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# STABILIZATION OF A LAYERED PIEZOELECTRIC 3-D BODY BY BOUNDARY DISSIPATION

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**Abstract.** We consider a linear coupled system of quasi-electrostatic equations which govern the evolution of a 3-D layered piezoelectric body. Assuming that a dissipative effect is effective at the boundary, we study the uniform stabilization problem. We prove that this is indeed the case, provided some geometric conditions on the region and the interfaces hold. We also assume a monotonicity condition on the coefficients. As an application, we deduce exact controllability of the system with boundary control via a classical result due to Russell.

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## 1. Introduction

The quasi-electrostatic equations are perhaps the most standard model for piezoelectricity. Here we consider the evolution problem of a piezoelectric structure whose 3-D displacement field $u = u(x,t) = (u_1, u_2, u_3)$ and scalar electric potential $\varphi = \varphi(x,t)$ is given by a model on a bounded domain $\Omega$ of $\mathbb{R}^3$ with smooth boundary $\partial\Omega = S_0 \cup S_1$. Here $S_0$ and $S_1$ are two disjoint closed surfaces and the system read as follows

$$\begin{cases} u_{tt} - \operatorname{div} T(u,\varphi) = 0 \\ -\operatorname{div} \mathcal{D}(u,\varphi) = 0 \end{cases} \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.1}$$

where $u$ is the mechanical displacement and $\varphi$ is the electric potential. Some classical references where such models were deduced are [5,6]. From now on summation convention with respect to repeated indices will be used. In this quasi-electrostatic piezoelectric system $T(u,\varphi)$ is the mechanical stress, $\mathcal{D}$ is the electric displacement

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and $T$ and $\mathcal{D}$ satisfy the constitutive equations

$$T^{ij}(u,\varphi) = c_{ijk\ell}\,\varepsilon_{k\ell}(u) + e_{kij}\,\frac{\partial\varphi}{\partial x_k}$$
$$\mathcal{D}^{i}(u,\varphi) = -e_{ik\ell}\,\varepsilon_{k\ell}(u) + d_{ij}\,\frac{\partial\varphi}{\partial x_j}$$

where the stress tensor $T = (T^{ij})$ and the electric displacement $\mathcal{D} = (\mathcal{D}^i)$ are related to the linearized deformation tensor $\varepsilon(u) = (\varepsilon_{ij}(u))$ whose components are given by $\varepsilon_{k\ell}(u) = \frac{1}{2}\big(\frac{\partial u^\ell}{\partial x_k} + \frac{\partial u^k}{\partial x_\ell}\big)$. The 4th-order elasticity tensor $(c_{ijk\ell})$ is symmetric and positive, the 3rd-order coupling tensor $(e_{kij})$ is symmetric and the 2nd-order dielectric tensor $(d_{ij})$ is symmetric and positive.

In this article we prefer to rewrite the coupled system (1.1) in a more convenient form which will make more transparent our discussion of the so-called transmission problem. Let $A_i = [e_{k\ell i}]$, $D(x) = [d_{k\ell}(x)]$ and $A_{ij} = [a^{ij}_{k\ell}]$ be $3\times 3$ matrices where $a^{ij}_{k\ell}$ is given by

$$a^{ij}_{kh} = (1-\delta_{ih}\delta_{jk})c_{ikjh} + \delta_{ik}\delta_{jh}\,c_{ihjk}$$

where $\delta_{ij}$ denotes the standard Kronecker delta. In these notations we have that

$$\operatorname{div}\left\{A_k\,\frac{\partial u}{\partial x_k}\right\} = \frac{\partial}{\partial x_i}\,\{e_{ik\ell}\,\varepsilon_{k\ell}(u)\}$$
$$\frac{\partial}{\partial x_i}\{A_i^*\nabla\varphi\} = \frac{\partial}{\partial x_j}\,\left\{e_{k1j}\,\frac{\partial\varphi}{\partial x_k}, e_{k2j}\,\frac{\partial\varphi}{\partial x_k}, e_{k3j}\,\frac{\partial\varphi}{\partial x_k}\right\}$$

and

$$\frac{\partial}{\partial x_i}\left\{A_{ij}\,\frac{\partial u}{\partial x_j}\right\} = \frac{\partial}{\partial x_j}\{c_{1jkh}\varepsilon_{kh}(u), c_{2jkh}\varepsilon_{kh}(u), c_{3jkh}\varepsilon_{kh}(u)\}$$

where $\nabla$ denotes the (spatial) usual gradient operator.

Using the above notation we rewrite system (1.1) in the form

$$\begin{cases} u_{tt} - \dfrac{\partial}{\partial x_i}\left\{A_{ij}\,\dfrac{\partial u}{\partial x_j} + A_i^*\nabla\varphi\right\} = 0 & (1.2)\\[2ex] \operatorname{div}\left\{A_k\,\dfrac{\partial u}{\partial x_k} - D\nabla\varphi\right\} = 0 & (1.3)\end{cases}$$

in $\Omega\times(0,+\infty)$. In (1.2) $A_i^*$ denotes the adjoint of $A_i$. As we mentioned above we are concerned with a transmission problem associated with system (1.2)–(1.3): we assume that the bounded domain $\Omega = \mathcal{O}_0\backslash\overline{\mathcal{O}}_1$ where $\mathcal{O}_0$ and $\mathcal{O}_1$ are open bounded domains with $\overline{\mathcal{O}}_1\subset\mathcal{O}_0$ where $\overline{\mathcal{O}}_1$ denotes the closure of $\mathcal{O}_1$, $\partial\mathcal{O}_0 = S_0$ and $\partial\mathcal{O}_1 = S_1$. Let us fix an integer $n>1$ and $k = 1,2,\dots,n$. For each $k$, let $B_k$ be an open subset with smooth boundary and such that $\overline{\mathcal{O}}_1\subset B_k\subset\mathcal{O}_0$, $\overline{B}_k\subset B_{k+1}$. We set $\Omega_0 = B_1\backslash\overline{\mathcal{O}}_1$ and $\Omega_k = B_{k+1}\backslash\overline{B}_k$ for $k = 1,2,\dots,n-1$. Also $\Omega_n = \mathcal{O}_0\backslash\overline{B}_n$.

We consider problem (1.2)–(1.3) with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \tag{1.4}$$

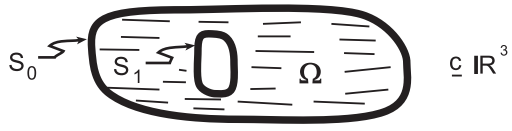


Figure 1. $n = 1$.

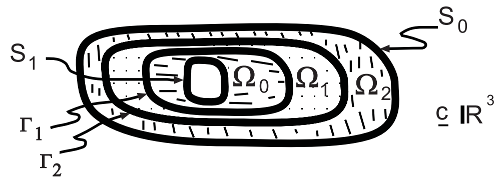


Figure 2. $n = 3$.

and boundary conditions

$$\begin{cases} \left(A_{ij}\dfrac{\partial u}{\partial x_j} + A_i^*\nabla\varphi\right)\eta_i + \alpha(x)u_t = 0 \quad \text{on} \quad S_0 \times [0,+\infty) \\ \varphi = 0 \quad \text{on} \quad S_0 \times [0,+\infty) \end{cases} \tag{1.5}$$

$$\begin{cases} \left(A_k\dfrac{\partial u}{\partial x_k} - D\nabla\varphi\right) \bullet \eta = 0 \quad \text{on} \quad S_1 \times [0,+\infty) \\ u = 0 \quad \text{on} \quad S_1 \times [0,+\infty). \end{cases} \tag{1.6}$$

From now on the dot $\bullet$ denotes the usual inner product in $\mathbb{R}^3$.

Finally we require the transmission conditions

$$\left[D^{(m-1)}\nabla\varphi^{(m-1)} - A_i\frac{\partial u^{(m-1)}}{\partial x_i}\right] \bullet \eta = \left[D^{(m)}\nabla\varphi^{(m)} - A_i\frac{\partial u^{(m)}}{\partial x_i}\right] \bullet \eta \tag{1.7}$$

$$u^{(m-1)} = u^{(m)}, \quad \varphi^{(m-1)} = \varphi^{(m)} \tag{1.8}$$

$$\left[A_{ij}^{(m-1)}\frac{\partial u^{(m-1)}}{\partial x_j} + A_i^*\nabla\varphi^{(m-1)}\right]\eta_i = \left[A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right]\eta_i. \tag{1.9}$$

All these transmission conditions should hold at the interfaces $\Gamma_m \times [0,+\infty)$, $m = 1,2,\ldots,n$. From here on $\eta = (\eta_1,\eta_2,\eta_3)$ will always denote the unit normal vector pointing the exterior of $B_m$ or $\Omega$ and $D^{(m)}$, $A_{ij}^{(m)}$, $\varphi^{(m)}$ or $u^{(m)}$ are the restrictions of the corresponding matrices or functions on $\Omega_m$.

Figures 1 and 2 illustrate simple such situations when $n = 1$ or $n = 3$.

The aim of this work is to show that under suitable assumptions on the elastic and dielectric tensors such as symmetry, coercivity and monotonicity as well as symmetry of the coupling tensor given in Hypothesis 1 and Hypotesis 3 together with geometrical assumptions given in Hypothesis 2 a result on uniform stabilization holds (Th. 3.1). As an application, we deduce a controllability result given in Theorem 4.1.

In order to mention the main result of this paper we give the assumptions on the coefficients or the matrices in problem (1.2)–(1.9):

**Hypothesis 1.**

1) We assume that the coefficients $c_{ijk\ell}$ and $d_{ij}$ (which are the cartesian components of the piezoelectric and electric permitivity tensors respectively) are $L^\infty(\Omega)$ and satisfy the following assumptions
   $c_{ijk\ell} = c_{k\ell ij} = c_{jik\ell}, \quad d_{ij} = d_{ji}$
   $d_{ij}\xi_j\xi_i \geq d_0|\xi|^2$ for some $d_0 > 0$ and any vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$
   $c_{ijk\ell}\,\lambda_{k\ell}\,\lambda_{ij} \geq c_0\,\lambda_{ij}\,\lambda_{ij}$ for some $c_0 > 0$ and any real symmetric tensor $[\lambda_{ij}]$ of order 3.
2) The $3 \times 3$ matrices $A_{ij}(x) = [a^{ij}_{k\ell}(x)]$ (which satisfy $A^*_{ij} = A_{ji}$ due to the symmetry of $c_{ijk\ell}$) are such that
$$A_{ij}\, r_j \bullet r_i \geq c_1 \sum_{i=1}^{3} |r_i|^2$$
   for some $c_1 > 0$ and any vector $r_i = (r_i^1, r_i^2, r_i^3) \in \mathbb{R}^3$.
3) The matrices $A_i = [e_{k\ell i}]_{3\times 3}$ are constant matrices with $e_{k\ell i} = e_{ki\ell}$, $D = [d_{k\ell}(x)]$ with $d_{ij} = d_{ij}(x)$ as well as $c_{ijk\ell} = c_{ijk\ell}(x)$ are piecewise constant functions which lose continuity only on $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$.
4) $\alpha(x) > 0 \quad \forall\, x \in S_0, \quad \alpha \in C(S_0)$.

**Observation 1.** For a linear material with Saint-Venant Kirchhoff mechanic behaviour the terms $c_{ijkh}$ are given by
$$c_{ijkh} = \lambda\,\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$$
where $\lambda$ and $\mu$ are constants such that $\lambda + \mu > 0$ and $\mu > 0$. In this situation, item 2) of Hypothesis 1 holds with the constant $c_1 = \mu$. In fact, in this case
$$A_{ij}\, v_j \bullet v_i = (\lambda + \mu)\left(\sum_{i=1}^{3} v_i^i\right)^2 + \mu \sum_{i,j=1}^{3} (v_i^j)^2 \geq \mu \sum_{i=1}^{3} |v_i|^2.$$

Assuming Hypothesis 1, we consider the total energy of the structure $E(t)$ associated with problem (1.2)–(1.9) is given by
$$E(t) = \sum_{m=0}^{n} \int_{\Omega_m} \left\{ |u_t^{(m)}|^2 + A_{ij}^{(m)}\, \frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m)}}{\partial x_i} + D^{(m)} \nabla\varphi^{(m)} \bullet \nabla\varphi^{(m)} \right\} \mathrm{d}x. \tag{1.10}$$
Formal calculations show that for every (smooth) solution of problem (1.2)–(1.9) the following identity holds
$$\frac{\mathrm{d}E}{\mathrm{d}t} + 2\int_{S_0} \alpha(x)|u_t|^2\, \mathrm{d}\Gamma = 0 \qquad \forall t \geq 0 \tag{1.11}$$
where the integral $\int_{S_0} \alpha|u_t|^2\, \mathrm{d}\Gamma$ means the surface integral of $\alpha|u_t|^2$ over the surface $S_0$.

Observe that when the structure is totally clamped, that is, when $S_1 = \partial\Omega$ then the energy is constant along a trajectory. This case was considered by Miara in [14]. The main result of this article shows that the total energy given by (1.10) decays exponentially to zero as $t \to +\infty$ provided suitable geometric conditions are imposed on $\Omega$ and $\Gamma_m$ and monotonicity assumptions on the coefficients of the system. The need for the above requirements were already noticed by Lions in [13] in the treatment of certain transmission problems. Later on, Lagnese [10] also used those type of assumptions to prove controllability results for a class of second order hyperbolic problems.

Results on control or stabilization of physical systems are quite important specially in the case of systems driven by coupled equations like thermo-elasticity (see [11]) or magneto-elasticity (see [4]). For those models be may mention other approaches such as microlocal techniques. An easy way to check whether our monotonicity conditions (see Hypothesis 3) are optimal or not would be to consider the case when there is no interaction in the piezoelectric system (that is when $A_i = 0$). In this case the potential $\varphi$ has to be zero and $u$ satisfies a wave-like equation that may be chosen to be a scalar wave equation including boundary dissipation. It is well known (and follows from [10]) that in this situation Hypothesis 3 is optimal. In the general case, that is when the coupling tensor $(e_{ijk})$ does not vanish, the optimality of Hypothesis 3 would require further study.

There are a large number of contributions concerning piezoelectric equations and/or quasi-electrostatic equations (see [1, 14] and the references therein). However, as far as we know, a transmission problem for such class of equations was treated only for similar systems (see [7, 10] and the references therein). Uniform stabilization results for model (1.2)–(1.9) are interesting while studying exact controllability because give us an explicit expression of the feedback control instead of the difficult computation of an exact control.

In a forthcoming article [8] we consider the case when no dissipation is included (that is, when $\alpha \equiv 0$). In that case we obtain a "boundary observation" inequality and use the HUM to solve the exact controllability problem.

Let us briefly describe the sections of this paper: Solvability of the initial boundary value problem (1.2)–(1.9) in the appropriate class of functions is outlined in Section 2. This is done *via* semigroup theory. In Section 3 we prove the exponential decay of the energy *via* the multiplier method. At this point, we needed to assume suitable geometric conditions on $\Omega$, the interfaces $\Gamma_m$ as well as monotonicity assumptions on the coefficients of the system.

We use standard notations, for example $H^r(\Omega)$ or $H^s(\partial\Omega)$ will denote the Sobolev spaces of order $r$ and $s$ on $\Omega$ and $\partial\Omega$ respectively. The norm of a vector $v \in \mathbb{R}^3$ will be denote by $|v|$. Given a real-valued function $g$ the notation $\int_S g\,\mathrm{d}\Gamma$ means "the surface integral of $g$ over the surface $S$".

## 2. Well-posedness

In this section we outline the function spaces where the solution pair $\{u, \varphi\}$ of problem (1.2)–(1.9) is considered. In order to obtain the main results in the next section it is sufficient to work with smooth solutions.

Let $\Omega$ be a bounded region as in the introduction. In $\Omega$ we consider the following problem

$$\begin{cases} \operatorname{div}(D\nabla\varphi) = \operatorname{div} F \quad \text{in} \quad \Omega_m \qquad m = 0, 1, \ldots, n & (2.1)\\ \varphi = 0 \text{ on } S_0,\ D\nabla\varphi \bullet \eta = F \bullet \eta \text{ on } S_1 & (2.2)\\ \varphi^{(m-1)} = \varphi^{(m)} \quad \text{on} \quad \Gamma_m,\ m = 1, 2, \ldots, n & (2.3)\\ D^{(m-1)}\nabla\varphi^{(m-1)} \bullet \eta - D^{(m)}\nabla\varphi^{(m)} \bullet \eta = F^{(m-1)} \bullet \eta - F^{(m)} \bullet \eta \text{ on } \Gamma_m & (2.4)\\ m = 1, 2, \ldots, n \end{cases}$$

where $F = (F_k)$ is a given function belonging to $[H^1(\Omega_m)]^3$. By elliptic theory, there exists a unique solution $\varphi$ of (2.1)–(2.4) which we denote by $\varphi = \beta(F)$. Denote by $X$ the real Hilbert space of pairs $\{u, v\}$ of three-component vector-valued functions such that $v^{(m)} \in [L^2(\Omega_m)]^3$, $u^{(m)} \in [H^2(\Omega_m)]^3$ and $u = 0$ on $S_1$. The inner product in $X$ is given by

$$\langle W, W_1\rangle_X = \sum_{m=0}^{n} \int_{\Omega_m} \left\{ v_1 \bullet v_2 + A_{ij}\frac{\partial u_1}{\partial x_j} \bullet \frac{\partial u_2}{\partial x_i} + + D\nabla\beta\left(A_k\frac{\partial u_1}{\partial x_k}\right) \bullet \nabla\beta\left(A_k\frac{\partial u_2}{\partial x_k}\right) \right\} \mathrm{d}x$$

whenever $W = (u_1, v_1)$ and $W_1 = (u_2, v_2)$ belong to $X$. We will denote by $|| \cdot ||_X$ the norm in $X$. In $X$ we define the unbounded operator $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ which consists of all the elements $(u, v) \in X$ such that

$$\begin{cases} v^{(m)} \in [H^1(\Omega_m)]^3, \quad v\big|_{S_1} = 0 \\ \left[A_{ij}\dfrac{\partial u}{\partial x_j} + A_i^*\nabla\beta\left(A_k\dfrac{\partial u}{\partial x_k}\right)\right]\eta_i + \alpha v = 0 \text{ on } S_0 \\ u^{(m-1)} = u^{(m)}, \quad v^{(m-1)} = v^{(m)} \text{ on } \Gamma_m \\ A_{ij}^{(m-1)}\dfrac{\partial u^{(m-1)}}{\partial x_j} + A_i^*\nabla\beta\left(A_k\dfrac{\partial u^{(m-1)}}{\partial x_k}\right)\eta_i \\ \qquad = A_{ij}^{(m)}\dfrac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\beta\left(A_k\dfrac{\partial u^{(m)}}{\partial x_k}\right)\eta_i \text{ on } \Gamma_m \end{cases} \tag{2.5}$$

$m = 1, 2, \ldots, n$. In the domain of $\mathcal{A}$, the operator is given by

$$\mathcal{A}(u, v) = \left(v, \frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j} + A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right\}\right).$$

**Lemma 1.** *Assume Hypothesis 1 given in the introduction, then the operator $\mathcal{A}$ is dissipative, that is $\langle\mathcal{A}(W), W\rangle_X \le 0 \quad \forall W \in \mathcal{D}(\mathcal{A})$.*

*Proof.* Let $W = (u, v) \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} \langle\mathcal{A}(W), W\rangle_X &= \left\langle\left(v, \frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j} + A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right\}\right), (u, v)\right\rangle_X \\ &= \sum_{m=0}^{n}\int_{\Omega_m}\left[\left(\frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j} + A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right\}\right)\bullet v\right. \\ &\quad \left. + A_{ij}\frac{\partial v}{\partial x_j}\bullet\frac{\partial u}{\partial x_i} + D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)\bullet\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right]\mathrm{d}x. \end{aligned} \tag{2.6}$$

Using the boundary and interface conditions we get

$$\begin{aligned} \sum_{m=0}^{n}\int_{\Omega_m}\left[\left(\frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j} + A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right\}\right)\bullet v\,\mathrm{d}x = -\sum_{m=0}^{n}\int_{\Omega_m}A_{ij}\frac{\partial u}{\partial x_j}\bullet\frac{\partial v}{\partial x_i}\,\mathrm{d}x \\ - \sum_{m=0}^{n}\int_{\Omega_m}A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\bullet\frac{\partial v}{\partial x_i}\,\mathrm{d}x - \int_{S_0}\alpha|v|^2\,\mathrm{d}\Gamma. \end{aligned} \tag{2.7}$$

Observe that the following identities are valid

$$\sum_{m=0}^{n}\int_{\Omega_m}\left\{D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)\bullet\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)-A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\bullet\frac{\partial v}{\partial x_i}\right\}\mathrm{d}x$$
$$=\sum_{m=0}^{n}\int_{\Omega_m}\left(\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\bullet\left(D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)-A_k\frac{\partial v}{\partial x_k}\right)\right]\mathrm{d}x$$
$$=\sum_{m=1}^{n}\int_{\Gamma_m}\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\left\{D^{(m-1)}\nabla\beta^{(m-1)}\left(A_k\frac{\partial v}{\partial x_k}\right)-A_k\frac{\partial v^{(m-1)}}{\partial x_k}\right\}\bullet\eta\,\mathrm{d}\Gamma$$
$$-\sum_{m=1}^{n}\int_{\Gamma_m}\left[D^{(m)}\nabla\beta^{(m)}\left(A_k\frac{\partial v}{\partial x_k}\right)-A_k\frac{\partial v}{\partial x_k}\right]\bullet\eta\,\mathrm{d}\Gamma$$
$$+\int_{S_1}\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\left[D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)-A_k\frac{\partial v}{\partial x_k}\right]\bullet\eta\,\mathrm{d}\Gamma$$
$$-\sum_{m=0}^{m}\int_{\Omega_m}\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\left\{\mathrm{div}\left(D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)\right)-\mathrm{div}\left(A_k\frac{\partial v}{\partial x_k}\right)\right\}\mathrm{d}x.$$

From (2.1)–(2.4) it follows that

$$\sum_{m=0}^{n}\int_{\Omega_m}\left\{\left(D\nabla\beta\left(A_k\frac{\partial v}{\partial x_k}\right)\bullet\left(\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right)\right)-\left(A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\bullet\frac{\partial v}{\partial x_i}\right)\right\}\mathrm{d}x=0.$$

Consequently, from (2.4)–(2.8) we obtain that

$$\langle\mathcal{A}(u,v),(u,v)\rangle_X=-\int_{S_0}\alpha|v|^2\,\mathrm{d}\Gamma\le 0.$$

□

Now, we consider the adjoint operator $\mathcal{A}^*$. It can be verified that the domain of $\mathcal{A}^*$ consists of all elements $(u,v)\in X$ satisfying (2.5) except that we should change $\alpha(x)$ by $-\alpha(x)$. For $(u,v)\in\mathcal{D}(\mathcal{A}^*)$ we have

$$\mathcal{A}^*(u,v)=-\left(v,\frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j}+A_i^*\nabla\beta\left(A_k\frac{\partial u}{\partial x_k}\right)\right\}\right)\cdot$$

We can verify that $\mathcal{A}^*$ is dissipative. Obviously $\mathcal{A}$ is closed and densely defined. It follows by a well known criteria (see Pazy [15], Cor. I.44) that the operator $\mathcal{A}$ generates a strongly continuous semigroup of contractions $\{U(t)\}_{t\ge 0}$. Furthermore, for $f=(f_1,f_2)\in\mathcal{D}(\mathcal{A})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)f=\mathcal{A}U(t)f\,,\quad U(0)f=f$$

and $U(t)f$ is the unique (strong) solution of problem (1.2)–(1.9). Let $f=(f_1,f_2)\in X$, $f^{(n)}=\left(f_1^{(n)},f_2^{(n)}\right)\in\mathcal{D}(\mathcal{A})$ such that $\lim_{n\to+\infty}||f-f^{(n)}||_X=0$. Then, $U(t)f^{(n)}$ satisfies the following identity

$$\int_0^T\left\{\left\langle U(t)f^{(n)},\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle_X+\langle U(t)f^{(n)},\mathcal{A}^*\psi\rangle_X\right\}\mathrm{d}t=-\langle f^{(n)},\psi(0)\rangle_X \tag{2.9}$$

where $\psi\in L^2(0,T;\mathcal{D}(\mathcal{A}^*))$, $\psi_t\in L^2(0,T;X)$ with $\psi(T)=0$. Passing to the limit in (2.9) we obtain

$$\int_0^T\left\{\left\langle U(t)f,\frac{\mathrm{d}\psi}{\mathrm{d}t}\right\rangle_X+\langle U(t)f,\mathcal{A}^*\psi\rangle_X\right\}\mathrm{d}t=-\langle f,\psi(0)\rangle_X$$

that is, $U(t)f$ is the weak solution in $X$ of the abstract initial-boundary value problem

$$W_t = \mathcal{A}(W), \quad W(0) = f. \qquad \square$$

## 3. Stabilization

In this section we prove the boundary stabilization result. The proof is based on the theory of multipliers and it is motivated by the invariance of system (1.2)–(1.3) with constant coefficients relative to the one-parameter group of dilations in all variables. A good reference for the use of this technique is Komornik's book [9]. The multipliers have to be conveniently modified in such a way the extra boundary terms appearing in the identities can be estimated by appropriate bounds. Let $g = g(x)$ be an auxiliary scalar smooth function on $\overline{\Omega}$ which we will choose later. Let us fix $t_0 > 0$ and consider the multiplier $L_1$ given by

$$L_1 u = (t + t_0)u_t + (\nabla g \bullet \nabla)u + u \tag{3.1}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$,

$$\nabla g \bullet \nabla = \frac{\partial g}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial g}{\partial x_3}\frac{\partial}{\partial x_3}$$

and $u = (u_1, u_2, u_3)$. Let $L_2$ the operator

$$L_2\varphi = (t + t_0)\varphi\,\frac{\partial}{\partial t} - \nabla g \cdot \nabla\varphi. \tag{3.2}$$

We take the inner product (in $\mathbb{R}^3$) of $L_1 u$ with equation (1.2) and apply the operator $L_2\varphi$ to equation (1.3). Since $\{u, \varphi\}$ is a (smooth) solution of (1.2)–(1.9) then adding the identities we obtain

$$\frac{\partial F}{\partial t} = \operatorname{div} G + \frac{\partial H_i}{\partial x_i} + J \tag{3.3}$$

where

$$F = (t + t_0)\left[|u_t|^2 + A_{ij}\frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + D\nabla\varphi \bullet \nabla\varphi\right] + 2\,u_t \bullet \left[(\nabla g \bullet \nabla)u + u\right] \tag{3.4}$$

$$\begin{aligned} G = {} & 2(t + t_0)\varphi\,\frac{\partial}{\partial t}\left\{D\nabla\varphi - A_k\frac{\partial u}{\partial x_k}\right\} + \nabla g D\nabla\varphi \bullet \nabla\varphi \\ & - 2(\nabla\varphi \bullet \nabla g)D\nabla\varphi - 2\nabla g\,A_k\frac{\partial u}{\partial x_k} \bullet \nabla\varphi + 2(\nabla g \bullet \nabla\varphi)A_k\,\frac{\partial u}{\partial x_k} \end{aligned} \tag{3.5}$$

$$H_i = 2\left\{(t + t_0)u_t + (\nabla g \bullet \nabla)u + u\right\} \bullet \left\{A_{ij}\frac{\partial u}{\partial x_j} + A_i^*\nabla\varphi\right\} + \frac{\partial g}{\partial x_i}\left[|u_t|^2 - A_{pq}\,\frac{\partial u}{\partial x_q} \bullet \frac{\partial u}{\partial x_p}\right] \tag{3.6}$$

and

$$\begin{aligned} J = {} & (\Delta g - 1)A_{ij}\frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} - 2\,\frac{\partial^2 g}{\partial x_p \partial x_i}A_{ij}\frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_p} + (3 - \Delta g)|u_t|^2 + 2\,\frac{\partial^2 g}{\partial x_i \partial x_k}\,d_{ij}\,\frac{\partial\varphi}{\partial x_j}\,\frac{\partial\varphi}{\partial x_k} \\ & + (1 - \Delta g)D\nabla\varphi \bullet \nabla\varphi + 2\nabla\varphi \bullet \left\{(\Delta g - 1)A_k\frac{\partial u}{\partial x_k} - \frac{\partial^2 g}{\partial x_i \partial x_k}A_k\frac{\partial u}{\partial x_i} - \left(A_k\frac{\partial u}{\partial x_k} \bullet \nabla\right)\nabla g\right\}. \end{aligned} \tag{3.7}$$

**Observation 2.** If we consider $g(x) = \frac{1}{2}\,|x - x_0|^2$ for some fixed $x_0 \in \mathbb{R}^3$ then $J \equiv 0$. In this case (3.3) will be a conservation law. However, due to the expressions of $G$ and $H_i$ we would require to have a definite sign for $\frac{\partial g}{\partial \eta}$.

Thus, later on we will choose $g$ as an "small" perturbation of $\frac{1}{2}|x-x_0|^2$. Let $\{u,\varphi\}$ be a smooth solution of (1.2)–(1.9). Integration over $\Omega_m \times (0,T)$ of identity (3.3) and summation over $m$ implies that

$$(T+t_0)E(T) + 2\sum_{m=0}^{n}\int_{\Omega_m} u_t^{(m)} \bullet \left\{(\nabla g \bullet \nabla)u^{(m)} + u^{(m)}\right\}\mathrm{d}x\Big]_{t=0}^{t=T} = t_0\,E(0) + \sum_{m=1}^{n}\int_0^T\int_{\Gamma_m}(\boldsymbol{V}_{m-1} - \boldsymbol{V}_m)\mathrm{d}\Gamma\,\mathrm{d}t$$
$$+ \int_0^T\int_{S_0}\boldsymbol{V}_n\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_0^T\int_{S_1}\boldsymbol{V}_0\,\mathrm{d}\Gamma\,\mathrm{d}t + \sum_{m=0}^{n}\int_0^T\int_{\Omega_m} J_m(x,t)\mathrm{d}x\mathrm{d}t \tag{3.8}$$

where $E(t)$ is given by (1.10), $J_m = J_m(u,\varphi,g)$ denotes the restriction of $J$ (in (3.7)) to the region $\Omega_m$ and

$$V_m = 2\left\{(t+t_0)u_t^{(m)} + (\nabla g\bullet\nabla)u^{(m)} + u^{(m)}\right\}\bullet\left\{\left(A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right)\eta_i\right\}$$
$$+ \frac{\partial g}{\partial\eta}\left(|u_t^{(m)}|^2 - A_{pq}^{(m)}\frac{\partial u^{(m)}}{\partial x_q}\bullet\frac{\partial u^{(m)}}{\partial x_p}\right)$$
$$+ \frac{\partial g}{\partial\eta}D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)} + 2(t+t_0)\varphi^{(m)}\left(D^{(m)}\nabla\varphi_t^{(m)} - A_k\frac{\partial^2 u^{(m)}}{\partial x_k\partial t}\right)\bullet\eta$$
$$- 2(\nabla\varphi^{(m)}\bullet\nabla g)\left(D^{(m)}\nabla\varphi^{(m)} - A_k\frac{\partial u^{(m)}}{\partial x_k}\right)\bullet\eta - 2\,\frac{\partial g}{\partial\eta}\left(A_k\frac{\partial u^{(m)}}{\partial x_k}\bullet\nabla\varphi^{(m)}\right). \tag{3.9}$$

Here $\frac{\partial g}{\partial\eta}$ denotes the normal derivative of $g$ at $x\in\Gamma_m$ (or $S_0$, $S_1$). Next lemma tell us that the differences $V_{m-1} - V_m$ will have "good" sign if we choose $g$ conveniently and assume a monotonicity condition on $\{A_{ij}^{(m)}\}$ and $\{D^{(m)}\}$:

**Lemma 2.** *Let $\{u,\varphi\}$ be a smooth solution of (1.2)–(1.9). Then, the identity*

$$V_{m-1} - V_m = -\frac{\partial g}{\partial\eta}\Bigg\{\left(A_{ij}^{(m-1)} - A_{ij}^{(m)}\right)\frac{\partial u^{(m-1)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i} + A_{ij}^{(m)}\left[\frac{\partial u^{(m)}}{\partial x_j} - \frac{\partial u^{(m-1)}}{\partial x_j}\right]\bullet\left[\frac{\partial u^{(m)}}{\partial x_i} - \frac{\partial u^{(m-1)}}{\partial x_i}\right]$$
$$+ (D^{(m)} - D^{(m-1)})\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)} + D^{(m-1)}(\nabla\varphi^{(m-1)} - \nabla\varphi^{(m)})\bullet(\nabla\varphi^{(m-1)} - \nabla\varphi^{(m)}\Bigg\} \tag{3.10}$$

*holds.*

*Proof.* The idea is to use the interface conditions (1.7)–(1.9). In fact, direct calculations using (3.9) and the interfaces conditions imply that

$$V_{m-1} - V_m = 2(\nabla g\bullet\nabla)(u^{(m-1)} - u^{(m)})\bullet\left(A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right)\eta_i$$
$$- \frac{\partial g}{\partial\eta}\Bigg\{\left(A_{pq}^{(m-1)}\frac{\partial u^{(m-1)}}{\partial x_q}\bullet\frac{\partial u^{(m-1)}}{\partial x_p}\right) - \left(A_{pq}^{(m)}\frac{\partial u^{(m)}}{\partial x_q}\bullet\frac{\partial u^{(m)}}{\partial x_p}\right)$$
$$- (D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m-1)}) + (D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)})$$
$$+ 2\left(A_k\frac{\partial u^{(m-1)}}{\partial x_k}\bullet\nabla\varphi^{(m-1)}\right) - 2\left(A_k\frac{\partial u^{(m)}}{\partial x_k}\bullet\nabla\varphi^{(m)}\right)\Bigg\}$$
$$- 2\left(D^{(m)}\nabla\varphi^{(m)} - A_k\frac{\partial u^{(m)}}{\partial x_k}\right)\bullet\eta\left(\nabla g\bullet\nabla\varphi^{(m-1)} - \nabla g\bullet\nabla\varphi^{(m)}\right). \tag{3.11}$$

Now, we use the identity

$$(\nabla g \bullet \nabla)(u^{(m-1)} - u^{(m)}) = \frac{\partial g}{\partial x_i}\left(\frac{\partial u^{(m-1)}}{\partial x_i} - \frac{\partial u^{(m)}}{\partial x_i}\right) = \frac{\partial g}{\partial x_i}\eta_i\left(\frac{\partial u^{(m-1)}}{\partial \eta} - \frac{\partial u^{(m)}}{\partial \eta}\right) = \frac{\partial g}{\partial \eta}\left(\frac{\partial u^{(m-1)}}{\partial \eta} - \frac{\partial u^{(m)}}{\partial \eta}\right)$$

in order to obtain

$$\begin{aligned}
&2\left\{(\nabla g \bullet \nabla)(u^{(m-1)} - u^{(m)})\right\} \bullet \left\{\eta_i\left(A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right)\right\}\\
&\quad = 2\frac{\partial g}{\partial \eta}\left\{\left(\frac{\partial u^{(m-1)}}{\partial \eta} - \frac{\partial u^{(m)}}{\partial \eta}\right)\eta_i \bullet \left(A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right)\right\}\\
&\quad = 2\frac{\partial g}{\partial \eta}\left\{\left(\frac{\partial u^{(m-1)}}{\partial x_i} - \frac{\partial u^{(m)}}{\partial x_i}\right) \bullet \left(A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} + A_i^*\nabla\varphi^{(m)}\right)\right\}\\
&\quad = 2\frac{\partial g}{\partial \eta}\Bigg\{A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m-1)}}{\partial x_i} - A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m)}}{\partial x_i}\\
&\qquad + A_i\frac{\partial u^{(m-1)}}{\partial x_i} \bullet \nabla\varphi^{(m)} - A_i\frac{\partial u^{(m)}}{\partial x_i} \bullet \nabla\varphi^{(m)}\Bigg\}.
\end{aligned} \tag{3.12}$$

In a similar way we obtain

$$\begin{aligned}
&-2\left(D^{(m)}\nabla\varphi^{(m)} - A_k\frac{\partial u^{(m)}}{\partial x_k}\right) \bullet \eta\left\{\nabla g \bullet \nabla\varphi^{(m-1)} - \nabla g \bullet \nabla\varphi^{(m)}\right\}\\
&\quad = -2\left(D^{(m)}\nabla\varphi^{(m)} - A_k\frac{\partial u^{(m)}}{\partial x_k}\right) \bullet \eta\left\{\nabla g \bullet \eta\left(\frac{\partial \varphi^{(m-1)}}{\partial \eta} - \frac{\partial \varphi^{(m)}}{\partial \eta}\right)\right\}\\
&\quad = 2\frac{\partial g}{\partial \eta}\left(D^{(m)}\nabla\varphi^{(m)} - A_k\frac{\partial u^{(m)}}{\partial x_k}\right) \bullet (\nabla\varphi^{(m)} - \nabla\varphi^{(m-1)})\\
&\quad = 2\frac{\partial g}{\partial \eta}\Bigg\{(D^{(m)}\nabla\varphi^{(m)} \bullet \nabla\varphi^{(m)}) - (D^{(m)}\nabla\varphi^{(m)} \bullet \nabla\varphi^{(m-1)})\\
&\qquad + A_k\frac{\partial u^{(m)}}{\partial x_k} \bullet \nabla\varphi^{(m-1} - A_k\frac{\partial u^{(m)}}{\partial x_k} \bullet \nabla\varphi^{(m)}\Bigg\}.
\end{aligned} \tag{3.13}$$

From (3.11)–(3.13) we obtain the identity

$$\begin{aligned}
V_{m-1} - V_m = \frac{\partial g}{\partial \eta}\Bigg\{&2\,A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m-1)}}{\partial x_i} - A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m)}}{\partial x_i}\\
&- A_{ij}^{(m-1)}\frac{\partial u^{(m-1)}}{\partial x_j} \bullet \frac{\partial u^{(m-1)}}{\partial x_i} - 2\,D^{(m)}\nabla\varphi^{(m)} \bullet \nabla\varphi^{(m-1)}\\
&+ D^{(m)}\nabla\varphi^{(m)} \bullet \nabla\varphi^{(m)} + D^{(m-1)}\nabla\varphi^{(m-1)} \bullet \nabla\varphi^{(m-1)}\\
&+ 2\,A_k\frac{\partial u^{(m-1)}}{\partial x_k} \bullet \nabla\varphi^{(m)} - 2\,A_k\frac{\partial u^{(m)}}{\partial x_k} \bullet \nabla\varphi^{(m)}\\
&+ 2\,A_k\frac{\partial u^{(m)}}{\partial x_k} \bullet \nabla\varphi^{(m-1)} - 2\,A_k\frac{\partial u^{(m-1)}}{\partial x_k} \bullet \nabla\varphi^{(m-1)}\Bigg\}.
\end{aligned} \tag{3.14}$$

Using the interface conditions (1.7) it follows that

$$\left(A_k\frac{\partial u^{(m)}}{\partial x_k} - A_k\frac{\partial u^{(m-1)}}{\partial x_k}\right) \bullet \eta = \left(D^{(m)}\nabla\varphi^{(m)} - D^{(m-1)}\nabla\varphi^{(m-1)}\right) \bullet \eta.$$

Therefore

$$
\begin{aligned}
&2\,\frac{\partial g}{\partial \eta}\Big\{A_k\,\frac{\partial u^{(m-1)}}{\partial x_k}\bullet\nabla\varphi^{(m)} - A_k\,\frac{\partial u^{(m)}}{\partial x_k}\bullet\nabla\varphi^{(m)} + A_k\,\frac{\partial u^{(m)}}{\partial x_k}\bullet\nabla\varphi^{(m-1)} - A_k\,\frac{\partial u^{(m-1)}}{\partial x_k}\bullet\nabla\varphi^{(m-1)}\Big\}\\
&\quad = 2\,\frac{\partial g}{\partial \eta}\left(A_k\,\frac{\partial u^{(m)}}{\partial x_k} - A_k\,\frac{\partial u^{(m-1)}}{\partial x_k}\right)\bullet\Big(\nabla\varphi^{(m-1} - \nabla\varphi^{(m)}\Big)\\
&\quad = 2\,\frac{\partial g}{\partial \eta}\left(A_k\,\frac{\partial u^{(m)}}{\partial x_k} - A_k\,\frac{\partial u^{(m-1)}}{\partial x_k}\right)\bullet\eta\left\{\frac{\partial\varphi^{(m-1)}}{\partial\eta} - \frac{\partial\varphi^{(m)}}{\partial\eta}\right\}\\
&\quad = 2\,\frac{\partial g}{\partial \eta}\Big(D^{(m)}\nabla\varphi^{(m)} - D^{(m-1)}\nabla\varphi^{(m-1)}\Big)\bullet\eta\left\{\frac{\partial\varphi^{(m-1)}}{\partial\eta} - \frac{\partial\varphi^{(m)}}{\partial\eta}\right\}\\
&\quad = 2\,\frac{\partial g}{\partial \eta}\Big(D^{(m)}\nabla\varphi^{(m)} - D^{(m-1)}\nabla\varphi^{(m-1)}\Big)\bullet\Big\{\nabla\varphi^{(m-1)} - \nabla\varphi^{(m)}\Big\}\\
&\quad = 2\,\frac{\partial g}{\partial \eta}\Big\{-D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)} + D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m-1)}\\
&\qquad - D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m-1)} + D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m)}\Big\}.
\end{aligned}
\tag{3.15}
$$

From (3.14) and (3.15) we deduce that

$$
\begin{aligned}
V_{m-1} - V_m = \frac{\partial g}{\partial \eta}\Big\{&2\,A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i} - A_{ij}^{(m)}\,\frac{\partial u^{(m)}}{\partial x_j}\bullet\frac{\partial u^{(m)}}{\partial x_i}\\
&- A_{ij}^{(m-1)}\,\frac{\partial u^{(m-1)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i} - D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)}\\
&- D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m-1)} + 2\,D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m)}\Big\}.
\end{aligned}
\tag{3.16}
$$

The conclusion of Lemma 3.1 follows from (3.16) observing the validity of the identities

$$
\begin{aligned}
A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j}\bullet\frac{\partial u^{(m)}}{\partial x_i} + A_{ij}^{(m-1)}\,\frac{\partial u^{(m-1)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i} - 2\,A_{ij}^{(m)}\,\frac{\partial u^{(m)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i}&\\
= \Big(A_{ij}^{(m-1)} - A_{ij}^{(m)}\Big)\,\frac{\partial u^{(m-1)}}{\partial x_j}\bullet\frac{\partial u^{(m-1)}}{\partial x_i}&\\
+ A_{ij}^{(m)}\left(\frac{\partial u^{(m)}}{\partial x_j} - \frac{\partial u^{(m-1)}}{\partial x_j}\right)\bullet\left(\frac{\partial u^{(m)}}{\partial x_i} - \frac{\partial u^{(m-1)}}{\partial x_i}\right)&
\end{aligned}
$$

and

$$
\begin{aligned}
&D^{(m)}\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)} + D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m-1)} - 2D^{(m-1)}\nabla\varphi^{(m-1)}\bullet\nabla\varphi^{(m)}\\
&\quad = D^{(m-1)}\Big(\nabla\varphi^{(m-1)} - \nabla\varphi^{(m)}\Big)\bullet\Big(\nabla\varphi^{(m-1)} - \nabla\varphi^{(m)}\Big) + (D^{(m)} - D^{(m-1)})\nabla\varphi^{(m)}\bullet\nabla\varphi^{(m)}. \quad \square
\end{aligned}
$$

Let us choose a convenient function $g(x)$: Let $\Phi(x)$ be a solution of the elliptic problem

$$\begin{cases} \Delta\Phi = 1 \quad \text{in} \quad \Omega \\ \dfrac{\partial\Phi}{\partial\eta} = 2\,\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{area}(S_0)} \quad \text{on} \quad S_0 \\ \dfrac{\partial\Phi}{\partial\eta} = -\dfrac{\mathrm{Vol}(\Omega)}{\mathrm{area}(S_1)} \quad \text{on} \quad S_1 \end{cases} \tag{3.17}$$

which admits a solution $\Phi(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Let $\delta > 0$, $x_0 \in \mathbb{R}^3$ (to be chosen later) and define

$$g(x) = \delta\,\Phi(x) + \frac{1}{2}\,|x - x_0|^2. \tag{3.18}$$

Now we concentrate our discussion in estimating the term $\sum_{m=0}^{n} \int_0^T \int_{\Omega_m} J_m \,\mathrm{d}x\mathrm{d}t$ in (3.8).

**Lemma 3.** *Under the assumptions of Lemma 2 Hypothesis 1 and choosing $g(x)$ as in (3.18) we have*

$$\sum_{m=0}^{n} \int_0^T \int_{\Omega_m} J_m \,\mathrm{d}x\mathrm{d}t \le \delta\,\tilde{c} \int_0^T E(t)\,\mathrm{d}t$$

*for any $\delta > 0$ and some positive constant $\tilde{c}$ which depends only on $\Phi$ and the norms of the matrices $A_{ij}$, $A_i$ and $D$.*

*Proof.* The index $m$ will be omitted in order to simplify notations. With our choice of $g(x)$, straightforward calculations show that $J_m$ (given by (3.7)) can be written as

$$\begin{aligned} J_m = 2\delta &\left\{ A_k \frac{\partial u}{\partial x_k} - \frac{\partial^2\Phi}{\partial x_i \partial x_k} A_k \frac{\partial u}{\partial x_i} - \left( A_k \frac{\partial u}{\partial x_k} \bullet \nabla \right) \nabla\Phi \right\} \bullet \nabla\varphi \\ &+ 2\delta \left\{ \left( A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \right) - \frac{\partial^2\Phi}{\partial x_p \partial x_i} A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_p} + \frac{\partial^2\Phi}{\partial x_i \partial x_k} d_{ij} \frac{\partial\varphi}{\partial x_j} \frac{\partial\varphi}{\partial x_k} \right\} \\ &- \delta \left\{ |u_t|^2 + A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + D\nabla\varphi \bullet \nabla\varphi \right\}. \end{aligned} \tag{3.19}$$

Let us estimate each term on the right hand side of (3.19). We consider $v_i = \frac{\partial^2\Phi}{\partial x_p \partial x_i} \frac{\partial u}{\partial x_p}$ and $\varepsilon > 0$, then, we can write

$$\begin{aligned} -2\,A_{ij} \frac{\partial u}{\partial x_j} \bullet v_i &= -A_{ij} \left( \sqrt{\varepsilon} \frac{\partial u}{\partial x_j} + \frac{1}{\sqrt{\varepsilon}} v_j \right) \bullet \left( \sqrt{\varepsilon} \frac{\partial u}{\partial x_i} + \frac{1}{\sqrt{\varepsilon}} v_i \right) \\ &\quad + \varepsilon\,A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + \varepsilon^{-1} A_{ij}\, v_j \bullet v_i \\ &\le \varepsilon\,A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + \varepsilon^{-1} A_{ij}\, v_j \bullet v_i \end{aligned} \tag{3.20}$$

because $A_{ij}$ satisfies Assumption 2) in Hypothesis 1.

Let $c_3$ and $c_4$ be the following numbers

$$c_3 = \max_{\substack{x\in\overline{\Omega} \\ i,j=1,2,3}} ||A_{ij}(x)||, \qquad c_4 = \max_{\substack{x\in\overline{\Omega} \\ i,j=1,2,3}} \left| \frac{\partial^2\Phi(x)}{\partial x_i \partial x_j} \right|,$$

where $||A_{ij}||$ denotes the norm of the matrix $A_{ij}$. With the above notations, we have

$$|v_i| \leq C_4 \sum_{j=1}^{3} \left| \frac{\partial u}{\partial x_j} \right|$$

and

$$\begin{aligned} |A_{ij} v_j \bullet v_i| &\leq ||A_{ij}|| \, |v_j| \, |v_i| \leq c_3 \left( \sum_{j=1}^{3} |v_j| \right)^2 \\ &\leq 9 \, c_3 \, c_4^2 \left\{ \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right| \right\}^2 \leq 27 \, c_3 \, c_4^2 \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right|^2 \\ &\leq 27 \, c_1^{-1} \, c_3 \, c_4^2 \, A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \end{aligned} \tag{3.21}$$

where $c_1 > 0$ is as in item 2) of Hypothesis 1.

From (3.20) and (3.21) we obtain the inequality

$$-2\delta \, \frac{\partial^2 \Phi}{\partial x_p \partial x_i} \, A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_p} \leq \delta(\varepsilon + 27(\varepsilon \, c_1)^{-1} \, c_3 \, c_4^2) A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \cdot \tag{3.22}$$

Similarly, we estimate

$$2\delta \, \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \, d_{ij} \, \frac{\partial \varphi}{\partial x_j} \, \frac{\partial \varphi}{\partial x_k} \leq \delta \, \varepsilon_1 \, D \nabla \varphi \bullet \nabla \varphi + \delta \, \varepsilon_1^{-1} \, D \psi \bullet \psi$$

for any $\varepsilon_1 > 0$ where $\psi_i = \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \frac{\partial \varphi}{\partial x_k}$, $\quad \psi = (\psi_1, \psi_2, \psi_3)$ in order to obtain

$$2\delta \, \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \, d_{ij} \, \frac{\partial \varphi}{\partial x_j} \, \frac{\partial \varphi}{\partial x_k} \leq \delta(\varepsilon_1 + 9(d_0 \varepsilon_1)^{-1} \, c_5 c_4^2) D \nabla \varphi \bullet \nabla \varphi \tag{3.23}$$

where $d_0 > 0$ is as in Hypothesis 1 and $c_5 = ||D||$.

Following the same reasoning as above we get the estimates

$$-2\delta \, \nabla \varphi \bullet \left( A_k \frac{\partial u}{\partial x_k} \bullet \nabla \right) \nabla \Phi \leq \delta \left( 3\varepsilon_2 \, d_0^{-1} \, c_4^2 \, D \nabla \varphi \bullet \nabla \varphi \right) + \delta \left( 3(c_0 \varepsilon_2)^{-1} \, c_6^2 \, A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \right) \tag{3.24}$$

where $\varepsilon_2 > 0$ and $c_6 = \max_{k=1,2,3} ||A_k||$.

Also

$$2\delta \, \nabla \varphi \bullet A_k \, \frac{\partial u}{\partial x_k} \leq \delta \left( \varepsilon_3 d_0^{-1} c_6 \, D \nabla \varphi \bullet \nabla \varphi + 3c_6 (c_0 \varepsilon_3)^{-1} \, A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \right) \tag{3.25}$$

$$\begin{aligned} -2\delta \, \nabla \varphi \bullet \frac{\partial^2 \Phi}{\partial x_i \partial x_k} A_k \, \frac{\partial u}{\partial x_i} \leq \delta \Big( & \varepsilon_4 \, d_0^{-1} \, c_6 \, D \nabla \varphi \bullet \nabla \varphi \\ & + 27 (c_0 \varepsilon_4)^{-1} \, c_6 \, c_4^2 \, A_{ij} \, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \Big) \end{aligned} \tag{3.26}$$

for any $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$.

From the above estimates (3.22)–(3.26) we deduce from (3.19) that

$$J_m \leq \delta \left\{ c_7\, A_{ij}\, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + c_8\, D\nabla\varphi \bullet \nabla\varphi \right\} - \delta \left\{ |u_t|^2 + A_{ij}\, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} + D\, \nabla\varphi \bullet \nabla\varphi \right\} \tag{3.27}$$

where $c_7$ and $c_8$ are positive constants. Integration of inequality (3.27) in $\Omega_m \times (0,T)$ and adding in $m$ completes the proof of Lemma 3.2 by taking $\tilde{c} = \max\{c_7, c_8\}$. □

From now on we fix $\delta_0 > 0$ such that $\delta_0 \tilde{c} < 1$ where $\tilde{c}$ is as in Lemma 3.2. Thus, we will work with the auxiliary function

$$g(x) = \delta_0\, \Phi(x) + \frac{1}{2}\, |x - x_0|^2.$$

Our next goal is to estimate the surface integrals (over $S_0 \times (0,T)$ and $S_1 \times (0,T)$ in (3.8)).

Now, we will impose geometric conditions on $\Omega$ and $\Gamma_m$.

**Hypothesis 2.** There exists a point $x_0 \in \mathbb{R}^3$ such that

a) $(x - x_0) \bullet \eta > -2\delta_0\, \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{area}(S_0)}$ for all $x \in S_0$

b) $(x - x_0) \bullet \eta \leq \delta_0\, \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{area}(S_1)}$, for all $x \in S_1$

c) $\delta_0\, \dfrac{\partial \Phi}{\partial \eta} + (x - x_0) \bullet \eta \geq 0$ for all $x \in \Gamma_m$

$m = 1, 2, \ldots, n$, where $\eta = \eta(x)$ denotes the unit outward normal to $S_0$, $S_1$ or $\Gamma_m$.

**Remark 1.** We note that the above assumptions on Hypothesis 2 are valid when $\delta_0 = 0$ for star-shaped surfaces $S_1, \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and strongly star-shaped surface $S_0$, *i.e.*

$$(x - x_0) \bullet \eta > 0, \quad \forall\, x \in S_0.$$

Moreover, if $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are strongly star-shaped with respect to a point $x_0$, then the above conditions hold with $\delta_0 > 0$ for a class of domains which includes star-shaped domains.

Let $\lambda_0 > 0$ be such that

$$\frac{\partial g}{\partial \eta} \geq \lambda_0 |\nabla g| \quad \text{for any} \quad x \in S_0$$

which is possible since $\frac{\partial g}{\partial \eta} = \delta_0\, \frac{\mathrm{Vol}(\Omega)}{\mathrm{Area}(S_0)} + (x - x_0) \bullet \eta > 0$ on $S_0$ and $S_0$ is compact. Using the boundary conditions (1.5)–(1.6) we find that

$$V_n = -\frac{\partial}{\partial t} \left\{ \alpha |u|^2 \right\} - \left[ 2\alpha(t + t_0) - \frac{\partial g}{\partial \eta} \right] |u_t|^2 - \frac{\partial g}{\partial \eta} \left\{ A_{pq}\, \frac{\partial u}{\partial x_q} \bullet \frac{\partial u}{\partial x_p} + D\, \nabla\varphi \bullet \nabla\varphi \right\} - 2\alpha\, u_t \bullet (\nabla g \bullet \nabla) u \tag{3.28}$$

on $S_0$. Also

$$V_0 = \frac{\partial g}{\partial \eta} \left\{ \left( A_{ij}\, \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \right) + (D\, \nabla\varphi \bullet \nabla\varphi) \right\} \quad \text{on} \quad S_1. \tag{3.29}$$

Observe that

$$-2\alpha\, u_t \bullet (\nabla g \bullet \nabla) u \leq 2\alpha |u_t|\, |(\nabla g \bullet \nabla) u| \leq \alpha\, \varepsilon_0^{-1} |u_t|^2\, |\nabla g| + \alpha\, \varepsilon_0 |\nabla g| \left( \sum_{i=1}^{3} |\nabla u_i|^2 \right)$$
$$\leq \alpha\, \varepsilon_0^{-1} |\nabla g|\, |u_t|^2 + \alpha\, \varepsilon_0\, c_1^{-1} |\nabla g| A_{ij} \frac{\partial u}{\partial x_j} \bullet \frac{\partial u}{\partial x_i} \tag{3.30}$$

for any $\varepsilon_0 > 0$ where $c_1 > 0$ is as in Hypothesis 1.

Let $\varepsilon_0 = c_1 \lambda_0\, \gamma_1^{-1}$ where $\gamma_1 = \max_{x \in S_0} \alpha(x)$. It follows from (3.28) and (3.30) that

$$V_n \leq -\frac{\partial}{\partial t} \left\{ \alpha |u|^2 \right\} - \left[ 2\alpha(t + t_0) - \frac{\partial g}{\partial \eta} - \alpha \gamma_1 (c_1 \lambda_0)^{-1} |\nabla g| \right] |u_t|^2 \tag{3.31}$$

on $S_0$. We choose $t_0 > 0$ large enough so that $\left[ 2\alpha(t + t_0) - \frac{\partial g}{\partial \eta} - \alpha \gamma_1 (c_1 \lambda_0)^{-1} |\nabla g| \right] > 0$ for any $t \geq 0$ and $x \in S_0$. Also from Hypothesis 2 we obtain that

$$V_0 \leq 0 \quad \text{on} \quad S_1. \tag{3.32}$$

The following inequality can be proved by standard arguments

$$2 \sum_{m=0}^{n} \int_{\Omega_m} u_t^{(m)} \bullet \left\{ (\nabla g \bullet \nabla) u^{(m)} + u^{(m)} \right\} \mathrm{d}x + \int_{S_0} \alpha |u|^2 \, \mathrm{d}\Gamma$$
$$\leq c_{10} \sum_{m=0}^{n} \int_{\Omega_m} \left[ |u_t^{(m)}|^2 + A_{ij}^{(m)} \frac{\partial u^{(m)}}{\partial x_j} \bullet \frac{\partial u^{(m)}}{\partial x_i} \right] \mathrm{d}x \leq c_{10}\, E(t) \tag{3.33}$$

for some positive constant $c_{10}$ (which can be chosen as $c_{10} = \max\{2, c_1^{-1}(k + \max_{x \in \overline{\Omega}} |\nabla g(x)|)\}$ where $k > 0$ is such that

$$\sum_{m=0}^{n} \int_{\Omega_m} |u^{(m)}|^2 \, \mathrm{d}x + \int_{S_0} \alpha |u|^2 \, \mathrm{d}\Gamma \leq k \sum_{m=0}^{n} \int_{\Omega_m} |\nabla u_i^{(m)}|^2 \, \mathrm{d}x$$

where $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}) \in [H^1(\Omega_m)]^3$ with $u^{(m-1)} = u^{(m)}$ on $\Gamma_m$, $m = 1, 2, \ldots, n$.)

**Hypothesis 3.** The coefficients of system (1.2)–(1.3) satisfy the following monotonicity conditions

$$\left( A_{ij}^{(m-1)} - A_{ij}^{(m)} \right) v_j \bullet v_i \geq 0 \quad \text{for any} \quad v_i \in \mathbb{R}^3, \ 1 \leq m \leq n$$
$$\left( D^{(m)} - D^{(m-1)} \right) v \bullet v \geq 0 \quad \text{for any} \quad v \in \mathbb{R}^3, \ 1 \leq m \leq n.$$

**Theorem 3.1.** *Let us assume Hypothesis 1, 2 and 3. Let $\{u, \varphi\}$ be the unique solution of problem (1.2)–(1.9) as shown in Section 2. Then, there exists positive constants $c$ and $w$ such that*

$$E(t) \leq c \exp(-wt) E(0) \quad \forall t \geq 0$$

*where $E(t)$ is given by (1.10).*

*Proof.* It follows from identity (3.8) using Lemmas 3.1 and 3.2 together with (3.31), (3.32), (3.33) and Hypothesis 3 that

$$(T+t_0)E(T) \leq (2c_{10}+t_0)E(0) + \delta_0\tilde{c}\int_0^T E(t)\mathrm{d}t \tag{3.34}$$

where $\delta_0\tilde{c} < 1$ (as in Lem. 3.2), holds for any $T > 0$.

Let us denote by $h(T)$ the right hand side of (3.34). Clearly $\frac{h'(T)}{h(T)} \leq \frac{\delta_0\tilde{c}}{T+t_0}$ therefore $h(T) \leq \frac{(T+t_0)^p}{t_0^p}\,h(0)$ where $p = \delta_0\tilde{c} < 1$. Returning to (3.34) we obtain that

$$E(T) \leq \frac{c_{11}}{(T+t_0)^{1-p}}\,E(0) \tag{3.35}$$

where $c_{11} = (2c_{10}+t_0)t_0^{-p}$. Now, we choose $T = \widetilde{T} > 0$ large enough in (3.35) so that $T^* = \frac{c_{11}}{(\widetilde{T}+t_0)^{1-p}} < 1$. The semigroup (see Pazy [15]) property implies the conclusion of Theorem 3.1. □

## 4. Application: exact controllability

In this section, we use the main result obtained above in order to prove exact boundary controllability to an arbitrary state of solutions of (1.2)–(1.9) where instead of the first boundary condition in (1.5) we consider

$$\left(A_{ij}\,\frac{\partial u}{\partial x_j} + A_i^*\nabla\varphi\right)\eta_i = h(x,t) \quad \text{on} \quad S_0\times[0,+\infty) \tag{4.1}$$

where $f = (u_0, u_1)$ (in (1.4)) is an arbitrary element of the space $X$ (defined in Sect. 2). The formulation of the exact boundary control for the above system is the following: given the initial distribution $f = (u_0, u_1)$, a time $T > 0$ and a desired terminal state $g = (g_1, g_2)$, find a vector-valued function $h = h(x,t)$ such that the solution of (1.2)–(1.9) with condition (4.1) instead of the first boundary condition in (1.5), satisfies the conditions

$$u(x,T) = g_1(x), \qquad u_t(x,T) = g_2(x).$$

Let $\{U(t)\}_{t\geq 0}$ be the semigroup associated with problem (1.2)–(1.9) (with condition (4.1) instead of the first boundary condition in (1.5)). Consider the following equation in $X$:

$$w - U^*(T)U(T)w = f - U^*(T)g.$$

The operator $F(T) = U^*(T)U(T)$ takes $X$ into itself and $||F(T)|| < 1$ for any $T > T^*$ where $T^*$ was chosen in the proof of Theorem 3.1 as $T^* = \frac{c_{11}}{(\widetilde{T}+t_0)^{1-p}}$ with $\widetilde{T}$ large enough so that $T^* < 1$. Thus, we can solve this equation for any $f$ and $g \in Y$ with $w$ satisfying

$$||w|| \leq C(||f|| + ||g||).$$

Consequently, if we choose $w = (I - F(T))^{-1}(f - U^*(T)g)$ where $I$ denotes the identity and set

$$V(x,t) = U(t)w - U^*(T-t)(U(T)w - g) \equiv (\tilde{u},\tilde{v}) - (\tilde{\tilde{u}},\tilde{\tilde{v}}).$$

It follows that

$$V(x,0) = f(x), \qquad V(x,T) = g(x)$$

and $(u,v) = V(x,t)$ is a weak solution of (1.2)–(1.9) (with condition (4.1) instead of the first boundary condition in (1.5)) with $h(x,t) = -\alpha(\tilde{v} + \tilde{\tilde{v}})$. We observe that by the energy identity

$$||h||^2_{L^2(S_0\times(0,T))} \leq C\big(||f||^2_X + ||g||^2_X\big).$$

Thus, we arrive to the following assertion:

**Theorem 4.1.** *Assume that $\Omega$, $\Gamma_m$ and the coefficients satisfy the assumptions of Theorem 3.1. Then, for any $T > T^*$, given any initial data $f = (u_0, u_1) \in X$ and any $g = (g_1, g_2) \in X$ there exists a control $h(x,t) \in L^2(S_0 \times (0,T))$ such that the corresponding solution of (1.2)–(1.9) (with the above mentioned modification in (1.5)) satisfies*

$$u(x,T) = g_1(x), \qquad u_t(x,T) = g_2(x).$$

*Moreover*

$$||h||_{L^2(S_0\times(0,T))} \leq \widetilde{C}\big(||f||_X + ||g||_X\big)$$

*for some positive constant $\widetilde{C}$.*

## 5. Conclusions

In this work we consider a three-dimensional layered piezoelectric body with a dissipative mechanism effective at the boundary and appropriate transmission conditions at the interfaces. Using the multiplier technique we conclude that the total energy $E(t)$ decays exponentially as $t \to +\infty$, provided that the coefficients of the model satisfy a monotonicity condition and the domain as well as the interfaces also satisfy geometric requirements. As an application of our result we deduce exact controllability of the system with boundary control via a classical result due to Russell [17].

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