) = c_{3,p}$ and $F(u(\cdot, y_n) + t v_n) > c_{3,p}$ for any $t \in [0, t_n)$. Then, using (3.1), it is not difficult to recognize that for any $n \in \mathbb{N}$ the function

$$u_n(x, y) = \begin{cases} u(x, y + y_n) & \text{if } y < 0 \\ u(x, y_n) + y v_n & \text{if } 0 \leq y < t_n \\ u(x, y_n) + t_n v_n & \text{if } y \geq t_n \end{cases}$$

belongs to $\mathcal{M}_p$, that $F(u_n(\cdot, 0)) \to c_{3,p}$ and that $u_n(\cdot, 0) \in \Gamma_{3,\bar{p}}$. By Lemma 4.3 we obtain that

$$m_p \leq \liminf_{n\to\infty} \varphi_{p,(-\infty,0)}(u_n) = \liminf_{n\to\infty} \varphi_{p,(-\infty,y_n)}(u) = \varphi_{p,(-\infty,y_0)}(u),$$

and the lemma follows. □

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