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A NONLOCAL SINGULAR PERTURBATION PROBLEM WITH PERIODIC WELL POTENTIAL

Matthias Kurzke[1]

Abstract. For a one-dimensional nonlocal nonconvex singular perturbation problem with a noncoercive periodic well potential, we prove a Γ-convergence theorem and show compactness up to translation in all L^p and the optimal Orlicz space for sequences of bounded energy. This generalizes work of Alberti, Bouchitté and Seppecher (1994) for the coercive two-well case. The theorem has applications to a certain thin-film limit of the micromagnetic energy.

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1. Introduction

Alberti, Bouchitté and Seppecher [1] considered on $L^1(I)$, $I \subset \mathbb{R}$ an interval, the functionals

$$F_\varepsilon(u) = \varepsilon \iint_{I\times I} \left| \frac{u(x)-u(y)}{x-y} \right|^2 \mathrm{d}x\mathrm{d}y + \lambda_\varepsilon \int_I W(u)\mathrm{d}x, \tag{1.1}$$

where $W : \mathbb{R} \to [0,\infty]$ is continuous, $W^{-1}(0) = \{\alpha,\beta\}$, $W(t) \geq C(t^2-1)$ with some $C > 0$, and λ_ε satisfies $\varepsilon \log \lambda_\varepsilon \to K \in (0,\infty)$ as $\varepsilon \to 0$.

Here, the double integral represents (up to constants) the nonlocal $H^{1/2}$ seminorm of u. Similar functionals with local energies were studied before, see *e.g.* Modica [8], where the Dirichlet integral is used instead of the $H^{1/2}$ seminorm, and the scaling $\lambda_\varepsilon \sim \frac{1}{\varepsilon}$ leads to a Γ-convergence result. The study of (1.1) is motivated by the research [2], where Alberti *et al.* combine interior and boundary phase transitions. Regarding the Dirichlet integral as a functional on the boundary leads to the $H^{1/2}$ seminorm.

We study a different problem that also leads to essentially the same functional, just with a periodic potential W: Kohn and Slastikov [5] derived a reduced model for thin soft ferromagnetic films, and could show that certain rescalings of the full micromagnetic functional Γ-converge to functionals of the type

$$\mathcal{E}^\alpha(m) = \alpha \int_\Omega |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\Omega} (m\cdot n)^2, \tag{1.2}$$

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[1] Institute for Mathematics and its Applications, University of Minnesota, 400 Lind Hall, 207 Church Street SE, Minneapolis, MN 55455, USA; kurzke@ima.umn.edu

where n denotes the normal to $\partial\Omega$, in the space of $m \in H^1(\Omega, S^1)$, for a simply connected domain $\Omega \subset \mathbb{R}^2$. We will analyze the behavior of $\frac{1}{\alpha|\log\alpha|}\mathcal{E}^\alpha$ as $\alpha \to 0$. To simplify the analytic setting, we set $m = \mathrm{e}^{\mathrm{i}u}$ with $u \in H^1(\Omega)$ and $n = \mathrm{i}\mathrm{e}^{\mathrm{i}g}$, with a function g that is as smooth as n except for a single jump of height -2π. This leads to the functionals

$$\frac{1}{|\log\alpha|}\int_\Omega |\nabla u|^2 + \frac{1}{2\pi\alpha|\log\alpha|}\int_{\partial\Omega}\sin^2(u-g).$$

Considering this functional only on harmonic functions (which corresponds to replacing the Dirichlet integral by the $H^{1/2}$ seminorm of the boundary values) and generalizing to arbitrary periodic wells, we have the following result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected $C^{1,\beta}$ domain and denote the harmonic extension of a function $v : \partial\Omega \to \mathbb{R}$ to Ω by $h_v : \Omega \to \mathbb{R}$. Set for $u \in L^1(\partial\Omega)$*

$$\mathcal{G}^\eta(u) := \begin{cases} \eta\displaystyle\int_\Omega |\nabla h_u|^2 + \mu_\eta \int_{\partial\Omega} W(u-g) & \text{if } u \in H^{1/2}(\partial\Omega) \\ +\infty & \text{else,} \end{cases} \tag{1.3}$$

where $W : \mathbb{R} \to [0,\infty)$ is a continuous, π-periodic function with $W^{-1}(0) = \pi\mathbb{Z}$, $\eta, \mu_\eta > 0$, and $g : \partial\Omega \to \mathbb{R}$ is a function with a jump of height $2\pi d$ such that $\mathrm{e}^{\mathrm{i}g}$ can be extended as a $H^1(N, S^1)$ map to a neighborhood N of $\partial\Omega$, so g has (after possibly moving the jump point) extensions to $H^1(\Omega \setminus B_\rho(a))$ for any $a \in \partial\Omega$, $\rho > 0$. Assume that $\eta \log \mu_\eta \to K \in (0,\infty)$ as $\eta \to 0$ and set

$$\mathcal{G}(u) = \begin{cases} K\,\|D(u-g)\|\,(\partial\Omega) & \text{if } u - g \in BV(\partial\Omega, \pi\mathbb{Z}) \\ +\infty & \text{else.} \end{cases} \tag{1.4}$$

Then we have:

(i) Compactness up to translation:
If $\mathcal{G}^\eta(u_\eta) \le M < \infty$ then there exists a sequence of $z_\eta \in \pi\mathbb{Z}$ such that for $1 \le p < \infty$

$$\|u_\eta - z_\eta\|_{L^p(\partial\Omega)} \le C(p) < \infty. \tag{1.5}$$

Furthermore, $(u_\eta - z_\eta)$ is relatively compact in the strong topology of $L^1(\partial\Omega)$, and every cluster point u has the property that $u - g \in BV(\partial\Omega, \pi\mathbb{Z})$.

(ii) Lower bound:
If $u_\eta \to u$ in $L^1(\partial\Omega)$, then

$$\mathcal{G}(u) \le \liminf_{\eta\to 0} \mathcal{G}^\eta(u_\eta). \tag{1.6}$$

(iii) Upper bound / Construction:
Let $u \in L^1(\partial\Omega)$. Then there exists a sequence $u_\eta \to u$ in $L^1(\partial\Omega)$ such that

$$\mathcal{G}(u) = \lim_{\eta\to 0} \mathcal{G}^\eta(u_\eta). \tag{1.7}$$

Here we have replaced $\frac{1}{|\log\alpha|}$ of our previous notation by $\eta \to 0$ and $\frac{1}{2\pi\alpha|\log\alpha|}$ by $\mu_\eta \to \infty$.

Note that this is an extension of the result in [1], since the energy of a harmonic function can be calculated *via* the $H^{1/2}$ norm of its boundary trace, see Section 2 where we reduce the functional to a form more similar to (1.1). Unlike the two-well potential in [1], our periodic potential W cannot yield any *a priori* coercivity. However, we can still obtain compactness up to translation in all L^p and even determine up to constants an optimal Orlicz type space in which compactness holds, see Proposition 2.11 and Remark 2.12. The proof uses a more elaborate rearrangement result than the simple two-set rearrangement used in [1].

It is also possible to derive the Γ-convergence part of Theorem 1.1 from the result of [1] by a cutoff argument like in [3], but this approach does not lead to the compactness results obtained here.

Corollary 1.2. *The functionals $\mathcal{G}^\eta$ are equicoercive (this means "compactness") up to translation and Γ-converge to $\mathcal{G}$ with respect to all strong L^p topologies, $1 \le p < \infty$.*

Proof. Γ-convergence and equicoercivity in L^1 is the content of Theorem 1.1. For L^p, we note that strong compactness in L^p is by interpolation a consequence of strong compactness in L^1 and weak compactness in L^q for $q > p$, which holds by (i). The construction used for the proof of the upper bound part holds in all L^p. □

2. Localization of the functional

We look at the case $\Omega = B_1(0)$, in which case we have an explicit expression for the energy of the harmonic extension, *i.e.* the $H^{1/2}$ seminorm of the boundary trace.

Proposition 2.1. *If the results of Theorem 1.1 hold for $B_1(0)$, they hold for every simply connected $C^{1,\beta}$ domain.*

Proof. Let $u : \partial\Omega \to \mathbb{R}$ with harmonic continuation $h_u : \partial\Omega \to \mathbb{R}$. Let $\psi : \overline{B_1(0)} \to \overline{\Omega}$ be a conformal map. By the Kellogg-Warschawski theorem (see *e.g.* [12], Th. 3.6), $\psi \in C^{1,\beta}(\overline{B_1(0)})$.

Since the Dirichlet integral is invariant under conformal transformations, we have for $\tilde{u} = u \circ \psi$ that $h_{\tilde{u}} = \widetilde{h_u}$ and can calculate by the change of variables formula

$$\mathcal{G}^\eta(u) = \eta \int_{B_1(0)} |\nabla h_{\tilde{u}}|^2 + \mu_\eta \int_{S^1} W(\tilde{u} - \tilde{g}) \left| \frac{\partial}{\partial \tau} \psi \right|.$$

Now there are $c_1, c_2 > 0$ with $c_1 \le \left|\frac{\partial}{\partial\tau}\psi\right| \le c_2$ since ψ and its inverse are C^1 on the boundary. Thus we have that $\mathcal{G}^\eta$ is bounded from above and below by functionals

$$\eta \int_{B_1(0)} |\nabla h_{\tilde{u}}|^2 + c_i \mu_\eta \int_{S^1} W(\tilde{u} - \tilde{g}),$$

and since $\eta \log(c_i \mu_\eta) \to K$ as $\eta \to 0$ for $i = 1, 2$, we obtain the equality of the Γ-limits for these functionals. From this we can deduce the theorem for the $\tilde{u}_\eta$, but these converge if and only if the corresponding u_η converge. □

Proposition 2.2. *Let $u \in H^{1/2}(S^1)$ and $h_u \in H^1(B_1)$ be its harmonic continuation. Then*

$$\int_{B_1(0)} |\nabla h_u|^2 = \frac{1}{8\pi} \int_{S^1 \times S^1} \left| \frac{u(x) - u(y)}{\sin \frac{1}{2}(x - y)} \right|^2. \tag{2.1}$$

This can be proved by expanding u as a Fourier series and doing some clever summations, see *e.g.* [10], Section 311. Another proof by using the periodic Hilbert transform can be found in [14], Section 3.

Definition 2.3. For $\eta > 0$, $A \subset S^1$, and $u \in L^1(A)$, set

$$\mathcal{F}_g^\eta(u; A) = \begin{cases} \dfrac{\eta}{8\pi} \displaystyle\int_A \int_A \left| \frac{u(x) - u(y)}{\sin \frac{1}{2}(x - y)} \right|^2 \mathrm{d}x\mathrm{d}y + \mu_\eta \int_A W(u(x) - g(x))\mathrm{d}x & \text{if this is finite} \\ +\infty & \text{else,} \end{cases} \tag{2.2}$$

and

$$\mathcal{F}_g(u; A) = \begin{cases} K \left\| D(u - g) \right\| (A) & \text{if } u - g \in BV(A, \pi\mathbb{Z}) \\ +\infty & \text{else.} \end{cases} \tag{2.3}$$

We also set $\mathcal{F}^\eta := \mathcal{F}_0^\eta$ and $\mathcal{F} := \mathcal{F}_0$, and write

$$\mathcal{J}(u; A) = \frac{1}{8\pi} \int_A \int_A \left| \frac{u(x) - u(y)}{\sin \frac{1}{2}(x-y)} \right|^2 \mathrm{d}x\mathrm{d}y$$

for the localized form of the $H^{1/2}$ norm.

For these functionals we will prove the results corresponding to those for $\mathcal{G}^\eta = F_g^\eta(\cdot; S^1)$ and $\mathcal{G} = F_g(\cdot; S^1)$. Our main tool will be a rearrangement inequality. We use in the following the terms "decreasing" and "increasing" in the weak sense, *i.e.* denoting what is often called "non-increasing" and "non-decreasing", respectively.

Definition 2.4. For a measurable $f : A \to \mathbb{R}$ we define its *distribution function* λ_f by

$$\lambda_f(s) = |\{x \in A : |f(x)| > s\}|.$$

Definition 2.5. For a function $u : A \to \mathbb{R}$, $A = (a, b) \subset \mathbb{R}$ an interval, its *decreasing rearrangement* u^* is given by

$$u^*(x) = \inf \{s : \lambda_u(s) \le x - a\}.$$

Similarly the *increasing rearrangement* u_* is defined by

$$u_*(x) = \inf \{s : \lambda_u(s) \le b - x\}.$$

Clearly, u^* is decreasing and u_* increasing. Also, the rearrangement is equimeasurable, *i.e.* $\lambda_u = \lambda_{u^*} = \lambda_{u_*}$. See *e.g.* [7], Chapter 3.3.

Theorem 2.6. *Let $A \subset S^1$ be an interval of length $|A| < \pi$. Then*

$$\mathcal{J}(u_*; A) = \mathcal{J}(u^*; A) \le \mathcal{J}(u; A). \tag{2.4}$$

Proof. This follows from Theorem I.1 in Garsia and Rodemich [4]. □

2.1. L^p and Orlicz space estimates

Proposition 2.7. *Let $A \subset S^1$ be an interval of length $|A| < \pi$. Assume $\eta \to 0$ and let u_η be a sequence in $L^1(A)$ such that $\mathcal{F}^\eta(u_\eta) \le M < \infty$. Then there exist $z_\eta \in \pi\mathbb{Z}$ such that*

$$\|u_\eta - z_\eta\|_{L^p(A)} \le C(p, A, K, M, W). \tag{2.5}$$

Proof. We choose a sequence of $z_\eta \in \pi\mathbb{Z}$ such that $|\{u_\eta < z_\eta\}| \ge \frac{|A|}{4}$ and $|\{u_\eta > z_\eta - \pi\}| \ge \frac{|A|}{4}$. It suffices to show the L^p bounds for $v_\eta := (u_\eta - z_\eta)_+$ and $w_\eta := (u_\eta - (z_\eta - \pi))_-$. As this cutoff obviously decreases energy by the assumptions on W, we have $\mathcal{F}^\eta(v_\eta) \le \mathcal{F}^\eta(u_\eta) \le M$ and $\mathcal{F}^\eta(w_\eta) \le \mathcal{F}^\eta(u_\eta) \le M$. It therefore suffices to assume $u_\eta \ge 0$ and $|\{u_\eta = 0\}| \ge \frac{|A|}{4}$. Finally, since $\int_A W(u) = \int_A W(u_*)$ and by Theorem 2.6, we can assume all u_η to be increasing.

We will assume u_η to be nonnegative, increasing, and satisfying the bound $|\{u_\eta = 0\}| \ge \frac{|A|}{4}$ for the rest of this subsection.

Let λ_η denote the distribution function of u_η. The L^p norm of u_η over A can then be calculated as

$$\|u_\eta\|_p^p = p \int_0^\infty t^{p-1} \lambda_\eta(t) \mathrm{d}t.$$

Now by the Orlicz space estimate of Proposition 2.11 this is estimated as

$$\|u_\eta\|_p^p \le C_1 \int_0^\infty t^{p-1} \exp(-C_2 t) \mathrm{d}t \le C(p, M, K, W, A).$$ □

The following lemma contains the main computations that lead to the lower bound and compactness results.

Lemma 2.8. *Let* $\delta \in (0, \frac{\pi}{2})$ *and* $s \in \mathbb{N}$. *For* $u \in H^{1/2}(A)$, *set* $a_0 := \big|\{x : u(x) < \delta\}\big|$, $a_s := \big|\{x : u(x) > s\pi - \delta\}\big|$, *and* $\rho := \big|\{x : \mathrm{dist}(u(x), \pi\mathbb{Z}) > \delta\}\big|$. *Let*

$$L(z) := \log \sin \frac{z}{2} - \log \sin \frac{|A|}{2}.$$

Then

$$\mathcal{J}(u; A) \geq \pi s^2 (L(a_0 + \rho) + L(a_s + \rho)) - \pi s \left(1 - \frac{2\delta}{\pi}\right)^2 L(\rho). \tag{2.6}$$

Proof. By (2.4) we can assume u to be increasing. We set

$$\begin{aligned} A_0 &:= \{x : u(x) < \delta\}, \\ A_j &:= \{x : u(x) \in (j\pi - \delta, j\pi + \delta)\} \quad \text{for } j = 1, \dots, s-1, \\ A_s &:= \{x : u(x) > s\pi - \delta\} \end{aligned}$$

and

$$P_j := \{x : u(x) \in [j\pi + \delta, (j+1)\pi - \delta]\} \quad \text{for } j = 1, \dots, s-1.$$

We also define $a_k := |A_k|$ and $\rho_j = |P_j|$ for $k = 0, \dots, s$ and $j = 1, \dots, s-1$ respectively. By assumption we have $a_0 \geq \frac{1}{4}|A|$.

Using the monotonicity of u, we can then estimate the $H^{1/2}$ norm as follows:

$$\mathcal{J}(u; A) \geq \frac{1}{4\pi} \sum_{0 \leq j < k \leq s} \int_{A_j} \int_{A_k} \frac{(u(x) - u(y))^2}{\sin^2(\frac{1}{2}(x-y))} \mathrm{d}x\mathrm{d}y, \tag{2.7}$$

and using the definitions of A_k we arrive at

$$\mathcal{J}(u; A) \geq \frac{1}{4\pi} \sum_{0 \leq j < k \leq s} (\pi(k-j) - 2\delta)^2 \int_{A_j} \int_{A_k} \frac{1}{\sin^2(\frac{1}{2}(x-y))} \mathrm{d}x\mathrm{d}y. \tag{2.8}$$

For $\beta_1 < \beta_2 < \alpha_1 < \alpha_2$, we evaluate the integral

$$\int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \frac{1}{\sin^2(\frac{x-y}{2})} \mathrm{d}x\mathrm{d}y = 4 \log \frac{\sin(\frac{\alpha_1 - \beta_1}{2}) \sin(\frac{\alpha_2 - \beta_2}{2})}{\sin(\frac{\alpha_1 - \beta_2}{2}) \sin(\frac{\alpha_2 - \beta_1}{2})}. \tag{2.9}$$

As u is an increasing function, the positions of the A_j and P_j are determined by their measures only, and so (2.8) and (2.9) lead to the estimate

$$\begin{aligned} \mathcal{J}(u; A) \geq \pi \sum_{0 \leq j < k \leq s} (k - j - 2\frac{\delta}{\pi})^2 \Big(& L(a_j + \rho_j + \dots + a_{k-1} + \rho_{k-1}) \\ & + L(\rho_j + a_{j+1} + \dots + \rho_{k-1} + a_k) - L(a_j + \rho_j + \dots + \rho_{k-1} + a_k) \\ & - L(\rho_j + a_{j+1} + \dots + a_{k-1} + \rho_{k-1}) \Big), \end{aligned}$$

which can be further estimated below by replacing all terms of type $\sum_{i=j}^{k-1} \rho_i$ by $\rho := \sum_{i=1}^{s-1} \rho_i$, as follows from (2.9) since this essentially amounts to moving A_j and A_k further apart, and $z \mapsto \frac{1}{\sin^2 \frac{z}{2}}$ is decreasing in z.

We introduce the further abbreviations

$$Q_d := \left(d - 2\frac{\delta}{\pi}\right)^2$$

and

$$T_j^k := L\left(\rho + \sum_{i=j}^{k} a_i\right).$$

Note that by definition of the empty sum, we have $T_j^k = L(\rho)$ if $j > k$.

We now calculate

$$\begin{aligned}
\frac{1}{\pi}\mathcal{J}(u;A) &\geq \sum_{j=0}^{s-1}\sum_{d=1}^{s-j} Q_d(T_j^{j+d-1} + T_{j+1}^{j+d} - T_j^{j+d} - T_{j+1}^{j+d-1}) \\
&= \sum_{j=0}^{s-1}\sum_{d=0}^{s-j-1} Q_{d+1}T_j^{j+d} + \sum_{j=1}^{s}\sum_{d=1}^{s-j+1} Q_d T_j^{j-1+d} - \sum_{j=0}^{s-1}\sum_{d=1}^{s-j} Q_d T_j^{j+d} - \sum_{j=1}^{s}\sum_{d=1}^{s-j+1} Q_d T_j^{j+d-2} \\
&= \sum_{j=0}^{s-1}\sum_{d=0}^{s-j-1} Q_{d+1}T_j^{j+d} + \sum_{j=1}^{s}\sum_{d=0}^{s-j} Q_{d+1}T_j^{j+d} - \sum_{j=0}^{s-1}\sum_{d=1}^{s-j} Q_d T_j^{j+d} - \sum_{j=1}^{s}\sum_{d=-1}^{s-j-1} Q_{d+2} T_j^{j+d} \\
&= \sum_{j=1}^{s-1}\sum_{d=1}^{s-j-1} (2Q_{d+1} - Q_d - Q_{d+2})T_j^{j+d} + \sum_{j=1}^{s-1} Q_1 T_j^j + \sum_{d=0}^{s-1} Q_{d+1}T_0^d + \sum_{j=1}^{s-1} Q_{s-j+1}T_j^s \\
&\quad + Q_1 T_s^s - \sum_{j=1}^{s-1} Q_{s-j}T_j^s - \sum_{d=1}^{s} Q_d T_0^d - \sum_{j=1}^{s-1}\sum_{d=-1}^{0} Q_{d+2}T_j^{j+d} - Q_1 T_s^{s-1} \\
&= \sum_{j=1}^{s-1}\sum_{d=1}^{s-j-1} (2Q_{d+1} - Q_d - Q_{d+2})T_j^{j+d} + \sum_{j=1}^{s-1}(Q_1 - Q_2)T_j^j - \sum_{j=1}^{s} Q_1 T_j^{j-1} + Q_1 T_0^0 \\
&\quad + Q_1 T_s^s - Q_s T_0^s + \sum_{j=1}^{s-1}(Q_{j+1} - Q_j)T_0^j + \sum_{j=1}^{s-1}(Q_{s-j+1} - Q_{s-j})T_j^s.
\end{aligned}$$

Taking into account that $2Q_{d+1} - Q_d - Q_{d+2} = -2$ and $Q_{k+1} - Q_k = 2k + 1 - 4\frac{\delta}{\pi}$ this can be further simplified to

$$\begin{aligned}
\frac{1}{\pi}\mathcal{J}(u;A) &\geq -2\sum_{j=1}^{s-1}\sum_{d=1}^{s-j-1} T_j^{j+d} - (3 - 4\tfrac{\delta}{\pi})\sum_{j=1}^{s-1} T_j^j - (s - 2\tfrac{\delta}{\pi})^2 T_0^s \\
&\quad + \sum_{j=1}^{s-1}(2j + 1 - 4\tfrac{\delta}{\pi})(T_0^j + T_{s-j}^s) + (1 - 2\tfrac{\delta}{\pi})^2(T_0^0 + T_s^s) - sQ_1 L(\rho).
\end{aligned} \tag{2.10}$$

As $T_j^k \leq 0$, the inequality still holds when we omit the first three terms in (2.10). For the same reason, we can omit δ in all terms but the last one. Using further $1 + \sum_{j=1}^{s-1}(2j+1) = s^2$ and estimating $T_0^j \geq T_0^0$ and $T_{s-j}^s \geq T_s^s$, we obtain

$$\mathcal{J}(u;A) \geq \pi s^2(T_0^0 + T_s^s) - \pi s Q_1 L(\rho) \tag{2.11}$$

and by the definitions of T and L we arrive at the claim. □

Lemma 2.9. *There is a constant $\eta_1 = \eta_1(A) > 0$ such that for all $\eta < \eta_1$, the distribution function λ_η of u_η satisfies for all $s \in \mathbb{N}$ with $s < \frac{1}{\pi\eta}$ the inequality*

$$\lambda_\eta(\pi s) \leq 8|A|^{1-\frac{1}{s}} \exp \frac{M + C_0 - sK}{\pi\eta s^2} \tag{2.12}$$

for some $C_0 = C_0(W) > 0$.

Proof. Choose a $\delta > 0$ small and set $\sigma = \min\{W(t) : \delta \leq t \leq \pi - \delta\} > 0$. Using Lemma 2.8 and the notation used there, we can estimate

$$\begin{aligned} M \geq \mathcal{F}^\eta(u_\eta; A) &= \eta\mathcal{I}(u; A) + \mu_\eta \int_A W(u_\eta) \\ &\geq \eta\pi s^2(T_0^0 + T_s^s) - s\eta\pi Q_1 L(\rho) + \mu_\eta\sigma\rho. \end{aligned}$$

From the estimate $\log x \leq x$ we deduce

$$Bz \geq \log \frac{2Bz}{2} = \log(2B) + \log\frac{z}{2} \geq \log\sin\frac{z}{2} + \log(2B)$$

so setting $L_0 := \log\sin\frac{|A|}{2}$ we have $-L(z) + Bz \geq \log(2B) + L_0$, and we obtain

$$M \geq \pi s^2\eta(T_0^0 + T_s^s) + \pi s\eta Q_1 \log\frac{2\mu_\eta\sigma}{\pi s\eta Q_1} + \pi s\eta Q_1 L_0$$

from which it follows that

$$T_0^0 + T_s^s \leq \frac{1}{\pi\eta s^2}\left(M - \pi s Q_1\eta\log\mu_\eta - \pi s\eta Q_1\log\frac{1}{\pi s\eta Q_1} - \pi s\eta Q_1(L_0 + \log(2\sigma))\right). \tag{2.13}$$

By the inequality $x\log\frac{1}{x} > 0$ for $0 < x < 1$, we can omit the term $\pi s\eta Q_1\log\frac{1}{\pi s\eta Q_1}$ in (2.13) as long as $s < \frac{1}{\pi Q_1\eta}$, in particular for $s < \frac{1}{\pi\eta}$. We choose δ sufficiently small so $\pi Q_1 > \frac{4}{3}$ (this also defines σ) and η_1 so small that $\eta\log\mu_\eta < \frac{3}{4}K$ for $\eta < \eta_1$, so $\pi Q_1\eta\log\mu_\eta > K$. For $s < \frac{1}{\pi Q_1\eta}$, we can also estimate $-\pi s Q_1\log(2\sigma) < -\log(2\sigma)$. Using the definitions of T, L, and L_0, we obtain that

$$\begin{aligned} \sin\frac{1}{2}(a_s + \rho)\sin\frac{1}{2}(a_0 + \rho) &\leq \sin^2\frac{|A|}{2}\exp\left(\frac{M - \log(2\sigma) - sK}{\pi\eta s^2} - \frac{Q_1 L_0}{s}\right) \\ &\leq \left(\sin\tfrac{|A|}{2}\right)^{2-\frac{1}{s}}\exp\frac{M - \log(2\sigma) - sK}{\pi\eta s^2}. \end{aligned}$$

Since $\frac{1}{4}z \leq \sin\frac{1}{2}z < \frac{1}{2}z$ for $z < \pi$ and $a_0 \geq \frac{1}{4}|A|$, this shows for $s \geq 1$ that

$$a_s < 8|A|^{1-\frac{1}{s}}\exp\frac{M - \log(2\sigma) - sK}{\pi\eta s^2} \tag{2.14}$$

and this finishes the proof (with $C_0 = -\log(2\sigma)$) since $\lambda_\eta(\pi s) \leq \lambda_\eta(\pi s - \delta) = a_s$. □

Lemma 2.10 (Trudinger-Moser inequality)**.** *There are constants $\gamma, C > 0$ such that every function $u \in H^{1/2}(S^1)$ with $\operatorname{supp} u \subset A \subset S^1$, A a small interval, satisfies the inequality*

$$\int_A \exp\left(\frac{\gamma u^2}{\mathcal{I}(u; A)}\right) \leq C|A|. \tag{2.15}$$

Proof. For a function v supported in a fixed interval, say $[0,1]$, the Trudinger-Moser inequality (see *e.g.* [13], Chap. 13.4) yields

$$\int_{[0,1]} \exp\left(\frac{\gamma v^2}{\|v\|^2_{H^{1/2}(\mathbb{R})}}\right) \le C.$$

Using an appropriate Poincaré inequality, we can replace, by changing γ appropriately, the full $H^{1/2}$ norm by the seminorm $\|\cdot\|_{\dot H^{1/2}}$. From the scaling invariance of this seminorm, we obtain for a function supported in $[0,r]$ that

$$\int_{[0,r]} \exp\left(\frac{\gamma v^2}{\|v\|^2_{\dot H^{1/2}(\mathbb{R})}}\right) \le Cr, \tag{2.16}$$

and this estimate stays valid if we calculate the seminorm on $[0,r]$ instead or all of $\mathbb{R}$. For $|A| = r$ sufficiently small, the square of this seminorm is equivalent to $\mathcal{J}(u;A)$, and we obtain (2.15). □

Proposition 2.11. *There are constants $C_1, C_2 > 0$ depending on A, M, K, W such that the distribution function λ_η of u_η satisfies for η sufficiently small the estimate*

$$\lambda_\eta(t) \le C_1 \mathrm{e}^{-C_2 t}. \tag{2.17}$$

Proof. For $t > 4\frac{M+C_0}{K}$, C_0 the constant from Lemma 2.9, we set $s = t - 2\frac{M+C_0}{K} \ge \frac{t}{2}$.

From Lemma 2.9 that we use on a suitable integer N close to $2\frac{M+C_0}{K}$ and Lemma 2.10 applied to $(u_\eta - N)_+$ on the interval $\{u_\eta \ge N\}$, we then obtain

$$\lambda_\eta(t) \le c_1 \exp\left(-\frac{c_2}{\eta} - c_3 \eta s^2\right) \le c_1 \exp(-c_4 s) \le c_1 \exp\left(-\frac{c_4}{2} t\right)$$

by the inequality $\frac{a}{\eta} + b\eta \ge 2\sqrt{ab}$. Combining this with the trivial estimate $\lambda_\eta(t) \le |A|$ for $t \le 4\frac{M+C_0}{K}$, we arrive at (2.17). □

Remark 2.12. It is possible to construct examples showing that there can be no uniform L^∞ bounds for sequences of bounded energy, and that the decay estimate given in Proposition 2.11 is essentially optimal. We define for $k \in \mathbb{Z}$ the sequence $u_k : \mathbb{R}^2 \to \mathbb{R}$ by

$$u_k(z) = \begin{cases} k & \text{if} \quad |z-1| \le \mathrm{e}^{-2k}, \\ \log \frac{1}{|z-1|} - k & \text{if} \quad \mathrm{e}^{-2k} < |z-1| < \mathrm{e}^{-k}, \\ 0 & \text{if} \quad |z-1| \ge \mathrm{e}^{-k}. \end{cases} \tag{2.18}$$

It is easy to check that $\|\nabla u_k\|^2_{L^2(\mathbb{R}^2)} = k$. With $g = 0$ and $v_k = u_k\big|_{\partial B_1(0)}$, we obtain for any W satisfying the hypotheses of Theorem 1.1 that

$$\mathcal{H}^1(\{x \in S^1 : W(v_k(x) - g(x)) \ne 0\} \le c\mathrm{e}^{-k}.$$

We set $\eta = \frac{1}{k}$ and $\mu_\eta = \mathrm{e}^k$ so $\eta \log \mu_\eta = 1$. The functions v_k now satisfy

$$\mathcal{F}^\eta(v_k) \le \frac{1}{k} \|\nabla u_k\|^2_{L^2(\mathbb{R}^2)} + \mathrm{e}^k c\mathrm{e}^{-k} \sup W \le c \sup W + 1,$$

so their energy is uniformly bounded, but the L^∞ norm converges to $+\infty$. The distribution function of λ_k of v_k satisfies $\lambda_k(k) \approx \mathrm{e}^{-2k}$, which corresponds up to constants to the result of Proposition 2.11.

2.2. The lower bound

Proposition 2.13. *Let $A \subset S^1$ and $u_\eta \in L^1(A)$ be a sequence such that $\mathcal{F}^\eta(u_\eta) \le M < \infty$ and $u_\eta \rightharpoonup u$ in some L^p, $1 \le p < \infty$. Then (u_η) is relatively compact in the strong topology of $L^1(A)$.*

Additionally, we have that for every sequence $u_\eta \to u$ in $L^1(A)$,

$$\mathcal{F}(u) \le \liminf_{\eta\to 0} \mathcal{F}^\eta(u_\eta), \tag{2.19}$$

so every cluster point u belongs to $BV(A, \pi\mathbb{Z})$.

Proof. Let $(\nu_x)_{x\in A}$ be the Young measure generated by u_η. Since $\int_A W(u_\eta) \le \frac{M}{\mu_\eta} \to 0$, the sequence $W(u_\eta)$ is relatively compact in $L^1(A)$, and so we can apply the fundamental theorem on Young measures (see [11], Th. 6.2 or [9], Th. 3.1) which shows

$$\int_{\mathbb{R}} W(t)\mathrm{d}\nu_x(t) = 0 \quad \text{for } a.e.\ x \in A. \tag{2.20}$$

and by the assumptions on u_η we also have

$$u(x) = \int_{\mathbb{R}} t\mathrm{d}\nu_x(t) \quad \text{for } a.e.\ x \in A. \tag{2.21}$$

As $W \ge 0$, $W(z) = 0$ exactly for $z \in \pi\mathbb{Z}$, (2.20) shows that $\operatorname{supp}\nu_x \subset \pi\mathbb{Z}$ for $a.e.$ $x \in A$. Since ν_x is a probability measure $a.e.$, we can find for each $j \in \mathbb{Z}$ a measurable function

$$\theta_j : S^1 \to [0,1] \tag{2.22}$$

such that

$$\sum_{j\in\mathbb{Z}} \theta_j(x) = 1 \quad \text{for } a.e.\ x \in S^1 \tag{2.23}$$

and

$$\nu_x = \sum_{j\in\mathbb{Z}} \theta_j(x)\delta_{\pi j}. \tag{2.24}$$

We will show that these functions θ_j are of class $BV(A, \{0,1\})$. To this end, we define the set

$$S := \left\{ x \in A : \text{ there is a } j \in \mathbb{Z} \text{ such that } \operatorname*{ap\,lim}_{y\to x} \theta_j(y) \notin \{0,1\} \right\} \tag{2.25}$$

and consider an $x_0 \in S$. By (2.22) and (2.23) it is clear that there are $s_1 < s_2 \in \mathbb{Z}$ such that the corresponding approximate limits of θ_{s_1} and θ_{s_2} are neither 0 nor 1. In a small interval $J \subset A$ centered around x_0, we use Lemma 2.8 with

$$\begin{aligned} s &= s_2 - s_1, \\ a_\eta^0 &= |\{x \in J : u_\eta(x) < \pi s_1 + \delta\}|, \\ a_\eta^s &= |\{x \in J : u_\eta(x) > \pi s_2 - \delta\}|, \\ \rho_\eta &= |\{x \in J : \operatorname{dist}(u_\eta(x), \pi\mathbb{Z}) \ge \delta \text{ and } u_\eta(x) \in (s_1\pi, s_2\pi)\}|. \end{aligned}$$

We obtain with $Q_1 = (1 - \frac{2\delta}{\pi})^2$ and $L(z) := \log\sin\frac{z}{2} - \log\sin\frac{|J|}{2}$ the inequality

$$\liminf_{\eta\to 0} \mathcal{F}^\eta(u_\eta, J) \ge \eta(\pi s^2(L(a_\eta^0 + \rho_\eta) + L(a_\eta^s + \rho_\eta)) - \pi s Q_1 L(\rho_\eta) + \mu_\eta \sigma \rho_\eta.$$

As can be seen by suitable integrations over ν_x (take a continuous function that is 1 for $x < \pi s_1$ and 0 for $x > \pi s_1 + \delta$), $\liminf_{\eta\to 0} a_\eta^0 \geq \int_J \theta_{s_1} > 0$ and similarly $\liminf_{\eta\to 0} a_\eta^s > 0$, and so we have we have $\lim_{\eta\to 0} \eta L(a_\eta^0 + \rho_\eta) = \lim_{\eta\to 0} \eta L(a_\eta^s + \rho_\eta) = 0$. The limit estimate thus can be simplified to

$$\liminf_{\eta\to 0} \mathcal{F}^\eta(u_\eta, J) \geq \liminf_{\eta\to 0}(-\pi s Q_1 \eta L(\rho_\eta) + \mu_\eta \sigma \rho_\eta).$$

Using the estimate $-L(z) + Bz \geq \log(2B) + \log\sin\frac{|J|}{2}$, this shows

$$\begin{aligned}\liminf_{\eta\to 0} \mathcal{F}^\eta(u_\eta, J) &\geq \liminf_{\eta\to 0} \pi s Q_1 \eta \left(-L(\rho_\eta) + \frac{\mu_\eta \sigma}{\pi s Q_1 \eta}\rho_\eta\right) \\ &\geq \liminf_{\eta\to 0} \pi s Q_1 \eta \log \frac{2\mu_\eta \sigma \sin\frac{|J|}{2}}{\pi s Q_1 \eta},\end{aligned}$$

where the last term converges for $\eta \to 0$ since $\eta \log \mu_\eta \to K$ and $\eta \log \frac{C}{\eta} \to 0$ for any $C > 0$, so we obtain

$$\liminf_{\eta\to 0} \mathcal{F}^\eta(v_\eta, J) \geq \pi s Q_1 K. \tag{2.26}$$

Letting $\delta \to 0$ we have $Q_1 \to 1$ so we even have

$$\liminf_{\eta\to 0} \mathcal{F}^\eta(v_\eta, J) \geq \pi s K. \tag{2.27}$$

By the assumption $\mathcal{F}^\eta(v_\eta) \leq M$, we see that $s = s_2 - s_1$ must be bounded. Using the superadditivity of $\mathcal{F}^\eta$, we also see that S must be finite. This also shows that at almost any $x \in S^1$, only one of the functions θ_j can be nonzero. In particular, ν_x is a Dirac measure everywhere. This shows $u \in BV(S^1, \pi\mathbb{Z})$, and the limit estimate follows from adding up (2.27) with the maximum possible s around every $x \in S_u$.

If u_η has only been converging weakly in some L^p, then the fact that ν_x is Dirac improves this to strong convergence in L^1 as claimed. □

3. Extension to $g \neq 0$

Here we show how the lower bound from Theorem 1.1 (in its localized form) follows from the special case for $g = 0$ that was treated above.

Let $A \subset S^1$ be an intervals of length $|A| < \pi$. We can choose a representative for g that has no jump in A. Setting $v_\eta := u_\eta - g$, we have that

$$\mathcal{F}_g^\eta(u_\eta; A) = \mathcal{F}^\eta(v_\eta) + \eta \int_A \int_A \frac{(u_\eta(x) - u_\eta(y))^2 - (v_\eta(x) - v_\eta(y))^2}{\sin^2 \frac{1}{2}(x-y)} \mathrm{d}x\mathrm{d}y.$$

Now we calculate (with $u_\eta(x) =: u_1$ and $u_\eta(y) =: u_2$ etc.)

$$(u_1 - u_2)^2 - (u_1 - g_1 - (u_2 - g_2))^2 = 2(u_1 - u_2)(g_1 - g_2) - (g_1 - g_2)^2. \tag{3.1}$$

By Cauchy-Schwarz inequality, we estimate

$$\begin{aligned}&\left|\int_A \int_A \frac{(u(x) - u(y))(g(x) - g(y))}{\sin^2 \frac{1}{2}(x-y)} \mathrm{d}x\mathrm{d}y\right| \\ &\qquad \leq \left(\int_A \int_A \frac{(u(x) - u(y))^2}{\sin^2 \frac{1}{2}(x-y)} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_A \int_A \frac{(g(x) - g(y))^2}{\sin^2 \frac{1}{2}(x-y)} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \leq \sqrt{\frac{M}{\eta}} c(g),\end{aligned}$$

since $\mathcal{F}^\eta(u) \leq M$ and g has a H^1 extension to a domain containing A in its boundary, so the g-integral is bounded. This and (3.1) show

$$\mathcal{F}^\eta(u_\eta; A) - \sqrt{\eta M} - \eta c(g)\mathcal{F}_g^\eta(u; A) \leq \mathcal{F}^\eta(u_\eta; A) + \sqrt{\eta M} + \eta c(g) \tag{3.2}$$

so $\mathcal{F}_g^\eta(\cdot; A)$ and $\mathcal{F}^\eta(\cdot; A)$ have the same compactness behaviour and Γ-limits.

We can now obtain the Γ-lim inf and compactness results on S^1 by covering it with small intervals A_i on which we use the lower bound from Proposition 2.13. This yields a lower bound for the functional on S^1 since $\mathcal{F}_g^\eta$ is superadditive.

4. The upper bound

Here we prove part (iii) of Theorem 1.1 in the case of S^1, which by Proposition 2.1 is enough to prove the general case. Let u be such that $v = u - g \in BV(S^1, \pi\mathbb{Z})$ is a function with jump set S. Let $x_0 \in S$ be a jump point with approximate limits $v(x-) = \pi s_1$, $v(x+) = \pi s_2$, $s_1, s_2 \in \mathbb{Z}$, where we can assume w.l.o.g. $s_2 - s_1 = r > 0$. For $\delta_\eta \to 0$ and $\varkappa_\eta \to 0$ to be chosen later, we define v_η in a neighborhood of x_0 as

$$v_\eta(x) = \begin{cases} \pi s_1 & \text{if } x < x_0 \\ \pi(s_1 + j) & \text{if } x \in (x_0 + j(\delta_\eta + \varkappa_\eta), x_0 + j(\delta_\eta + \varkappa_\eta) + \varkappa_\eta) \quad (1 \leq j \leq r-1) \\ \pi s_2 & \text{if } x > x_0 + r(\delta_\eta + \varkappa_\eta), \end{cases} \tag{4.1}$$

and linear interpolation in the remaining parts. Proceeding like this around every $x_0 \in S$, it is easy to see that we obtain a sequence (v_η) with $u_\eta = v_\eta + g \to u$ in all L^p, $1 \leq p < \infty$.

Calculating $\mathcal{F}^\eta(u_\eta)$, we obtain for the single integral a bound

$$\int_{S^1} W(u_\eta - g)\mathrm{d}x \leq C\delta_\eta, \tag{4.2}$$

where $C = C(S, \|u\|_\infty)$.

We split the double integral over $S^1 \times S^1$ for the $H^{1/2}$ norm up into integrations over the finitely many pairs of definition intervals. Analogously to what we did in (3.2) we can use v_η instead of u_η for the calculations as long as the $H^{1/2}$-norms stay bounded.

Most of the integrals over two definition intervals of v_η are easily seen to be $O(1)$ in δ_η, so they will go to 0 when multiplied with η. The only interesting terms are those arising from the constancy intervals of v_η near a jump point. Their contribution around one jump point can then be written (by appropriate change of variables and using the shorthand $\delta = \delta_\eta$, $\varkappa = \varkappa_\eta$) as

$$\frac{\pi}{2} \sum_{0 \leq j < k \leq r} (k-j)^2 \int_{j(\delta+\varkappa)}^{j(\delta+\varkappa)+\varkappa} \int_{k(\delta+\varkappa)}^{k(\delta+\varkappa)+\varkappa} \frac{1}{\sin^2(\frac{x-y}{2})} \mathrm{d}x\mathrm{d}y, \tag{4.3}$$

which can be approximated using $\sin z \sim z$ as

$$2\pi \sum_{0 \leq j < k \leq r} (k-j)^2 \log \frac{(k-j)^2(\varkappa+\delta)^2}{(k-j)^2(\varkappa+\delta)^2 - \varkappa^2}.$$

We can rewrite

$$\frac{(k-j)^2(\varkappa+\delta)^2}{(k-j)^2(\varkappa+\delta)^2 - \varkappa^2} = \frac{1}{1 - \frac{1}{(k-j)^2(1+\frac{\delta}{\varkappa})^2}},$$

so we see that for $\frac{\delta}{\varkappa} \to 0$, the terms in (4.3) with $k - j > 1$ will be $O(1)$. Considering the $k - j = 1$ terms gives us

$$\log \frac{(\varkappa+\delta)^2}{(2\varkappa+\delta)\delta} = \log\left(\frac{(1+\frac{\delta}{\varkappa})^2}{(2+\frac{\delta}{\varkappa})}\frac{\varkappa}{\delta}\right)\cdot$$

Calculating for $r > 1$ the contribution of the integral over the "long" intervals on both sides of a multiple jump, we have a term of the form

$$\frac{\pi r^2}{2}\int_{-a}^{0}\int_{r(\delta_\eta+\varkappa_\eta)}^{a}\frac{1}{\sin^2(\frac{x-y}{2})}\mathrm{d}x\mathrm{d}y \sim 2\pi r^2 \log\frac{a}{2r(\delta_\eta+\varkappa_\eta)} = 2\pi r^2\log\frac{1}{\varkappa_\eta} + O(1).$$

Combining everything, we see we arrive at the assertion of the theorem if only

$$\varkappa_\eta \to 0, \frac{\delta_\eta}{\varkappa_\eta} \to 0, \eta\log\frac{1}{\varkappa_\eta} \to 0 \text{ and } \eta\log\frac{\varkappa_\eta}{\delta_\eta} \to K.$$

A possible choice is

$$\varkappa_\eta = \eta \text{ and } \delta_\eta = \frac{\eta}{\mu_\eta}\cdot \tag{4.4}$$

This finishes the proof of the upper bound part of Theorem 1.1. □

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