AN ELLIPTIC EQUATION WITH NO MONOTONICITY CONDITION ON THE NONLINEARITY

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\textbf{Abstract.} An elliptic PDE is studied which is a perturbation of an autonomous equation. The existence of a nontrivial solution is proven via variational methods. The domain of the equation is unbounded, which imposes a lack of compactness on the variational problem. In addition, a popular monotonicity condition on the nonlinearity is not assumed. In an earlier paper with this assumption, a solution was obtained using a simple application of topological (Brouwer) degree. Here, a more subtle degree theory argument must be used.

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1. INTRODUCTION

In this paper we consider an elliptic equation of the form

\[ -\Delta u + u = f(x, u), \quad x \in \mathbb{R}^N, \]  

(1.1)

where \( f \) is a “superlinear” function of \( u \). For large \( |x| \), the equation resembles an autonomous equation

\[ -\Delta u + u = f_0(u), \quad x \in \mathbb{R}^N. \]  

(1.2)

Under weak assumptions on \( f \) and \( f_0 \), we prove the existence of a nontrivial solution \( u \) of (1.1) with \( |u(x)| \to 0 \) as \( |x| \to \infty \).

Let \( f \) satisfy

\( (f_1) \) \( f \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \).

\( (f_2) \) \( f(x, 0) = 0 = f_q(x, 0) \) for all \( x \in \mathbb{R}^N \), where \( f \equiv f(x, q) \).

\( (f_3) \) If \( N > 2 \), there exist \( a_1, a_2 > 0 \), \( s \in (1, (N + 2)/(N - 2)) \) with \( |f_q(x, q)| \leq a_1 + a_2|q|^{s-1} \) for all \( q \in \mathbb{R}, \quad x \in \mathbb{R}^N \). If \( N = 2 \), there exist \( a_1 > 0 \) and a function \( \varphi : \mathbb{R}^+ \to \mathbb{R} \) with \( |f_q(x, q)| \leq a_1 \exp(\varphi(|q|)) \) for all \( q \in \mathbb{R}, \quad x \in \mathbb{R}^N \) and \( \varphi(t)/t^2 \to 0 \) as \( t \to \infty \).

\textbf{Keywords and phrases.} Mountain-pass theorem, variational methods, Nehari manifold, Brouwer degree, concentration-compactness.

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There exists $\mu > 2$ such that
\[ 0 < \mu F(x, q) \equiv \mu \int_0^q f(x, s) \, ds \leq f(x, q) q \]
for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$.

Let $f_0 \in C^2(\mathbb{R}, \mathbb{R})$ with satisfy (f1)-(f4) (except there is no dependence on $x$). Let $f$ also satisfy
\[ (f_5) \quad (f(x, q) - f_0(q))/f_0(q) \to 0 \text{ as } |x| \to \infty, \text{ uniformly in } q \in \mathbb{R}^N \setminus \{0\}. \]

In order to state the theorem, we need to outline the variational framework of the problem. Define functionals $I_0, I \in C^2(W^{1,2}(\mathbb{R}^N, \mathbb{R}), \mathbb{R})$ by
\[
I_0(u) = \frac{1}{2} ||u||^2 - \int_{\mathbb{R}^N} F_0(u(x)) \, dx, \quad (1.4)
\]
\[
I(u) = \frac{1}{2} ||u||^2 - \int_{\mathbb{R}^N} F(x, u(x)) \, dx, \quad (1.5)
\]
where $||u||$ is the standard norm on $W^{1,2}(\mathbb{R}^N, \mathbb{R})$ given by
\[
||u||^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + u(x)^2 \, dx. \quad (1.6)
\]

Critical points of $I_0$ correspond exactly to solutions $u$ of (1.2) with $u(x) \to 0$ as $|x| \to \infty$, and critical points of $I$ correspond exactly to solutions $u$ of (1.1) with $u(x) \to 0$ as $|x| \to \infty$.

By (f4), $F_0$ and $F$ are "superquadratic" functions of $q$, with, for example, $F(x, q)/q^2 \to 0$ as $q \to 0$ and $F(x, q)/q^2 \to \infty$ as $|q| \to \infty$ for all $x \in \mathbb{R}^N$, uniformly in $x$. Therefore $I(0) = I_0(0) = 0$, and there exists $r_0 > 0$ with $I(u) \geq ||u||^2/3$ and $I_0(u) \geq ||u||^2/3$ for all $u \in W^{1,2}(\mathbb{R}^N)$ with $||u|| \leq r_0$, and there also exist $u, u_0 \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$ with $I_0(u_0) < 0$ and $I(u) < 0$. So the sets of "mountain-pass curves" for $I_0$ and $I$,

\[
\Gamma_0 = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I_0(\gamma(1)) < 0 \}, \quad (1.7)
\]
\[
\Gamma = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I(\gamma(1)) < 0 \}, \quad (1.8)
\]
are nonempty, and the mountain-pass values
\[
c_0 = \inf_{\gamma \in \Gamma_0} \max_{\theta \in [0,1]} I_0(\gamma(\theta)) \quad (1.9)
\]
\[
c = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I(\gamma(\theta)) \quad (1.10)
\]
are positive.

We are now ready to state the theorem.

**Theorem 1.1.** If $f_0$ and $f$ satisfy (f1)-(f4) and $f$ satisfies (f5), and if there exists $\alpha > 0$ such that
\[
I_0 \text{ has no critical values in the interval } [c_0, c_0 + \alpha) \quad (1.11)
\]
then there exists $\epsilon_0 = \epsilon_0(f_0) > 0$ with the following property: if $f$ satisfies
\[
|f(x, q) - f_0(q)| < \epsilon_0 f_0(q) \quad (1.12)
\]
for all $x \in \mathbb{R}^N$, $q \in \mathbb{R}$, then (1.2) has a nontrivial solution $u \not\equiv 0$ with $u(x) \to 0$ as $|x| \to \infty$.

As shown in [9], (1.12) holds in a wide variety of situations.
The missing monotonicity assumption

One interesting aspect of Theorem 1.1 is a condition that is not assumed. We do not assume

\[ F_0(q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q > 0; \]
\[ F_0(q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q < 0; \quad (1.13) \]
\[ F(x,q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q > 0; \text{ or} \]
\[ F(x,q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q < 0. \]

This condition holds in the power case, \( F_0(q) = |q|^\alpha/\alpha, \alpha > 2 \). The condition is due to Nehari.

If (1.13) were case, then for any \( u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \), the mapping \( s \mapsto I(su) \) would begin at 0 at \( s = 0 \), increase to a positive maximum, then decrease to \(-\infty\) as \( s \to \infty \). Defining

\[ S = \{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \mid I'(u)u = 0 \}, \quad (1.14) \]

\( S \) would be a codimension-one submanifold of \( E \), homeomorphic to the unit sphere in \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) via radial projection. \( S \) is known as the Nehari manifold in the literature. Any ray of the form \( \{su \mid s > 0\} (u \neq 0) \) intersects \( S \) exactly once. All nonzero critical points of \( I \) are on \( S \). Conversely, under suitable smoothness assumptions on \( F \), any critical point of \( I \) constrained to \( S \) would be a critical point of \( I \) (in the large) (see [17]). Therefore, one could work with \( S \) instead of \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and look for, say, a local minimum of \( I \) constrained to \( S \) (which may be easier than looking for a saddle point of \( I \)). There is another way to use (1.13): for any \( u \in S \), the ray from 0 passing through \( u \) can be used (after rescaling in \( \theta \)) as a mountain-pass curve along which the maximum value of \( I \) is \( I(u) \). Conversely, any mountain-pass curve \( \gamma \in \Gamma \) intersects \( S \) at least once [6]. Therefore, one may work with points on \( S \) instead of paths in \( \Gamma \). Since assumption (1.13) is so helpful, it is found in many papers, such as [1,5,20], and [18].

In the paper [17], a result similar to Theorem 1.1 was proven for the \( N = 1 \) (ODE) case. The proof of Theorem 1.1 is similar except that a simple connectivity argument must be replaced by a degree theory argument [18]. proves a version of Theorem 1.1 under the assumption (1.13). Without 1.13, the manifold \( S \) must be replaced by a set with similar properties.

Define \( B_1(0) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \), and \( \overline{\Omega} \) and \( \partial \Omega \) to be, respectively, the topological closure and topological boundary of \( \Omega \). It is a simple consequence of the Brouwer degree [7] that for any continuous function \( h : B_1(0) \to \mathbb{R}^N \) with \( h(x) = x \) for all \( x \in \partial B_1(0) \), there exists \( x \in B_1(0) \) with \( h(x) = 0 \). We will need the following generalization:

**Lemma 1.2.** Let \( h \in C(\overline{B_1(0)} \times [0,1], \mathbb{R}^N) \) with, for all \( x \in \overline{B_1(0)} \) and \( t \in [0,1] \),

(i) \( h(x,0) = x = h(x,1) \).
(ii) \( x \in \partial B_1(0) \Rightarrow h(x,t) = x \).

Then there exists a connected subset \( C_0 \subset \overline{B_1(0)} \times [0,1] \) with \( (0,0), (0,1) \in C_0 \) and \( h(x,t) = 0 \) for all \( (x,t) \in C_0 \).

Using the Brouwer degree, it is clear that under the hypotheses of Lemma 1.2, for each “horizontal slice” \( \overline{B_1(0)} \times \{t\} \) of the cylinder \( \overline{B_1(0)} \times [0,1] \), there exists \( x \in B_1(0) \) with \( h(x,t) = 0 \). The conclusion of Lemma 1.2 does not follow from this observation. A generalization of Lemma 1.2 is known [16]: however, the reference may be difficult to find, so a proof is given here.

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1. The proof of Lemma 1.2 is deferred until Section 3.
2. Proof of Theorem 1.1

It is fairly easy to show that
\[ c \leq c_0, \]  
where \( c \) and \( c_0 \) are from (1.9)–(1.10): it is proven in [11] that there exists \( \gamma_1 \in \Gamma_0 \) with \( \max_{\theta \in [0,1]} I_0(\gamma_1(\theta)) = c_0 \). Define the translation operator \( \tau \) as follows: for a function \( u \) on \( \mathbb{R}^N \) and \( a \in \mathbb{R}^N \), define let \( \tau_u \) be \( u \) shifted by \( a \), that is, \( (\tau_u)(x) = u(x-a) \). Let \( \epsilon > 0 \). Let \( \epsilon = \frac{1}{10} \), \( \epsilon > 0 \) and define \( \tau_{\epsilon, \gamma_1} \) by \( \tau_{\epsilon, \gamma_1}(\theta) = \tau_{\epsilon, \gamma_1}(\gamma_1(\theta)) \). Then for large \( R > 0 \), by (f), \( \tau_{\epsilon, \gamma_1} \in \Gamma \) and \( \max_{\theta \in [0,1]} \| I((\tau_{\epsilon, \gamma_1}(\theta)) < c_0 + \epsilon \). Since \( \epsilon > 0 \) was arbitrary, \( c \leq c_0 \).

A Palais-Smale sequence for \( I \) is a sequence \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( (I(u_m)) \) convergent and \( \| I'(u_m) \| \to 0 \) as \( m \to \infty \). It is well-known that \( I \) fails the “Palais-Smale condition”. That is, a Palais-Smale sequence need not converge. However, the following proposition states that a Palais-Smale sequence “splits” into the sum of a critical point of \( I \) and translates of critical points of \( I_0 \):

**Proposition 2.1.** If \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( I'(u_m) \to 0 \) and \( I(u_m) \to a > 0 \), then there exist \( k \geq 0 \), \( v_0, v_1, \ldots, v_k \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and sequences \( (x_m^i)_{i=1}^k \subset \mathbb{R}^N \), such that

(i) \( I'(v_0) = 0 \);  
(ii) \( I_0(v_i) = 0 \) for all \( i = 1, \ldots, k \),

and along a subsequence (also denoted \( (u_m) \))

(iii) \( \| u_m - (v_0 + \sum_{i=1}^k \tau_{x_i^0}) \| \to 0 \) as \( m \to \infty \);

(iv) \( |x_i^0| \to \infty \) as \( m \to \infty \) for \( i = 1, \ldots, k \);

(v) \( |x_i^0 - x_j^0| \to \infty \) as \( m \to \infty \) for all \( i \neq j \);

(vi) \( I(v_0) + \sum_{i=1}^k I_0(v_i) = a \).

A proof for the case of \( x \)-periodic \( F \) is found in [6], and essentially the same proof works here. Similar propositions for nonperiodic coefficient functions, for both ODE and PDE, are found in [1,5], and [19], for example. All are inspired by the “concentration-compactness” theorems of P.-L. Lions [12].

If \( c < c_0 \), then by standard deformation arguments [15], there exists a Palais-Smale sequence \( (u_m) \) with \( I(u_m) \to c \). By [11], the smallest nonzero critical value of \( I_0 \) is \( c_0 \). Applying Proposition 2.1, we obtain \( k = 0 \), and \( (u_m) \) has a convergent subsequence, proving Theorem 1.1. So assume from now on that

\[ c = c_0. \]  

For \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \) and \( i \in \{1, \ldots, N\} \), define \( \mathcal{L}_i \), the \( i \)th component of the “location” of \( u \), by

\[ \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_i - \mathcal{L}_i(u)) \, dx = 0 \]  

and the “location” of \( u \) by

\[ \mathcal{L}(u) = (\mathcal{L}_1(u), \ldots, \mathcal{L}_N(u)) \in \mathbb{R}^N. \]

The following lemma establishes the existence and continuity of \( \mathcal{L} \).

**Lemma 2.2.** \( \mathcal{L} \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \).

**Proof.** It suffices to show that \( \mathcal{L}_1 \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). Let \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). By Leibniz’s Theorem, the mapping \( \phi : s \mapsto \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - s) \, dx \) is continuous, differentiable, and strictly decreasing, with

\[ \phi'(s) = -\int_{\mathbb{R}^N} u^2(x)/((x_1 - s)^2 + 1) \, dx < 0. \]  

\( \phi(s) \to +\infty \) as \( s \to -\infty \). Therefore \( \mathcal{L}_1(u) \) is unique and well-defined. Let \( \epsilon > 0 \) and \( u_m \to u \). Now \( \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - (\mathcal{L}_1(u) + \epsilon)) \, dx < 0 \). Since \( u_m^2 \to u^2 \) in \( L^1(\mathbb{R}^N, \mathbb{R}) \), \( \int_{\mathbb{R}^N} u_m^2 \tan^{-1}(x_1 - (\mathcal{L}(u) + \epsilon)) \, dx < 0 \) for
large $m$, so for large $m$, $\mathcal{L}_1(u_m) < \mathcal{L}_1(u) + \epsilon$. Similarly, for large $m$, $\mathcal{L}_1(u_m) > \mathcal{L}_1(u) - \epsilon$. Since $\epsilon$ is arbitrary, $\mathcal{L}_1(u_m) \to \mathcal{L}_1(u)$.

We are ready to begin the minimax argument. First we construct a mountain-pass curve $\gamma_0$ with some special properties:

**Lemma 2.3.** There exists $\gamma_0 \in \Gamma_0$ such that for all $\theta \in [0, 1]$,

(i) $I_0(\gamma_0(\theta)) \leq c_0$.

(ii) $\theta > 0 \Rightarrow \gamma_0(\theta) \neq 0$.

(iii) $\theta \leq 1/2 \Rightarrow I_0(\gamma(\theta)) \leq c_0/2$.

(iv) $\theta > 0 \Rightarrow \mathcal{L}(\gamma(\theta)) = 0$.

**Proof.** By [10], there exists $\gamma_1 \in \Gamma_0$ with $\max_{\theta \in [0, 1]} I_0(\gamma_1(\theta)) = c_0$. Assume without loss of generality that $\gamma_1(\theta) \neq 0$ for $\theta > 0$. By rescaling in $\theta$ if necessary, assume that $I_0(\gamma_1(\theta)) \leq c_0/2$ for $\theta \leq 1/2$. Finally, define $\gamma_0$ by $\gamma_0(0) = 0$, $\gamma_0(\theta) = \tau - \mathcal{L}(\gamma_1(\theta))\gamma_1(\theta)$ for $\theta > 0$.

Assume $c_0$ in (1.12) is small enough so that for all $x \in \mathbb{R}^N$ and $\theta \in [0, 1]$,

$$I(\tau_x(\gamma_0(\theta))) < \min(2c_0, c_0 + \alpha) \text{ and } I(\tau_x(\gamma_0(1))) < 0,$$  \hspace{1cm} (2.6)

where $\alpha$ is from (1.11).

**A substitute for $S$**

Using the mountain-pass geometry of $I$ and the fact that Palais-Smale sequences of $I$ are bounded in norm [6], we construct a set which has similar properties to $S$, described in Section 1. Let $\nabla I$ denote the gradient of $I$, that is, $(\nabla I(u), w) = I'(u)w$ for all $u, w \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$. Here, $(\cdot, \cdot)$ is the usual inner product defined by $(u, w) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla w + uw \, dx$. Let $\varphi : W^{1,2}(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ be locally Lipschitz, with $I(u) \geq -1 \Rightarrow \varphi(u) = 1$ and $I(u) \leq -2 \Rightarrow \varphi(u) = 0$. Let $\eta$ be the solution of the initial value problem

$$\frac{d\eta}{ds} = -\varphi(\eta)\nabla I(u), \quad \eta(0, u) = u. \hspace{1cm} (2.7)$$

In [19] it is proven that $\eta$ is well-defined on $\mathbb{R}^+ \times W^{1,2}(\mathbb{R}^N)$. Let $B$ be the basin of attraction of 0 under the flow $\eta$, that is,

$$B = \{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \mid \eta(s, u) \to 0 \text{ as } s \to \infty \} \hspace{1cm} (2.8)$$

$B$ is an open neighborhood of 0 [19]. Let $\partial B$ be the topological boundary of $B$ in $W^{1,2}(\mathbb{R}^N, \mathbb{R})$. $\partial B$ has some properties in common with $S$. For example, for any $\gamma \in \Gamma$, $\gamma([0, 1])$ intersects $\partial B$ at least once.

A pseudo-gradient vector field for $I$ may be used in place of $\nabla I$, in which case $B$ and $\partial B$ would be different, but the ensuing arguments would be the same.

Let

$$c^+ = \inf \{ I(u) \mid u \in \partial B, \ | \mathcal{L}(u) | \leq 1 \}. \hspace{1cm} (2.9)$$

The reason for the label “$c^+$” will become apparent in a moment. From now on, let us assume

$$I \text{ has no critical values in } (0, c_0] = (0, c]. \hspace{1cm} (2.10)$$

This will lead to the conclusion that $I$ has a critical value greater than $c_0$.

We claim that under assumptions (2.2) and (2.10),

$$c^+ > c_0. \hspace{1cm} (2.11)$$

We use arguments that are sketched here and found in more detail in [19] and [5].
To prove the claim, suppose first that $c^+ < c_0$. Then there exists $u_0 \in \partial B$ with $I(u_0) < c_0$. By arguments in [19], there exists a large positive constant $P$ with

$$I(u) \leq c_0 \text{ and } \|u\| \geq 2P \Rightarrow I(\eta(s, u)) < 0 \text{ for some } s > 0, \text{ and } \|\eta(s, u)\| > P$$

(2.12)

for all $s > 0$. Suppose $a > 0$ and $\|I'(\eta(s_m, U_0))\| \geq a$ for some sequence $(s_m)$ with $s_m \to \infty$. Since $u_0 \in \partial B$, $\|\eta(u_0)\| < 2P$ and all $s > 0$, $I'$ is bounded on bounded subsets of $W^{1,2}(\mathbb{R})$, so $I'$ is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R})$. Therefore $I(\eta(s, u_0)) < 0$ for some $s > 0$. This is impossible since $u_0 \in \partial B$. Therefore $I'(\eta(s, u_0)) \to 0$ as $s \to \infty$.

Define $u_n = \eta(n, u_0)$. Since $I'(u_n) \to 0$ and $u_n \in \partial B$, there exists $b \in (0, c_0)$ with $I(u_n) \to b$. By [11], $I_0$ has no critical values between 0 and $c_0$. Therefore, Proposition 2.1, with $k = 0$, implies that $(u_n)$ converges along a subsequence to a critical point $w$ of $I$ with $0 < I(w) < c_0$. This contradicts assumption (2.10).

Next, suppose that $c^+ = c_0$. Then there exists a sequence $(u_n) \subset \partial B$ with $|\mathcal{L}(u_n)| \leq 1$ for all $n$ and $I(u_n) \to c_0$ as $n \to \infty$. As above, $I'(u_n) \to 0$ as $n \to \infty$; to prove, suppose otherwise. Then there exist $a > 0$ and a subsequence of $(u_n)$ (also called $(u_n)$) along which $\|I'(u_n)\| > a$. Since $\partial B$ is forward-$\eta$-invariant [19], $\eta(I, u_n) \in \partial B$ for all $n$. Since $(\eta(I, u_n))_{n \geq 1}$ is bounded and $I'$ is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R}^N, \mathbb{R})$, for large $n$, $\eta(I, u_n) \in \partial B$ with $I(\eta(I, u_n)) < c_0$. By the argument above, this implies that $I$ has a critical value in $(0, c_0)$, contradicting assumption (2.2). Thus $I'(u_n) \to 0$ as $n \to \infty$. Applying Proposition 2.1 and using the fact that $|\mathcal{L}(u_n)| \leq 1$ for all $n$, $(u_n)$ converges along a subsequence to a critical point of $I$, contradicting assumption (2.10). (2.11) is proven.

Let $R > 0$ be big enough so that for all $x \in \partial B_R(0) \subset \mathbb{R}^N$ and $\theta \in [0, 1]$,

$$I(\tau_x \gamma_0(\theta)) < c^+.$$  

(2.13)

This is possible by (1.12), (2.11), and Lemma 2.3(i). Define the minimax class

$$\mathcal{H} = \{ h \in C(\overline{B_R(0)} \times [0, 1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \text{ for all } x \in \overline{B_R(0)} \text{ and } t \in [0, 1], \}
\begin{align*}
& t > 0 \Rightarrow h(x, t) \neq 0 \\
& 0 \leq t \leq 1/2 \Rightarrow h(x, t) = \tau_x \gamma_0(t) \\
& x \in \partial B_R(0) \Rightarrow h(x, t) = \tau_x \gamma_0(t) \\
& h(x, 1) = \tau_x \gamma_0(1) \}
\end{align*}

and the minimax value

$$h_0 = \inf_{h \in \mathcal{H}} \max_{(x, t) \in \overline{B_R(0)} \times [0, 1]} I(h(x, t)).$$

(2.14)

We claim

$$c_0 < c^+ \leq h_0 < \min(2c_0, c_0 + \alpha).$$

(2.15)

**Proof of Claim.** Define $\tilde{h} \in \mathcal{H}$ by $\tilde{h}(x, t) = \tau_x \gamma_0(t)$. Then $\tilde{h} \in \mathcal{H}$ and by (2.6), $\max_{(x, t) \in \overline{B_R(0)} \times [0, 1]} \tilde{h}(x, t) < \min(2c_0, c_0 + \alpha)$. Therefore $h_0 < \min(2c_0, c_0 + \alpha)$.

Next, let $h \in \mathcal{H}$. By Lemma 1.2, and a suitable rescaling of $x$ and $t$, there exists a connected set $C_2 \subset B_R(0) \times [1/2, 1]$ with $(0, 1/2), (0, 1) \in C_2$ and along which for all $(x, t) \in C_2$,

$$\mathcal{L}(h(x, t)) = 0.$$  

(2.16)

Joining $C_2$ with the segment $\{0\} \times [0, 1/2]$, we obtain a connected set $C_3 \subset B_R(0) \times [0, 1]$ such that $(0, 0), (0, 1) \in C_3$ and for all $(x, t) \in C_3$, $\mathcal{L}(h(x, t)) = 0$. $C_3$ is not necessarily path-connected, so let $r > 0$ be small enough so
that for all
\[(x, t) \in N_r(C_3) \equiv \{(y, s) \in B_R(0) \times [0, 1] \mid
\exists (x', t') \in B_R(0) \times [0, 1] \text{ with } |y - x'|^2 + (s - t')^2 < r^2\}, \]
(2.17)
\[|\mathcal{L}(h(x, t))| < 1.\]

\[N_r(C_3) \text{ is path-connected [21], so there exists a path } g \in C([0, 1], N_r(C_3)) \text{ with } g(0) = (0, 0), g(1) = (0, 1), \text{ and } \]
g(\theta) \in N_r(C_3) \text{ for all } \theta \in [0, 1]. \text{ If we define } \tilde{\gamma} = \Gamma by \tilde{\gamma}(\theta) = h(g(\theta)), \text{ then } |\mathcal{L}(\tilde{\gamma}(\theta))| < 1 \text{ for all } \theta \in [0, 1]. \text{ Since } \]
\[\tilde{\gamma}(0) = 0 \text{ and } I(\tilde{\gamma}(1)) < 0, \text{ there exists } \theta^* \in [0, 1] \text{ with } \tilde{\gamma}(\theta^*) \in \partial \mathcal{B}. \text{ By the definition of } c^+ (2.9), I(\tilde{\gamma}(\theta^*)) \geq c^+. \]

Since \(h\) was an arbitrary element of \(\mathcal{H}, h_0 \geq c^+. \)

By standard deformation arguments, such as described in [15], there exists a Palais-Smale sequence \((u_n) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \text{ with } I'(u_n) \to 0 \text{ and } I(u_n) \to h_0 \text{ as } n \to \infty. c_0 < h_0 < \min(2c_0, c_0 + \alpha). \text{ Apply Proposition 2.1 to } (u_n). \text{ Since } I_0 \text{ has no positive critical values smaller than } c_0 [11], k \leq 1. \text{ By (2.10), } (u_n) \text{ converges along a subsequence to a critical point } z \text{ of } I, \text{ with } I(z) = h_0. \text{ Theorem 1.1 is proven.} \]

3. A DEGREE-THEORETIC LEMMA

Here, we prove Lemma 1.2. Let \(h\) be as in the hypotheses of the lemma. For \(l > 0\), define \(\mathcal{A}_l \subset \overline{B_1(0)} \times [0, 1]\) by
\[\mathcal{A}_l = \{(x, t) \in \overline{B_1(0)} \times [0, 1] \mid |f(x, t)| < l\}. \]
(3.1)

\(\mathcal{A}_l\) is an open neighborhood of \((0, 0)\). Let \(C_1\) be the component of \(\mathcal{A}_l\) containing \((0, 0)\). We will prove the following claim:

For all \(\epsilon > 0, (0, 1) \in C_\epsilon\).
(3.2)

Then we will use the \(C_\epsilon\)’s to construct \(C_0\). For \(l > 0\) and \(t \in [0, 1]\), define
\[C_\epsilon^l = \{x \in \overline{B_1(0)} \mid (x, t) \in C_\epsilon\}. \]
(3.3)

Fix \(\epsilon \in (0, 1). \) Define \(\phi : [0, 1] \to \mathbb{Z}\) by
\[\phi(t) = d(h(\cdot, t), C_\epsilon^l, 0), \]
(3.4)

where \(d\) is the topological Brouwer degree [7]. We will prove \(\phi(t) = 1\) for all \(t \in [0, 1]\), in particular \(\phi(1) = 1\), so (3.2) is satisfied.

\(f\) is continuous on a compact domain, so \(f\) is uniformly continuous. Let \(\rho > 0\) be small enough so that for all \(x \in \overline{B_1(0)}\) and \(t_1, t_2 \in [0, 1],\)
\[|t_1 - t_2| < \rho \Rightarrow |h(x, t_1) - h(x, t_2)| < \epsilon/4. \]
(3.5)

Clearly
\[\phi(0) = d(id, B_1(0), 0) = 1. \]
(3.6)

Let \(0 \leq t_1 < t_2 \leq 1\) with \(t_2 - t_1 < \rho\). We will show \(\phi(t_1) = \phi(t_2)\), proving that \(\phi\) is constant, which by (3.6), implies (3.2).

\(\Omega\) is nonempty. For all \(x \in \partial C_\epsilon^t, |h(x, t_1)| = \epsilon, \) so by (3.5),
\[x \in \partial C_\epsilon^t \Rightarrow |h(x, t_1)| \geq 3\epsilon/4. \]
(3.7)

By the additivity property of \(d\) [7],
\[\phi(t_2) \equiv d(f(\cdot, t_2), C_\epsilon^{t_2}, 0) \]
\[= d(f(\cdot, t_2), C_\epsilon^{t_2} \setminus C_\epsilon^t, 0) + d(f(\cdot, t_2), C_\epsilon^t \cap C_\epsilon^{t_2}, 0). \]
(3.8)
We will show:

There does not exist $x \in C_t^2 \setminus \overline{C_t^1}$ with $h(x, t_2) = 0$. \hfill (3.9)

Suppose such an $x$ exists. Then by (3.5), $|h| < \epsilon/4$ on the segment \{x$\} \times [t_1, t_2]$. $x \in C_t^2$, so $(x, t_2) \in \mathcal{C}_\epsilon$, and by the definition of $C_\epsilon$, $(x, t_1) \in \mathcal{C}_\epsilon$, and $x \in \overline{C_t^1 \setminus \mathcal{C_t}^1}$. So (3.9) is true. Therefore by (3.8),

$$\phi(t_2) = d(f(\cdot, t_2), C_t^1 \cap C_t^2, 0). \hfill (3.10)$$

By the same argument, switching the roles of $t_1$ and $t_2$,

$$\phi(t_1) = d(f(\cdot, t_1), C_t^1 \cap C_t^2, 0). \hfill (3.11)$$

For all $t \in [t_1, t_2]$ and $x \in \partial C_t^1 \cup \partial C_t^2$, (3.5) gives \(|h(x, t_1)| > 3\epsilon/4 \) and \(|h(x, t) - h(x, t_1)| < \epsilon/4 \). Therefore by the homotopy invariance property of the degree \cite{7},

$$\phi(t_1) = d(f(\cdot, t_1), C_t^1 \cap C_t^2, 0) = d(f(\cdot, t_2), C_t^1 \cap C_t^2, 0) = \phi(t_2). \hfill (3.12)$$

$\phi(0) = 1$ and $\phi(t_1) = \phi(t_2)$ for any $t_1 < t_2$ with $t_1, t_2 \in [0, 1]$ and $t_2 - t_1 < \rho$. Therefore $\phi$ is constant, and $\phi(1) = 1$. Therefore $(0, 1) \in C_\epsilon$.

Now let

$$C_0 = \bigcap_{\epsilon > 0} C_\epsilon. \hfill (3.13)$$

Each $C_\epsilon$ is a connected set containing $(0, 0)$ and $(0, 1)$, so it is easy to show that $C_0$ is a connected set containing $(0, 0)$ and $(0, 1)$, and clearly for all $(x, t) \in C_0$, $h(x, t) = 0$.

References


