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BOUNDARY-INFLUENCED ROBUST CONTROLS: TWO NETWORK EXAMPLES *

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Abstract. We consider the differential game associated with robust control of a system in a compact state domain, using Skorokhod dynamics on the boundary. A specific class of problems motivated by queueing network control is considered. A constructive approach to the Hamilton-Jacobi-Isaacs equation is developed which is based on an appropriate family of extremals, including boundary extremals for which the Skorokhod dynamics are active. A number of technical lemmas and a structured verification theorem are formulated to support the use of this technique in simple examples. Two examples are considered which illustrate the application of the results. This extends previous work by Ball, Day and others on such problems, but with a new emphasis on problems for which the Skorokhod dynamics play a critical role. Connections with the viscosity-sense oblique derivative conditions of Lions and others are noted.

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Introduction

Robust control of nonlinear systems generally leads to differential games. Soravia [33] has given a rather general treatment of this, including the characterization of the (lower) value of the game as a viscosity solution of the Hamilton-Jacobi-Isaacs equation (HJI). In Lions' study [26] of Neumann-type boundary conditions for viscosity solutions the connection is made between oblique-derivative boundary conditions in the viscosity sense and problems in which the state dynamics use a Skorokhod problem mechanism to model boundary reflections which serve to restrict the state to a closed domain. We consider here a particular class of differential games, motivated by recent work in queueing control, which involve Skorokhod dynamics and a compact state domain.

Two somewhat different points of view have suggested deterministic differential games in the context of optimal service control strategies for queueing networks. Deterministic fluid processes, like $x(t)$ in our (8), arise as functional strong law limits $\frac{1}{n}X_{nt} \Rightarrow x(t)$ of discrete state stochastic queueing processes X_s; see [12]. It has been shown that stability properties of the zero-state of these fluid limits determine stochastic stability properties of the original stochastic queueing processes; see Dai [13] and Meyn [29]. Thus it is natural to pose various control problems for the limiting fluid systems in an effort to identify effective service strategies for the stochastic queueing processes. Formulations which minimize the mean time to empty and the mean holding cost until empty have been studied with considerable success; see Weiss [34] and Avram, Bertsimas, Ricard [5]

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(and [27, 35] regarding methods of calculation). A natural next step in this direction is to seek controls (for the fluid model) which are more robust with respect to modeling parameters. Following the successful game approach to robust control for nonlinear systems [9], one can consider a version of the fluid system which, in addition to the control player, also includes a second "disturbance" player whose role is to perturb the arrival and/or service rates. A game can then be posed using a cost criterion which balances some measure of system performance against some measure of the size of the disturbances. The particular game (11) we consider here originated from this point of view. Proposed by Day, Ball and others [6, 7, 15], it uses simple quadratic cost criteria similar to those of classical linear control theory.

A second source of differential game problems for queueing models is the risk-averse limit of risk-sensitive control problems. Whittle's [36] original idea of risk-sensitive control is to "exponentially sensitize" a stochastic control problem to large values of the cost by seeking to minimize $E\exp(\frac{1}{\epsilon}\text{cost})$, where $\epsilon > 0$ is a sensitivity parameter. When the sensitivity parameter is coupled to the scaling parameter of the fluid limit, $\epsilon = \frac{1}{n}$, the "risk-averse limit" as $\epsilon \to 0$ typically produces a deterministic differential game, in which the large-deviations structure of the stochastic process appears in the resulting cost. For standard control formulations (using Gaussian noise) this produces a nice connection of H^∞ ideas to stochastic control; see Fleming and James [21] and Fleming and McEneaney [22] for some results in this direction. The idea has been applied to policy optimization for queueing processes by Atar, Dupuis, and Schwartz [2, 3] leading to games for fluid queueing processes which are similar to ours, but with an entropy-like cost associated with the disturbance player, rather than the quadratic cost we use. For additional examples and ideas in the use of deterministic control problems in queueing networks, see Avram [4].

The Skorokhod problem mechanism is natural in modeling queueing systems because it effectively accounts for the inherent nonnegativity of queue lengths and the associated changes in the state dynamics when one or more queues becomes empty. This is true in both stochastic and deterministic (fluid) models; see [11,13,18,23,31,37] for instance. Its presence in the differential game gives rise to the boundary conditions and boundary extremals that are the focus of our work below.

Given the formulation of a differential game, the usual approach to its solution by means of its HJI equation consists of the following steps:

- prove that the value function $V(x)$ solves the HJI (typically in the viscosity sense);
- prove that solutions to the HJI (in the appropriate sense) are unique;
- construct (by some method) a solution to the HJI.

It is *not* our intent to carry out that general approach here. (The second step is particularly difficult for the games arising in robust control, and not adequately resolved even for more classical examples.) Rather we will pursue an approach more akin to that used by Isaacs [25] to explore illustrative examples. This consists of constructing the value $V(x)$ in specific examples using families of extremal trajectories covering the state domain, and which satisfy various equations or inequalities on the curves where two of the families coincide. This is not a theoretically general approach. However the solution to an example by this approach, when it exists, provides more detailed information about the structure of the differential game than any other method. Many of the papers we have cited above explore games for specific low dimensional queueing models in which this constructive approach is feasible. The examples of [6, 7, 15] in particular are addressed by such an approach. However, the construction of the extremals for the examples of those papers did not directly involve the Skorokhod dynamics. Only *after* the construction was complete were the Skorokhod dynamics considered, as part of the verification argument. We are concerned here with problems for which the Skorokhod dynamics play a more prominent role in the construction.

The reason that Skorokhod dynamics could be ignored in the constructions of [6,7,15] was essentially because the cases considered were all single-server models. Under the optimal strategy, service effort is applied only to nonempty queues, which keeps the system out of the Skorokhod dynamics regime. When multiple servers are involved it is possible for all the queues at one of the servers to be empty so that it is forced to operate at reduced capacity (which is the effect of the Skorokhod mechanism), while other servers still have work to do. The optimal strategy for the other servers may well be influenced by the reduced output from the server facing empty queues. In such examples we expect the Skorokhod dynamics to play a more important role in the design of the optimal policy. We will see that this is the case in the examples below.

Multiple server examples necessarily involve 3 or more state dimensions. We consider one (briefly) in Section 7. However as a tutorial to illustrate the applicability of our results, a 2-dimensional example is easier to describe and discuss. A 2-dimensional example for which the Skorokhod dynamics figure prominently in the construction is obtained if we choose some parameters in a way which would not be realistic for an actual network. The example of Section 6 is of this type, and will be our primary example. Although it is not a realistic network example, our purpose is to illustrate the constructive approach, and for that it is more suitable than a higher dimensional multiple server example.

Section 1 will describe the class of models under consideration and their motivation from queueing networks. Some technicalities associated with the Skorokhod problem are presented as well.

Section 2 will describe the differential game that we consider and its HJI equation. Our overall approach is to develop results which are sufficient to identify the value of the game using features found in simple examples, *but which are stronger than necessary in general.* In particular we will work with a version of the HJI in which the Skorokhod dynamics are included directly in the Hamiltonian (12):

$$H^{\pi}(x, \mathrm{D}V(x)) = 0.$$

This is appropriate for verification theorem of Section 4, but is different than the typical approach which employs a simpler Hamiltonian (14) together with separate oblique derivative boundary conditions, combined as in (15). Theorem 2.1 shows that a viscosity solution of our equation $-H^{\pi} = 0$ is necessarily a viscosity solution in the typical formulation (15). (The minus sign is important for the viscosity interpretation.) Theorem 2.1 also shows that for smooth functions $V(x)$, the classical and viscosity notions of solution to $-H^{\pi} = 0$ coincide. This is not the case for the typical formulation; a smooth function V which satisfies (15) *need not* satisfy the boundary conditions $d_i \cdot \mathrm{D}V(x) = 0$ in the classical sense. Although we will be working exclusively with smooth solutions of $-H^{\pi} = 0$, the connection with the oblique derivative boundary condition is used in Section 5.

Section 3 provides a number of purely technical sufficient conditions for a smooth function V to satisfy $H^{\pi}(x, \mathrm{D}V(x)) = 0$ at boundary points. These results are specific to the particular dynamics (8) and cost structure (11) of the games we are considering. When we construct the solution in an example using extremals and boundary extremals, the structure of the extremals will provide hypotheses needed to invoke these results. The results themselves are stated without any reference to extremals, however.

Section 4 provides the "structured" verification result Theorem 4.1 which identifies a smooth solution $V(x)$ of $H^{\pi}(x, \mathrm{D}V(x)) = 0$ as the value of the game. The theorem depends on the existence of a family $\mathcal{E}$ of extremals associated with V and having special properties. Although we cannot expect such structure to exist in general, it is present in some examples, as Section 6 will show.

Section 5 turns to the issue of how we might find extremals as needed for the family $\mathcal{E}$ needed by Theorem 4.1. In particular it motives a simple system of differential equations (50) which will produce an appropriate extremal running along the boundary of Ω for which the Skorokhod dynamics are active, what we call boundary extremals. There are no theorems in this section. Some presumptions are imposed for the sake of explaining why equations (50) are natural. But those presumptions do not need to be checked to apply the equations. Rather the equations are used in the construction of an extremal family $\mathcal{E}$ for which the properties needed to invoke Theorem 4.1 are either manifest in the equations themselves or by means of the results of Section 3. The fact that we have made restrictive presumptions in discussing the boundary extremal equations means only that they do not necessarily describe all possible boundary extremals. They will be enough for our examples, however.

Section 6 brings these various results to bear on a simple example in two dimensions. A smooth V and associated family of extremals $\mathcal{E}$ is described, which is built up starting from a pair of boundary extremals produced using the equations of Section 5. The numerous hypotheses needed to invoke Theorem 4.1 are discussed, using the technical results of Section 3 in particular to verify the saddle point conditions required at

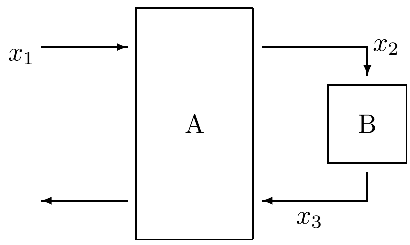

FIGURE 1. 2-Server example.

boundary points. The applicability of Theorem 4.1 implies that V is indeed the value of the game. In addition Theorem 2.1 implies that V is a viscosity solution of the typical HJI with boundary conditions (15).

Finally, in Section 7 we look at a few features of a family of extremals for the 2-server example of Figure 1. The purpose is only to illustrate those results from Section 3 that were not needed in Section 6. We do not present a full discussion of that example. In fact we believe this example produces a nonsmooth value function, though we do not pursue that here. We simply use it to illustrate remaining results from Section 3 by applying them locally.

We use the usual notation $x = (x_1, \ldots, x_n)$ to indicate $x \in \mathbb{R}^n$ in terms of its coordinates. For purposes of matrix expressions we consider all vectors $x \in \mathbb{R}^n$ to be column vectors; see (3) below for example. The usual scalar product of $x, y \in \mathbb{R}^n$ is thus $x \cdot y = x^T y$. The standard basis vectors in $\mathbb{R}^n$ are $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, with the dimension n determined by the context.

1. Model dynamics

We consider systems of the type arising in the fluid queueing models of [15] and [6] but with a more general compact domain Ω. As an example of our class of systems, consider the simple network of Figure 1. The server at A must allocate a total effort of $u_1(t) + u_3(t) = 1$ between serving queues x_1 and x_3, while the server at B has no service allocation decision and always serves x_2 at maximum capacity ($u_2 \equiv 1$). With maximum service rates $s_i > 0$ and time dependent arrival rates $q_i(t)$, the state or queue length vector $x(t) \in \mathbb{R}^3$ is described (nominally) by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} - s_1 u_1(t)\begin{bmatrix}1\\ -1\\ 0\end{bmatrix} - s_2\begin{bmatrix}0\\ 1\\ -1\end{bmatrix} - s_3 u_3(t)\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}. \tag{1}$$

The three vectors

$$d_1 = \begin{bmatrix}1\\ -1\\ 0\end{bmatrix},\ d_2 = \begin{bmatrix}0\\ 1\\ -1\end{bmatrix},\ d_3 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \tag{2}$$

describe the the contributions to $\dot{x}$ resulting from of service at each of the three queues. With $u(t) = (u_1(t), u_3(t))$ we can rewrite (1) as

$$\dot{x}(t) = q(t) - Gu(t), \tag{3}$$

where

$$G = \begin{bmatrix} s_1 & 0\\ s_2 - s_1 & s_2\\ -s_2 & s_3 - s_2\end{bmatrix}.$$

This only describes the state nominally because it does not account for the changes to the dynamics when one or more of the queues becomes empty. To maintain $x_i \geq 0$ we must add to the right side of (3) a positive multiple of d_i whenever $x_i = 0$ and $\dot{x}_i < 0$. This corresponds to simply reducing the requested service rate $s_i u_i$ to the level that can actually be implemented, given x_i, the arrival rate q_i, and any throughput from upstream queues. This is, in intuitive terms, the mechanism of Skorokhod dynamics, denoted $\dot{x} = \pi(x, q - Gu)$ with $\pi(\cdot,\cdot)$ as described below. The result, given input $q(\cdot)$ and control $u(\cdot)$, is a state $x(t)$ remaining in the nonnegative orthant $\mathbb{R}^3_+$.

We may also wish to impose finite buffer capacities. For instance if we impose limits $x_i \le B_i$, then we would also add to $\dot{x}$ positive multiples of "overflow" vectors $\tilde{d}_i$ when $x_i = B_i$ and $\dot{x}_i > 0$. For instance if customers overflowing buffer 2 are simply lost to the system we could take $\tilde{d}_2 = (0, -1, 0)$. If they are forced back into queue 1 we could take $\tilde{d}_2 = (1, -1, 0)$. Thus $\tilde{d}_i$ can be chosen to model various overflow assumptions. Once chosen, we would have a model of the form (8) below producing a state $x(t)$ in the closed box Ω consisting of x with of $0 \le x_i \le B_i$.

1.1. **System hypotheses**

To describe a general class of systems including such examples, we consider a state space $\Omega \subseteq \mathbb{R}^n$ which is assumed to be a compact, convex polyhedron defined by a system of linear inequalities

$$x \cdot n_i \ge c_i, \quad i = 1, \dots, N. \tag{4}$$

The n_i are assumed to be unit vectors. We require $0 \in \Omega$, which is equivalent to $c_i \le 0$. For $x \in \Omega$,

$$I(x) = \{i : \; x \cdot n_i = c_i\}$$

will denote the set of indices of active constraints. The *faces* of Ω are

$$\partial_i \Omega = \{x \in \Omega : \; n_i \cdot x = c_i\}.$$

Thus $x \in \partial_i \Omega$ iff $i \in I(x)$. For x in the relative interior of $\partial_i \Omega$, $I(x) = \{i\}$. More generally,

$$\partial_F \Omega = \cap_{i \in F} \partial_i \Omega = \{x \in \Omega : \; n_i \cdot x = c_i, \text{ all } i \in F\}.$$

Note that $x \in \partial_F \Omega$ implies $F \subseteq I(x)$, but F could be a proper subset of $I(x)$.

Associated with each constraint i is a *restoration vector* d_i which we assume to be normalized by

$$n_i \cdot d_i = 1.$$

Given a locally integrable $\psi(t) \in \mathbb{R}^n$ and initial point $x(0) \in \Omega$, the Skorokhod Problem consists of finding an absolutely continuous $x(t) \in \Omega$ described roughly by

$$\dot{x} = \psi(t) + \sum \beta_i(t) d_i \tag{5}$$

where $\beta_i(t)$ are nonnegative with $\beta_i(t) > 0$ only if $x(t) \in \partial_i \Omega$. If we take $\psi(t) = q(t) - Gu(t)$ in the context of our example (3) above, we see that the effect of the $\beta_i(t) d_i$ is simply to reduce the service rates to those levels which can be implemented without producing any negative state variables x_i.

The description (5) contemplates only absolutely continuous $x(t)$. For a careful and more general formulation in the context of paths with jump discontinuities, and the associated existence and uniqueness theory of the Skorokhod Problem, see Dupuis and Ishii [16], and also [18]. Technical hypotheses on Ω and the d_i are needed to insure that the problem is well-posed. Specifically we impose Assumption 2.1 of [16]: there exists a compact, convex set $B \subseteq \mathbb{R}^n$ with $0 \in B^\circ$, such that for each $i = 1, \dots N$ and $z \in \partial B$, and any inward normal v to B at z,

$$|z \cdot n_i| < 1 \quad \text{implies} \quad v \cdot d_i = 0. \tag{6}$$

(See Sect. 6 below for an example.) This hypothesis insures uniqueness and other continuity properties of the Skorokhod Problem.

For each $F \subseteq \{1, \dots, N\}$ let N_F (respectively D_F) denote the $n \times |F|$ matrix whose columns are n_i (respectively d_i), $i \in F$. We assume that for each $F = I(x)$, $x \in \partial\Omega$,

$$N_F^T D_F \text{ is coercive,} \tag{7}$$

namely that for any $a_i \in \mathbb{R}$, $i \in F$ not all zero, $(\sum_F a_i n_i) \cdot (\sum_F a_i d_i) > 0$. Notice that (7) only refers to those $F \subseteq \{1, \dots, N\}$ which actually occur as $F = I(x)$ for some $x \in \partial\Omega$. Although the number of constraints N might be large compared to the dimension n, (7) implies that $\{n_i : i \in I(x)\}$ and $\{d_i : i \in I(x)\}$ are both linearly independent for any $x \in \partial\Omega$ and thus the number $|I(x)|$ of active constraints can be at most n. Under these hypotheses, the solution $x(t)$ of the Skorokhod problem with a locally integrable $\psi(\cdot)$ is described precisely by an ordinary differential equation

$$\dot{x}(t) = \pi(x(t), \psi(t)),$$

holding almost surely, where $\pi(x, v)$ is the *velocity projection* defined in [16]. Section 1.2 below describes π and the significance of (7) in more detail.

The systems we consider are thus of the general form

$$\dot{x}(t) = \pi(x(t), q(t) - Gu(t)), \quad x(0) = x_0. \tag{8}$$

Here $x(t) \in \Omega$ is the queue-length vector, or state. $q(t) \in \mathbb{R}^n$ represents an unspecified external load, such as arrival rates of new "customers", to which the system must respond. The matrix G is fixed and the control variable $u(t)$ belongs to the convex hull $\mathcal{U}$ of some finite set U_0 of basic control settings. Taken together, $-Gu$, $u \in \mathcal{U}$, give the possible queue depletion rates that can be achieved by the various control settings. The choice of G, U_0 and d_i is made based on the particular network of interest, as illustrated by the example above. Other examples were treated in [6,15].

Before moving on, we want to record the following technical facts related to $\partial\Omega$, which are a consequence of (7).

Lemma 1.1. *Given $x \in \partial_F\Omega$ there exist vectors ν_j, $j \in F^c$ ($= \{1, \dots, N\} \setminus F$), such that the following hold:*

a) *n_i, $i \in F$ together with ν_j, $j \in F^c$ form a basis of $\mathbb{R}^n$.*
b) *For $i \in I(x)$ and $j \in F^c$, $n_i \cdot \nu_j = \delta_{ij}$.*
c) *For some $\epsilon > 0$,*

$$I\left(x + \sum_{j \in F^c} a_j \nu_j\right) = F \quad \textit{for all } 0 \le a_j < \epsilon.$$

Note that in b) it is possible that $i = j$ if F is a proper subset of $I(x)$. The point of c) is that given any $F \subseteq I(x)$ it is possible to construct $x' \in \Omega$ arbitrarily close to x with $I(x') = F$. This will be needed in the arguments of Section 5 below.

Proof. We establish the existence of ν_j, $j \in F^c$ first for $j \in I(x) \setminus F$. Let $m = |I(x)|$. The independence of n_i, $i \in I(x)$ means that $N_{I(x)}^T$ has full row rank m. Consequently, for each $j \in I(x) \setminus F$ there exists a solution ν_j to $N_{I(x)}^T \nu_j = e_j$ ($e_j \in \mathbb{R}^m$). We can assume moreover that each ν_j is in the span of n_i, $i \in I(x)$. This accounts for b) if $j \in I(x) \setminus F$. The next step is to show that n_i, $i \in F$ together with ν_j, $j \in I(x) \setminus F$ are linearly independent. To prove the linear independence, suppose α_i, β_j are scalars such that

$$\sum_{i \in F} \alpha_i n_i + \sum_{j \in I(x) \setminus F} \beta_j \nu_j = 0.$$

Taking the scalar product of this with $\sum_{i\in F}\alpha_i n_i$ and using b) implies that $\|\sum_{i\in F}\alpha_i n_i\|^2 = 0$. The linear independence of n_i implies that $\alpha_i = 0$. We are left with $\sum_{j\in I(x)\setminus F}\beta_j\nu_j = 0$. The scalar product of this with n_j, $j \in I(x)\setminus F$ now implies $\beta_j = 0$. It now follows that n_i, $i \in F$ together with ν_j, $j \in I(x)\setminus F$ are a basis of the span of n_i, $i \in I(x)$.

We need complete the selection of ν_j, for $j \in I(x)^c$. We take these to be any orthonormal basis of the orthogonal complement of the span of n_i, $i \in I(x)$. Part a) is now satisfied. Part b) remains true for the new ν_j.

We turn now to part c). Let

$$y = x + \sum_{j\in F^c} a_j\nu_j.$$

For $i \in F$, since $n_i \cdot x = c_i$, it follows from b) that

$$n_i \cdot y = n_i \cdot x = c_i.$$

For $j \in I(x)\setminus F$, since $n_j \cdot x \geq c_j$ and $a_j \geq 0$ we have

$$n_j \cdot y = n_j \cdot x + a_j \geq c_j.$$

For $j \in I(x)^c$ we have

$$n_j \cdot y \geq n_j \cdot x - \sum |a_j|\|\nu_j\| \geq n_j \cdot x - \left(\sum a_j\right)\max\|\nu_j\|.$$

Since $n_j \cdot x > c_j$ for such j, it is clear that $\epsilon > 0$ exists as claimed. □

1.2. The projection map and reflection matrices

In [16] the Skorokhod Problem is considered under two hypotheses. The first is our (6), which insures uniqueness of solutions and Lipschitz continuity properties. The second is Assumption 3.1 of [16], which concerns existence of solutions. It is generally recognized that Assumption 3.1 is equivalent to the solvability of a collection of complementarity problems associated with $x \in \partial\Omega$. One development of this is presented in [14]. These complementarity problems provide the representation of the velocity projection map $\pi(x, v)$ that we will use below. For x in the interior of Ω, $\pi(x, v)$ is simply v. For $x \in \partial\Omega$ the value $w = \pi(x, v)$ is described by

$$w = v + \sum_{i\in I(x)}\beta_i d_i, \tag{9}$$

where for each $i \in I(x)$, β_i satisfies the following complementarity conditions:

$$\beta_i \geq 0,\ n_i \cdot w \geq 0,\ \text{and}\ \beta_i(n_i \cdot w) = 0. \tag{10}$$

Our hypothesis (7) is a simple sufficient condition for the existence of a unique solution to the complementarity problem (9) and (10). Note that a consequence of the complementarity conditions is that

$$n_i \cdot \pi(x, v) \geq 0,\ \text{for all}\ i \in I(x).$$

We record a number of facts related to $\pi(x, v)$ which will be used in later sections.

Lemma 1.2. *The map $v \mapsto \pi(x, v)$ is Lipschitz continuous, uniformly over $x \in \Omega$: there exists a constant K so that*

$$\|\pi(x, v_2) - \pi(x, v_1)\| \leq K\|v_2 - v_1\|.$$

This follows from the hypothesis (6) as in [16]. It also can be derived directly from the coercivity (7). A simple consequence is the bound

$$\|\pi(x, v)\| \leq K\|v\|.$$

In solving (9), (10) if one knew $F = \{i \in I(x) : \beta_i > 0\}$ *a priori*, then solving $n_i \cdot w = 0$, $i \in F$ for β_i leads to $\pi(x, v) = R_F v$ where R_F are the *reflection matrices*

$$R_F = I + D_F B_F,$$

and $B_F = -(N_F^T D_F)^{-1} N_F^T$. (Notice that $N_F^T D_F$ is indeed invertible under our hypothesis (7).) In general one does not know F in advance and must solve the complementarity problem to determine F. The following simple modification of Lemma 1 of [15] describes $\pi(x, v)$ directly in terms of the reflection matrices. (Vector inequalities are to be interpreted coordinatewise.)

Lemma 1.3. *Given $x \in \Omega$ and $F \subseteq I(x)$ then $\pi(x, v) = R_F v$ if and only if*

a) $B_F v \geq 0$, *and*
b) $N_{I(x)\setminus F}^T R_F v \geq 0$.

(We consider a) [respectively b)] to hold vacuously if $F = \emptyset$ [$I(x) = F$].) Moreover, $\pi(x, v) = R_F v$ determines $F \subseteq I(x)$ uniquely if and only if

$$B_F v > 0 \quad \text{and} \quad N_{I(x)\setminus F}^T R_F v > 0$$

hold with strict inequality in all components.

Note that $\pi(x, v) = R_F v$ holds for $F = F_0 = \{i \in I(x) : \beta_i > 0\}$ in particular, where β_i are as in the complementarity problem (10). And certainly $\pi(x, v) = R_F v$ implies $F_0 \subseteq F$. Thus F_0 is the minimal F for which $\pi(x, v) = R_F v$. The following properties of the reflection matrices will be useful.

Lemma 1.4. *For each $F = I(x)$, $x \in \partial\Omega$, the following hold:*

a) *The kernel of R_F has basis $\{d_i : i \in F\}$.*
b) *The kernel of R_F^T has basis $\{n_i : i \in F\}$.*
c) $p = R_F^T p$ *iff* $d_i \cdot p = 0$ *all* $i \in F$.
d) $v = R_F v$ *iff* $n_i \cdot v = 0$ *all* $i \in F$.

Proof. It is easy to see from the formula for R_F that $R_F D_F = 0$ and $N_F^T R_F = 0$. Therefore the rank of R_F is at most $n - k$ where $k = |F|$. For any p orthogonal to all d_i, $i \in F$, one also easily checks that $p = R_F^T p$. This implies that the rank of R_F is at least $n - k$, and therefore equal to $n - k$. This implies a) and b), as well as the "if" assertion of c). If $p = R_F^T p$ then p is in the range of R_F^T. Since the range of R_F^T is the same as the orthogonal complement of the kernel of R_F, a) implies $d_i \cdot p = 0$ for all $i \in F$. This completes the proof of c). Part d) is similar: one easily checks that $n_i \cdot v = 0$ for all $i \in F$ implies $v = R_F v$. Conversely, $v = R_F v$ implies v is in the range of R_F, which is the same as the orthogonal compliment of the kernel of R_F^T, which by b) implies $n_i \cdot v = 0$ all $i \in F$. □

Lemma 1.5. *For any subsets $F_1 \subseteq F_2 \subseteq I(x)$, $x \in \partial\Omega$,*

$$R_{F_1} R_{F_2} = R_{F_2} R_{F_1} = R_{F_2}.$$

In particular, for any $F \subseteq I(x)$, R_F is an (oblique) projection: $R_F = R_F R_F$.

Proof. By Lemma 1.4b, the rows of $N_{F_1}^T$ are a subset of the kernel of $R_{F_2}^T$. Therefore $N_{F_1}^T R_{F_2} = 0$, and so

$$R_{F_1} R_{F_2} = R_{F_2} - D_{F_1}(N_{F_1}^T D_{F_1})^{-1} N_{F_1}^T R_{F_2} = R_{F_2}.$$

Similarly, Lemma 1.4a implies $R_{F_2} D_{F_1} = 0$, so that

$$R_{F_2} R_{F_1} = R_{F_2} - R_{F_2} D_{F_1}(N_{F_1}^T D_{F_1})^{-1} N_{F_1}^T = R_{F_2}.$$

□

2. L_2 robust control formulation and Hamilton-Jacobi-Isaacs equations

Using the system equations (8) we consider the differential game formulated in [15]. Specifically, we seek the lower value function

$$V(x(0)) = \inf_{\alpha(\cdot)} \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t. \tag{11}$$

We now describe the qualifications of the objects on the right side. First, T ranges over $(0,\infty)$ and $q(\cdot)$ is an arbitrary square integrable function on $[0,T]$. Next, $\alpha(\cdot)$ refers to an arbitrary nonanticipating strategy. By this we mean that for each $x(0)\in\Omega$ and $q\in \mathrm{L}^2[0,T]$, α is a rule which determines at least one (possibly more) measurable control function $u(t)=\alpha[x(0),q(\cdot)](t)\in\mathcal{U}$ for $0\le t\le T$ such that the solution of (8) exists on $[0,T]$. For technical reasons we allow a policy to determine more than one control. The necessity of this is explained in some detail in Section 4; see the discussion preceding Theorem 4.1. Nonanticipating means that if $q(s)=\tilde q(s)$ for $s\le t$, then $\alpha[x(0),q(\cdot)](s)=\alpha[x(0),\tilde q(\cdot)](s)$ for $s\le t$. More properly, since we are allowing a policy to determine more than one control, our definition of nonanticipating should interpret $\alpha[x(0),q(\cdot)](s)$ as the *set* of associated $u(s)$ on $0\le s\le t$. We mean this nonanticipating property to include the usual *Markov-extension property*: if $q\in\mathrm{L}^2[0,T]$ and $\tilde q\in\mathrm{L}^2[0,\tilde T]$ with $T<\tilde T$ and $q(t)=\tilde q(t)$ for $0\le t\le T$, then any $u(\cdot)$ in $\alpha[x(0),q(\cdot)]$ for $[0,T]$ is the restriction to $[0,T]$ of some $\tilde u(\cdot)$ in $\alpha[x(0),\tilde q(\cdot)]$ on $[0,\tilde T]$. In other words, if $u(t)$ is one of the controls that a policy α associates with $q(t)$ on $[0,T]$, and we extend q to $\tilde q$ on $[0,\tilde T]$, then $u(t)$ has an extension $\tilde u(t)$ to $[0,\tilde T]$ still associated with the same policy α.

The possibility of multiple $u(t)$ for a given α is handled in (11) by putting the multiple $u(\cdot)$ into the supremum. Thus for each nonanticipating policy α and given $x(0)\in\Omega$ the supremum in (11) is over all $q\in\mathrm{L}^2[0,T]$, all $u(t)=\alpha[x(0),q(\cdot)](t)$ produced by the policy and solutions $x(t)$ of (8) corresponding to $q(\cdot),u(\cdot)$. The outer infimum is then over all nonanticipating strategies.

Aside from the allowance of multiple controls per policy, and the presence of Skorokhod dynamics in (8), this is the general formulation of robust control as described by Soravia [33]; see his equation (2.6) in particular. In [15], Section 2.4, additional technicalities are discussed which arise because only $\Omega=\mathbb{R}^n_+$ (unbounded) was considered, while (11) can only be finite in a bounded set. We have avoided that complication here by assuming Ω to be compact. Soravia and other treatments of robust control typically include a gain parameter γ^2 multiplying the $\|q\|^2$ in (11). The reasoning of [15, Section 2.4] showed that for problems of our structure, γ can be scaled out of the problem by considering $\gamma^3V(\gamma^{-1}x)$ (in $\Omega_\gamma=\gamma\Omega$). Thus we simply take $\gamma=1$ for our discussion.

Based on the results of Soravia, and the general theory and heuristics of differential games [8], we expect $V(x)$ to be a viscosity solution of the following Hamilton-Jacobi-Isaacs equation (HJI):

$$-H^\pi(x,\mathrm{D}V(x))=0,\quad x\in\Omega. \tag{12}$$

The Hamiltonian here is defined by

$$H^\pi(x,p)=\sup_q \inf_{u\in\mathcal{U}}\left(p\cdot\pi(x,q-Gu)-\frac{1}{2}\|q\|^2+\frac{1}{2}\|x\|^2\right). \tag{13}$$

As explained in the introduction, our goal is not to prove (12) as a theoretical necessity. Rather we want to develop a constructive approach which uses (12) as part of sufficient conditions to identify the value (11). This is the same approach as in [6,15]. The new features here relate to problems in which the Skorokhod dynamics play a more prominent role.

The papers [1,2] are among the few treatments in the literature of differential games which include Skorokhod dynamics on polyhedral domains such as our Ω. They prove that the value function is necessarily a viscosity solution of a HJI equation with oblique derivative boundary conditions. In that approach the Hamiltonian is defined independently of the reflection map $\pi(x,v)$. In the remainder of this section we observe some simple relationships between viscosity solutions of (12) and the boundary condition formulation. After this section V

will always be C^1. But here we temporarily relax our hypothesis to $V \in C(\Omega)$ for the purpose of making these connections to the general theory of viscosity solutions of Hamilton-Jacobi equations.

For both HJI formulations we need to be precise about the definition of the super- and sub-differentials of $V(x)$. $D^+_\Omega V(x)$ is defined to be the set of $\phi^+ = D\psi(x)$ where $\psi(\cdot)$ is a smooth test function such that, for some $\epsilon > 0$, $V(x) - \psi(x) \geq V(y) - \psi(y)$ for all $y \in \Omega$ with $\|y - x\| < \epsilon$. In other words, $V - \psi$ has a local maximum at x *relative to* Ω. The definition of $D^-_\Omega V(x)$ is analogous, with local relative minimum replacing maximum.

We say that a continuous function V is a viscosity solution of $-H^\pi(x, DV(x)) = 0$ if for each $x \in \Omega$, $\phi^+ \in D^+_\Omega V(x)$ and $\phi^- \in D^-_\Omega V(x)$, the following hold:

$$-H^\pi(x, \phi^+) \leq 0, \quad \text{and } -H^\pi(x, \phi^-) \geq 0.$$

The formulation using boundary conditions uses only the the *interior Hamiltonian*,

$$\begin{aligned} H(x,p) &= \sup_q \inf_{u \in \mathcal{U}} \left(p \cdot (q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right) \\ &= \frac{1}{2}\|p\|^2 - \sup_{u \in \mathcal{U}} p \cdot Gu + \frac{1}{2}\|x\|^2. \end{aligned} \tag{14}$$

A continuous function V is a *viscosity solution of* $-H(x, DV(x)) = 0$ *in* Ω *with oblique derivative boundary conditions* $-d_i \cdot DV(x) = 0$, $i \in I(x)$, if for each $x \in \Omega$, $\phi^+ \in D^+_\Omega V(x)$ and $\phi^- \in D^-_\Omega V(x)$ the following hold:

$$\min_{i \in I(x)} (-H(x, \phi^+), -d_i \cdot \phi^+) \leq 0, \quad \text{and } \max_{i \in I(x)} (-H(x, \phi^-), -d_i \cdot \phi^-) \geq 0. \tag{15}$$

The theorem below provides the simple connections between viscosity solutions of our original HJI (12) and the boundary condition formulation (15). Note in particular that for $C^1(\Omega)$ functions the classical and viscosity notions for (12) coincide. (This is not true for the formulation (15).)

Theorem 2.1.

a) *Suppose* $V \in C^1(\Omega)$. *Then* V *is a classical solution of* $-H^\pi(x, DV(x)) = 0$ *in* Ω *if and only if it is a viscosity solution of* $-H^\pi(x, DV(x)) = 0$ *in* Ω.
b) *If* V *is a continuous viscosity solution of* $-H^\pi(x, DV(x)) = 0$ *in* Ω, *then* V *is a viscosity solution of* $-H(x, DV(x)) = 0$ *in* Ω *with boundary conditions* $-d_i \cdot DV(x) = 0$ *for all* $i \in I(x)$.

Proof. For x in the interior of Ω the hypothesis in part a) that $V \in C^1(\Omega)$ means that $D^\pm_\Omega V(x) = \{DV(x)\}$, so that the viscosity and classical notions of solution coincide. Consider then $x \in \partial\Omega$ and $\phi^+ \in D^+_\Omega V(x)$. We claim that $\phi^+ = DV(x) + \sum_{I(x)} b_i n_i$, for some $b_i \geq 0$. This boils down to the conjugacy relationship between a convex cone and its polar cone: [32], Theorem 14.1. Let

$$\mathcal{N}(x) = \left\{ \sum_{i \in I(x)} b_i n_i, \; b_i \geq 0 \right\}.$$

By definition of Ω, the vectors v such that $x + tv \in \Omega$ for all t in some interval $[0, \epsilon)$ are precisely those for which $v \cdot n \geq 0$ for all $n \in \mathcal{N}(x)$. In other words $-v \in \mathcal{N}(x)^\circ$, the polar cone to $\mathcal{N}(x)$. The definition of $\phi^+ \in D^+_\Omega V(x)$ implies that $(\phi^+ - DV(x)) \cdot v \geq 0$ for all such v. This means that $\phi^+ - DV(x) \in \mathcal{N}(x)^{\circ\circ}$, which is the same as $\mathcal{N}(x)$ by the conjugacy relationship. This proves our claim that $\phi^+ = DV(x) + \sum_{I(x)} b_i n_i$, for some $b_i \geq 0$.

For any $v \in \mathbb{R}^m$ we know $\pi(x, v) \cdot n_i \geq 0$, all $i \in I(x)$. Therefore $\phi^+ \cdot \pi(x, v) \geq DV(x) \cdot \pi(x, v)$, which implies

$$-H^\pi(x, \phi^+) \leq -H^\pi(x, DV(x)).$$

So if V is a classical solution of $H^\pi(x, \mathrm{D}V(x)) = 0$ then it is a viscosity subsolution. Analogous reasoning implies the supersolution property. Conversely if V is a viscosity solution of $-H^\pi(x, \mathrm{D}V(x)) = 0$, then since $V \in \mathrm{C}^1(\Omega)$ we know $\mathrm{D}V(x)$ belongs to both $\mathrm{D}^\pm_\Omega V(x)$, which implies $H^\pi(x, \mathrm{D}V(x)) = 0$. Hence V is also a classical solution.

We turn now to b). Again, for x in the interior of Ω the two notions of solution coincide, so we consider an $x \in \partial\Omega$ and $\phi^+ \in \mathrm{D}^+_\Omega V(x)$. By hypothesis $-H^\pi(x, \phi^+) \le 0$. We want to show that

$$-H(x, \phi^+) \wedge \min_{I(x)}(-d_i \cdot \phi^+) \le 0.$$

We may suppose that $d_i \cdot \phi^+ < 0$ for all $i \in I(x)$, else the above is trivial. Now we know that for any $v \in \mathbb{R}^m$, $\pi(x, v) = v + \sum_{I(x)} \beta_i d_i$ for some $\beta_i \ge 0$. It follows that $\phi^+ \cdot \pi(x, v) \le \phi^+ \cdot v$, and consequently,

$$-H(x, \phi^+) \le -H^\pi(x, \phi^+) \le 0.$$

This shows that the implication of b) is true for subsolutions of the two problems. The argument for supersolutions is analogous. □

3. Sufficient conditions for saddle point solutions of $H^\pi = 0$

Our goal in the sections to follow is to produce a $V \in \mathrm{C}^1(\Omega)$ satisfying (12). (The negative sign is significant for the notion of viscosity solution, but is irrelevant for smooth solutions.) At interior points of Ω, $H(x, p) = H^\pi(x, p)$ for all p. So the V we seek will satisfy $H(x, \mathrm{D}V(x)) = 0$ at all $x \in \Omega$, including boundary points by continuity. The results of this section are aimed at the converse conclusion. Specifically we assume throughout this section that $p(x)$ $(= \mathrm{D}V(x))$ is a continuous $\mathbb{R}^n$-valued function on Ω which satisfies

$$H(x, p(x)) = 0, \quad \text{all } x \in \Omega. \tag{16}$$

We develop a number of purely technical sufficient conditions under which we can conclude that $H^\pi(x, p(x)) = 0$ for $x \in \partial\Omega$ as well. Notice that in the definition (14) of $H(x, p(x))$, the $\sup_q$ and $\inf_{u \in \mathcal{U}}$ separate, and the value is always given by a saddle point (q^*, u_*) determined by

$$q^* = p(x) \quad \text{and} \quad p(x) \cdot Gu_* = \max_{u \in \mathcal{U}} p(x) \cdot Gu. \tag{17}$$

We will call (17) the *interior saddle point conditions* at x.

We will be interested specifically in *saddle point solutions* of $H^\pi(x, p(x)) = 0$. By this we mean that the $\sup_q \inf_{u \in \mathcal{U}}$ of (13) is given by a saddle point (q^*, u_*):

$$u_* \in \mathcal{U} \text{ minimizes } p(x) \cdot \pi(x, q^* - Gu) \tag{18}$$

$$q^* \in \mathbb{R}^m \text{ maximizes } p(x) \cdot \pi(x, q - Gu_*) - \frac{1}{2}\|q\|^2, \tag{19}$$

with the saddle value

$$0 = p(x) \cdot \pi(x, q^* - Gu_*) - \frac{1}{2}\|q^*\|^2 + \frac{1}{2}\|x\|^2. \tag{20}$$

Notice that for $H^\pi(x, p(x)) = 0$ one would only need that q^* maximizes

$$\inf_{u \in \mathcal{U}} \left(p(t) \cdot \pi(x(t), q - Gu) - \frac{1}{2}\|q\|^2 \right),$$

which is weaker than (19). However the stronger property (19) will be important in the verification theorem of the next section. We will refer to (18) as the *u_*-saddle condition* and (19) as the *q^*-saddle condition.* When

$x \in \partial\Omega$ the saddle conditions (18), (19), and (20) are more complicated than their interior versions (17) due to the presence of the velocity projection $\pi(x, v)$. In general it is conceivable that (18)–(20) hold for a different saddle point than the interior saddle conditions. However, in our results we will always have the same $q^* = p(x)$ and u_* as in (17).

As a first result we observe that if the boundary conditions $d_i \cdot \mathrm{D}V(x) = 0$ are satisfied in the classical sense at $x \in \partial\Omega$, then the saddle conditions for $H^\pi(x, p(x)) = 0$ hold there as well. While elementary, this is significant because the boundary extremals of Section 5 below will produce values of $p(x)$ which satisfy the boundary conditions classically; see (49) and Lemma 1.4.

Theorem 3.1. *Assume* (16) *and that* $d_i \cdot p(x_0) = 0$ *for all* $i \in I(x_0)$. *Then all* (q^*, u_*) *satisfying the interior saddle point conditions* (17) *at* x_0 *satisfy the saddle conditions* (18) *and* (19) *at* x_0 *as well.*

Proof. Since $d_i \cdot p(x_0) = 0$ for all $i \in I(x_0)$ it follows from (9) that $p(x_0) \cdot \pi(x_0, v) = p(x_0) \cdot v$ for all v. Thus the interior saddle point conditions are equivalent to (18) and (19). □

Next we collect several sufficient conditions for the q^*-saddle condition.

Theorem 3.2. *Assume* (16), $x_0 \in \partial\Omega$, *that* q^* *and* u_* *satisfy the interior saddle point conditions* (17) *there, and that*

$$\pi(x_0, q^* - Gu_*) = q^* - Gu_*. \tag{21}$$

The following are each sufficient for the boundary q^**-saddle condition* (19) *at* x_0, *using the same* u_*.

a) $d_i \cdot p(x_0) \leq 0$ *for all* $i \in I(x_0)$.
b) $I(x_0) = \{i\}$ *and*

$$p(x_0) \cdot d_i \leq 2n_i \cdot (p(x_0) - Gu_*). \tag{22}$$

c) *For some* $\epsilon > 0$ *and each* $\|x - x_0\| < \epsilon$, $\bar{q}(x) = R^T_{I(x)} p(x)$ *fails to satisfy one of the following:*
  i) $\pi(x, \bar{q}(x) - Gu_*) = R_{I(x)}(\bar{q}(x) - Gu_*)$, *and*
  ii) $p(x) \cdot \pi(x, \bar{q}(x) - Gu_*) + \frac{1}{2}\|\bar{q}(x)\| > p(x) \cdot \pi(x, p(x) - Gu_*) + \frac{1}{2}\|p(x)\|$.

Several comments are in order. First observe that (21) is equivalent to $n_i \cdot (q^* - Gu_*) \geq 0$, all $i \in I(x_0)$, *i.e.* that the velocity $v = q^* - Gu_*$ points into Ω from x_0. When we consider extremals in Section 5 we will find that in the interior of Ω they will satisfy $\dot{x} = p - Gu_*$. Thus Theorem 3.2 (and Th. 3.3 below) will apply at $x_0 \in \partial\Omega$ at which an extremal moves from x_0 into the interior in positive time.

Part b) provides a definitive test for the q^* saddle condition on a face. (In fact (22) is necessary as well, though we have only presented the sufficiency.) Notice that in light of the hypothesis $n_i \cdot (p(x_0) - Gu_*) \geq 0$, the inequality

$$d_i \cdot p(x_0) \leq n_i \cdot (p(x_0) - Gu_*) \tag{23}$$

implies (22). In particular if $d_i = n_i$, then b) applies whenever $n_i \cdot Gu_* \leq 0$. Or if $d_i = n_i - e_{i+1}$ we find that $n_i \cdot Gu_* \leq e_{i+1} \cdot p(x_0)$ is sufficient. These observations will be useful in Section 7.

Part c) is useful on edges. Consider a point on an edge: $I(x_0) = \{i, j\}$. Suppose we have already verified the saddle conditions on the adjoining faces $\partial_i\Omega$ and $\partial_j\Omega$. This implies that ii) does not hold on those faces because otherwise $\bar{q}(x)$ would violate the saddle condition there. Thus we would only need to check that i) or ii) fails for x on the edge $I(x) = \{i, j\}$ and sufficiently close to x_0. In particular, by Lemma 1.3, if one coordinate of $B_{I(x)}(q(x) - Gu_*)$ is negative at $x = x_0$, then by continuity i) cannot hold. The q^* saddle condition would follow as a consequence.

Proof. Part a) is the same as Lemma 2.2 of [6]. The proof is elementary, being analogous to that of Theorem 3.3 part a), which is written out below.

We turn to the proof of b). Define the "fixed-control" Hamiltonians

$$H_{u_*}(x,p) = \sup_q \left(p\cdot(q-Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right)$$
$$= \frac{1}{2}\|p\|^2 - p\cdot Gu_* + \frac{1}{2}\|x\|^2,$$

and

$$H^\pi_{u_*}(x,p) = \sup_q \left(p\cdot\pi(x,q-Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right).$$

Our hypotheses imply $H_{u_*}(x_0,p(x_0)) = 0$. Our goal is to show that $H^\pi_{u_*}(x_0,p(x_0)) = 0$. We begin by writing

$$p(x_0) = \mu n_i + \tau,$$

where τ is a vector orthogonal to n_i and $\mu = n_i\cdot p(x_0)$. Define the scalar function

$$h(w) = H_{u_*}(x, wn_i + \tau).$$

We know $h(w)$ is quadratic, strictly convex, with $h'(w) = n_i\cdot(wn_i+\tau-Gu_*) = w-w_0$, where

$$w_0 = n_i\cdot Gu_*.$$

Thus $h(w)$ takes its unique minimum at w_0. By hypothesis $h(\mu) = H_{u_*}(x_0,p(x_0)) = 0$. So $h(w_0)\le 0$. If $\mu\ne w_0$ the other root of $h(w)=0$ is

$$\tilde\mu = w_0 - (\mu - w_0) = 2w_0 - \mu.$$

Note that from (21) we know $n_i\cdot(p(x_0)-Gu_*)\ge 0$, which implies $\mu\ge w_0$. Therefore

$$\tilde\mu \le w_0 \le \mu. \tag{24}$$

Since $R_{\{i\}} = I - d_i n_i^T$ we find

$$R^T_{\{i\}}p(x_0) = p(x_0) - (p(x_0)\cdot d_i)\ n_i$$
$$= \mu_R n_i + \tau, \tag{25}$$

where

$$\mu_R = \mu - p(x_0)\cdot d_i.$$

Observe that the hypothesis $p(x_0)\cdot d_i \le 2n_i\cdot(p(x_0)-Gu_*)$ means $\mu-\mu_R\le 2(\mu-w_0)$, which on rearrangement implies that

$$\tilde\mu \le \mu_R. \tag{26}$$

With these preliminaries we can turn to the main argument. The $\sup_q$ defining $H^\pi_{u_*}(x,p(x_0))$ can be broken into two parts. The first part considers just those q for which $\pi(x,q-Gu_*) = q-Gu_*$, *i.e.* those q for which $n_i\cdot(q-Gu_*)\ge 0$, which is to say $n_i\cdot q\ge w_0$. The second part considers those q for which $\pi(x,q-Gu_*) = R_{\{i\}}(q-Gu_*)$, *i.e.* q for which $n_i\cdot q\le w_0$. (The q with $n_i\cdot q = w_0$, so that $R_{\{i\}}(q-Gu_*) = (q-Gu_*)$, are included in both parts of the supremum.) Thus we can write

$$H^\pi_{u_*}(x,p(x_0)) = \max(Q_1,Q_2), \tag{27}$$

where

$$Q_1 = \sup_{n_i \cdot q \geq w_0} \left(p(x_0) \cdot (q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right)$$
$$Q_2 = \sup_{n_i \cdot q \leq w_0} \left(p(x_0) \cdot R_{\{i\}}(q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right),$$

Consider Q_1. The global supremum (*i.e.* unconstrained by $n_i \cdot q \geq w_0$) is attained at $q^* = p(x_0)$. By hypothesis, $n_i \cdot q^* - w_0 = n_i \cdot (p(x_0) - Gu_*) \geq 0$. Thus q^* satisfies the constraint, so that

$$Q_1 = H_{u_*}(x, p(x_0)) = h(\mu) = 0.$$

Next consider Q_2. Observe that we can complete the square to rewrite the expression to be minimized as:

$$p(x_0) \cdot R_{\{i\}}(q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 = C - \frac{1}{2}\|q - R_{\{i\}}^T p(x_0)\|^2,$$

where C is a collection of terms that do not depend on q. The global maximum with respect to q (*i.e.* ignoring the constraint to $n_i \cdot q \leq w_0$) is attained at $\bar{q} = R_{\{i\}}^T p(x_0)$. If $\mu_R \leq w_0$ then (25) implies that $\bar{q}$ satisfies the constraint. In that case $Q_2 = H_{u_*}(x, R_{\{i\}}^T p(x_0)) = h(\mu_R)$. Moreover (26) and (24) imply that μ_R is in the interval $[\tilde{\mu}, \mu]$ on which $h(w) \leq 0$. Therefore $Q_2 = h(\mu_R) \leq 0$.

Now suppose $\mu_R > w_0$. Then the maximum Q_2 will be attained at that q closest to $R_{\{i\}}^T p(x_0) = \mu_R n_i + \tau$ with $n_i \cdot q \leq w_0$. Thus the maximizer is $\bar{q} = w_0 n_i + \tau$. Since $n_i \cdot (\bar{q} - Gu_*) = w_0 - w_0 = 0$, it follows that $R_{\{i\}}(\bar{q} - Gu_*) = (\bar{q} - Gu_*)$ and therefore

$$Q_2 = p(x_0) \cdot (\bar{q} - Gu_*) - \frac{1}{2}\|\bar{q}\|^2 + \frac{1}{2}\|x\|^2.$$

But notice that

$$(p(x_0) - \bar{q}) \cdot (\bar{q} - Gu_*) = (\mu - w_0)\, n_i \cdot (\bar{q} - Gu_*) = 0.$$

Thus, when $\mu_R > w_0$ we have

$$\begin{aligned} Q_2 &= \bar{q} \cdot (\bar{q} - Gu_*) - \frac{1}{2}\|\bar{q}\|^2 + \frac{1}{2}\|x\|^2 \\ &= H_{u_*}(x, \bar{q}) \\ &= h(w_0) \leq 0. \end{aligned}$$

Thus, in either case, $Q_2 \leq 0$ and so

$$H_{u_*}^{\pi}(x, p(x_0)) = \max(Q_1, Q_2) = Q_1 = 0.$$

Moreover the supremum over q defining $H_{u_*}^{\pi}(x, p(x_0))$ is attained at the $q^* = p(x_0)$ of Q_1. This b).

Now we prove c). The hypotheses imply that $q^* = p(x)$ satisfies

$$p(x_0) \cdot \pi(x_0, q^* - Gu_*) - \frac{1}{2}\|q^*\|^2 + \frac{1}{2}\|x_0\|^2 = 0.$$

Suppose the q^*-saddle condition fails at x_0. We must show that there exist x arbitrarily close to x_0 at which both i) and ii) hold. Since the q^*-saddle condition fails there must exist a q with

$$p(x_0) \cdot \pi(x_0, q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x_0\|^2 > 0.$$

We can assume q maximizes this expression. Because (q^*, u_*) satisfy the interior saddle point conditions, it must be that $\pi(x_0, q - Gu_*) = R_F(q - Gu_*)$ for some $\emptyset \subsetneq F \subseteq I(x_0)$. By choosing the minimal such F we may suppose that $B_F(q - Gu_*) > 0$. (See the comments following Lem. 1.3.) If it were the case that $F = I(x_0)$, then $B_F v \geq 0$ would be sufficient for $\pi(x_0, v) = R_F v$ (Lem. 1.3), and so q would be a local maximum of

$$p(x_0) \cdot R_F(q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x_0\|^2$$

which would imply $q = \bar{q}(x_0)$, satisfying i) and ii) for $x = x_0$.

Suppose that $F \subsetneq I(x_0)$. Lemma 1.1 implies that for all $0 < \delta$ sufficiently small $x' = x_0 + \sum_{i\in F^c} \delta\nu_j$ will belong to Ω, have $I(x') = F$, $B_F(q - Gu_*)) > 0$ and

$$p(x') \cdot R_F(q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x'\|^2 > 0.$$

Since $F = I(x')$, $B_F(q - Gu_*) > 0$ implies that $\pi(x', q - Gu_*) = R_F(q - Gu_*)$, so that the above says

$$p(x') \cdot \pi(x', q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x'\|^2 > 0. \tag{28}$$

We now repeat the argument from above. Take q to maximize (28). It must be that $\pi(x', q-Gu_*) = R_{F'}(q-Gu_*)$ for some $\emptyset \subsetneq F' \subseteq I(x') = F$. Take F' to be minimal: $B_{F'}(q - Gu_*) > 0$. If $F' = I(x')$, then $\pi(x', v) = R_{F'}v$ for v in a neighborhood of $q - Gu_*$, which implies $q = \bar{q}(x')$, satisfying i) and ii) for $x = x'$.

If on the other hand $F' \subsetneq I(x')$ then iterate the preceding paragraph again. The process must terminate because at each stage F' is a proper subset of F, but must be nonempty. □

Lastly we collect sufficient conditions for the u_*-saddle condition.

Theorem 3.3. *Assume* (16), $x_0 \in \partial\Omega$, *that* $q^* = p(x_0)$ *and* u_* *satisfy the interior saddle point conditions* (17) *there, and that* (21) *holds:*

$$\pi(x_0, q^* - Gu_*) = q^* - Gu_*.$$

Each of the following is sufficient for u_* *to satisfy the* u_**-saddle condition* (18) *at* x_0, *using the same* q^*.

a) $d_i \cdot p(x_0) \geq 0$ *for all* $i \in I(x_0)$.

b) *There exists a single* $F \subseteq I(x_0)$ *such that for all* $u \in \mathcal{U}$

$$\pi(x_0, p(x_0) - Gu) = R_F(p(x_0) - Gu), \text{ and}$$
$$p(x_0) \cdot R_F G u_* \geq p(x_0) \cdot R_F G u.$$

c) $I(x_0) = \{i\}$ *(i.e. x is on a face), and for all* $u \in U_0$

$$p(x_0) \cdot \pi(x_0, p(x_0) - Gu_*) \leq p(x_0) \cdot \pi(x_0, p(x_0) - Gu).$$

The point of c) is that on a face $\partial_i\Omega$, in order to extend the u-saddle condition from the interior to the boundary, we only need to check the basic control values $u \in U_0$ against u_*; we don't need to check all convex combinations $u \in \mathcal{U}$.

Proof. For part a), using $d_i \cdot p(x_0) \geq 0$ in (9) implies $p(x_0) \cdot \pi(x_0, v) \geq p(x_0) \cdot v$. Thus

$$\begin{aligned} p(x_0) \cdot \pi(x_0, q^* - Gu) &\geq p(x_0) \cdot (q^* - Gu) \\ &\geq p(x_0) \cdot (q^* - Gu_*) \\ &= p(x_0) \cdot \pi(x_0, q^* - Gu_*), \end{aligned}$$

the last equality by (21). This is the u_*-saddle condition.

Part b) is trivial. Part c) was argued at the end of Section 4.2 of [15]. □

We pause briefly to note how the above results apply to examples considered in previous work [6]. For the non re-entrant single servers of [6], Section 3, the construction produced $p \cdot d_i = 0$ and $n_i \cdot (p(x) - Gu_*) \geq 0$ for all $i \in I(x)$. Thus the saddle conditions for both $q^* = p(x)$ and u_* follow from Theorem 3.1.

In the Loop Model of [6], Section 4, on the vertical face $\partial_1\Omega$ it turned out that $d_1 \cdot p(x) \leq 0$, so that Theorem 3.2a applies for the q^*-saddle condition. The u_*-saddle condition was essentially a development of our Theorem 3.3b. On the horizontal face $\partial_2\Omega$ the model used $d_2 = n_2 = e_2$. By the remarks after Theorem 3.2, $n_2 \cdot Gu_* \leq 0$ is sufficient to invoke b) for the q^*-saddle condition. Since $Gu_* = Ge_1 = [\begin{smallmatrix} s_1 \\ -s_1 \end{smallmatrix}]$, we see that this is indeed satisfied. Regarding the u_*-saddle condition on $\partial_2\Omega$, it was observed on [6, pg. 343] that $p \cdot d_2 \geq 0$, which implies

$$p \cdot \pi(x, p - Ge_2) \geq p \cdot (p - Ge_2) \geq p \cdot (p - Ge_1) = p \cdot \pi(x, p - Ge_1),$$

so Theorem 3.3c applies.

4. A structured verification theorem

We turn now to the structured verification result, Theorem 4.1. As explained in the introduction, our intent is to construct the value V of our game using a family of extremals. Our verification theorem uses the structure of such a family of extremals to identify a smooth solution of $H^\pi(x, \mathrm{D}V(x)) = 0$ as the value of the game. We emphasize once more that there is no claim that the structure described below is necessary, only that it is sufficient.

We will say that $V \in \mathrm{C}^1(\Omega)$ is *associated with a regular family $\mathcal{E}$ of extremals* when the following all hold.

(1) $\mathcal{E}$ consists of a collection of triples $(x(t), q^*(t), u_*(t))$, called *extremals*, each defined and satisfying (8) on some interval $\tau \leq t \leq 0$, and terminating at the origin:

$$\dot{x}(t) = \pi(x(t), q^*(t) - Gu_*(t)); \quad x(0) = 0, \tag{29}$$

x absolutely continuous and both q^* and u_* piecewise continuous.

(2) $\mathcal{E}$ provides a simple covering of Ω in the sense that for each $y \in \Omega$ there exists $(x(t), q^*(t), u_*(t)) \in \mathcal{E}$ with $y = x(s)$ for some $\tau \leq s \leq 0$, and any two such extremals through y agree on $[s, 0]$.

(3) For each $(x(t), q^*(t), u_*(t))$ in $\mathcal{E}$, the disturbance component is given specifically by $q^*(t) = p(t)$, where $p(t) = \mathrm{D}V(x(t))$ is the corresponding costate.

(4) $(q^*(t), u_*(t))$ satisfy the saddle conditions,

$$u_*(t) \in \mathcal{U} \text{ minimizes } p(t) \cdot \pi(x(t), q^*(t) - Gu) \text{ over } u \in \mathcal{U} \tag{30}$$

$$q^*(t) \in \mathbb{R}^m \text{ maximizes } p(t) \cdot \pi(x(t), q - Gu_*(t)) - \frac{1}{2}\|q\|^2 \text{ over } q \in \mathbb{R}^n, \tag{31}$$

with saddle value

$$0 = p(t) \cdot \pi(x(t), q^*(t) - Gu_*(t)) - \frac{1}{2}\|q^*(t)\|^2 + \frac{1}{2}\|x(t)\|^2. \tag{32}$$

(5) For each $x \in \Omega$ with $x \neq 0$, the *positive storage condition* holds:

$$\|\mathrm{D}V(x)\|^2 < \|x\|^2. \tag{33}$$

Notice that the above imply that $V(x)$ is a saddle point solution of $H^\pi(x, \mathrm{D}V(x)) = 0$, as defined in Section 3. To apply the verification theorem in an example we will need to exhibit such a solution and an associated family $\mathcal{E}$ with all the listed properties. We will do just that in the example of Section 6. The results of Section 3 will assist us in verifying the saddle point properties (30)–(32) on $\partial\Omega$. The hypothesis that $q^*(t) = p(t)$ is consistent with those results. The significance of the positive storage condition (33) will appear in the proof of Theorem 4.1.

An important assertion of the theorem is the existence of an optimal strategy α_*. One would hope to produce a state-feedback policy which, when at state x, specifies a control value u_* which is optimal for the "worst case" disturbance $q^* = \mathrm{D}V(x)$ (optimal meaning that the saddle conditions hold). However there are serious technical difficulties in defining such a policy. The easiest resolution of these difficulties is the use of multiple-valued polices, as indicated in Section 2. To explain this, for each $x \in \Omega$ let $A(x) \subseteq \mathcal{U}$ denote the set (or a nonempty subset) of control values u_0 satisfying the saddle conditions for $q^* = \mathrm{D}V(x)$:

$$(u_0, q^*) \text{ satisfies the saddle point conditions (18), (19) for every } u_0 \in A(x). \tag{34}$$

In case $A(x)$ is only a subset of the possible saddle point controls at x we need to also require that

$$u_*(t) \in A(x(t)) \text{ for any } (x(\cdot), q^*(\cdot), u_*(\cdot)) \in \mathcal{E}, \tag{35}$$

in order that α_* be consistent with our family $\mathcal{E}$. To be well-defined our policy α_* must accept *any* $q(\cdot) \in \mathrm{L}^2[0,T]$ and $x_0 \in \Omega$ and produce one (or more) controls $u(\cdot) \in \alpha_*[x_0, q(\cdot)]$ on $[0,T]$ so that the solution $x(t)$ of (8) satisfies the desired feedback relation:

$$u(t) \in A(x(t)). \tag{36}$$

As a consequence, the state $x(t)$ corresponding the $q(\cdot)$ and the control $u(\cdot)$ prescribed by the policy α_* must obey the differential inclusion

$$\dot{x} \in \pi(x(t), q(t) - GA(x(t)); \quad x(0) = x_0. \tag{37}$$

To define $\alpha_*[x_0, q(\cdot)]$ we must solve the differential inclusion, and take $u(t) \in A(x(t))$ as the control selection actually used by the solution. Naturally, we would prefer that the policy prescribe a single control $u(\cdot)$ for a given x_0 and $q(\cdot)$. For this we would like to take $A(x) = \{a(x)\}$ to be a singleton, using some function $a : \Omega \to \mathcal{U}$. Then (37) would be a differential equation and, if uniquely solvable, would produce a unique control $u(t) = a(x(t))$. Unfortunately, depending on the solution V, the saddle point requirement (34) may rule out the existence of a continuous $a(x)$. Indeed in the example of Section 6 below $a(x)$ would be forced to have jump discontinuities; see (64). Without continuity properties of $a(x)$ (or some more detailed knowledge about its jumps) we would not able to assert the existence (much less uniqueness) of solutions to (37) in general, creating an obstacle for the definition of α_*. For an adequate existence theory in the presence of such discontinuities we are forced to the level of differential inclusions. Filippov [20] provides some general theory. Sufficient conditions for existence typically include closed graph properties of the right side. In [6, §1.4] an existence argument for (37) specifically was outlined. The essential hypotheses are that $A(x)$ is nonempty and closed for each x, and upper semi-continuous in x. (Upper semi-continuity means that $A(x)$ contains all limits points of $A(x_n)$ for sequences $x_n \to x$. Closure and upper semi-continuity together are equivalent to closed graph.) That is the existence argument that our proof below appeals to. We note that for examples such as (64) below, the upper semi-continuity property makes a set-valued $A(x)$ unavoidable.

Closure and upper semi-continuity of $A(x)$ resolve the existence of solutions to (37). However with such weak continuity hypotheses we cannot expect any uniqueness of solutions in general. (Solutions to (8) with $u(\cdot)$, $q(\cdot)$, and x_0 given *are* unique. It is the inclusion of state feedback in the differential inclusion which creates the nonuniqueness.) Defining a single-valued policy α_* would require selecting one among the many possible solutions to (37) for each prescribed $q(\cdot)$. The difficult aspect is to make the selection so that the nonanticipating requirement is satisfied. To do this carefully would require the design of some explicit selection mechanism, whose discussion would be a detraction from our main purpose here. It is far easier to accept the idea of multivalued strategies. Given $A(x)$ satisfying hypotheses sufficient for existence of solutions to the initial value problem (37), simply define $\alpha_*[x_0, q(\cdot)]$ to consist of *all* controls $u(\cdot)$ with the property that the corresponding solution $x(t)$ of (8) satisfies the feedback condition (36). The nonanticipating (Markov-extension) property of α_* then follows easily from the existence of solutions to (37).

To invoke the theorem we will need to exhibit $A(x)$ with the above properties along with the extremal family $\mathcal{E}$. Another hypothesis of the theorem is Lipschitz continuity of $p(x) = \mathrm{D}V(x)$, which will be needed for the approximation construction in the proof. The notation $V \in \mathrm{C}^{1,\mathrm{Lip}}(\Omega)$ means that $\mathrm{D}V(x)$ is Lipschitz continuous (uniformly) in Ω.

Theorem 4.1. *Suppose $V \in \mathrm{C}^{1,Lip}(\Omega)$ is associated with a regular family of extremals $\mathcal{E}$, as described above, that $V(0) = 0$ and that $A(x)$ is a nonempty, closed, upper semi-continuous set-valued function with $A(x) \subseteq \mathcal{U}$ for all $x \in \Omega$. Assume $A(x)$ satisfies both* (35) *and* (34) *above. Then $V(x)$ is the lower value of the differential game of Section* 2*. Specifically, α_* as described above in terms of $A(x)$ is a valid policy, and for each $x(0) \in \Omega_0$,*

$$V(x(0)) = \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t.$$

For an arbitrary control policy,

$$V(x(0)) \le \sup_{q(\cdot),T} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t. \tag{38}$$

The proof is a typical dynamic programming argument. The details for our specific setting are as in [6,15], with some simplifications.

Proof. We define α^* as described preceding the proof: $\alpha_*[x_0, q(\cdot)]$ consists of all controls $u(\cdot)$ such that the resulting solution $x\cdot)$ of (8) satisfies the state-feedback relationship (36). The closure and upper semi-continuity of $A(x)$ provides the technical structure necessary to invoke the existence argument described in [6], Section 1.4, which is based in turn on [17, Theorem 3]. This is enough to imply that α_* is well-defined as a multi-valued policy and satisfies the nonanticipating, Markov-extension property.

We continue to use $p(x)$ to denote $\mathrm{D}V(x)$. By hypothesis, $p(x)$ is Lipschitz continuous. For an extremal $(x(t), q^*(t), u_*(t))$ from the family $\mathcal{E}$ we know by hypothesis that $q^*(t) = p(x(t))$. It will be convenient to use $p(t) = p(x(t))$ for the associated costate trajectory. That $u_*(t)$ is one of the controls associated with the policy α_* for $q^*(\cdot)$ is evident from (30) and the description of $A(x)$ above. Given $x_0 \in \Omega$ consider the extremal with initial point $x_0 = x(\tau)$, $\tau < 0$, and $x(0) = 0$, the value of $V(x_0)$ is given by

$$V(x_0) = V(0) + \int_\tau^0 p(t) \cdot \dot{x}(t) \,\mathrm{d}t = \int_\tau^0 \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q^*(t)\|^2 \,\mathrm{d}t,$$

by virtue of (29) and (32). We have used the time parameter $t \in [\tau, 0]$ here because that is the way we described our family $\mathcal{E}$ in (29). But to write this in the form most compatible with the rest of the proof, we make a simple shift of time variable, so that $x(t)$ is defined on $[0, T]$ $(T = -\tau)$ with $x(0) = x_0$, $x(T) = 0$. Then, using $V(0) = 0$, the above becomes

$$V(x(0)) = \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q^*(t)\|^2 \,\mathrm{d}t. \tag{39}$$

Note at this point that the positive storage condition (33) implies that $V(x_0) > 0$ for $x_0 \neq 0$.

Next, for the same initial point x_0, consider an arbitrary disturbance $q(t)$ and *any* control $u_*(t)$ produced by α_*: $u_*(t) \in A(x(t))$. By definition of $A(x)$ the saddle conditions are satisfied. The q^*-saddle condition implies that

$$0 = p(x) \cdot \pi(x, q^* - Gu_*) - \frac{1}{2}\|q^*\|^2 + \frac{1}{2}\|x\|^2 \ge p(x) \cdot \pi(x, q - Gu_*) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2.$$

So if $x(t)$ is a state trajectory of the system with control policy α_* for an arbitrary load $q(t)$ (and any control $u_*(t) \in A(x(t))$ associated with α_*), we have

$$\mathrm{D}V(x(t)) \cdot \dot{x}(t) \le \frac{1}{2}\|q(t)\|^2 - \frac{1}{2}\|x(t)\|^2.$$

Thus on any interval $[0,T]$ we have

$$V(x(T)) - V(x(0)) \le \int_0^T \frac{1}{2}\|q(t)\|^2 - \frac{1}{2}\|x(t)\|^2 \,\mathrm{d}t.$$

Since $V(x(T)) \ge 0$, it follows that

$$V(x(0)) \ge \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t.$$

When we use the extremal from our family with the prescribed $x(0)$, and on the interval $[0,T]$ such that $x(T)=0$, the above holds with equality: (39). This shows that using α_* we have

$$V(x(0)) = \sup_{T,q(\cdot)} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t. \tag{40}$$

Recall that this supremum includes all $u(t)$ produced by α_* for each $q(\cdot)$.

Now we consider an arbitrary control policy α. We would like to consider the behavior of the controlled system when given a load such that $q(t) = q^*(x(t))$, where $x(t)$ is the controlled state trajectory produced by the policy α. However, we can't assume that such a $q(t)$ exists; that would require the solvability of the system equations subject to the policy α but with a feedback loop $q(t) = p(x(t))$ added to the dynamics. We have only asked that control policies to respond to *open loop* loads $q(t)$. Following the construction of [15], Section 5, we can approximate the desired $q(t)$. The idea is to get around the solvability issue by introducing a small time lag:

$$q(t) = \begin{cases} p(x(t-\epsilon \mathrm{e}^{-t})) & \text{if } \epsilon \mathrm{e}^{-t} < t \\ p(x(0)) & \text{if } t \le \epsilon \mathrm{e}^{-t}. \end{cases}$$

We can solve the system incrementally on a sequence of intervals $[t_{n-1}, t_n]$ such that $q(t)$ for each interval depends only on $x(t)$ for the preceding interval $[t_{n-2}, t_{n-1}]$. The basic existence and Markov-extension hypotheses for α subject to a prescribed $q(t)$ on an interval $[0,T]$ is used inductively to insure the existence of $x(t)$ with $q(t)$ as above and some control $u(t)$ produced by the policy α. Note that $q(t)$ is always a value of $p(x) = \mathrm{D}V(x)$ at some $x \in \Omega$, although not $x = x(t)$.

We claim that for some constant C_2 (independent of ϵ and T),

$$V(x(0)) - V(x(T)) \le C_2\epsilon + \frac{1}{2}\int_0^T \|x(t)\|^2 - \|q(t)\|^2 \,\mathrm{d}t \tag{41}$$

holds for all T. The argument for this starts by observing that $p(x) - Gu$ is bounded over $x \in \Omega$, $u \in \mathcal{U}$. It follows from Lemma 1.2 that there is a uniform bound on $\|\dot{x}\|$:

$$\|\dot{x}\| = \|\pi(x, q(t) - Gu(t))\| \le B.$$

Consequently,

$$\|x(t) - x(t-\epsilon \mathrm{e}^{-t})\| \le B\,\epsilon \mathrm{e}^{-t}.$$

Next, since $p(x)$ is Lipschitz, it follows that

$$\|q(t) - \mathrm{D}V(x(t))\| = \|\mathrm{D}V(x(t-\epsilon \mathrm{e}^{-t})) - \mathrm{D}V(x(t))\| \le C_1\,\epsilon \mathrm{e}^{-t}, \tag{42}$$

for some constant C_1 (independent of ϵ). Since $\pi(x,v)$ is Lipschitz in v it follows that for some constant C_2 (independent of ϵ) and all $t > 0$

$$\mathrm{D}V(x)\cdot\pi(x, q(t) - Gu(t)) - \frac{1}{2}\|q(t)\|^2 + C_2\epsilon \mathrm{e}^{-t} \ge \mathrm{D}V(x)\cdot\pi(x, \mathrm{D}V(x) - Gu(t)) - \frac{1}{2}\|\mathrm{D}V(x)\|^2.$$

(Because by hypothesis Ω is compact, $\|\mathrm{D}V(x)\|^2$ is Lipschitz as well.) Since V is a saddle point solution of $H^\pi(x, \mathrm{D}V(x)) = 0$ with saddle point $q^*(x) = \mathrm{D}V(x)$ and some u_*, we know that

$$0 \leq \mathrm{D}V(x) \cdot \pi(x, \mathrm{D}V(x) - Gu(t)) - \frac{1}{2}\|\mathrm{D}V(x)\|^2 + \frac{1}{2}\|x\|^2.$$

So it follows that

$$-\mathrm{D}V(x) \cdot \pi(x, q(t) - Gu(t)) \leq C_2 \epsilon \mathrm{e}^{-t} + \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2.$$

Since $\dot{x} = \pi(x, q(t) - Gu(t))$, integrating both sides over $[0, T]$ yields (41).

To finish the argument we want to show that the $V(x(T))$ can be ignored when taking $\sup_T$ of the right side of (41). Suppose there exists a sequence of $T_n > 0$ with $x(T_n) \to 0$. Then $V(x(T_n)) \to 0$ and it follows that

$$V(x(0)) \leq C_2\epsilon + \limsup_{n\to\infty} \int_0^{T_n} \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t \leq C_2\epsilon + \sup_T \frac{1}{2}\int_0^T \|x(t)\|^2 - \|q(t)\|^2 \,\mathrm{d}t.$$

If no such T_n exist, then there must exist a compact set $K \subseteq \Omega \setminus \{0\}$ with $x(t) \in K$ for all $t \geq 0$. In that case the positive storage condition (33) implies that (for $\epsilon > 0$ sufficiently small) there is a positive lower bound:

$$\frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \geq k > 0.$$

Consequently from (42),

$$\int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t \geq C_2\epsilon + kT \to +\infty \text{ as } T \to +\infty.$$

Thus in either case we find that

$$V(x(0)) \leq C_2\epsilon + \sup_T \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \,\mathrm{d}t.$$

Since $\epsilon > 0$ was arbitrary, this proves (38). □

5. Boundary extremals

Consider now the task of constructing a regular family $\mathcal{E}$ of extremals satisfying the hypotheses of Theorem 4.1 above. The purpose of this section is to identify differential equations which can be used in such a construction. Specifically, we need to adjoin to the state equation

$$\dot{x}(t) = \pi(x(t), p(t) - Gu_*(t))$$

an equation for $\dot{p}(t)$, where $p(t) = \mathrm{D}V(x(t))$, $V(x)$ being the solution of $H^\pi(x, \mathrm{D}V(x)) = 0$ under construction. We are particularly interested in the appropriate $\dot{p}$ equation when $x(t)$ runs along $\partial\Omega$ with active Skorokhod dynamics. We do not intend an exhaustive derivation of all theoretically possible types of extremals. Rather we are merely trying to predict the typical forms of boundary extremals away from the complicated structures which might occur at various types of singularities, such as discontinuities in the optimal control or in the $F \subseteq I(x)$ for which $\pi(x, q^* - Gu_*) = R_F(q^* - Gu_*)$. To that end we make a number of simplifying assumptions (marked by bullets • below) to help us anticipate equations (50) for boundary extremals. These particular types of extremals will be enough to discover a regular family $\mathcal{E}$ as desired for Theorem 4.1 in our examples. Our simplifying assumptions begin with the following.

- Assume $V \in \mathrm{C}^2(\Omega)$ is a saddle-point solution of $H^\pi(x, \mathrm{D}V(x)) = 0$.

As always $p(x)$ will denote $\mathrm{D}V(x)$. First consider an extremal $(x(t), q^*(t), u_*(t))$ (29)–(32) as it passes through an interior point $x_0 = x(t_0)$ of Ω.

- Assume there is a unique $u_0 \in \mathcal{U}$ maximizing $p(x_0) \cdot Gu$.

Recall that $\mathcal{U}$ is the convex hull of a finite set U_0. This assumption means u_0 must be one of the extreme points U_0 and must remain the maximizer of $p(x) \cdot Gu$ for x in a neighborhood of x_0. Thus we must have $q^*(t) = p(x(t)) = \mathrm{D}V(x(t))$ and $u_*(t) = u_0$ for t in a neighborhood of t_0. In particular $\dot{x} = p(x) - Gu_0$. Moreover in a neighborhood of x_0 we have

$$0 = H(x, \mathrm{D}V(x)) = \frac{1}{2}\|p(x)\|^2 - p(x) \cdot Gu_0 + \frac{1}{2}\|x\|^2.$$

Differentiating with respect to x yields

$$0 = p(x) \cdot Dp(x) - (Gu_0) \cdot Dp(x) + x.$$

The matrix $Dp(x)$ is the Hessian of $V(x)$ and is therefore symmetric. So the above can be rewritten as $\dot{p}(x(t)) = p_x(x(t)) \cdot (p(x(t)) - Gu_0) = -x$. Thus (under our bulleted assumptions) the state $x(t)$ and costate $p(t) = p(x(t))$ along an *interior extremal* will satisfy

$$\dot{x} = p - Gu_*, \quad \dot{p} = -x, \tag{43}$$

with $u_*(t) \equiv u_0$. These are the equations used to build the solution families in [6, 15].

Next we want to develop some comparable equations to describe extremals for which the action of $\pi(\cdot, \cdot)$ is nontrivial, *boundary extremals* as we will call them. To this end we make the following assumptions.

- $x_0 \in \partial_F\Omega$ and there is an extremal $(x^*(t), u_*(t), q^*(t))$ as in (30)–(32) through $x(t_0) = x_0$ with $x(t) \in \partial_F\Omega$ all t in a neighborhood of t_0.
- There is constant control $u_\partial \in \mathcal{U}$ and a function $q^\partial(x)$ such that for all $x \in \partial_F\Omega$ sufficiently close to x_0,
  - (i) the saddle point (18), is given uniquely by $q^\partial(x), u_\partial$, for some function $q^\partial(x)$; and
  - (ii) F is the unique subset of $I(x)$ for which

$$\pi(x, q^\partial(x) - Gu_\partial) = R_F(q^\partial(x) - Gu_\partial).$$

We now draw out some consequences of these and the continuing assumption that V is a C^2 saddle point solution of $H^\pi(x, \mathrm{D}V(x)) = 0$. First, the uniqueness of F in (ii) means, by Lemma 1.3, that for each x as in the second bullet we have strict inequalities:

$$B_F(q^\partial(x) - Gu_\partial) > 0, \text{ and } N^T_{I(x)\setminus F} R_F(q^\partial(x) - Gu_\partial) > 0.$$

These inequalities must be preserved for all q sufficiently close to $q^\partial(x)$ and $u \in \mathcal{U}$ sufficiently close to u_∂, so that $\pi(x, q - Gu) = R_F(q - Gu)$. In particular $q^\partial(x), u_\partial$ is a local saddle point of

$$p(x) \cdot R_F(q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2.$$

The $\sup_q \inf_u$ separate for this expression, allowing us to deduce that $q^\partial(x) = R_F^T p(x)$. For all such x, including x_0, we have that

$$\begin{aligned}
0 &= H^\pi(x, p(x)) \\
&= p(x) \cdot R_F(q^\partial(x) - Gu_\partial) - \frac{1}{2}\|q^\partial(x)\|^2 + \frac{1}{2}\|x\|^2 \\
&= H_{u_\partial}(x, R_F^T p(x)),
\end{aligned} \tag{44}$$

where, as before, H_{u_∂} denotes the fixed control Hamiltonian,

$$H_{u_\partial}(x,p) = \frac{1}{2}\|p\|^2 - p \cdot Gu_\partial + \frac{1}{2}\|x\|^2.$$

Now consider the saddle point extremal $(x^*(t), u_*(t), q^*(t))$. According to the above, it must be based on $q^*(t) = R_F^T p(x)$, and $u_*(t) \equiv u_\partial$:

$$\dot{x} = \pi(x, q^*(t) - Gu_\partial) = R_F(q^*(x) - Gu_\partial); \quad x(0) = x_0.$$

Notice that this is the same as

$$\dot{x} = \frac{\partial}{\partial p}\tilde{H}(x,p), \quad \text{where } \tilde{H}(x,p) = H_{u_\partial}(x, R_F^T p(x)). \tag{45}$$

We want to adjoin to this an equation for $\dot{q}^*(t)$. We know from (44) that

$$\tilde{H}(x, p(x)) = 0$$

for all $x \in \partial_F\Omega$ close to x_0. We can differentiate this in any of the complementary directions $\nu = \nu_j$ of Lemma 1.1: $0 = \frac{\partial}{\partial\nu}\tilde{H}(x,p(x))$. In Cartesian coordinates, with $\nu = (\gamma_1, \ldots, \gamma_n)$, we can write out the implication as follows:

$$\begin{aligned}
-\nu \cdot \tilde{H}_x = -\sum_j \gamma_j \frac{\partial}{\partial x_j}\tilde{H} &= \sum_{j,k,l} \gamma_j \frac{\partial \tilde{H}}{\partial p_l}(R_F)_{kl}\frac{\partial p_k}{\partial x_j} \\
&= \sum_{j,k,l} \gamma_j \frac{\partial p_j}{\partial x_k}(R_F)_{kl}\frac{\partial \tilde{H}}{\partial p_l}, \quad \text{because } \frac{\partial p_k}{\partial x_j} = \frac{\partial p_j}{\partial x_j}, \\
&= \sum_j \gamma_j \frac{\mathrm{d}}{\mathrm{d}t} p_j(x(t)), \quad \text{because of (45).} \\
&= \nu \cdot \dot{p}.
\end{aligned}$$

This holds for each vector $\nu = \nu_j$, $j \in F^c$ complementary to n_i, $i \in F$ as in Lemma 1.1. This identifies $\dot{p}$ up to a vector from the span of n_i, $i \in F$. However such n_i are a basis of the kernel of R_F^T, so we know just enough to identify

$$\dot{q}^* = R_F^T \dot{p} = -R_F^T \tilde{H}_x(x, p(x)) = -R_F^T x.$$

Thus under our hypothesis we are led to the following equations describing the extremal:

$$\dot{x} = R_F(q^* - Gu_\partial), \quad \dot{q}^* = -R_F^T x. \tag{46}$$

The equations (46) give us something to work with to construct extremals on edges with $F \subseteq I(x)$. However tracking $q^* = R_F^T p(x)$ only identifies $p(x)$ up to the the span of n_i, $i \in F$. There are some circumstances which imply that $p(x) = q^*(x)$. According to Lemma 1.4 part c), this is equivalent to the classical oblique derivative conditions $d_i \cdot p(x) = 0$ all $i \in F$. We claim this to be a consequence of the following hypotheses.

- There is a unique u_0 which maximizes $p(x_0) \cdot Gu$ over $u \in \mathcal{U}$, and
- $n_i \cdot (p(x_0) - Gu_0) < 0$ for all $i \in F$.

Note that we are not assuming that $u_0 = u_\partial$, but will deduce that as a byproduct.

First consider the situation in a single face $\partial_i\Omega$: $F = I(x_0) = \{i\}$. As before, u_0 must be one of the extreme points $u_0 \in U_0$, and so for all $\|\xi - p(x_0)\|$ sufficiently small u_0 will remain the maximizer of $\xi \cdot Gu$. We are assuming that $V \in C^2(\Omega)$ solves $H^\pi(x, DV(x)) = 0$ in Ω. For all $\|\xi - p(x_0)\|$ sufficiently small we know that $H(x,\xi) = H_{u_0}(x,\xi)$, where

$$H_{u_0}(x,\xi) = \frac{1}{2}\|\xi\|^2 - \xi \cdot Gu_0 + \frac{1}{2}\|x\|^2.$$

In particular,

$$H_{u_0}(x_0, p(x_0)) = 0. \tag{47}$$

Theorem 2.1 tells us that V is a viscosity solution of $-H(x, DV(x)) = 0$ with the boundary condition $-d_i \cdot DV(x) = 0$ on $\partial_i\Omega$. Now $D^{\pm}_{\Omega}V(x_0) = \{p(x_0) \pm cn_i : \ c \geq 0\}$. Suppose $d_i \cdot p(x_0) < 0$. Then for all $0 \leq c < -d_i \cdot p(x_0)$ we will have $\xi = p(x_0) + cn_i \in D^+_\Omega V(x_0)$ and $d_i \cdot \xi < 0$. Thus the subsolution property requires that $H(x_0, p(x_0) + cn_i) \geq 0$ for all $0 \leq c < -d_i \cdot p(x_0)$. Moreover, for $c > 0$ sufficiently small $H = H_{u_*}$ so that

$$H_{u_*}(x_0, p(x_0) + cn_i) \geq 0. \tag{48}$$

From (47) and (48), by differentiation with respect to c it follows that

$$0 \leq n_i \cdot \frac{\partial}{\partial p} H_{u_*}(x_0, p(x_0)) = n_i \cdot (p(x_0) - Gu_*).$$

Supposing $d_i \cdot p(x_0) > 0$ we can make an analogous argument using that V is a supersolution of the boundary conditions property to again deduce that $0 \leq n_i \cdot (p(x_0) - Gu_*)$. Thus if $0 > n_i \cdot (p(x_0) - Gu_*)$ we are forced to conclude that $d_i \cdot p(x_0) = 0$.

This conclusion can be extended to higher order edges by a continuity argument. Consider any $i \in F$. Applying Lemma 1.1 (with $F = \{i\}$) we can perturb x_0 slightly to obtain $\tilde{x}_0$ arbitrarily close to x_0 but with $I(\tilde{x}_0) = \{i\}$. By continuity u_0 will remain the unique maximizer of $p(\tilde{x}_0) \cdot Gu$ and $n_i \cdot (p(\tilde{x}_0) - Gu_0) < 0$. The above argument applies at $\tilde{x}_0$ to imply $d_i \cdot p(\tilde{x}_0) = 0$. Passing to the limit as $\tilde{x}_0 \to x_0$ we see that $d_i \cdot p(x_0) = 0$.

Returning to our extremal $(x(t), q^*(t), u_*(t))$, the inequality $n_i \cdot (p - Gu_*) < 0$ will remain true for all $x(t)$ close to $x(0)$ implying that for t near 0, $d_i \cdot p(x(t)) = 0$, all $i \in F$, which is equivalent to

$$p(x(t)) = R_F^T p(x(t)). \tag{49}$$

or $q^\partial(t) = p(x(t))$. In that case the $\dot{q}^*$ equation of (46) becomes $\dot{p} = -R_F^T x$. Observe that $n_i \cdot (p(\tilde{x}_0) - Gu_0) < 0$ is what we would expect if the extremals associated with V move from the interior of Ω to the boundary $\partial\Omega$ and then follow the boundary extremal (46). Thus in attempting to construct V in the interior by following extremals from $\partial\Omega$ backwards into the interior, (49) is a natural feature to expect. We also note that (49) implies that $u_\partial = u_0$; the optimal control is the same in the interior and on the boundary.

Although we have made some strong presumptions (•) for this discussion, we have identified the following equations describing boundary extremals for $F \subseteq I(x)$:

$$\dot{x} = R_F(p - Gu_*), \quad \dot{p} = -R_F^T x, \tag{50}$$

with $q^\partial(t) = p(x(t))$. (Observe that (43) is just the special case of $F = \emptyset$.) Along such an extremal we also need the following conditions to insure the validity of the velocity projection $R_F(p - Gu_*) = \pi(x, p - Gu_*)$:

$$F \subseteq I(x), \tag{51}$$

$$B_F(p - Gu_*) \geq 0, \tag{52}$$

$$N_{I(x)}^T R_F(p - Gu_*) \geq 0. \tag{53}$$

Notice that, given $x_0 \in \partial_F\Omega$ and $x \in \Omega$, an equivalent condition for $x \in \partial_F\Omega$ is that $n_i \cdot (x - x_0) = 0$ for all $i \in F$. This in turn is equivalent to $(x - x_0) = R_F(x - x_0)$ by Lemma 1.4d. Thus $\partial_F\Omega$ consists precisely of those $x \in \Omega$ which are fixed points of

$$\Phi_F(x) = x_0 + R_F(x - x_0).$$

Now observe that (50) coincides with the Hamiltonian system with Hamiltonian

$$\tilde{H}(x,p) = H_{u_*}(\Phi_F(x), R_F^T p). \tag{54}$$

This observation is significant for two reasons. One is that $\tilde{H}$ is constant along solutions to (50). Using Lemma 1.5 we see that both $x = \Phi_F(x)$ and $p = R_F^T p$ are invariant with respect to (50). Therefore

$$H_{u_*}(x(t), p(t)) \text{ is constant in } t$$

along solutions of (50). Thus if we can confirm that $p(t) = \mathrm{D}V(x(t))$ and the saddle point conditions for $q^* = p$ and u_*, then we will indeed have a saddle point solution of $H^\pi(x, \mathrm{D}V(x)) = 0$. (The constant will be 0 because by hypothesis the extremals in a family $\mathcal{E}$ all terminate at $x(0) = p(0) = 0$.)

Secondly, as is standard in the method of characteristics for first order PDEs, a Hamiltonian system preserves exactness of the differential form $p(x) \cdot dx$, which implies that $p(x)$ is indeed the gradient of some function $V(x)$. See [24] for details. To be more precise, if integrals of $p \cdot dx$ are independent of path within a manifold of initial conditions for (50), then $p \cdot dx$ remains independent of path in the region covered by the solutions of the Hamiltonian system with the given initial conditions. This is what guarantees that a function $p(x)$ defined implicitly by $p(x(t)) = p(t)$ for a family of solutions to (50) will be the gradient of a function $p(x) = \mathrm{D}V(x)$, provided p is compatible with a gradient within the manifold of initial conditions. In an example this reasoning needs to be applied piecewise over several subregions making up Ω. When we have the specific example of the following section in front of us this will be easier to explain; see Section 6.3 below.

6. An example in 2 dimensions

In this section we will illustrate how the various results above are used to study a specific example. This will show that the many hypotheses of Theorem 4.1 and of the results of Section 3 are indeed reasonable and appropriate for the class of problems under consideration. The example is described in Section 6.1, and technical hypotheses associated with the Skorokhod problem are verified. Then, in the remaining subsections, we will carry out the following.

- Section 6.2 We describe the family $\mathcal{E}$ of extremals (29) providing a simple covering of Ω. This will include extremals of both types: (43) and (50) for Section 5. Requirements 1 and 2 of Section 4 will be manifest once $\mathcal{E}$ is exhibited.
- Section 6.3 We show that $\mathcal{E}$ implicitly determines a (Lipschitz) function $p(x)$ on Ω by $p(t) = p(x(t))$. We will explain why $p(x)$ is the gradient $\mathrm{D}V(x)$ for a function $V(x)$ solving $H(x, \mathrm{D}V(x)) = 0$. We will verify requirements 3 and 5 of Section 4, as well as requirement 4 on the interior.
- Section 6.4 Using the results of Section 3, we show that $\mathcal{E}$ does in fact satisfy the saddle point conditions (30)–(32) on $\partial\Omega$. It follows that V satisfies $H^\pi(x, \mathrm{D}V(x)) = 0$. This will account for requirement 4 of Section 4 at boundary points. Moreover, by Theorem 2.1 V is a viscosity solution of $-H(x, \mathrm{D}V(x)) = 0$ with viscosity sense oblique derivative boundary conditions $-\mathrm{D}V(x) \cdot d_i = 0$, $x \in \partial_i\Omega$.
- Section 6.5 We will exhibit the set valued function $A(x)$ to complete the hypotheses needed to invoke Theorem 4.1. It will follow that the V constructed is indeed the value of the game. We will exhibit the optimal control policy.

Closed form expressions will describe all the extremals in $\mathcal{E}$, but some of the extremals involve one or more changes of form due to switching of the control value $u_*(\cdot)$ or the transition from an interior extremal (43) to a boundary extremal (50). This renders the full expressions for some extremals quite cumbersome. Rather than work with those expressions "by hand", we have used *Mathematica* to manage, manipulate and evaluate them as needed to confirm the hypotheses of the various results from Section 3 and to produce Figure 3, showing the resulting value function $V(x)$. In a few places, where algebraic verification would be excessively burdensome, we have resorted to numerical evaluation to confirm inequalities. We emphasize however that this is a shortcut to ease the exposition. Our primary goal is to exhibit the feasibility and success of the constructive approach in this example. The example itself is not of enough significance to merit the effort of writing out explicit, closed-form verifications of every feature. Our extremal family $\mathcal{E}$ provides a complicated though explicit *parametric* representation of V (*not* a numerical approximation), the parameters being the time parameters along the different sections of the different extremals. We have not made the effort to eliminate these parameters from the representation to obtain an explicit expression for $V(x)$. Although in principle that may be possible, it would be excessively burdensome, and of little conceptual value.

6.1. **The example**

We consider an example of (8) in $\mathbb{R}^2$. We take

$$G = \begin{bmatrix} 2 & 1/2 \\ 1 & 2 \end{bmatrix}$$

with two basic control settings $U_0 = \{e_1, e_2\}$. Thus

$$\mathcal{U} = \{(u_1, u_2) : \ u_1 + u_2 = 1, \ 0 \le u_i\}.$$

We take Ω to be the triangular region defined by

$$x_1 \ge 0, \quad x_2 \ge 0, \quad 2x_1 + 3x_2 \le 7/2.$$

To describe Ω in the form (4) we take $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (-2,-3)/\sqrt{13}$; $c_1 = c_2 = 0$, and $c_3 = \frac{-7}{2\sqrt{13}}$. To complete the system description we specify the following restoration vectors:

$$d_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad d_2 = n_2, \quad d_3 = n_3.$$

We hasten to note that this example is artificial. Unlike the example of Figure 1, d_1 and d_2 do *not* correspond to the reductions of the two service options (here just the columns of G) associated with empty queues. Thus this does not model any realistic network. However the differential game associated with an optimal service policy still makes sense mathematically. The mismatch between the d_i and G gives rise to boundary extremals, which are not typical in 2-dimensional examples. The example is interesting for that reason.

Before proceeding, we need to confirm the hypotheses (6) and (7) for the Skorokhod problem. Regarding (6), a set B satisfying the hypotheses in pictured in Figure 2. The point a is $a = (0, 9)$. The strip between a and $-a$ bounded by the dashed lines are the points in B for which $|z \cdot n_1| < 1$. The sides of the boundary of B containing $\pm a$ are parallel to d_1, so that (6) is satisfied for $i = 1$. The strips surrounding $\pm b$ with $b = (7, 0)$ and $\pm c$ with $c = (6, -4)$ are constructed likewise, for $i = 2, 3$ respectively. It is obvious from the figure that the hypothesis is satisfied. Though easily checked explicitly, we omit the calculations. For the coercivity hypothesis (7), it is sufficient to consider the three corners:

$$\text{for } F = \{1,2\}, \quad N_F^T D_F = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$\text{for } F = \{1,3\}, \quad N_F^T D_F = \begin{bmatrix} 1 & \frac{-2}{\sqrt{13}} \\ \frac{1}{\sqrt{13}} & 1 \end{bmatrix},$$

$$\text{for } F = \{2,3\}, \quad N_F^T D_F = \begin{bmatrix} 1 & \frac{-3}{\sqrt{13}} \\ \frac{-3}{\sqrt{13}} & 1 \end{bmatrix}.$$

Each of there is easily checked to be coercive.

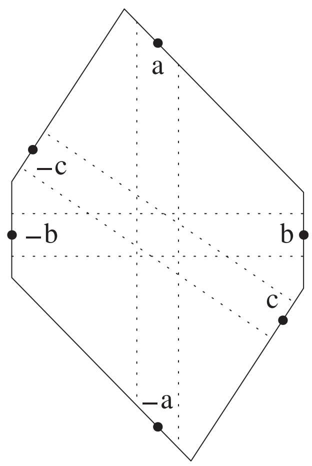

FIGURE 2. The set B of hypothesis (6).

6.2. **The extremal family**

The construction of the extremal family is based on two boundary extremals, one along each coordinate axis. Along the vertical axis $\partial_1\Omega$ ($x_1 = 0$) we take

$$x^{\partial_1}(t) = \frac{-3}{2\sqrt{2}}\sin(\sqrt{2}t)\begin{bmatrix}0\\1\end{bmatrix}, \quad p^{\partial_1}(t) = \frac{3}{2}(1-\cos(\sqrt{2}t))\begin{bmatrix}1\\1\end{bmatrix}, \quad \tau^{\partial_1} \le t \le 0. \tag{55}$$

These are the solutions of the boundary extremal equations (50) with $F = \{1\}$, $u_* = e_1$, and terminal conditions $x(0) = p(0) = 0$. The value

$$\tau^{\partial_1} = -\frac{\arcsin(\frac{7}{9\sqrt{2}})}{\sqrt{2}} \approx -.41177$$

is simply the lower bound of $t < 0$ for which $x^{\partial_1}(t) \in \Omega$, *i.e.* $x^{\partial_1}(\tau_1) \cdot n_3 = c_3$. To confirm the validity as a boundary extremal one must also check (52) and (53). The expression for (52) works out as

$$B_F(p^{\partial_1}(t) - Gu_*) = \frac{1}{\sqrt{2}}(1 + 3\cos(\sqrt{2}t)),$$

which is seen to be nonnegative for $\tau^{\partial_1} \le t \le 0$. For $\tau^{\partial_1} < t$, (53) follows from Lemma 1.4b since $I(x) = F$ so that $N_{I(x)}^T R_F = 0$.

Anticipating that $p^{\partial_1}(t) = \mathrm{D}V(x^{\partial_1}(t))$, we define $v^{\partial_1}(t)$ according to $\frac{\mathrm{d}}{\mathrm{d}t}v^{\partial_1} = p^{\partial_1} \cdot \dot{x}^{\partial_1}$; $v^{\partial_1}(0) = 0$. We can integrate the expression for $p^{\partial_1}(t) \cdot \dot{x}^{\partial_1}(t)$ to obtain the parametric expression for $v^{\partial_1}(t) = V(x^{\partial_1}(t))$ along x^{∂_1}:

$$v^{\partial_1}(t) = \frac{9t}{4} - \frac{9\sin(\sqrt{2}t)}{2\sqrt{2}} + \frac{9\sin(2\sqrt{2}t)}{8\sqrt{2}}\cdot$$

The solid curve along the $x_1 = 0$ coordinate plane in Figure 3 is the parametric plot of $x^{\partial_1}(t), v^{\partial_1}(t)$, thus rendering graph of V along the x_2-axis.

The saddle condition (18) will be verified in Section 6.4 below. We should note here that (50) can also be solved along the vertical face using the other control value, $u_* = e_2$. However it turns out that the u_*-saddle condition (18) fails for the resulting extremal. Many such exploratory calculations are behind the discovery of the extremal family being described here.

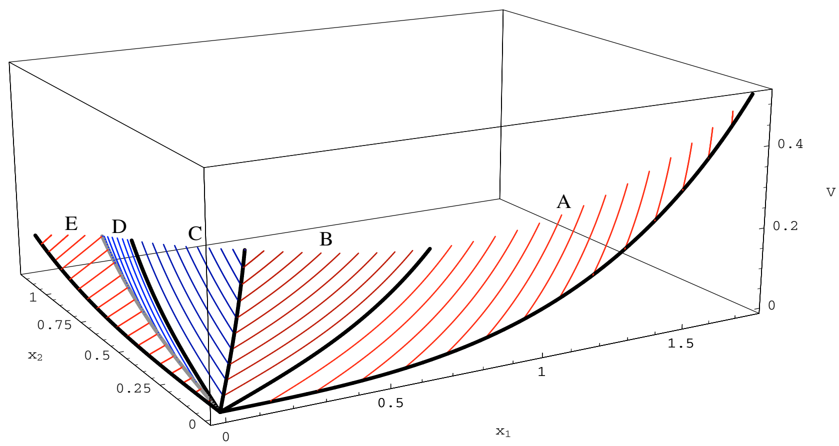

FIGURE 3. Extremal family for planar example.

Similar calculations lead to a boundary extremal $x^{\partial_2}(t)$, $p^{\partial_2}(t)$ on the horizontal boundary $\partial_2\Omega$ ($x_2 = 0$), again using $u_* = e_1$. The resulting expressions are

$$x^{\partial_2}(t) = \begin{bmatrix} -2\sin(t) \\ 0 \end{bmatrix}, \quad p^{\partial_2}(t) = \begin{bmatrix} 2(1-\cos(t)) \\ 0 \end{bmatrix}, \quad \tau^{\partial_2} \le t \le 0, \tag{56}$$

with

$$\tau^{\partial_2} = -\arcsin(7/8) \approx -1.06544$$

and

$$v^{\partial_2}(t) = V(x^{\partial_2}(t)) = 2t - 4\sin(t) + \sin(2t).$$

Again (52) and (53) are all verifiable explicitly to confirm the validity of this as a boundary extremal. The parametric plot of these expressions produces the solid curve on the $x_2 = 0$ face in Figure 3.

The next important member of our extremal family is a special interior extremal (43) using the relaxed control $u_* = (\frac{5}{13}, \frac{8}{13})$, resulting in:

$$x^{\mathrm{a}}(t) = \frac{-7}{13}\sin(t)\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad p^{\mathrm{a}}(t) = \frac{7}{13}(1-\cos(t))\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \tau^{\mathrm{a}} \le t \le 0,$$

with $\tau^{\mathrm{a}} = -\arcsin(1/2) \approx -0.523599$ and the following parametric expression for the solution of $\frac{\mathrm{d}}{\mathrm{d}t}v^{\mathrm{a}} = p^{\mathrm{a}} \cdot \dot{x}^{\mathrm{a}}$; $v^{\mathrm{a}}(0) = 0$:

$$v^{\mathrm{a}}(t) = \frac{49t}{26} - \frac{49\sin(t)}{13} + \frac{49\sin(2t)}{52},$$

which will give $v^{\mathrm{a}}(t) = V(x^{\mathrm{a}}(t))$. The parametric plot of $x^{\mathrm{a}}(t), v^{\mathrm{a}}(t)$ is the solid curve between regions B and C in Figure 3.

To describe the next stage of construction, consider the collection of extremals marked A in Figure 3. These are interior extremals (43) using $u_* = e_1$, starting with terminal conditions taken from the boundary extremal $x^{\partial_2}(s)$ at some $\tau^{\partial_2} < s \le 0$:

$$x(0) = x^{\partial_2}(s), \quad p(0) = p^{\partial_2}(s), \tag{57}$$

and solving (43) for $t < 0$ as long as the resulting $x(t)$ remains in Ω: $\tau^{\mathrm{A}}(s) \le t \le 0$ for some $\tau^{\mathrm{A}}(s)$ depending on the value of s. The general solution formulae for (43) with $u_*(t) \equiv u$ and terminal conditions $x(0) = x_0$,

$p(0) = p_0$ are

$$x(t) = \cos(t)x_0 + \sin(t)(p_0 - Gu)$$
$$p(t) = -\sin(t)x_0 + \cos(t)(p_0 - Gu) + Gu.$$

We will use an operator notation to denote the state components of the solution:

$$x(t) = \mathcal{X}_t^{(i)} x_0,$$

with $i = 1, 2$ for the respective control values $u = e_i$. This notation does not indicate the dependence of $x(t)$ on p_0, but it is to be understood that the state component $x(t)$ of every extremal is implicitly accompanied by a costate counterpart $p(t)$. With this notation, an extremal $x(t)$ with terminal conditions (57) is

$$x(t) = \mathcal{X}_t^{(1)} x^{\partial_2}(s).$$

This produces the two parameter family covering region A:

$$x^{\mathrm{A}}(s,t) = \mathcal{X}_t^{(1)} x^{\partial_2}(s), \quad \tau^{\partial_2} \le s \le 0,\ \tau^{\mathrm{A}}(s) \le t \le 0. \tag{58}$$

Strictly speaking, to describe an extremal in A using a single parameterization $x(t)$ from $x(T) = x^{\mathrm{A}}(s_0, t_0)$ to $x(0) = 0$ we should write $T = s_0 + t_0$,

$$x(t) = \begin{cases} \mathcal{X}_{t-s_0}^{(1)} x^{\partial_2}(s_0) & T \le t \le s_0 \\ x^{\partial_2}(t) & s_0 \le t \le 0, \end{cases}$$

or $x(t) = x^{\mathrm{A}}(t \wedge s_0, (t - s_0) \vee 0)$. However to view x^{A} as a covering of A it is more convenient to use two parameters.

In like manner, the extremals marked B in the figure use $u_* = e_1$ but take terminal values $x^{\mathrm{a}}(s)$:

$$x^{\mathrm{B}}(s,t) = \mathcal{X}_t^{(1)} x^{\mathrm{a}}(s), \quad \tau_a \le s \le 0,\ \tau^{\mathrm{B}}(s) \le t \le 0. \tag{59}$$

Those marked C use $u_* = e_2$:

$$x^{\mathrm{C}}(s,t) = \mathcal{X}_t^{(2)} x^{\mathrm{a}}(s), \quad \tau^{\mathrm{a}} \le s \le 0,\ \tau^{\mathrm{C}}(s) \le t \le 0.$$

Those marked E begin the same way, with terminal values from $x^{\partial_1}(s)$ and $u_* = e_1$: $x^{\mathrm{E}}(s,t) = \mathcal{X}_t^{(1)} x^{\partial_1}(s)$. However we find that the u_*-saddle condition (17) only holds for $\sigma(s) \le t \le 0$, where $\sigma(s)$ is obtained by solving for $t = \sigma(s)$ in

$$p^{\mathrm{E}}(s,t) \cdot Ge_1 = p^{\mathrm{E}}(s,t) \cdot Ge_2.$$

This leads to an explicit but complicated expression:

$$\sigma(s) = -\arcsin\left(\frac{8\sqrt{2}}{\sqrt{131 + 60\cos(\sqrt{2}\,s) - 63\cos(2\sqrt{2}\,s)}}\right) - \arctan\left(\frac{\left(5 + 3\cos(\sqrt{2}\,s)\right)\csc(\sqrt{2}\,s)}{6\sqrt{2}}\right). \tag{60}$$

Thus we only consider $x^{\mathrm{E}}(s,t)$ defined for $\sigma(s) \le t \le 0$. To extend to $t < \sigma(s)$ we use the values of $x^E(s, \sigma(s))$ (and their implicit costate counterparts) as terminal values but switch to $u_* = e_2$:

$$x^D(s,t) = \mathcal{X}_t^{(2)} \mathcal{X}_{\sigma(s)}^{(1)} x^{\partial_1}(s).$$

The t-parameter range here is $\tau^{\mathrm{D}}(s) \le t \le 0$, where $\tau^{\mathrm{D}}(s)$ is the lower limit of t such that $x^D(t,s) \in \Omega$. Note that we have "reset" the t parameter to start at $t = 0$ in x^D, so $x^{\mathrm{D}}(s,0) = x^{\mathrm{E}}(s,\sigma(s))$. The switching curve which separates D and E in the figure consists of the points $x^{\mathrm{E}}(s,\sigma(s))$.

6.3. Coverage of Ω and interior verifications

The five 2-parameter families described above, $x^{\mathrm{A}}(s,t)$ – $x^{\mathrm{E}}(s,t)$ (and their costate counterparts $p^{\mathrm{A}}(s,t)$ – $p^{\mathrm{E}}(s,t)$) comprise the regular extremal family $\mathcal{E}$ associated with the solution to the differential game for our example, with the time resets described above. To confirm this requires several issues to be checked. First we need to recognize that Ω is simply covered by this family. Each of our 2-parameterizations covers a specific subset of Ω. For instance $x^{\mathrm{A}}(s,t)$ accounts the subset of $x \in \Omega$ for which

$$2x_2 \le x_1.$$

In fact the expression for $x^{\mathrm{A}}(s,t)$ simplifies to

$$x^{\mathrm{A}}(s,t) = \begin{bmatrix} -2\sin(s+t) \\ -\sin(t) \end{bmatrix}.$$

The parameter range of (58) is contained in the enlarged range $-\pi/2 \le s+t \le t \le 0$, on which the expression for $x^{\mathrm{A}}(s,t)$ is easily seen to be one–to–one. The region so covered is the triangle with vertices $(0,0)$, $(1,\frac{1}{2})$ and $(\frac{7}{4},0)$.

The interface between the A and B regions is the line $x_1 = 2x_2$, which are the points $x^{\mathrm{A}}(0,t) = x^{\mathrm{B}}(0,t)$. The region covered by x^{B} consists of those $x \in \Omega$ with

$$x_1 \le 2x_2 \le 3x_1.$$

Just as for x^{A}, each x in this part of Ω is $x = x^{\mathrm{B}}(s,t)$ for a unique (s,t) in the parameter range of (59). We find the same situation for the regions of Ω associated with C, D, E, and F. The expressions for $x^{\mathrm{B}}(s,t)$, $x^{\mathrm{C}}(s,t)$, and $x^{\mathrm{E}}(s,t)$ are relatively simple, as was $x^{\mathrm{A}}(s,t)$ above. $x^{\mathrm{D}}(s,t)$ is quite complicated, because of the expression for $\sigma(s)$ in (60). Consequently the expression for $x^{\mathrm{D}}(s,t)$ is not easy to work with in closed form. However the simple coverage is clear from the figure.

Now, convinced of the simple covering of Ω by our family, the next issue is to see that each of $x^{\mathrm{A}}(s,t)$ — $x^{\mathrm{E}}(s,t)$ provides a nonsingular parameterization of the respective region, *i.e.* that their Jacobians are nonvanishing in the appropriate parameter ranges. This will imply that each has a smooth inverse. In the case of x^{A} for instance it will follow that

$$x = x^{\mathrm{A}}(s,t) \mapsto (s,t) \mapsto p^{\mathrm{A}}(s,t) = p(x)$$

is smooth in A. *I.e.* the function $p(x)$ defined implicitly by the (s,t) parameterization of A will be smooth in each subregion of Ω and continuous throughout. (Note that $x^{\mathrm{A}}(s,t)$, $p^{\mathrm{A}}(s,t)$, and each of the other 2-parameterizations is smooth in its parameter range, as can be seen from the various explicit solution formulae above.) The nonvanishing of Jacobians can be checked as follows (described for the case of x^{A}). An expression for the Jacobian $J(s,t) = \partial x^{\mathrm{A}}/\partial(s,t)$ is computed, and from that, an expression for the maximal $t = \chi(s)$ for which $J(s,t) = 0$ obtained (in closed form, even in the case of x^{D}, because the t-dependence is always relatively simple). Then $\chi(s)$ and $\tau^{\mathrm{A}}(s)$ are plotted with respect to s. (Recall that $\tau^{\mathrm{A}}(s)$ is the lower limit of t for which $x^{\mathrm{A}}(s,t) \in \Omega$.) In all cases (A – E) these graphs confirm that $\chi(s) < \tau^{\mathrm{A}}(s)$, so that $J(s,t) \ne 0$ in the appropriate parameter range. Again, these calculations are not feasible by hand, but with *Mathematica* they are quite manageable. In this way we find that our family $\mathcal{E}$ determines a continuous function $p : \Omega \to \mathbb{R}^2$ with piecewise continuous derivatives. In particular, $p(x)$ is Lipschitz continuous, as needed for Theorem 4.1.

Next we need to consider why there exists $V \in \mathrm{C}^1$ with $V(0) = 0$ such that $\mathrm{D}V(x) = p(x)$ and why $H(x,p(x)) = 0$ holds there. This was explained in [6], Section 3.2.1, and [15], p. 286, but we repeat the reasoning here. In each of the regions A–E, as well as along the two coordinate boundaries $\partial_i\Omega$, the values of

$p(x)$ are obtained by solving a Hamiltonian system (54). As explained at the end of the last section, that means that if $p(x)\cdot dx$ is (locally) independent of path within the manifold of initial points for one of these regions, then it will be independent of path through the region covered by the solution family to (54). For instance $p(x)\cdot dx$ is (trivially) independent of path within the single point initial manifold $\{x=0, p=0\}$ for the extremal x^{∂_1}, p^{∂_1} which covers the vertical axis $\partial_1\Omega$. Thus $p(x)\cdot dx$ is independent of path within $\partial_1\Omega$. (Since $\partial_1\Omega$ is one dimensional we could have started here, rather than with the single point manifold at the origin.) Next, $\partial_1\Omega$ is the initial manifold for the x^{E} family covering region E, and hence independence of path holds in E. In particular independence of path holds within the curve forming the boundary between E and D. This curve is the initial manifold for the family of extremals covering region D, so independence of path extends to all of D. Likewise we argue that $p(x)\cdot dx$ is independent of path within each of the individual regions A–E. With continuity of $p(x)$ across the interfaces, independence of path for the full region Ω follows. A second consequence of (54) is that $H_{u_*(t)}(x(t), p(t))$ is constant along all extremals in our family. Since they all lead ultimately to $x(0)=0$, $p(0)=0$, it follows that $H_{u_*(t)}(x(t), p(t))=0$. Thus once we justify the u_*-saddle conditions, it will follow that $H(x, p(x))=0$ throughout Ω as well.

Next we need to consider the saddle conditions and positive storage condition in the interior of Ω. The interior saddle conditions (18) and (19) are largely implicit in the construction above. The interior q_*-saddle condition is simply that $q_*=p(t)$, which was the case for our boundary as well as interior extremals. From the formulae (55) and (56) for p^{∂_1} and p^{∂_2} one easily checks that $p^{\partial_i}\cdot Ge_1 \geq p^{\partial_i}\cdot Gu$ for all $u\in\mathcal{U}$. Since $u_*=e_1$ on both boundary faces this accounts for the interior saddle conditions there. In the region covered by x^{E} the u_*-saddle point condition is implicit because by construction these extremals are only used to the first time $\sigma(s)=t$ when $p(t)\cdot(Ge_1-Ge_2)\geq 0$ fails; that was the relationship that defined $\sigma(s)$ and led to (60). Consider the extension into the region covered by x^{A}. The explicit expression for p^{A} works out to be

$$p^{\mathrm{A}}(s,t)=(2(1-\cos(s+t)), 1-\cos(t)).$$

Now calculate that

$$p^{\mathrm{A}}(s,t)\cdot(Ge_1-Ge_2)=3(1-\cos(s+t))-(1-\cos(t)).$$

We have already observed that the parameter range (58) is contained in

$$-\frac{\pi}{2}\leq s+t\leq t\leq 0.$$

Since $1-\cos(x)$ is decreasing on $[-\frac{\pi}{2},0]$ it follows that $1-\cos(t)\leq 1-\cos(s+t)$ and therefore

$$p^{\mathrm{A}}(s,t)\cdot(Ge_1-Ge_2)\geq 2(1-\cos(s+t))\geq 0.$$

This implies the u_*-saddle condition for $u_*=e_1$ in the x^{A} section of Ω. Similar calculations are possible for the regions associated with x^{B} and x^{C}. Explicit expressions for x^{D} and p^{D} are quite cumbersome however, due to (60). The u_*-saddle condition there is most easily verified numerically.

This is a good place to make another observation about our construction. Since $\dot{p}=-x$ and the coordinates of x are nonnegative, it follows that $p(x)\geq 0$. Moreover, based on our explicit formulae for p on the two coordinate faces, we see that $p_1(x)=0$ only at the origin, and $p_2(x)=0$ only on the horizontal axis $\partial_2\Omega$. For reference below we indicate this briefly as

$$p(x)>0 \text{ with equality in both coordinates only for } x=0. \tag{61}$$

While the saddle-conditions are essentially insured by the correct choice of extremal family $\mathcal{E}$, the positive storage condition (33) has not influenced the choice of extremals at all. Indeed it arises as a somewhat fortuitous feature of the geometry of the extremal family. Observe that $H_{u_*}(x, p(x))=0$ implies that

$$\frac{1}{2}(\|p(x)\|^2-\|x\|^2)=p(x)\cdot(p(x)-Gu_*).$$

So to check the positive storage condition, since $p(x) \geq 0$ it is enough to check that both coordinates of $\dot{x} = p(x) - Gu_*$ are negative. This is visually obvious in Figure 3. Direct algebraic confirmation is possible as well. For instance in the region for x^{A} we can use the simple expression for $p^{\mathrm{A}}(s,t)$ above to see that

$$p^{\mathrm{A}} - Gu_* = (-2\cos(s+t), -\cos(t)).$$

The horizontal axis itself corresponds to $s < t = 0$, so that $\dot{x}_1 < 0$. Since $p_1 > 0$ we still get strict inequality $p(x) \cdot (p(x) - Gu_*) < 0$, except at the origin. Algebraic confirmation is not feasible in the x^{D} region because of the complicated expressions. In addition to the visual evidence of Figure 3, numerical verification also confirms the positive storage condition in region D.

We make one last observation before proceeding to the boundary saddle point conditions. Consider any $x \in \partial_3\Omega$, except the corners. If $x(t)$, $T < t < 0$ is the extremal from our family that reaches $x(T) = x$, we know that $x(t) \in \Omega$ for $T < t$ and so

$$0 \leq n_3 \cdot \dot{x}(T+) = n_3 \cdot (p(x) - Gu_*(T+)). \tag{62}$$

6.4. **Boundary conditions**

Now we consider the saddle point conditions for $V(x)$ for $x \in \partial\Omega$, including the corners. This is where the results of Section 3 are valuable. Our discussion above verifies the interior saddle conditions, which are a hypothesis for those results.

Along the two coordinate axes, the values of $q^*(x) = p(x)$ are determined by our two boundary extremals: $x^{\partial_i}(t), p^{\partial_i}(t)$. We observed following (54) that $p = R_F^T p$ is invariant for the boundary extremal equations. Since $p^{\partial_i}(0) = 0$ it follows that $R_{\{i\}}^T p^{\partial_i}(x) = p^{\partial_i}(x)$ along each of $\partial_i\Omega$, $i = 1, 2$ respectively. By Lemma 1.4 this means or $d_i \cdot p(x) = 0$ along $\partial_1\Omega$ and $\partial_2\Omega$. (This of course can be checked directly from the explicit formulae (55) and (56), but we want to point out that this a feature of boundary extremals in general.) Thus Theorem 3.1 applies to show that the saddle point conditions are satisfied on both of the coordinate faces $\partial_1\Omega$ and $\partial_2\Omega$, excepting possibly the two corners $(0, 3.5/3) \in \partial_{\{1,3\}}\Omega$ or $(3.5/2, 0) \in \partial_{\{2,3\}}\Omega$. We will come back to those after considering the other face.

The third face is the diagonal boundary $\partial_3\Omega$, where $(2,3) \cdot x = 7/2$. For each such x, $p(x)$ is the value of $p(T)$ along an extremal $\dot{p}(t) = -x(t)$, $T < t < 0$, with $x(t) \in \Omega$. As observed in (61), $p_i(x) \geq 0$. Both coordinates of d_3 are negative. Therefore $d_3 \cdot p(x) \leq 0$ for all $x \in \partial_3\Omega$. Since we know from (62) that $n_3 \cdot \dot{x} \geq 0$, Theorem 3.2a applies to assure us that the q^*-saddle condition holds along this portion of the boundary. This applies to the corners as well because $p \cdot d_i = 0$ for $i = 1, 2$ respectively. As for the u_*-saddle condition here, Theorem 3.3 part b, will provide what we need. To see this, consider any $x \in \partial_3\Omega$, except the corners. Observe that our choice of n_3 has the property that $n_3 \cdot Gu = -7/\sqrt{5}$ for all $u \in \mathcal{U}$. Thus it follows from (62) that $n_3 \cdot (p(x) - Gu) \geq 0$ for all $u \in \mathcal{U}$, and so Theorem 3.3 part b applies with $F = \emptyset$, the inequality $p(x_0) \cdot Gu_0 \geq p(x_0) \cdot Gu$ being a consequence of the interior saddle point conditions, which we have already checked.

Now we turn to the u_*-saddle condition at the two (nonzero) corners. Consider the upper-left corner first:

$$x_0 = \begin{bmatrix} 0 \\ 7/6 \end{bmatrix} = x^{\partial_1}(\tau_1), \quad p(x_0) = p^{\partial_1}(\tau_1) = \frac{18 - \sqrt{226}}{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We will now check explicitly, using Lemma 1.3 that $\pi(x_0, p(x_0) - Gu) = R_F(p(x_0) - Gu)$ with $F = \{1\}$ for all $u \in \mathcal{U}$, as needed to apply Theorem 3.3 part b). Using $u = (u_1, 1 - u_1)$ we compute that

$$B_{\{1\}}(p(x_0) - Gu) = \frac{-12 + \sqrt{226} + 18\, u_1}{6\sqrt{2}},$$

which is clearly nonnegative for $0 \le u_1 \le 1$. Next, we calculate that

$$R_{\{1\}}(p(x_0) - Gu) = \begin{bmatrix} 0 \\ \frac{3-\sqrt{226}-3\,u_1}{6} \end{bmatrix}. \tag{63}$$

From this we check that $n_i \cdot R_{\{1\}}(p(x_0) - Gu) \ge 0$ for both $i = 1, 3$. Thus $\pi(x_0, p(x_0) - Gu) = R_{\{1\}}(p(x_0) - Gu)$ for all u. So to apply Theorem 3.3 part b) we only need to verify that

$$p(x_0) \cdot R_{\{1\}}(p(x_0) - Ge_1) \le p(x_0) \cdot R_{\{1\}}(p(x_0) - Gu),$$

since $u_* = e_1$ here. But this is clear from (63) since both coordinates of $p(x_0)$ are nonnegative and both components of $R_{\{1\}}(p(x_0) - Gu)$ are nonincreasing in u_1. Thus, Theorem 3.3 part b) implies the u_*-saddle condition at the upper-left corner x_0.

The lower-right corner $x_0 = x^{\partial_2}(\tau^{\partial_2})$ is more complicated:

$$x_0 = \begin{bmatrix} 7/4 \\ 0 \end{bmatrix}, \quad p(x_0) = p^{\partial_2}(\tau^{\partial_2}) = \begin{bmatrix} 2 - \frac{\sqrt{15}}{4} \\ 0 \end{bmatrix}.$$

Theorem 3.3 part b) does not apply here, the F involved in $\pi(x_0, p(x_0) - Gu) = R_F(p(x_0) - Gu)$ is not constant over $u \in \mathcal{U}$. For smaller values of u_1 one finds that $F = \{2, 3\}$, but for larger values $F = \{2\}$. One can check the u_*-saddle condition for $u_* = e_1$ with explicit calculations, since $p(x_0) \cdot \pi(x_0, p(x_0) - Gu)$ is piecewise linear in u_1. We need to show that this is nonincreasing in u_1. For that it is enough to check that the derivative with respect to u_1 is nonpositive for each of $F = \{2, 3\}$ and $F = \{2\}$:

$$-p(x_0) R_{\{2,3\}} G \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] = 0, \quad \text{and} \quad - p(x_0) R_{\{2\}} G \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix} \right] = \frac{3\left(-8 + \sqrt{15}\right)}{8},$$

both of which are indeed nonpositive. Our confirmation of the saddle point conditions for $V(x)$ are now complete.

6.5. **Optimality**

We have now confirmed almost all the properties of $\mathcal{E}$ for Theorem 4.1 to be invoked for this example. We simply need to present a set-valued $A(x)$ as in the verification theorem. The region Ω is partitioned into three subregions by the value of u_* associated with out family $\mathcal{E}$. Referring to the regions A—E of Figure 3, we take

$$\begin{aligned} \Omega^{(*)} &= (B \cap C) \cup (D \cap E) \\ \Omega^{(1)} &= (A \cup B \cup E) \setminus \Omega^{(*)} \\ \Omega^{(2)} &= (D \cup C) \setminus \Omega^{(*)}. \end{aligned}$$

Note that $\Omega^{(*)}$ is made up of the two switching curves, $B \cap C$ consists of the points $x^{\mathrm{E}}(s, \sigma(s))$, and $D \cap E$ is the set of points on the relaxed extremal $x^{\mathrm{a}}(t)$. These provide the boundary between the two fixed-control regions, $\Omega^{(i)}$. In terms of these, $A(x)$ for the optimal feedback control is simply

$$A(x) = \begin{cases} \{e_1\} & \text{for } x \in \Omega^{(1)} \\ \{e_2\} & \text{for } x \in \Omega^{(2)} \\ \mathcal{U} & \text{for } x \in \Omega^{(*)}. \end{cases} \tag{64}$$

That $A(x)$ satisfies the consistency condition (35) is clear from the construction, as is the saddle point condition (34) in each of $\Omega^{(i)}$. Since for $x \in \Omega^{(*)}$ we have $p(x) \cdot Ge_1 = p(x) \cdot Ge_2$ it is clear that (34) is satisfied for those points as well. The closure and upper semi-continuity results are obvious. Thus Theorem 4.1 applies to confirm that $V(x)$ as defined parametrically by our family $\mathcal{E}$, and illustrated in Figure 3, is indeed that value of the game for this example.

In concluding this example we note that the optimal strategy is strongly influenced by the Skorokhod dynamics. This is evident in the dependence of our expressions above for regions A, D, and E on the boundary extremals x^{∂_i}. On the $x_1 = 0$ face, the optimal strategy takes advantage of the stronger emptying effect for x_2 of $R_{\{1\}}Ge_1 = (0, 3)$ as compared to $R_{\{1\}}Ge_2 = (0, \frac{5}{2})$. This advantage of $u_* = e_1$, made possible by the Skorokhod dynamics on $\partial_1\Omega$, influences the optimal strategy into region E of the interior.

7. A small network in 3 dimensions

In this final section we return to the 2-server network of Figure 1. This example has been considered in the literature previously, but with respect to different performance criteria. Weiss [34] uses this network (with $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = 2$, $q(t) \equiv 0$, and $x(0) = (1, 0, 1)$) as an example with draining time $T = \inf\{t \geq 0 : \ x(t) = 0\}$ as the performance criterion. He shows that the LBFS policy is not optimal, but that FBFS is. He also considers it using total inventory $\int_0^T \sum x_i(t)\,\mathrm{d}t$ as the performance criterion, taking a constant load of the form $q(t) \equiv (\alpha, 0, 0)$. In that regard he finds that a constant control is optimal, taking one of three possible values, depending on the signs of x_2 and $x_3 - \gamma x_1$ for a certain critical value γ. Eng, Humphrey and Meyn [19] are concerned with the total inventory problem more generally, and with the use of linear programming calculations to obtain upper and lower performance bounds. They consider this network as an example, using simulations to compare their bounds with actual performance. The network also appears as an example in Meyn [28, 30]. The issue there is optimal policies with respect to a long-run (ergodic) cost criterion $\lim_{T\to\infty} \frac{1}{T} \int_0^T \sum x_i(t)\,\mathrm{d}t$.

We use formulation given in Sections 1 and 2, with the specific service rates $s_1 = s_2 = s_3 = 1$:

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We take $\Omega = \mathbb{R}^3_+$. (Since we will only be concerned with features on a small portion of $\partial\Omega$, we will not bother to specify additional boundaries to reduce Ω to a compact set.) We take $n_i = e_i$ $(i = 1, 2, 3)$; $c_i = 0$ and restoration vectors d_i as specified in (2).

Figure 4 illustrates a family $\mathcal{E}$ of extremals for this problem, constructed along the same lines as in Section 6, although considerably more complicated in this case. Our intent here is merely to look at the portion of $\partial\Omega$ near the x_1-axis $(\partial_{\{2,3\}}\Omega)$, covered by the extremals labeled E in the figure, to illustrate the application of Theorems 3.2b and c, and 3.3c, since they were not used in Section 6.

The section of the extremal family of interest to us is

$$x^{\mathrm{E}}(s, t, w) = \mathcal{X}^{(1)}_w \mathcal{X}^{(2)}_{\sigma(s,t)} \mathcal{X}^{\partial_2}_t x^{\mathrm{a}}(s).$$

We describe the stages of this extremal very briefly. $x^{\mathrm{a}}(s)$ is a boundary extremal (50) in the face $\partial_2\Omega$, using a nonconstant relaxed control,

$$u_*(t) = \begin{bmatrix} \lambda(t) \\ 1 - \lambda(t) \end{bmatrix}, \quad \lambda(t) = \frac{5 + \cos(\sqrt{\frac{6}{5}}\,t)}{15}.$$

with terminal conditions $x_a(0) = p_a(0) = 0$:

$$x^{\mathrm{a}}(t) = -\frac{1}{\sqrt{30}} \sin\left(\sqrt{\frac{6}{5}}\,t\right) \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad p^{\mathrm{a}}(t) = \frac{1 - \cos(\sqrt{\frac{6}{5}}\,t)}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

defined for $\tau^{\mathrm{a}} \leq t \leq 0$, where $\tau^{\mathrm{a}} = -\frac{\sqrt{5}\pi}{2\sqrt{6}}$.

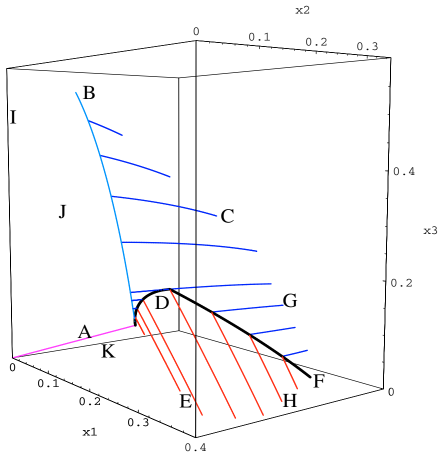

FIGURE 4. Anatomy of the extremal family.

Next, using terminal points $x^{\mathrm{a}}(s)$, $p^{\mathrm{a}}(s)$ we follow a boundary extremal

$$\mathcal{X}_t^{\partial_2} x^{\mathrm{a}}(s)$$

using $u_* = e_2$ into the upper portion of $\partial_2\Omega$, labeled B in the figure. From the points of $\mathcal{X}_t^{\partial_2} x^{\mathrm{a}}(s)$ we follow an interior extremal using $u_* = e_2$ into the interior of Ω:

$$\mathcal{X}_w^{(2)} \mathcal{X}_t^{\partial_2} x^{\mathrm{a}}(s).$$

Such extremals are labeled C in the figure. For small t we find that $\mathcal{X}_w^{(2)} \mathcal{X}_t^{\partial_2} x^{\mathrm{a}}(s)$ encounters a control switching surface D at $w = \sigma(s,t)$, much as in (60) above. From there we follow an interior extremal $x^{\mathrm{E}}(s,t,w)$ with control $u_* = e_1$:

$$\mathcal{X}_w^{(1)} \mathcal{X}_{\sigma(s,t)}^{(2)} \mathcal{X}_t^{\partial_2} x^{\mathrm{a}}(s)$$

Given suitable $s, t < 0$ these continue to $w = \tau(s,t)$ at which point $x^{\mathrm{E}}(s,t,w)$ contacts either $\partial_2\Omega$ or $\partial_3\Omega$. In particular, points on or near the edge $\partial_{\{2,3\}}\Omega$ are accounted for by $x_0 = x^{\mathrm{E}}(s,t,\tau(s,t))$. We want to consider the q^*- and u_*-saddle conditions at such boundary points.

One can verify all the saddle point conditions along these extremals, as we did in Section 6. We wish only to examine the points $x_0 = x^E(s,t,\tau(s,t))$ on $\partial\Omega$. First consider $x_0 = x^{\mathrm{E}}(s,t,\tau(s,t)) \in \partial_3\Omega$. The q^*-saddle condition for $q^* = p(x_0)$ on this face follows from the observations following Theorem 3.2 above, since $d_3 = n_3$, and

$$n_3 \cdot Gu_* = n_3 \cdot Ge_1 = -1 < 0.$$

The saddle conditions for $u_* = e_1$ also follow easily: observe that $n_3 \cdot Ge_i \le 0$ for both $i = 1, 2$. Therefore, for all $u \in \mathcal{U}$ we have

$$n_3 \cdot (p - Gu) \ge n_3 \cdot p \ge 0,$$

using the general nonnegativity of p (which follows by the same argument as (61)). Therefore $\pi(x, p - Gu) = p - Gu$ for all u, so that the boundary u_*-saddle condition is equivalent to the interior version. (This is Theorem 3.3 b).)

Next consider those x_0 in the face $\partial_2\Omega$. Regarding the q^*-saddle condition for $q^* = p(x)$ at such a point, since $d_2 = n_2 - n_3$, Theorem 3.2b says that $n_2 \cdot Ge_1 \le e_3 \cdot p$ is sufficient. Since $n_2 \cdot Ge_1 = 0$, this reduces to $e_3 \cdot p \ge 0$, which we know from general considerations. For the u_*-saddle condition on this face we resort to

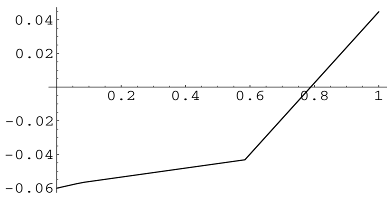

FIGURE 5. Example of u-saddle condition on x_1-axis.

numerical examination. According to Theorem 3.3 part c) we only need to compare the two extreme control values, to verify that

$$p(\tau) \cdot \pi(x(\tau), p(\tau) - Ge_1) \leq p(\tau) \cdot \pi(x(\tau), p(\tau) - Ge_2) \tag{65}$$

at such points. Numerical examination indicates that this condition does hold for $x_0 = x(\tau)$ sufficiently close to the x_1-axis.

We turn our attention lastly to points on the x_1-axis. These points are covered by extremals x^{E}, but additional considerations are necessary to verify the boundary saddle conditions. This is a situation in which Theorem 3.2c is useful. It allows us to extend the q^*-saddle condition from the two faces $\partial_2\Omega$ and $\partial_3\Omega$ to the axis, since the saddle point control value $u_* = e_1$ is constant in a neighborhood of this edge. As explained following the statement of Theorem 3.2, we just need to check that i) of the theorem fails on the edge $\partial_{\{2,3\}}\Omega$. One computes that $\bar{q} = R^T_{\{2,3\}}p = (p_1, 0, 0)^T$ and

$$B_{\{2,3\}}(\bar{q} - Ge_1) = (0, -1)^T,$$

which does have a negative component. Thus, by Lemma 1.3, i) of Theorem 3.2c is not true. Consequently, the q^* saddle condition *is* satisfied at such x_0. For the u_*-saddle condition with $u_* = e_1$ none of our theorems apply. We must resort to numerical examination. For instance, the extremal that leads to

$$x(\tau) = (0.423358, 0, 0)$$

produces

$$p(\tau) = (0.211544, 0.0761725, 0.092384).$$

One can now examine the graph of $p(\tau) \cdot \pi(x(\tau), p(\tau) - G \left[\begin{smallmatrix} 1-\lambda \\ \lambda \end{smallmatrix}\right])$ with respect to $0 \leq \lambda \leq 1$, presented in Figure 5. We see that the minimum value does indeed occur at $\lambda = 0$, corresponding to $u_* = e_1$. Similar calculations indicate that the u_*-saddle condition is satisfied at all points on this axis. The graph is obviously piecewise linear, corresponding the changes in the subset $F \subseteq \{2,3\}$ for which $\pi(x(\tau), p(\tau) - Gu) = R_F(p(\tau) - Gu)$. Note also that, as one can just barley see, the graph is non-convex near $\lambda = 0$, showing that the components of $\pi(x, v)$ need not be convex with respect to v in general.

The above serves to illustrate the applicability of Theorems 3.2 and 3.3c to verify saddle point conditions at boundary points. At present, the analysis of this example is not complete. It turns out that there is a slender region just beneath $x^{\mathrm{a}}(t)$ (A in the figure) in which the boundary u_*-saddle condition fails (for the family of Fig. 4). In fact further calculations suggest that the true value function for this example will be nonsmooth near that region, satisfying the HJI only in the viscosity sense there.

In closing let us come back to the comment in the introduction that the inclusion of boundary extremals is especially important for multiple server examples. Referring again to Figure 4, the switching curve/surface labeled F turns out to be described by the *naive policy*, which is simply to choose the control $u(t)$ to maximize the rate of decrease of $\|x(t)\|^2$ (ignoring Skorokhod effects):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = 2x(t) \cdot (q(t) - Gu(t)).$$

This policy would simply divide the states into two constant control regions: G (for $u = e_1$) and H (for $u = e_1$), separated by the hyperplane $x_1 = x_2 + x_3$. In our figure, F lies exactly on that hyperplane. In the introduction we anticipated that where one server becomes idle (here near the face $\partial_3\Omega$, where $x_2 = 0$) the optimal policy might be influenced by the Skorokhod dynamics as they modify the behavior of the non-idle server. Indeed we see this effect in Figure 4 in that the naive switching surface F does not extend all the way to $\partial_3\Omega$ but in fact doubles back abruptly as the curve D. Responsible for this is the complicated structure of boundary extremals x^{A} and x^{B} within the face $\partial_3\Omega$.

References

[1] R. Atar and P. Dupuis, A differential game with constrained dynamics and viscosity solutions of a related HJB equation. *Nonlinear Anal.* **51** (2002) 1105–1130.

[2] R. Atar, P. Dupuis and A. Shwartz, An escape criterion for queueing networks: Asymptotic risk-sensitive control via differential games. *Math. Op. Res.* **28** (2003) 801–835.

[3] R. Atar, P. Dupuis and A. Schwartz, Explicit solutions for a network control problem in the large deviation regime, *Queueing Systems* **46** (2004) 159–176.

[4] F. Avram, *Optimal control of fluid limits of queueing networks and stochasticity corrections*, in Mathematics of Stochastic Manufacturing Systems, G. Yin and Q. Zhang Eds., AMS, *Lect. Appl. Math.* **33** (1996).

[5] F. Avram, D. Bertsimas, M. Ricard, *Fluid models of sequencing problems in open queueing networks; and optimal control approach*, in Stochastic Networks, F.P. Kelly and R.J. Williams Eds., Springer-Verlag, NY (1995).

[6] J.A. Ball, M.V. Day and P. Kachroo, Robust feedback control of a single server queueing system. *Math. Control, Signals, Syst.* **12** (1999) 307–345.

[7] J.A. Ball, M.V. Day, P. Kachroo and T. Yu, Robust L_2-Gain for nonlinear systems with projection dynamics and input constraints: an example from traffic control. *Automatica* **35** (1999) 429–444.

[8] M. Bardi and I. Cappuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhäuser, Boston (1997).

[9] T. Basar and P. Bernhard, H^∞*-Optimal Control and Related Minimax Design Problems – A Dynamic game approach.* Birkhäuser, Boston (1991).

[10] A. Budhiraja and P. Dupuis, Simple necessary and sufficient conditions for the stability of constrained processes. *SIAM J. Appl. Math.* **59** (1999) 1686–1700.

[11] H. Chen and A. Mandelbaum, Discrete flow networks: bottleneck analysis and fluid approximations. *Math. Oper. Res.* **16** (1991) 408–446.

[12] H. Chen and D.D. Yao, *Fundamentals of Queueing Networks: Performance, Asymptotics and Optimization.* Springer-Verlag, N.Y. (2001).

[13] J.G. Dai, On the positive Harris recurrence for multiclass queueing networks: a unified approach via fluid models. *Ann. Appl. Prob.* **5** (1995) 49–77.

[14] M.V. Day, On the velocity projection for polyhedral Skorokhod problems. *Appl. Math. E-Notes* **5** (2005) 52–59.

[15] M.V. Day, J. Hall, J. Menendez, D. Potter and I. Rothstein, Robust optimal service analysis of single-server re-entrant queues. *Comput. Optim. Appl.* **22** (2002), 261–302.

[16] P. Dupuis and H. Ishii, On Lipschitz continuity of the solution mapping of the Skorokhod problem, with applications. *Stochastics and Stochastics Reports* **35** (1991) 31–62.

[17] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities. *Annals Op. Res.* **44** (1993) 9–42.

[18] P. Dupuis and K. Ramanan, Convex duality and the Skorokhod problem, I and II. *Prob. Theor. Rel. Fields* **115** (1999) 153–195, 197–236.

[19] D. Eng, J. Humphrey and S. Meyn, *Fluid network models: linear programs for control and performance bounds* in *Proc. of the 13th World Congress of International Federation of Automatic Control* **B** (1996) 19–24.

[20] A.F. Filippov, *Differential Equations with Discontinuous Right Hand Sides*, Kluwer Academic Publishers (1988).

[21] W.H. Fleming and M.R. James, The risk-sensitive index and the H_2 and H_∞ morms for nonlinear systems. *Math. Control Signals Syst.* **8** (1995) 199–221.

[22] W.H. Fleming and W.M. McEneaney, Risk-sensitive control on an infinite time horizon. *SAIM J. Control Opt.* **33** (1995) 1881–1915.

[23] J.M. Harrison, *Brownian models of queueing networks with heterogeneous customer populations*, in *Proc. of IMA Workshop on Stochastic Differential Systems.* Springer-Verlag (1988).

[24] P. Hartman, *Ordinary Differential Equations* (second edition). Birkh auser, Boston (1982).

[25] R. Isaacs, *Differential Games.* Wiley, New York (1965).

[26] P.L. Lions, Neumann type boundary conditions for Hamilton-Jacobi equations, *Duke Math. J.* **52** (1985) 793–820.

[27] X. Luo and D. Bertsimas, A new algorithm for state-constrained separated continuous linear programs. *SIAM J. Control Opt.* **37** (1998) 177–210.

[28] S. Meyn, *Stability and optimizations of queueing networks and their fluid models*, in Mathematics of Stochastic Manufacturing Systems, G. Yin and Q. Zhang Eds., *Lect. Appl. Math.* **33**, AMS (1996).

[29] S. Meyn, Transience of multiclass queueing networks via fluid limit models. *Ann. Appl. Prob.* **5** (1995) 946–957.

[30] S. Meyn, Sequencing and routing in multiclass queueing networks, part 1: feedback regulation. *SIAM J. Control Optim.* **40** (2001) 741–776.

[31] M.I. Reiman, Open queueing networks in heavy traffic. *Math. Oper. Res.* **9** (1984) 441–458.

[32] R.T. Rockafellar, *Convex Analysis.* Princeton Univ. Press, Princeton (1970).

[33] P. Soravia, H_∞ control of nonlinear systems: differential games and viscosity solutions. *SIAM J. Control Optim.* **34** (1996) 071–1097.

[34] G. Weiss, *On optimal draining of re-entrant fluid lines*, in Stochastic Networks, F.P. Kelly and R.J. Williams, Eds. Springer-Verlag, NY (1995).

[35] G. Weiss, *A simplex based algorithm to solve separated continuous linear programs*, to appear (preprint available at `http://stat.haifa.ac.il/~gweiss/`).

[36] P. Whittle, *Risk-sensitive Optimal Control.* J. Wiley, Chichester (1990).

[37] R.J. Williams, *Semimartingale reflecting Brownian motions in the orthant*, Stochastic Networks, Springer, New York *IMA Vol. Math. Appl.* **71** (1995) 125–137.