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BOUNDARY FEEDBACK STABILIZATION OF A THREE-LAYER SANDWICH BEAM: RIESZ BASIS APPROACH *

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Abstract. In this paper, we consider the boundary stabilization of a sandwich beam which consists of two outer stiff layers and a compliant middle layer. Using Riesz basis approach, we show that there is a sequence of generalized eigenfunctions, which forms a Riesz basis in the state space. As a consequence, the spectrum-determined growth condition as well as the exponential stability of the closed-loop system are concluded. Finally, the well-posedness and regularity in the sense of Salamon-Weiss class as well as the exact controllability are also addressed.

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1. Introduction and model formulation

The vibration suppression of elastic structures has been studied extensively in many years due to its wide range of applications. One widely used technique is to make use of laminated members such as beams which consist of a compliant middle layer sandwiched between two stiff layers. The advantage of such a structure is to make the compliant layer create relatively large shear deformation to promote the dissipation of the vibrational energy of the system. In [5, 8], several constrained three-layer sandwich beam models are developed based on the assumptions that the middle layer resists shear but no bending, and the thickness is assumed to be sufficiently small so that the mass may be neglected or included in the outer layers. The out layers are the usual Euler-Bernoulli beams. For the damped model, it was indicated in [5] that the system can have an analytic semigroup solution, and the optimal damping parameter is also derived in terms of the material parameters of the structure. When the damping is included in the middle layer so that the shear motions are resisted by a force proportional to the rate of shear, by the multiplier method, it is shown in [9], under some natural boundary conditions, that the system associates with an analytic semigroup and the vibrational energy is exponentially decay.

Keywords and phrases. Riesz basis, sandwich beam, exponential stability, exact controllability.

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In this paper, we shall focus on the following cantilevered laminated beam

$$\begin{cases} mw_{tt}(x,t) + Aw_{xxxx}(x,t) - B\gamma s_x(x,t) = 0, \ 0 < x < 1, \ t > 0, \\ C\gamma s(x,t) - s_{xx}(x,t) + Bw_{xxx}(x,t) = 0, \qquad 0 < x < 1, \ t > 0 \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} w(0,t) = 0, \ w_x(0,t) = 0, \ s(0,t) = 0, \\ w_{xx}(1,t) = 0, \ s_x(1,t) = 0, \ Aw_{xxx}(1,t) - B\gamma s(1,t) = u(t) \end{cases}$$

where $w(x,t)$ stands for the transverse displacement at time t and longitudinal spatial variable x and $s(x,t)$ is the proportion to the shear in the middle layer. The constant $m > 0$ is the density of the beam, $A, B, C > 0$ the stiff constants, $\gamma > 0$ the stiffness of the middle layer, and $u(t) \in L^2_{loc}(0,\infty)$ is the boundary damping control force. The initial conditions prescribed for the system are

$$w(x,0) = w_0(x), \ w_t(x,0) = w_1(x). \tag{1.2}$$

The conservative model ($u = 0$) of (1.1) is developed in [5], and is shown that the system admits a C_0-semigroup solution. However, in this model, the vibrational energy is a constant. To suppress the vibration, a control must be present in the system. In present paper, the control is imposed at the boundary $x = 1$ due to its easy implementation in engineering practice. For mathematical modelling process and the other physical background of the system, we refer to [5] for more details.

Suppose the output of the system (1.1) is $y(t) = w_t(1,t)$. We propose the boundary output feedback control $u(t) = ky(t)$ where k is the positive constant feedback gain. Then the boundary conditions of (1.1) become

$$\begin{cases} w(0,t) = 0, \ w_x(0,t) = 0, \ s(0,t) = 0, \\ w_{xx}(1,t) = 0, \ s_x(1,t) = 0, \ Aw_{xxx}(1,t) - B\gamma s(1,t) = kw_t(1,t). \end{cases} \tag{1.3}$$

Let us introduce a second order differential operator $\mathcal{T}$ by ([5])

$$\begin{cases} \mathcal{T}\varphi = \varphi'', \\ \mathcal{D}(\mathcal{T}) = \{\varphi \in H^2(0,1) \mid \varphi(0) = \varphi'(1) = 0\}. \end{cases} \tag{1.4}$$

One can check that $\mathcal{T}$ is densely defined and negative definite in $L^2(0,1)$. Set $\alpha := C\gamma > 0$. Then, it is easily verified that $(\alpha - \mathcal{T})^{-1}$ exists and is compact on $L^2(0,1)$. Now, let

$$\mathcal{J} = -I + \alpha(\alpha - \mathcal{T})^{-1} \tag{1.5}$$

where I is the identity operator on $L^2(0,1)$. Obviously, $\mathcal{J}$ is a non-positive bounded operator on $L^2(0,1)$ and

$$\mathcal{J}\varphi = (\alpha - \mathcal{T})^{-1}\mathcal{T}\varphi, \quad \forall \varphi \in D(\mathcal{T}).$$

With the operator $\mathcal{J}$ at hand, the closed-loop laminated beam system (1.1)–(1.3) can be rewritten as $s(x,t) = -B(\alpha - \mathcal{T})^{-1}w_{xxx}(x,t)$ with w satisfying

$$\begin{cases} mw_{tt}(x,t) + Aw_{xxxx}(x,t) + B^2\gamma(\mathcal{J}w_x)_x(x,t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ Aw_{xxx}(1,t) + B^2\gamma\mathcal{J}w_x(1,t) = kw_t(1,t). \end{cases} \tag{1.6}$$

The total energy of the system (1.6) is given by

$$E(t) = \frac{1}{2}\int_0^1 mw_t^2(x,t) + Aw_{xx}^2(x,t) - [B^2\gamma\mathcal{J}w_x(x,t)]w_x(x,t)\mathrm{d}x. \tag{1.7}$$

One of the purposes of present paper is to show that the energy decay rate of the system (1.6) is determined by its spectrum. To do this, we need an asymptotic behavior of the eigenpairs. However, compared with the single beam equations studied before (*e.g.* [6]), a big obstacle in the computation of the eigenvalue for the system (1.6) consists of solving of a system of ordinary differential equations (see (2.7)). In order to get the eigenvalue distribution, there are two ways: one way is to substitute one equation into another in system (2.7) which will make the computation much more complicated, and another way is to treat it as a matrix operator pencil motivated by the works in [15, 16]. In this paper, the second approach is adopted in investigation.

The main contribution of this paper are: a) to show that a set of generalized eigenfunctions of the closed loop system (1.6) forms a Riesz basis for the state space; b) to get the spectrum-determined growth condition, a hard problem in infinite-dimensional systems; c) to obtain the exponential stability of the system; d) to conclude the exact controllability and observability of the system. For the last point, many papers contribute to the exact controllability by nonharmonic analysis approach, see *e.g.* [1, 13, 14], name just a few.

Now let us briefly outline the content of this paper. In the next section, the well-posed of the system will be established. Asymptotic expansion of the eigenfrequencies will be given in Section 3. Section 4 is devoted to the asymptotic expansion of the corresponding eigenfunctions. In Section 5, we obtain a more profound result, namely, the existence of a sequence of generalized eigenfunctions, which forms a Riesz basis for the state space. Consequently, the spectrum-determined growth condition and the exponential stability are concluded. Finally, as a consequence of the asymptotic expansion of eigenpairs and exponential stability of the system, we conclude the exact controllability and observability in the last section.

2. Well-posedness of the system

We begin by formulating the problem (1.6) on the energy state Hilbert space $\mathcal{H}$:

$$\mathcal{H} := H_w^2(0,1)\times L^2(0,1), \qquad H_w^2(0,1) := \{\varphi\in H^2(0,1)\mid \varphi(0)=\varphi'(0)=0\} \tag{2.1}$$

where and henceforth the primes above symbols representing functions denote differentiation with respect to spatial variable x. Due to energy function (1.7), it is natural to define the following inner product induced norm $\|\cdot\|$ on $\mathcal{H}$ as

$$\|(w,z)\|^2 := \int_0^1 m|z(x)|^2 + A|w''(x)|^2 - B^2\gamma(\mathcal{J}w'(x))\overline{w'(x)}\mathrm{d}x,\ \forall(w,z)\in\mathcal{H} \tag{2.2}$$

which makes sense because $\mathcal{J}$ is negative on $L^2(0,1)$. Next, define a linear operator $\mathcal{A}:\mathcal{D}(\mathcal{A})(\subset\mathcal{H})\to\mathcal{H}$ by

$$\mathcal{A}(w,z) := \left(z, -\frac{1}{m}[Aw''' + B^2\gamma(\mathcal{J}w')]'\right),\ \forall(w,z)\in\mathcal{D}(\mathcal{A}) \tag{2.3}$$

where

$$\mathcal{D}(\mathcal{A}) := \left\{(w,z)\in\mathcal{H}\ \middle|\ \begin{array}{l} w'\in\mathcal{D}(\mathcal{T}),\ z\in H_w^2(0,1),\ Aw'''+B^2\gamma(\mathcal{J}w')\in H^1(0,1),\\ w''(1)=0,\ Aw'''(1)+B^2\gamma(\mathcal{J}w')(1)=kz(1)\end{array}\right\}. \tag{2.4}$$

Set $Y(t) := (w(\cdot,t), w_t(\cdot,t))$. Then the system (1.6) can be written as an evolution equation in $\mathcal{H}$:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}Y(t) = \mathcal{A}Y(t),\ t>0,\\ Y(0) := (w(\cdot,0),\ w_t(\cdot,0)).\end{cases} \tag{2.5}$$

Theorem 2.1. *Let $\mathcal{A}$ be the operator defined by (2.3) and (2.4). Then $\mathcal{A}$ is dissipative in $\mathcal{H}$. In addition, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$. Therefore, $\mathcal{A}$ generates a C_0-semigroup of contractions $\mathrm{e}^{\mathcal{A}t}$ on $\mathcal{H}$ and the spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues only.*

Proof. Since $\mathcal{J}$ is negative on L^2 and

$$\begin{aligned}
\langle \mathcal{A}(w,z),(w,z)\rangle_{\mathcal{H}} &= \left\langle \left(z, -\frac{1}{m}\left[Aw''' + B^2\gamma(\mathcal{J}w')\right]'\right), (w,z)\right\rangle_{\mathcal{H}} \\
&= -\left\langle \left[Aw''' + B^2\gamma(\mathcal{J}w')\right]',\ z\right\rangle_{L^2} + A\left\langle z'',\ w''\right\rangle_{L^2} - B^2\gamma\left\langle \mathcal{J}z', w'\right\rangle_{L^2} \\
&= -\left[Aw'''(x) + B^2\gamma(\mathcal{J}w')(x)\right]\overline{z(x)}\Big|_0^1 + Aw''(x)\overline{z'(x)}\Big|_0^1 \\
&\quad - \left\langle Aw'',\ z''\right\rangle_{L^2} + B^2\gamma\left\langle \mathcal{J}w',\ z'\right\rangle_{L^2} + A\left\langle z'',\ w''\right\rangle_{L^2} - B^2\gamma\left\langle \mathcal{J}z', w'\right\rangle_{L^2} \\
&= -k|z(1)|^2 - \left\langle Aw'',\ z''\right\rangle_{L^2} + B^2\gamma\left\langle \mathcal{J}w',\ z'\right\rangle_{L^2} + A\left\langle z'',\ w''\right\rangle_{L^2} - B^2\gamma\left\langle \mathcal{J}z', w'\right\rangle_{L^2},
\end{aligned}$$

it follows that

$$\mathrm{Re}\langle \mathcal{A}(w,z),(w,z)\rangle_{\mathcal{H}} = -k|z(1)|^2 \le 0.$$

Hence $\mathcal{A}$ is dissipative. We accomplish the proof by showing that $0 \in \rho(\mathcal{A})$ because from Theorem 4.6 of [12], if $\mathcal{A}^{-1}$ exists, $\mathcal{A}$ must be densely defined in $\mathcal{H}$. Therefore, the Lumer-Phillips theorem can be applied to conclude that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

To do so, given $G := (g_1, g_2) \in \mathcal{H}$, we seek $F := (f_1, f_2) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}F = G,$$

from which we obtain that $f_2(x) = g_1(x)$ with f_1 satisfying

$$\left[Af_1''' + B^2\gamma(\mathcal{J}f_1')\right]'(x) = -mg_2(x). \tag{2.6}$$

The above is equivalent to

$$[A + B^2\gamma(\alpha - \mathcal{T})^{-1}]f_1'''(x) = -m_2\int_1^x g_2(\tau)\mathrm{d}\tau + kg_1(1) =: \phi(x) \in L^2(0,1).$$

So,

$$f_1'''(x) = [A^{-1} - B^2\gamma(\alpha + A^{-1}B^2\gamma - \mathcal{T})^{-1}]\phi(x) := \psi(x) \in L^2(0,1).$$

Solve the above equation, to obtain

$$f_1(x) = \int_1^x \frac{(x-\tau)^2}{2}\psi(\tau)\mathrm{d}\tau - x\int_0^1 \tau\psi(\tau)\mathrm{d}\tau + \int_0^1 \frac{\tau^2}{2}\psi(\tau)\mathrm{d}\tau.$$

Therefore, there is a unique solution $f_1(x)$ to (2.6), which in return implies that $\mathcal{A}^{-1}$ exists. Finally, by the Sobolev embedding theorem, we see from above expression that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$ and hence the spectrum $\sigma(\mathcal{A})$ consists of isolated eigenvalues only [10]. □

Let us formulate the eigenvalue problem for $\mathcal{A}$. If $\lambda \in \sigma(\mathcal{A})$ and $Y_\lambda := (w,z)$ is a corresponding eigenfunction, then it is routine to verify that $z = \lambda w$ and w satisfies the following characteristic equation with supplement variable s:

$$\begin{cases} m\lambda^2 w(x) + Aw^{(4)}(x) - B\gamma s'(x) = 0, \\ C\gamma s(x) - s''(x) + Bw'''(x) = 0, \\ w(0) = 0,\ w'(0) = 0,\ s(0) = 0, \\ w''(1) = 0,\ s'(1) = 0, \\ Aw'''(1) - B\gamma s(1) = k\lambda w(1). \end{cases} \tag{2.7}$$

For brevity in notation, from now on, we set

$$r_1 := \sqrt[4]{\frac{m}{A}},\ \ d_1 := \frac{B}{A}\gamma,\ \ d_2 := B,\ \ d_3^2 := C\gamma,\ \ \tilde{k} := \frac{k}{A}\cdot \tag{2.8}$$

(2.7) then becomes

$$\begin{cases} r_1^4\lambda^2 w(x) + w^{(4)}(x) - d_1 s'(x) = 0, \\ s''(x) - d_2 w'''(x) - d_3^2 s(x) = 0, \\ w(0) = 0,\ w'(0) = 0,\ s(0) = 0, \\ w''(1) = 0,\ s'(1) = 0, \\ w'''(1) - d_1 s(1) = \tilde{k}\lambda w(1). \end{cases} \tag{2.9}$$

Clearly, (2.9) is a system of two ordinary differential equations. In order to solve this equation, one natural way is to solve s from the second equation in (2.9) and substitute it into the first one. However, this makes the problem quite complicated. To overcome this difficulty, we shall use the matrix operator pencil method. Let

$$w_1 := w,\ w_2 := w',\ w_3 := w'',\ w_4 := w''',\ s_1 := s,\ s_2 := s' \tag{2.10}$$

and

$$\Phi := [w_1,\ w_2,\ w_3,\ w_4,\ s_1,\ s_2]^\top \tag{2.11}$$

where the superscript "$\top$" stands for the transpose; then (2.9) becomes

$$\begin{cases} T^D(x,\lambda)\Phi(x) := \Phi'(x) + M(\lambda)\Phi(x) = 0, \\ T^R(\lambda)\Phi := W^0(\lambda)\Phi(0) + W^1(\lambda)\Phi(1) = 0 \end{cases} \tag{2.12}$$

where

$$W^0(\lambda) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ W^1(\lambda) := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\tilde{k}\lambda & 0 & 0 & 1 & -d_1 & 0 \end{bmatrix} \tag{2.13}$$

and

$$M(\lambda) := M_0 + \lambda^2 M_2, \tag{2.14}$$

$$M_0 := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d_1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -d_2 & -d_3^2 & 0 \end{bmatrix},\ M_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ r_1^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.15}$$

The following result is a direct consequence of the above arguments.

Theorem 2.2. $\lambda \in \sigma(\mathcal{A})$ *if and only if (2.12) possesses a nonzero solution* $\Phi(x)$.

3. Asymptotic behavior of eigenfrequencies

In this section, we seek an asymptotic expansion of the eigenvalues of $\mathcal{A}$. This would be accomplished by expanding the characteristic determinant with asymptotic expression of the fundamental matrix solution of (2.12). The technique used here is the modified standard one due to Birkhoff and Langer [2] (see also [15] or [16]). A key step is to finding an invertible matrix transformation which is very powerful and useful in solving coupled PDE problems.

Due to Theorem 2.1 and the fact that the eigenvalues are symmetric about the real axis, we consider only those λ which are located in the second quadrant of the complex plane:

$$\lambda := i\rho^2,\ \rho \in \mathcal{S} := \left\{\rho \in \mathbb{C} \mid 0 \le \arg\rho \le \frac{\pi}{4}\right\}.$$

Note that for any $\rho \in \mathcal{S}$, we have

$$\mathrm{Re}(-\rho) \le \mathrm{Re}(i\rho) \le \mathrm{Re}(-i\rho) \le \mathrm{Re}(\rho),$$

and

$$\begin{cases} \mathrm{Re}(-\rho) = -|\rho|\cos(\arg\rho) \le -\dfrac{\sqrt{2}}{2}|\rho| < 0, \\ \mathrm{Re}(i\rho) = -|\rho|\sin(\arg\rho) \le 0. \end{cases}$$

As we have mentioned in the beginning of this section, a key step to solving the eigenvalue problem (2.12) is to find an invertible matrix in $\rho \in \mathcal{S}$ of the following:

$$P(\rho) := \begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0 \\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0 \\ r_1^3\rho^3 & r_1^3\rho^3 & -r_1^3\rho^3 & -r_1^3\rho^3 & 0 & 0 \\ r_1^4\rho^4 & -r_1^4\rho^4 & -ir_1^4\rho^4 & ir_1^4\rho^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^3 & 0 \\ d_2r_1^3\rho^3 & d_2r_1^3\rho^3 & -d_2r_1^3\rho^3 & -d_2r_1^3\rho^3 & 0 & \rho^3 \end{bmatrix}. \tag{3.1}$$

So the matrix $P(\rho)$ is a polynomial of degree 4 in ρ. Such a trick of finding the matrix $P(\rho)$ is inspired by [16]. Now for any $\rho \neq 0$, a direct computation shows that

$$P(\rho)^{-1} := \begin{bmatrix} \frac{1}{4r_1\rho} & \frac{1}{4r_1^2\rho^2} & \frac{1}{4r_1^3\rho^3} & \frac{1}{4r_1^4\rho^4} & 0 & 0 \\ \frac{1}{4r_1\rho} & -\frac{1}{4r_1^2\rho^2} & \frac{1}{4r_1^3\rho^3} & -\frac{1}{4r_1^4\rho^4} & 0 & 0 \\ \frac{1}{4r_1\rho} & -i\frac{1}{4r_1^2\rho^2} & -\frac{1}{4r_1^3\rho^3} & i\frac{1}{4r_1^4\rho^4} & 0 & 0 \\ \frac{1}{4r_1\rho} & i\frac{1}{4r_1^2\rho^2} & -\frac{1}{4r_1^3\rho^3} & -i\frac{1}{4r_1^4\rho^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^{-3} & 0 \\ 0 & 0 & -d_2\rho^{-3} & 0 & 0 & \rho^{-3} \end{bmatrix}. \tag{3.2}$$

Define

$$\Psi(x) := P^{-1}(\rho)\Phi(x) \tag{3.3}$$

and $\widehat{T}^D(x,\rho) := P(\rho)^{-1}T^D(x,i\rho^2)P(\rho)$. Then we have

$$\widehat{T}^D(x,\rho)\Psi(x) = \Psi'(x) - \widehat{M}(\rho)\Psi(x) = 0 \tag{3.4}$$

where

$$\widehat{M}(\rho) := -P(\rho)^{-1}M(i\rho^2)P(\rho) = -P(\rho)^{-1}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -r_1^4\rho^4 & 0 & 0 & 0 & 0 & -d_1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -d_2 & -d_3^2 & 0 \end{bmatrix} P(\rho)$$

$$= -\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4r_1\rho} & -\frac{1}{4r_1^2\rho^2} & -\frac{1}{4r_1^3\rho^3} & 0 & -\frac{d_1}{4r_1^4\rho^4} \\ \frac{1}{4} & -\frac{1}{4r_1\rho} & \frac{1}{4r_1^2\rho^2} & -\frac{1}{4r_1^3\rho^3} & 0 & \frac{d_1}{4r_1^4\rho^4} \\ -i\frac{1}{4} & -\frac{1}{4r_1\rho} & i\frac{1}{4r_1^2\rho^2} & \frac{1}{4r_1^3\rho^3} & 0 & -i\frac{d_1}{4r_1^4\rho^4} \\ i\frac{1}{4} & -\frac{1}{4r_1\rho} & -i\frac{1}{4r_1^2\rho^2} & \frac{1}{4r_1^3\rho^3} & 0 & i\frac{d_1}{4r_1^4\rho^4} \\ 0 & 0 & 0 & 0 & 0 & -\rho^{-3} \\ 0 & 0 & 0 & 0 & -d_3^2\rho^{-3} & 0 \end{bmatrix} P(\rho)$$

$$= -\begin{bmatrix} -r_1\rho - \frac{d_1d_2}{4r_1\rho} & -\frac{d_1d_2}{4r_1\rho} & \frac{d_1d_2}{4r_1\rho} & \frac{d_1d_2}{4r_1\rho} & 0 & -\frac{d_1}{4r_1^4\rho} \\ \frac{d_1d_2}{4r_1\rho} & r_1\rho + \frac{d_1d_2}{4r_1\rho} & -\frac{d_1d_2}{4r_1\rho} & -\frac{d_1d_2}{4r_1\rho} & 0 & \frac{d_1}{4r_1^4\rho} \\ -i\frac{d_1d_2}{4r_1\rho} & -i\frac{d_1d_2}{4r_1\rho} & -ir_1\rho + i\frac{d_1d_2}{4r_1\rho} & i\frac{d_1d_2}{4r_1\rho} & 0 & -i\frac{d_1}{4r_1^4\rho} \\ i\frac{d_1d_2}{4r_1\rho} & i\frac{d_1d_2}{4r_1\rho} & -i\frac{d_1d_2}{4r_1\rho} & ir_1\rho - i\frac{d_1d_2}{4r_1\rho} & 0 & i\frac{d_1}{4r_1^4\rho} \\ -d_2r_1^3 & -d_2r_1^3 & d_2r_1^3 & d_2r_1^3 & 0 & -1 \\ 0 & 0 & 0 & 0 & -d_3^2 & 0 \end{bmatrix}.$$

It is seen from above that $\widehat{M}(\rho)$ can be written as

$$\widehat{M}(\rho) := \rho\widehat{M}_1 + \widehat{M}_0 + \rho^{-1}\widehat{M}_{-1} \tag{3.5}$$

where

$$\widehat{M}_1 := \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ir_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ir_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \widehat{M}_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d_2r_1^3 & d_2r_1^3 & -d_2r_1^3 & -d_2r_1^3 & 0 & 1 \\ 0 & 0 & 0 & 0 & d_3^2 & 0 \end{bmatrix} \tag{3.6}$$

and

$$\widehat{M}_{-1} := \begin{bmatrix} \frac{d_1d_2}{4r_1} & \frac{d_1d_2}{4r_1} & -\frac{d_1d_2}{4r_1} & -\frac{d_1d_2}{4r_1} & 0 & \frac{d_1}{4r_1^4} \\ -\frac{d_1d_2}{4r_1} & -\frac{d_1d_2}{4r_1} & \frac{d_1d_2}{4r_1} & \frac{d_1d_2}{4r_1} & 0 & -\frac{d_1}{4r_1^4} \\ i\frac{d_1d_2}{4r_1} & i\frac{d_1d_2}{4r_1} & -i\frac{d_1d_2}{4r_1} & -i\frac{d_1d_2}{4r_1} & 0 & i\frac{d_1}{4r_1^4} \\ -i\frac{d_1d_2}{4r_1} & -i\frac{d_1d_2}{4r_1} & i\frac{d_1d_2}{4r_1} & i\frac{d_1d_2}{4r_1} & 0 & -i\frac{d_1}{4r_1^4} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.7}$$

Based on these transformations, we are now in a position to find an asymptotic expansion for the fundamental matrix solution of the system (3.4) with respect to $\rho \in \mathcal{S}$.

Theorem 3.1. *Let $0 \neq \rho \in \mathcal{S}$, and let $\widehat{M}(\rho)$ be given by (3.5). For $x \in [0,1]$, set*

$$E(x,\rho) := \begin{bmatrix} e^{r_1\rho x} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-r_1\rho x} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{ir_1\rho x} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-ir_1\rho x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.8}$$

Then there exists a fundamental matrix solution $\widehat{\Psi}(x,\rho)$ for system (3.4), and for large enough $|\rho|$,

$$\widehat{\Psi}(x,\rho) := \left(\widehat{\Psi}_0(x) + \frac{\widehat{\Psi}_1(x)}{\rho} + \frac{\widehat{\Psi}_2(x)}{\rho^2} + \frac{\widetilde{\Theta}(x,\rho)}{\rho^3}\right) E(\cdot,\rho) \tag{3.9}$$

where

$$\widehat{\Psi}_0(x) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{d_3x} & e^{-d_3x} \\ 0 & 0 & 0 & 0 & d_3e^{d_3x} & -d_3e^{-d_3x} \end{bmatrix}, \tag{3.10}$$

$$\widehat{\Psi}_1(x) := \begin{bmatrix} \frac{d_1d_2}{4r_1}x & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d_1d_2}{4r_1}x & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\frac{d_1d_2}{4r_1}x & 0 & 0 & 0 \\ 0 & 0 & 0 & i\frac{d_1d_2}{4r_1}x & 0 & 0 \\ d_2r_1^2 & -d_2r_1^2 & id_2r_1^2 & -id_2r_1^2 & e^{d_3x} & e^{-d_3x} \\ 0 & 0 & 0 & 0 & d_3e^{d_3x} & -d_3e^{-d_3x} \end{bmatrix}, \tag{3.11}$$

and all $\widehat{\Psi}_2, \widetilde{\Theta}(x,\rho)$ and $\widetilde{\Theta}_x(x,\rho)$ are uniformly bounded in $x \in [0,1]$ for all sufficiently larger $|\rho|$.

Proof. By (3.4) and (3.5), the Assumption 2.1 of [15] on page 134 is satisfied and hence Theorem 2.2 of [15] on p. 135 can be directly applied (see also [2]) to our problem, that is to say, a fundamental matrix solution of (3.4) is of the following form

$$\widehat{\Psi}(x,\rho)=\left(\widehat{\Psi}_0(x)+\rho^{-1}\widehat{\Psi}_1(x)+\rho^{-2}\widehat{\Psi}_2(x)+\rho^{-3}\widetilde{\Theta}(x,\rho)\right)E(x,\rho).$$

Since $\widehat{M_1}$ given by (3.6) is a diagonal matrix, it follows that $E(x,\rho)$ given by (3.8) is a fundamental matrix solution to the equation (3.4) involving only the leading order terms, in other words,

$$E'(x,\rho)=\rho\widehat{M_1}E(x,\rho).$$

Next, computing $\widehat{\Psi}'(x,\rho)$ and $\widehat{M}(\rho)\widehat{\Psi}(x,\rho)$ yields

$$\begin{aligned}\widehat{\Psi}'(x,\rho) &= \left(\widehat{\Psi}_0'(x)+\rho^{-1}\widehat{\Psi}_1'(x)+\rho^{-2}\widehat{\Psi}_2'(x)+\rho^{-3}\widetilde{\Theta}_x(x,\rho)\right)E(x,\rho)\\ &\quad+\rho\left(\widehat{\Psi}_0(x)+\rho^{-1}\widehat{\Psi}_1(x)+\rho^{-2}\widehat{\Psi}_2(x)+\rho^{-3}\widetilde{\Theta}(x,\rho)\right)\widehat{M_1}E(x,\rho)\end{aligned}$$

and

$$\widehat{M}(\rho)\widehat{\Psi}(x,\rho)=\left(\rho\widehat{M_1}+\widehat{M_0}+\rho^{-1}\widehat{M_{-1}}\right)\left(\widehat{\Psi}_0(x)+\rho^{-1}\widehat{\Psi}_1(x)+\rho^{-2}\widehat{\Psi}_2(x)+\rho^{-3}\widetilde{\Theta}(x,\rho)\right)E(x,\rho).$$

Inserting the last two equations in (3.4) and equating the corresponding coefficients of $\rho^i, i=1,0,-1$, it follows

$$\widehat{\Psi}_0(x)\widehat{M_1}-\widehat{M_1}\widehat{\Psi}_0(x)=0, \tag{3.12}$$

$$\widehat{\Psi}_0'(x)-\widehat{M_0}\widehat{\Psi}_0(x)+\widehat{\Psi}_1(x)\widehat{M_1}-\widehat{M_1}\widehat{\Psi}_1(x)=0, \tag{3.13}$$

$$\widehat{\Psi}_1'(x)-\widehat{M_0}\widehat{\Psi}_1(x)-\widehat{M_{-1}}\widehat{\Psi}_0(x)+\widehat{\Psi}_2(x)\widehat{M_1}-\widehat{M_1}\widehat{\Psi}_2(x)=0. \tag{3.14}$$

It remains to show that the leading order term $\widehat{\Psi}_0(\cdot)$ and the second order term $\widehat{\Psi}_1(\cdot)$ are given by (3.10) and (3.11), respectively. Let us denote $c_{ij}^{[s]}(x)$ the (i,j)-entry of the matrix $\widehat{\Psi}_s(x)$ with $i,j=1,2,\ldots,6;\ s=0,1$. Since $\widehat{M_1}$ is diagonal, it follows from (3.12) that the entries $c_{ij}^{[0]}(x)$ of the matrix function $\widehat{\Psi}_0$ satisfy

$$\begin{cases} c_{ij}^{[0]}(x)=0, & \text{if} \quad i\neq j,\ 1\le i,j\le 4,\\ c_{ij}^{[0]}(x)=0, & \text{if} \quad i=5,6,\ 1\le j\le 4,\\ c_{ij}^{[0]}(x)=0, & \text{if} \quad j=5,6,\ 1\le i\le 4,\end{cases}$$

and the entries $c_{ii}^{[0]}(x)$ $(i=1,2,\ldots,6)$, $c_{56}^{[0]}(x)$ and $c_{65}^{[0]}(x)$ can be found through (3.13) that

$$\begin{cases} {c_{ii}^{[0]}}'(x)=0, \quad i=1,2,3,4,\\ {c_{55}^{[0]}}'(x)=c_{65}^{[0]}(x), \quad {c_{56}^{[0]}}'(x)=c_{66}^{[0]}(x),\\ {c_{65}^{[0]}}'(x)=d_3^2c_{55}^{[0]}(x), \quad {c_{66}^{[0]}}'(x)=d_3^2c_{56}^{[0]}(x).\end{cases} \tag{3.15}$$

Hence (3.10) follows in terms of the initial date $\widehat{\Psi}_0(0) = I$. Similarly, all entries of $\widehat{\Psi}_1(x)$, namely,

$$\begin{cases} c_{ij}^{[1]}(x) = 0, \quad 1 \le i, j \le 4,\ i \ne j, \\ c_{ij}^{[1]}(x) = 0, \quad 1 \le i \le 4,\ j = 5, 6, \\ c_{ij}^{[1]}(x) = 0, \quad 1 \le j \le 4,\ i = 6, \\ c_{51}^{[1]}(x) = d_2 r_1^2,\ c_{52}^{[1]}(x) = -d_2 r_1^2,\ c_{53}^{[1]}(x) = i d_2 r_1^2,\ c_{54}^{[1]}(x) = -i d_2 r_1^2 \end{cases}$$

can be obtained from (3.13) except $c_{ii}^{[1]}(x)$ $(i = 1, 2, \ldots, 6)$, $c_{56}^{[1]}(x)$ and $c_{65}^{[1]}(x)$, which, in turn, can be found from (3.14). These lead us to

$$\begin{cases} c_{11}^{[1]\prime}(x) = \dfrac{d_1 d_2}{4r_1},\ \ c_{22}^{[1]\prime}(x) = -\dfrac{d_1 d_2}{4r_1},\ \ c_{33}^{[1]\prime}(x) = -i\dfrac{d_1 d_2}{4r_1},\ \ c_{44}^{[1]\prime}(x) = i\dfrac{d_1 d_2}{4r_1}, \\ c_{55}^{[1]\prime}(x) = c_{65}^{[1]}(x),\ \ c_{65}^{[1]\prime}(x) = d_3^2 c_{55}^{[1]}(x),\ \ c_{56}^{[1]\prime}(x) = c_{66}^{[1]}(x),\ \ c_{66}^{[1]\prime}(x) = d_3^2 c_{56}^{[1]}(x). \end{cases}$$

Thus (3.11) is concluded. The proof is complete. □

By virtue of transformation (3.3), we have immediately the following result which shows the relationship between (2.12) and (3.4).

Corollary 3.1. *Let $0 \ne \rho \in \mathcal{S}$, and let $\widehat{\Psi}(x, \rho)$ given by (3.9) be a fundamental matrix solution to the system (3.4). Then*

$$\widehat{\Phi}(x, \rho) := P(\rho)\widehat{\Psi}(x, \rho) \tag{3.16}$$

is a fundamental matrix solution to the first equation of (2.12) with respect to x.

We are now ready to estimate asymptotically the distribution of eigenvalues of $\mathcal{A}$ in the sector $\mathcal{S}$. From (2.12), $\lambda = i\rho^2 \in \sigma(\mathcal{A})$ if and only if it is a zero of the characteristic determinant $\Delta(\rho)$:

$$\Delta(\rho) := \det\big(T^R(i\rho^2)\widehat{\Phi}\big), \quad \rho \in \mathcal{S} \tag{3.17}$$

where the operator T^R is defined in (2.12) and $\widehat{\Phi}$ is the fundamental matrix solution given by (3.16) [15]. Since

$$T^R(i\rho^2)\widehat{\Phi} = W^0(i\rho^2)P(\rho)\widehat{\Psi}(0, \rho) + W^1(i\rho^2)P(\rho)\widehat{\Psi}(1, \rho), \tag{3.18}$$

it follows from (2.13) and (3.1) that

$$W^0(i\rho^2)P(\rho) = \begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0 \\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$W^1(i\rho^2)P(\rho)=\rho^3\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\r_1^3&r_1^3&-r_1^3&-r_1^3&0&0\\d_2r_1^3&d_2r_1^3&-d_2r_1^3&-d_2r_1^3&0&1\\r_1(r_1^3\rho-i\tilde{k})&-r_1(i\tilde{k}+r_1^3\rho)&-ir_1(\tilde{k}+r_1^3\rho)&ir_1(r_1^3\rho-\tilde{k})&-d_1&0\end{bmatrix}.$$

Once again for brevity in notations, we set

$$[a]_2:=a+\mathcal{O}(\rho^{-2}).$$

Since $E(0,\rho)=I$, a direct computation gives

$$W^0(i\rho^2)P(\rho)\widehat{\Psi}(0,\rho)=\begin{bmatrix}r_1\rho&r_1\rho&r_1\rho&r_1\rho&0&0\\r_1^2\rho^2&-r_1^2\rho^2&ir_1^2\rho^2&-ir_1^2\rho^2&0&0\\d_2r_1^2\rho^2&-d_2r_1^2\rho^2&id_2r_1^2\rho^2&-id_2r_1^2\rho^2&\rho^3(1+\rho^{-1})&\rho^3(1+\rho^{-1})\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}+\mathcal{O}(\rho^{-2})$$

and

$$W^1(i\rho^2)P(\rho)\widehat{\Psi}(1,\rho)=\rho^3\left[\begin{matrix}0&0&0\\0&0&0\\0&0&0\\r_1^{-1}(1+\frac{d_1d_2}{4r_1\rho})E_1&-r_1^{-1}(1-\frac{d_1d_2}{4r_1\rho})E_2&-ir_1^{-1}(1-i\frac{d_1d_2}{4r_1\rho})E_3\\r_1^{-1}d_2\left(1+\frac{d_1d_2}{4r_1\rho}\right)E_1&-r_1^{-1}d_2\left(1-\frac{d_1d_2}{4r_1\rho}\right)E_2&-ir_1^{-1}d_2\left(1-i\frac{d_1d_2}{4r_1\rho}\right)E_3\\\rho\left[1+\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2E_1&\rho\left[1-\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2E_2&\rho\left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2E_3\end{matrix}\right.$$

$$\left.\begin{matrix}0&0&0\\0&0&0\\0&0&0\\ir_1^{-1}\left(1+i\frac{d_1d_2}{4r_1\rho}\right)E_4&0&0\\id_2r_1^{-1}\left(1+i\frac{d_1d_2}{4r_1\rho}\right)E_4&d_3e^{d_3}(1+\rho^{-1})&-d_3e^{-d_3}(1+\rho^{-1})\\\rho\left[1+i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2E_4&-d_1e^{d_3}(1+\rho^{-1})&-d_1e^{-d_3}(1+\rho^{-1})\end{matrix}\right]+\mathcal{O}(\rho^{-2})$$

where

$$E_1:=r_1^4e^{r_1\rho},\ E_2:=-r_1^4e^{-r_1\rho},\ E_3:=-ir_1^4e^{ir_1\rho},\ E_4:=ir_1^4e^{-ir_1\rho}. \tag{3.19}$$

Thus

$$T^R(i\rho^2)\widehat{\Phi} = \begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0 \\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0 \\ d_2r_1^2\rho^2 & -d_2r_1^2\rho^2 & id_2r_1^2\rho^2 & -id_2r_1^2\rho^2 & \rho^3(1+\rho^{-1}) & \rho^3(1+\rho^{-1}) \\ r_1^{-1}\left(1+\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_1 & -r_1^{-1}\left(1-\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_2 & -ir_1^{-1}\left(1-i\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_3 & ir_1^{-1}\left(1+i\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_4 & 0 & 0 \\ r_1^{-1}d_2\left(1+\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_1 & -r_1^{-1}d_2\left(1-\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_2 & -ir_1^{-1}d_2\left(1-i\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_3 & id_2r_1^{-1}\left(1+i\frac{d_1d_2}{4r_1\rho}\right)\rho^3E_4 & d_3\rho^3\mathrm{e}^{d_3}(1+\rho^{-1}) & -d_3\rho^3\mathrm{e}^{-d_3}(1+\rho^{-1}) \\ \left[1+\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2\rho^4E_1 & \left[1-\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2\rho^4E_2 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2\rho^4E_3 & \left[1+i\rho^{-1}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2\rho^4E_4 & -d_1\rho^3\mathrm{e}^{d_3}(1+\rho^{-1}) & -d_1\rho^3\mathrm{e}^{-d_3}(1+\rho^{-1}) \end{bmatrix} + \mathcal{O}(\rho^{-2}).$$

Therefore

$$\begin{aligned} \Delta(\rho) &= \det(T^R(i\rho^2)\widehat{\Phi}) \\ &= d_2^2r_1^3\rho^{16}\det\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & i & -i & 0 & 0 \\ \rho^{-1} & -\rho^{-1} & i\rho^{-1} & -i\rho^{-1} & \frac{1}{d_2r_1^2}(1+\rho^{-1}) & \frac{1}{d_2r_1^2}(1+\rho^{-1}) \\ \left(1+\frac{d_1d_2}{4r_1\rho}\right)E_1 & \left(-1+\frac{d_1d_2}{4r_1\rho}\right)E_2 & \left(-i-\frac{d_1d_2}{4r_1\rho}\right)E_3 & \left(i-\frac{d_1d_2}{4r_1\rho}\right)E_4 & 0 & 0 \\ \left(1+\frac{d_1d_2}{4r_1\rho}\right)E_1 & \left(-1+\frac{d_1d_2}{4r_1\rho}\right)E_2 & \left(-i-\frac{d_1d_2}{4r_1\rho}\right)E_3 & \left(i-\frac{d_1d_2}{4r_1\rho}\right)E_4 & \frac{d_3}{d_2}r_1\mathrm{e}^{d_3}(1+\rho^{-1}) & -\frac{d_3}{d_2}r_1\mathrm{e}^{-d_3}(1+\rho^{-1}) \\ \left[1+\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2E_1 & \left[1-\frac{1}{\rho}\left(\frac{d_1d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2E_2 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2E_3 & \left[1+i\rho^{-1}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{4r_1}\right)\right]_2E_4 & -d_1\rho^{-1}\mathrm{e}^{d_3}(1+\rho^{-1}) & -d_1\rho^{-1}\mathrm{e}^{-d_3}(1+\rho^{-1}) \end{bmatrix} \end{aligned}$$

$$
= 2d_2^2 r_1^3 \rho^{16} E_1 \det \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & i & -i & 0 & 0 \\
0 & -\rho^{-1} & i\rho^{-1} & -i\rho^{-1} & \frac{1}{d_2 r_1^2}(1+\rho^{-1}) & 0 \\
\left(1+\frac{d_1 d_2}{4r_1\rho}\right) & 0 & \left(-i-\frac{d_1 d_2}{4r_1\rho}\right) E_3 & \left(i-\frac{d_1 d_2}{4r_1\rho}\right) E_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{d_3}{d_2} r_1 \mathrm{e}^{d_3}(1+\rho^{-1}) & -\frac{d_3}{d_2} r_1 \cosh(d_3)(1+\rho^{-1}) \\
\left[1+\frac{1}{\rho}\left(\frac{d_1 d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2 & 0 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_3 & \left[1+i\rho^{-1}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_4 & -d_1\rho^{-1}\mathrm{e}^{d_3}(1+\rho^{-1}) & d_1 \sinh(d_3)\rho^{-1}(1+\rho^{-1})
\end{bmatrix}
$$

$$
= -2d_2^2 r_1^3 \rho^{16} E_1 \det \begin{bmatrix}
\frac{1}{d_2 r_1^2}(1+\rho^{-1}) & 0 \\
\frac{d_3}{d_2} r_1 \mathrm{e}^{d_3}(1+\rho^{-1}) & -\frac{d_3}{d_2} r_1 \cosh(d_3)(1+\rho^{-1})
\end{bmatrix}
$$

$$
\times \det \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & -1 & i & -i \\
\left(1+\frac{d_1 d_2}{4r_1\rho}\right) & 0 & \left(-i-\frac{d_1 d_2}{4r_1\rho}\right) E_3 & \left(i-\frac{d_1 d_2}{4r_1\rho}\right) E_4 \\
\left[1+\frac{1}{\rho}\left(\frac{d_1 d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2 & 0 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_3 & \left[1+i\rho^{-1}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_4
\end{bmatrix}
$$

$$
= -2r_1^2 \rho^{16} d_3 \cosh(d_3) E_1 (1+2\rho^{-1})
$$

$$
\times \det \begin{bmatrix}
0 & 1+i & 1-i \\
\left(1+\frac{d_1 d_2}{4r_1\rho}\right) & \left(-i-\frac{d_1 d_2}{4r_1\rho}\right) E_3 & \left(i-\frac{d_1 d_2}{4r_1\rho}\right) E_4 \\
\left[1+\frac{1}{\rho}\left(\frac{d_1 d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_3 & \left[1+i\rho^{-1}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_4
\end{bmatrix}
$$

$$
= -2(1-i) r_1^2 \rho^{16} d_3 \cosh(d_3) E_1 (1+2\rho^{-1})
$$

$$
\times \det \begin{bmatrix}
\left(1+\frac{d_1 d_2}{4r_1\rho}\right) & \left(-i-\frac{d_1 d_2}{4r_1\rho}\right) E_3 + \left(1+i\frac{d_1 d_2}{4r_1\rho}\right) E_4 \\
\left[1+\frac{1}{\rho}\left(\frac{d_1 d_2}{4r_1}-i\frac{\tilde{k}}{r_1^3}\right)\right]_2 & \left[1-i\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_3 + \left[-i+\frac{1}{\rho}\left(i\frac{\tilde{k}}{r_1^3}+\frac{d_1 d_2}{4r_1}\right)\right]_2 E_4
\end{bmatrix}
$$

$$
= -2(1-i) r_1^2 \rho^{16} d_3 \cosh(d_3) E_1 (1+2\rho^{-1})
$$

$$
\times \left\{ \left[1+i+\left(\frac{d_1 d_2}{2r_1}-i(1+i)\frac{\tilde{k}}{r_1^3}\right)\frac{1}{\rho}\right]_2 E_3 + \left[-(i+1)-\left(i\frac{d_1 d_2}{2r_1}-2i\frac{\tilde{k}}{r_1^3}\right)\frac{1}{\rho}\right]_2 E_4 \right\}.
$$

Hence

$$(-2^2 d_3 \cosh(d_3) r_1^2 \rho^{16} E_1)^{-1} \Delta(\rho) = \left[1 + \left(-i\frac{\tilde{k}}{r_1^3} + 2 + (1-i)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] E_3 - \left[1 + \left(-i(1-i)\frac{\tilde{k}}{r_1^3} + 2 + (i+1)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] E_4 + \mathcal{O}(\rho^{-2}) \quad (3.20)$$

where E_1, E_3 and E_4 are given by (3.19). With these preparations, we come to the proof of the asymptotic behavior of the eigenvalues.

Theorem 3.2. *Let $\Delta(\rho)$ be the characteristic determinant in the sector $\mathcal{S}$ of the system (2.12) with $\lambda = i\rho^2$. Then the following asymptotic expansion holds:*

$$\Delta(\rho) = i2^2 d_3 \cosh(d_3) r_1^{10} \rho^{16} \mathrm{e}^{r_1\rho} \left\{ \left[1 + \left(-i\frac{\tilde{k}}{r_1^3} + 2 + (1-i)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] \mathrm{e}^{ir_1\rho} + \left[1 + \left(-i(1-i)\frac{\tilde{k}}{r_1^3} + 2 + (i+1)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] \mathrm{e}^{-ir_1\rho} + \mathcal{O}(\rho^{-2}) \right\} \quad (3.21)$$

where r_1, d_1, d_2, d_3 and $\tilde{k}$ are given by (2.8), respectively. Moreover, the eigenvalues $\{\lambda_n, \overline{\lambda}_n\}$ of the system (2.12) have the following asymptotic expansion

$$\lambda_n = -\frac{\tilde{k}}{r_1^4} + i\frac{d_1 d_2}{2r_1^2} + i\frac{(\frac{1}{2}+n)^2\pi^2}{r_1^2} + \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty \quad (3.22)$$

where n are positive integers. Therefore

$$\mathrm{Re}\{\lambda_n, \overline{\lambda}_n\} \to -\frac{\tilde{k}}{r_1^4} = -\frac{k}{m} \quad \text{as } n \to \infty. \quad (3.23)$$

Proof. Obviously, (3.21) is a direct consequence of (3.20). Using now (3.21), it follows that $\rho \in \mathcal{S}$ satisfies

$$\left[1 - \left(i\frac{\tilde{k}}{r_1^3} - 2 - (1-i)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] \mathrm{e}^{ir_1\rho} + \left[1 - \left((1+i)\frac{\tilde{k}}{r_1^3} - 2 - (i+1)\frac{d_1 d_2}{4r_1}\right)\frac{1}{\rho}\right] \mathrm{e}^{-ir_1\rho} + \mathcal{O}(\rho^{-2}) = 0 \quad (3.24)$$

which can also be rewritten as

$$\mathrm{e}^{ir_1\rho} + \mathrm{e}^{-ir_1\rho} + \mathcal{O}(\rho^{-1}) = 0. \quad (3.25)$$

Since in the first quadrant, the solutions of the equation

$$\mathrm{e}^{ir_1\rho} + \mathrm{e}^{-ir_1\rho} = 0$$

are given by

$$\tilde{\rho}_n = \frac{\frac{1}{2}+n}{r_1}\pi, \quad n = 0, 1, 2, \ldots,$$

it follows from the Rouché's theorem that the solutions to equation (3.25) have the form of

$$\rho_n = \tilde{\rho}_n + \alpha_n = \frac{1}{r_1}\left(\frac{1}{2}+n\right)\pi + \alpha_n, \quad \alpha_n = \mathcal{O}(n^{-1}), \quad n = N, N+1, \ldots, \quad (3.26)$$

where N is a sufficiently large positive integer. Substitute ρ_n into (3.24) and use the fact that $e^{ir_1\tilde{\rho}_n} = -e^{-ir_1\tilde{\rho}_n}$, to obtain

$$\left[1-\left(i\frac{\tilde{k}}{r_1^3}-2-(1-i)\frac{d_1d_2}{4r_1}\right)\frac{1}{\rho_n}\right]e^{ir_1\alpha_n}-\left[1-\left((1+i)\frac{\tilde{k}}{r_1^3}-2-(i+1)\frac{d_1d_2}{4r_1}\right)\frac{1}{\rho_n}\right]e^{-ir_1\alpha_n}+\mathcal{O}(\rho_n^{-2})=0.$$

Expand the exponential function according to its Taylor series, to give

$$\alpha_n=\frac{1}{2(\frac{1}{2}+n)\pi}\left[i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{2r_1}\right]+\mathcal{O}(n^{-2}).$$

Substituting above into (3.26) produces

$$\rho_n=\frac{1}{r_1}\left(\frac{1}{2}+n\right)\pi+\frac{1}{2(\frac{1}{2}+n)\pi}\left[i\frac{\tilde{k}}{r_1^3}+\frac{d_1d_2}{2r_1}\right]+\mathcal{O}(n^{-2})\quad\text{as } n\to\infty. \tag{3.27}$$

Since $\lambda_n = i\rho_n^2$, we get eventually

$$\lambda_n=-\frac{\tilde{k}}{r_1^4}+i\frac{d_1d_2}{2r_1^2}+i\frac{(\frac{1}{2}+n)^2\pi^2}{r_1^2}+\mathcal{O}(n^{-1})\quad\text{as } n\to\infty.$$

The proof is complete. □

4. Asymptotic behavior of eigenfunctions

Theorem 4.1. *Let $\sigma(\mathcal{A}) = \{\lambda_n, \overline{\lambda}_n\}$ be the eigenvalues of $\mathcal{A}$ and let $\lambda = i\rho^2$ with λ_n and ρ_n being given by (3.22) and (3.27) respectively. Then the corresponding eigenfunctions $\{(w_n, \lambda_n w_n), (\overline{w}_n, \overline{\lambda}_n\overline{w}_n)\}$ have the following asymptotics:*

$$\begin{cases} w_n''(x) = e^{-ir_1\rho_n(x+1)}+(1+i)e^{-ir_1\rho_n}e^{-r_1\rho_n x} \\ \qquad +ie^{ir_1\rho_n(x-1)}-(i-1)e^{r_1\rho_n(x-1)}+\mathcal{O}(n^{-1}), \\ \lambda_n w_n(x) = -ir_1^{-2}e^{-ir_1\rho_n(x+1)}+(i-1)r_1^{-2}e^{-ir_1\rho_n}e^{-r_1\rho_n x} \\ \qquad +r_1^{-2}e^{ir_1\rho_n(x-1)}+(i+1)r_1^{-2}e^{r_1\rho_n(x-1)}+\mathcal{O}(n^{-1}), \\ \mathcal{J}w_n'(x) = s_n(x) = \mathcal{O}(n^{-1}), \\ w_n'(x) = \mathcal{O}(n^{-1}) \end{cases} \tag{4.1}$$

for sufficient large positive integer n. Moreover, $(w_n, \lambda_n w_n)$ are approximately normalized in $\mathcal{H}$ in the sense that there exist positive constants c_1 and c_2 independent of n such that

$$c_1 \le \|w_n''\|_{L^2(0,1)},\ \|\lambda_n w_n\|_{L^2(0,1)} \le c_2 \tag{4.2}$$

for all sufficient large positive integers n.

Proof. Since the characteristic determinant $\Delta(\rho)$ possesses only simple roots for sufficiently large modulus ρ, the corresponding eigenfunctions $\Phi(x) = [w_1(x), w_2(x)w_3(x), w_4(x), s_1(x),\ s_2(x)]^\top$ (see (2.11)–(2.12)) can be obtained by replacing one of the rows of $T^R\widehat{\Phi}$ in (3.18) by $e_j^\top(\widehat{\Phi}(x,\rho))$, where e_j is the jth column of the identity matrix. Indeed, we know from (3.16) that $\widehat{\Phi}(x,\rho) = P(\rho)\widehat{\Psi}(x,\rho)$ and hence

$$\widehat{\Phi}(x,\rho)=\begin{bmatrix}\widehat{\Phi}_{11}(x,\rho) & O_{4\times 2}\\ \widehat{\Phi}_{21}(x,\rho) & \widehat{\Phi}_{22}(x,\rho)\end{bmatrix} \tag{4.3}$$

where

$$\widehat{\Phi}_{11}(x,\rho) := \begin{bmatrix} r_1\rho[1]_1 e^{r_1\rho x} & r_1\rho[1]_1 e^{-r_1\rho x} & r_1\rho[1]_1 e^{ir_1\rho x} & r_1\rho[1]_1 e^{-ir_1\rho x} \\ r_1^2\rho^2[1]_1 e^{r_1\rho x} & -r_1^2\rho^2[1]_1 e^{-r_1\rho x} & ir_1^2\rho^2[1]_1 e^{ir_1\rho x} & -ir_1^2\rho^2[1]_1 e^{-ir_1\rho x} \\ r_1^3\rho^3[1]_1 e^{r_1\rho x} & r_1^3\rho^3[1]_1 e^{-r_1\rho x} & -r_1^3\rho^3[1]_1 e^{ir_1\rho x} & -r_1^3\rho^3[1]_1 e^{-ir_1\rho x} \\ r_1^4\rho^4[1]_1 e^{r_1\rho x} & -r_1^4\rho^4[1]_1 e^{-r_1\rho x} & -ir_1^4\rho^4[1]_1 e^{ir_1\rho x} & ir_1^4\rho^4[1]_1 e^{-ir_1\rho x} \end{bmatrix}, \tag{4.4}$$

$$\widehat{\Phi}_{21}(x,\rho) := \rho^3 \begin{bmatrix} [0]_1 e^{r_1\rho x} & [0]_1 e^{-r_1\rho x} & [0]_1 e^{ir_1\rho x} & [0]_1 e^{-ir_1\rho x} \\ d_2 r_1^3[1]_1 e^{r_1\rho x} & d_2 r_1^3[1]_1 e^{-r_1\rho x} & -d_2 r_1^3[1]_1 e^{ir_1\rho x} & -d_2 r_1^3[1]_1 e^{-ir_1\rho x} \end{bmatrix} \tag{4.5}$$

and

$$\widehat{\Phi}_{22}(x,\rho) := \begin{bmatrix} \rho^3[e^{d_3 x}]_1 & \rho^3[e^{-d_3 x}]_1 \\ \rho^3[d_3 e^{d_3 x}]_1 & \rho^3[-d_3 e^{-d_3 x}]_1 \end{bmatrix}. \tag{4.6}$$

Thus the first component of $\Phi = (w_1(x,\rho), w_2(x,\rho), w_3(x,\rho), w_4(x,\rho), s_1(x,\rho), s_2(x,\rho))$ is given by

$$w_1(x,\rho) = \frac{d_3^{-1} r_1^{-3}}{2\cosh(d_3)} E_1^{-1}\rho^{-13} \det \begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0 \\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0 \\ d_2 r_1^2\rho^2 & -d_2 r_1^2\rho^2 & id_2 r_1^2\rho^2 & -id_2 r_1^2\rho^2 & \rho^3[1]_1 & \rho^3[1]_1 \\ r_1^{-1}[1]_1\rho^3 E_1 & -r_1^{-1}[1]_1\rho^3 E_2 & -ir_1^{-1}[1]_1\rho^3 E_3 & ir_1^{-1}[1]_1\rho^3 E_4 & 0 & 0 \\ r_1^{-1} d_2[1]_1\rho^3 E_1 & -r_1^{-1} d_2[1]_1\rho^3 E_2 & -ir_1^{-1} d_2[1]_1\rho^3 E_3 & id_2 r_1^{-1}[1]_1\rho^3 E_4 & d_3\rho^3 e^{d_3}[1]_1 & -d_3\rho^3 e^{-d_3}[1]_1 \\ r_1\rho[1]_1 e^{r_1\rho x} & r_1\rho[1]_1 e^{-r_1\rho x} & r_1\rho[1]_1 e^{ir_1\rho x} & r_1\rho[1]_1 e^{-ir_1\rho x} & 0 & 0 \end{bmatrix},$$

where E_1, E_2, E_3 and E_4 are given by (3.19). After a simple calculation, we obtain

$$\begin{aligned} w_1(x,\rho) &= \frac{1}{2\cosh(d_3)} \det \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & e^{-r_1\rho} & -e^{ir_1\rho} & -e^{-ir_1\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{d_3} & -e^{-d_3} \\ e^{r_1\rho(x-1)} & e^{-r_1\rho x} & e^{ir_1\rho x} & e^{-ir_1\rho x} & 0 & 0 \end{bmatrix} + \mathcal{O}(\rho^{-1}) \\ &= \det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & i & -i \\ 1 & e^{-r_1\rho} & -e^{ir_1\rho} & -e^{-ir_1\rho} \\ e^{r_1\rho(x-1)} & e^{-r_1\rho x} & e^{ir_1\rho x} & e^{-ir_1\rho x} \end{bmatrix} + \mathcal{O}(\rho^{-1}) \\ &= (i+1)\Big\{ e^{-ir_1\rho x} - e^{-r_1\rho x} + i(e^{ir_1\rho x} - e^{-r_1\rho x}) + e^{r_1\rho(x-1)}\left(e^{-ir_1\rho} + ie^{ir_1\rho}\right) + \mathcal{O}(\rho^{-1}) \Big\}. \end{aligned}$$

By (3.25), it follows that

$$w_1(x,\rho) = (i+1)\mathrm{e}^{ir_1\rho}\Big\{\mathrm{e}^{-ir_1\rho(x+1)} - (1+i)\mathrm{e}^{-ir_1\rho}\mathrm{e}^{-r_1\rho x} + i\mathrm{e}^{ir_1\rho(x-1)} + (i-1)\mathrm{e}^{r_1\rho(x-1)} + \mathcal{O}(\rho^{-1})\Big\}. \tag{4.7}$$

Similarly, we have

$$w_3(x,\rho) = \frac{d_3^{-1}r_1^{-3}}{2\cosh(d_3)}E_1^{-1}\rho^{-13}\det\begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0 \\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0 \\ d_2r_1^2\rho^2 & -d_2r_1^2\rho^2 & id_2r_1^2\rho^2 & -id_2r_1^2\rho^2 & \rho^3[1]_1 & \rho^3[1]_1 \\ r_1^{-1}[1]_1\rho^3E_1 & -r_1^{-1}[1]_1\rho^3E_2 & -ir_1^{-1}[1]_1\rho^3E_3 & ir_1^{-1}[1]_1\rho^3E_4 & 0 & 0 \\ r_1^{-1}d_2[1]_1\rho^3E_1 & -r_1^{-1}d_2[1]_1\rho^3E_2 & -ir_1^{-1}d_2[1]_1\rho^3E_3 & id_2r_1^{-1}[1]_1\rho^3E_4 & d_3\rho^3\mathrm{e}^{d_3}[1]_1 & -d_3\rho^3\mathrm{e}^{-d_3}[1]_1 \\ r_1^3\rho^3[1]_1\mathrm{e}^{r_1\rho x} & r_1^3\rho^3[1]_1\mathrm{e}^{-r_1\rho x} & -r_1^3\rho^3[1]_1\mathrm{e}^{ir_1\rho x} & -r_1^3\rho^3[1]_1\mathrm{e}^{-ir_1\rho x} & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} w_3(x,\rho) &= \frac{r_1^2\rho^2}{2\cosh(d_3)}\det\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & \mathrm{e}^{-r_1\rho} & -\mathrm{e}^{ir_1\rho} & -\mathrm{e}^{-ir_1\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{d_3} & -\mathrm{e}^{-d_3} \\ \mathrm{e}^{r_1\rho(x-1)} & \mathrm{e}^{-r_1\rho x} & -\mathrm{e}^{ir_1\rho x} & -\mathrm{e}^{-ir_1\rho x} & 0 & 0 \end{bmatrix} + \mathcal{O}(\rho^{-1}) \\ &= r_1^2\rho^2\det\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & i & -i \\ 1 & \mathrm{e}^{-r_1\rho} & -\mathrm{e}^{ir_1\rho} & -\mathrm{e}^{-ir_1\rho} \\ \mathrm{e}^{r_1\rho(x-1)} & \mathrm{e}^{-r_1\rho x} & -\mathrm{e}^{ir_1\rho x} & -\mathrm{e}^{-ir_1\rho x} \end{bmatrix} + \mathcal{O}(\rho^{-1}) \\ &= -(i+1)r_1^2\rho^2\Big\{\mathrm{e}^{-ir_1\rho x} + \mathrm{e}^{-r_1\rho x} + i(\mathrm{e}^{ir_1\rho x} + \mathrm{e}^{-r_1\rho x}) \\ &\qquad -\mathrm{e}^{r_1\rho(x-1)}\left(\mathrm{e}^{-ir_1\rho} + i\mathrm{e}^{ir_1\rho}\right) + \mathcal{O}(\rho^{-1})\Big\}. \end{aligned}$$

This, together with (3.25), implies that

$$\begin{aligned} w_3(x,\rho) = -(i+1)r_1^2\rho^2\mathrm{e}^{ir_1\rho}\Big\{&\mathrm{e}^{-ir_1\rho(x+1)} + (1+i)\mathrm{e}^{-ir_1\rho}\mathrm{e}^{-r_1\rho x} \\ &+i\mathrm{e}^{ir_1\rho(x-1)} - (i-1)\mathrm{e}^{r_1\rho(x-1)} + \mathcal{O}(\rho^{-1})\Big\}. \end{aligned} \tag{4.8}$$

Also, along the same line, we can obtain

$$w_2(x,\rho)=\frac{d_3^{-1}r_1^{-3}}{2\cosh(d_3)}E_1^{-1}\rho^{-13}\det\begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0\\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0\\ d_2r_1^2\rho^2 & -d_2r_1^2\rho^2 & id_2r_1^2\rho^2 & -id_2r_1^2\rho^2 & \rho^3[1]_1 & \rho^3[1]_1\\ r_1^{-1}[1]_1\rho^3E_1 & -r_1^{-1}[1]_1\rho^3E_2 & -ir_1^{-1}[1]_1\rho^3E_3 & ir_1^{-1}[1]_1\rho^3E_4 & 0 & 0\\ r_1^{-1}d_2[1]_1\rho^3E_1 & -r_1^{-1}d_2[1]_1\rho^3E_2 & -ir_1^{-1}d_2[1]_1\rho^3E_3 & id_2r_1^{-1}[1]_1\rho^3E_4 & d_3\rho^3\mathrm{e}^{d_3}[1]_1 & -d_3\rho^3\mathrm{e}^{-d_3}[1]_1\\ r_1^2\rho^2[1]_1\mathrm{e}^{r_1\rho x} & -r_1^2\rho^2[1]_1\mathrm{e}^{-r_1\rho x} & ir_1^2\rho^2[1]_1\mathrm{e}^{ir_1\rho x} & -ir_1^2\rho^2[1]_1\mathrm{e}^{-ir_1\rho x} & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} w_2(x,\rho) &= \frac{r_1\rho}{2\cosh(d_3)}\det\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0\\ 0 & -1 & i & -i & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 1 & \mathrm{e}^{-r_1\rho} & -\mathrm{e}^{ir_1\rho} & -\mathrm{e}^{-ir_1\rho} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathrm{e}^{d_3} & -\mathrm{e}^{-d_3}\\ \mathrm{e}^{r_1\rho(x-1)} & -\mathrm{e}^{-r_1\rho x} & i\mathrm{e}^{ir_1\rho x} & -i\mathrm{e}^{-ir_1\rho x} & 0 & 0 \end{bmatrix}+\mathcal{O}(\rho^{-1})\\ &= r_1\rho\det\begin{bmatrix} 0 & 1 & 1 & 1\\ 0 & -1 & i & -i\\ 1 & \mathrm{e}^{-r_1\rho} & -\mathrm{e}^{ir_1\rho} & -\mathrm{e}^{-ir_1\rho}\\ \mathrm{e}^{r_1\rho(x-1)} & -\mathrm{e}^{-r_1\rho x} & i\mathrm{e}^{ir_1\rho x} & -i\mathrm{e}^{-ir_1\rho x} \end{bmatrix}+\mathcal{O}(\rho^{-1})\\ &= (i+1)r_1\rho\Big\{-i\mathrm{e}^{-ir_1\rho x}+\mathrm{e}^{-r_1\rho x}+i(i\mathrm{e}^{ir_1\rho x}+\mathrm{e}^{-r_1\rho x})\\ &\qquad +\mathrm{e}^{r_1\rho(x-1)}\left(\mathrm{e}^{-ir_1\rho}+i\mathrm{e}^{ir_1\rho}\right)+\mathcal{O}(\rho^{-1})\Big\}. \end{aligned}$$

By (3.25), we get

$$\begin{aligned} w_2(x,\rho)= &\ (i+1)r_1\rho\mathrm{e}^{ir_1\rho}\Big\{-i\mathrm{e}^{-ir_1\rho(x+1)}+(1+i)\mathrm{e}^{-ir_1\rho}\mathrm{e}^{-r_1\rho x}\\ &-\mathrm{e}^{ir_1\rho(x-1)}+(i-1)\mathrm{e}^{r_1\rho(x-1)}+\mathcal{O}(\rho^{-1})\Big\}. \end{aligned} \tag{4.9}$$

Furthermore, it has

$$s_1(x,\rho)=\frac{d_3^{-1}r_1^{-3}}{2\cosh(d_3)}E_1^{-1}\rho^{-13}\det\begin{bmatrix} r_1\rho & r_1\rho & r_1\rho & r_1\rho & 0 & 0\\ r_1^2\rho^2 & -r_1^2\rho^2 & ir_1^2\rho^2 & -ir_1^2\rho^2 & 0 & 0\\ d_2r_1^2\rho^2 & -d_2r_1^2\rho^2 & id_2r_1^2\rho^2 & -id_2r_1^2\rho^2 & \rho^3[1]_1 & \rho^3[1]_1\\ r_1^{-1}[1]_1\rho^3E_1 & -r_1^{-1}[1]_1\rho^3E_2 & -ir_1^{-1}[1]_1\rho^3E_3 & ir_1^{-1}[1]_1\rho^3E_4 & 0 & 0\\ r_1^{-1}d_2[1]_1\rho^3E_1 & -r_1^{-1}d_2[1]_1\rho^3E_2 & -ir_1^{-1}d_2[1]_1\rho^3E_3 & id_2r_1^{-1}[1]_1\rho^3E_4 & d_3\rho^3e^{d_3}[1]_1 & -d_3\rho^3e^{-d_3}[1]_1\\ \rho^3[0]_1e^{r_1\rho x} & \rho^3[0]_1e^{-r_1\rho x} & \rho^3[0]_1e^{ir_1\rho x} & \rho^3[0]_1e^{-ir_1\rho x} & \rho^3[e^{d_3x}]_1 & \rho^3[e^{-d_3x}]_1 \end{bmatrix}$$

and

$$s_1(x,\rho)=\frac{r_1^{-1}\rho^2}{2\cosh(d_3)}\det\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0\\ 0 & -1 & i & -i & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 1 & e^{-r_1\rho} & -e^{ir_1\rho} & -e^{-ir_1\rho} & 0 & 0\\ 0 & 0 & 0 & 0 & e^{d_3} & -e^{-d_3}\\ 0 & 0 & 0 & 0 & e^{d_3x} & e^{-d_3x} \end{bmatrix}+\mathcal{O}(\rho^{-1})=\mathcal{O}(\rho^{-1}). \tag{4.10}$$

Based on above computations, (4.1) can then be deduced from (4.7)–(4.10) by setting

$$w_n(x):=-\frac{1-i}{2}r_1^{-2}\rho_n^{-2}e^{-ir_1\rho_n}w_1(x,\rho_n) \tag{4.11}$$

in (4.7)–(4.10), respectively. Finally, it follows from (3.26) that

$$\begin{aligned}&\|e^{-ir_1\rho_n(x+1)}\|^2_{L^2(0,1)}=1+\mathcal{O}(n^{-1}),\quad \|e^{ir_1\rho_n(x-1)}\|^2_{L^2(0,1)}=1+\mathcal{O}(n^{-1}),\\ &\|e^{-r_1\rho_nx}\|^2_{L^2(0,1)}=\mathcal{O}(n^{-1}),\quad \|e^{r_1\rho_n(x-1)}\|^2_{L^2(0,1)}=\mathcal{O}(n^{-1}).\end{aligned} \tag{4.12}$$

These together with (4.1) yield (4.2). The proof is complete. □

5. Riesz basis property and exponential stability

Let us recall that for a closed operator $\mathbf{A}$ in a Hilbert space $\mathbf{H}$, a nonzero element $\phi\in\mathbf{H}$ is called a generalized eigenvector of $\mathbf{A}$, corresponding to an eigenvalue λ of $\mathbf{A}$, if there is a nonnegative integer n such that $(\lambda-\mathbf{A})^n\phi=0$. A sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathbf{H}$ is called a Riesz basis for $\mathbf{H}$ if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $\mathbf{H}$ and a linear bounded invertible operator T such that [18]

$$T\phi_n=e_n,\quad n=1,2,\dots$$

Let $\{\lambda_n\}_{n=1}^{\infty} = \sigma(\mathbf{A})$, the spectrum of $\mathbf{A}$. Suppose each λ_n has finite algebraic multiplicity m_n, and let $\{\psi_{n_i}\}_1^{m_n}$ be the set of generalized eigenvectors of $\mathbf{A}$ corresponding to λ_n. Then if $\{\psi_{n_i} \mid 1 \le i \le m_n,\ n = 1, 2, \ldots\}$ form a Riesz basis for $\mathbf{H}$, then the C_0-semigroup generated by $\mathbf{A}$ can be represented as

$$\mathrm{e}^{\mathbf{A}t}x = \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n t} \sum_{j=1}^{m_n} a_{nj} f_{nj}(t)\psi_{nj}, \quad \forall x = \sum_{n=1}^{\infty}\sum_{j=1}^{m_n} a_{nj}\psi_{nj} \in \mathbf{H} \tag{5.1}$$

where $f_{nj}(t)$ are the polynomials of t with order not greater than m_n. In particular, if $m_n = 1$ for all sufficiently large n, then the spectrum determined growth condition holds, *i.e.*, $\omega(\mathbf{A}) = s(\mathbf{A})$, where $\omega(\mathbf{A})$ is the growth bound of $\mathrm{e}^{\mathbf{A}t}$, and $s(\mathbf{A})$ is the spectral bound of $\mathbf{A}$ [6].

The following result from [6] provides a useful way to verify the Riesz basis property for the generalized eigenvectors of linear operators with compact resolvents in Hilbert spaces.

Theorem 5.1. *Let $\mathbf{A}$ be a densely defined discrete operator (i.e., there is a $\lambda \in \rho(\mathbf{A})$, the resolvent set of $\mathbf{A}$, such that $(\lambda - \mathbf{A})^{-1}$ is compact on $\mathbf{H}$) in a Hilbert space $\mathbf{H}$. Let $\{\phi_n\}_1^{\infty}$ be a Riesz basis for $\mathbf{H}$. If there are an integer $N \ge 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^{\infty}$ of $\mathbf{A}$ such that*

$$\sum_{N+1}^{\infty} \|\phi_n - \psi_n\|^2 < \infty,$$

then

(1) *there are integer $M > N$ and generalized eigenvectors $\{\psi_{n_0}\}_1^M$ of $\mathbf{A}$ such that $\{\psi_{n_0}\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ form a Riesz basis for $\mathbf{H}$;*
(2) *if $\{\psi_{n_0}\}_1^M \cup \{\psi_n\}_{M+1}^{\infty}$ are the generalized eigenvectors corresponding to eigenvalues $\{\sigma_n\}_1^{\infty}$ of $\mathbf{A}$, then $\sigma(\mathbf{A}) = \{\sigma_n\}_1^{\infty}$ where σ_n is accounted as many as times according to its algebraic multiplicity;*
(3) *if there is an integer $M_0 > 0$ such that $\sigma_n \ne \sigma_m$ for all $m, n > M_0$, then there is an integer $N_0 > M_0$ such that all σ_n are algebraically simple for all $n > N_0$.*

Now, we are ready to state the first main result.

Theorem 5.2. *There is a sequence of generalized eigenfunctions of the operator $\mathcal{A}$ defined by (2.3) and (2.4), which forms a Riesz basis for $\mathcal{H}$. Moreover, all eigenvalues with sufficient large modulus are algebraically simple.*

Proof. We show that $\{(w_n, \lambda_n w_n), (\overline{w}_n, \overline{\lambda}_n \overline{w}_n)\}$ obtained in Theorem 4.1 satisfies the hypotheses in Theorem 5.1 with respect to a suitably chosen reference Riesz basis of $\mathcal{H}$. To do this, we define another operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0)$ ($\subset \mathcal{H}$) $\to \mathcal{H}$ by

$$\mathcal{A}_0(w, z) := \left(z, -\frac{1}{m}[Aw''' + B^2\gamma(\mathcal{J}w')]'\right) \tag{5.2}$$

with

$$\mathcal{D}(\mathcal{A}_0) := \left\{(w, z) \in \mathcal{H} \;\middle|\; \begin{array}{l} w' \in \mathcal{D}(\mathcal{T}),\ z \in H_w^2(0,1),\ Aw''' + B^2\gamma(\mathcal{J}w') \in H^1(0,1), \\ w''(1) = Aw'''(1) + B^2\gamma(\mathcal{J}w')(1) = 0 \end{array}\right\} \tag{5.3}$$

where m, A and B are the same as in (1.1), and the operators $\mathcal{T}$ and $\mathcal{J}$ are that in (1.4) and (1.5) respectively. It is easy to verify that $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ with compact resolvents and hence the generalized eigenfunctions $\{(w_{n0}, \lambda_{n0} w_{n0}),\ (\overline{w}_{n0}, \overline{\lambda}_{n0} \overline{w}_{n0})\}$ of $\mathcal{A}_0$ form a Riesz basis for $\mathcal{H}$. Moreover, from the arguments in Sections 3 and 4, λ_{n0} and $(w_{n0}, \lambda_{n0} w_{n0})$ have the same asymptotics (3.22) and (4.1) with $k = \tilde{k} = 0$, respectively. Now,

$$\sum_{n \ge N}^{\infty} \|(w_n, \lambda_n w_n) - (w_{n0}, \lambda_{n0} w_{n0})\|^2 = \sum_{n \ge N}^{\infty} \mathcal{O}(n^{-2}) < \infty. \tag{5.4}$$

The same is true for their conjugates. Hence, all hypotheses of Theorem 5.1 are satisfied and the generalized eigenfunctions of $\mathcal{A}$ form a Riesz basis in $\mathcal{H}$. Finally, since for a skew-adjoint operator, the geometric multiplicity and algebraic multiplicity of each eigenvalue are the same, we see that all eigenvalues of $\mathcal{A}_0$ with sufficiently large modulus are algebraically simple. Since $\{(w_n, \lambda_n w_n), (\overline{w}_n, \overline{\lambda}_n \overline{w}_n)\}$ form a Riesz basis for $\mathcal{H}$, we also have that all eigenvalues of $\mathcal{A}$ with sufficiently large modulus are algebraically simple. The proof is complete. □

The second main result is:

Theorem 5.3. *Let $\mathcal{A}$ be defined by (2.3) and (2.4). Then the spectrum-determined growth condition $\omega(\mathcal{A}) = s(\mathcal{A})$ holds true for the C_0-semigroup generated by $\mathcal{A}$. Moreover, the system (2.5) is exponentially stable, that is to say, there exist two positive constants M and ω such that the C_0-semigroup $\mathrm{e}^{\mathcal{A}t}$ generated by $\mathcal{A}$ satisfies*

$$\|\mathrm{e}^{\mathcal{A}t}\| \leq M\mathrm{e}^{-\omega t}. \tag{5.5}$$

Proof. The spectrum-determined growth condition follows from Theorem 5.2. By Theorem 2.1, $\mathcal{A}$ is dissipative and hence there is no eigenvalue on the right half complex plane. Moreover, it is a simple task to check that the operator $\mathcal{A}$ has no eigenvalue on the imaginary axis. This, together with (3.23) and the spectrum-determined growth condition, shows that $\mathrm{e}^{\mathcal{A}t}$ is exponentially stable. The proof is complete. □

6. Exact controllability

Instead of (1.6), we consider the open loop system

$$\begin{cases} mw_{tt}(x,t) + Aw_{xxxx}(x,t) + B^2\gamma(\mathcal{J}w_x)_x(x,t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ Aw_{xxx}(1,t) + B^2\gamma\mathcal{J}w_x(1,t) = u(t), \\ y(t) = w_t(1,t) \end{cases} \tag{6.1}$$

where $u \in L^2_{loc}(0,\infty)$ is the control input and y is the output. Suppose $\mathcal{A}_0$ is defined as in (5.2) and (5.3) that is nothing but $\mathcal{A}$ with $k = 0$. Define an extension $\widehat{\mathcal{A}}_0$ of $\mathcal{A}_0$ as

$$\begin{cases} \widehat{\mathcal{A}}_0(w,z) := \left(z, -\frac{1}{m}[Aw''' + B^2\gamma(\mathcal{J}w')]'\right), \\ \mathcal{D}(\widehat{\mathcal{A}}_0) := \left\{(w,z) \in \mathcal{H} \;\middle|\; \begin{array}{l} w' \in \mathcal{D}(\mathcal{T}),\ z \in H^2_w(0,1), \\ Aw''' + B^2\gamma(\mathcal{J}w') \in H^1(0,1), w''(1) = 0, \end{array} \right\}. \end{cases} \tag{6.2}$$

Then for any $(w,z) \in D(\widehat{\mathcal{A}}_0), (\phi,\psi) \in D(\mathcal{A}_0^*) = D(\mathcal{A}_0)$,

$$\langle \widehat{\mathcal{A}}_0(w,z), (\phi,\psi)\rangle = \langle (w,z), \mathcal{A}_0^*(\phi,\psi)\rangle - [Aw''' + B^2\gamma(\mathcal{J}w')](1)\overline{\psi(1)}. \tag{6.3}$$

Next, define the natural extension $\widetilde{\mathcal{A}}_0 : \mathcal{H} \to \mathcal{H}_{-1} = [D(\mathcal{A}_0)]'$ of $\mathcal{A}_0$ as

$$\langle \widetilde{\mathcal{A}}_0 F, G\rangle = \langle F, \mathcal{A}_0^* G\rangle,\ \forall G \in D(\mathcal{A}_0^*),\ F \in D(\widetilde{\mathcal{A}}_0) = \mathcal{H}. \tag{6.4}$$

Then for any $F = (w,z) \in D(\widehat{\mathcal{A}}_0)$,

$$\widehat{\mathcal{A}}_0 F = \widetilde{\mathcal{A}}_0 F - [Aw''' + B^2\gamma(\mathcal{J})w'](1)b \quad \text{in } \mathcal{H}_{-1} \tag{6.5}$$

where

$$b := \delta(\cdot - 1) \text{ the Dirac delta}. \tag{6.6}$$

It is seen that $Y(t) = (y(\cdot,t), y_t(\cdot,t))$ satisfies the first two equations of (6.1) only then $\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \widehat{\mathcal{A}}_0 Y(t)$. Furthermore, if $Y(t)$ also meets the third condition of (6.1), then $\widehat{\mathcal{A}}_0 Y(t) = \tilde{\mathcal{A}}_0 Y(t) + \mathbf{b}u(t)$ in $\mathcal{H}_{-1}$, where

$\mathbf{b} := -(0, b)^\top$. In other words, the system (6.1) can be represented as

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \widetilde{\mathcal{A}}_0 Y(t) + \mathbf{b}u(t) \tag{6.7}$$

in $\mathcal{H}_{-1}$. Therefore, (6.1) is equivalent to

$$\begin{cases} mw_{tt}(x,t) + Aw_{xxxx}(x,t) + B^2\gamma(\mathcal{J}w_x)_x(x,t) + \delta(x-1)u(t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, \\ Aw_{xxx}(1,t) + B^2\gamma\mathcal{J}w_x(1,t) = 0, \\ y(t) = b^* w_t(\cdot,t). \end{cases} \tag{6.8}$$

Or in the form of

$$\begin{cases} mw_{tt} + \mathbb{A}w + bu(t) = 0, \\ y(t) = b^* w_t. \end{cases} \tag{6.9}$$

Since by the method of [11] on page 8, it is easily shown that $D(\mathbb{A}^{1/2}) \times L^2(0,1) = \mathcal{H}$, where $\mathbb{A}$ is a positive selfadjoint operator in $L^2(0,1)$ defined by

$$\begin{cases} \mathbb{A}w(x) = Aw^{(4)}(x) + B^2\gamma(\mathcal{J}w')'(x), \\ D(\mathbb{A}) = \{w \in H^4(0,1) \cap H^2_w | w''(1) = Aw'''(1) + B^2\gamma(\mathcal{J}w')(1) = 0\}. \end{cases} \tag{6.10}$$

In this way, the system (6.9) is a typical second order collocated system studied in [7]. It is well-known that the system is exactly controllable if and only if it is exactly observable [4].

Now we apply the abstract results of [7] to the system (6.1). Since

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -\mathbb{A} & 0 \end{pmatrix}, \tag{6.11}$$

$\mathcal{A}_0(w_{n0}, \lambda_{n0}w_{n0}) = \lambda_{n0}(w_{n0}, \lambda_{n0}w_{n0})$ if and only if

$$\mathbb{A}e_n = -\lambda_{n0}^2 e_n = (i\lambda_{n0})^2 e_n,\ \omega_n = i\lambda_{n0},\ e_n = \frac{w_{n0}}{\|w_{n0}\|_{L^2(0,1)}}. \tag{6.12}$$

By (3.22) and (4.1), it follows that

$$\begin{cases} \omega_n = i\lambda_{n0} = -\dfrac{d_1d_2}{2r_1^2} - \dfrac{(\frac{1}{2}+n)^2\pi^2}{r_1^2} + \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty, \\ \lambda_{n0}\|w_{n0}\|_{L^2(0,1)} e_n(x) = \lambda_{n0}w_{n0}(x) = -ir_1^{-2}\mathrm{e}^{-ir_1\rho_n(x+1)} + (i-1)r_1^{-2}\mathrm{e}^{-ir_1\rho_n}\mathrm{e}^{-r_1\rho_n x} \\ \qquad\qquad + r_1^{-2}\mathrm{e}^{ir_1\rho_n(x-1)} + (i+1)r_1^{-2}\mathrm{e}^{r_1\rho_n(x-1)} + \mathcal{O}(n^{-1}). \end{cases} \tag{6.13}$$

Theorem 6.1.

(i). Let $T > 0$ be any constant and C_T be some positive constant depending on T. For any given initial condition $(w(\cdot,0), w_t(\cdot,0)) = (w_0, w_1) \in \mathcal{H}$ and control input $u \in L^2(0,T)$, there exists a unique solution to equation (6.1) such that $(w, w_t) \in C(0,T;\mathcal{H})$ satisfying

$$\|(w(\cdot,T), w_t(\cdot,T))\|^2_{\mathcal{H}} + \|y\|^2_{L^2(0,T)} \le C_T \left[\|(w_0,w_1)\|^2_{\mathcal{H}} + \|u\|^2_{L^2(0,T)}\right].$$

(ii). System (6.1) is regular. Precisely, if the initial condition is zero that $(w(\cdot,0), w_t(\cdot,0)) = (0,0)$ *and* $u(t) \equiv u$ *is a step control input, then the corresponding output response* y *satisfies*

$$\lim_{\sigma\to 0}\left|\frac{1}{\sigma}\int_0^\sigma y(1,t)\mathrm{d}t\right|^2 = 0.$$

(iii). System (6.1) is exactly controllable and observable on some $[0,T], T>0$.

Proof. By (4.2) and (6.13), it follows that $|b_n| = |\langle b, e_n\rangle_{[D(\mathcal{A}_0)]\times[D(\mathcal{A}_0)]'}| = |e_n(1)| < C$ for some constant $C>0$ and $|\omega_{n+1}-\omega_n| \geq \delta_0$ for some $\delta_0 > 0$ and all $n \geq 1$. By Proposition 2 of [7], b is admissible [3]. Moreover, since $|\omega_{n+1}-\omega_n| \geq \delta\omega_{n+1}^{\beta}$ for some constants δ, β, the hypotheses of Theorem 4 of [7] is satisfied. This together with the admissibility of b gives the *(i)*. Moreover, the transfer function of (6.1) tends to zero along the positive axis, which deduces *(ii)* of Theorem 6.1 by equivalent condition proved in [17].

Furthermore, by virtue of Theorem 5.3 and Theorem 4 of [7], the system (6.1) is exactly observable on some $[0,T], T>0$. This is actually the Russell's principle of "*exact controllability via exponential stability*" for hyperbolic systems. The proof is complete. □

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