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AN ELLIPTIC EQUATION WITH NO MONOTONICITY CONDITION ON THE NONLINEARITY

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Abstract. An elliptic PDE is studied which is a perturbation of an autonomous equation. The existence of a nontrivial solution is proven *via* variational methods. The domain of the equation is unbounded, which imposes a lack of compactness on the variational problem. In addition, a popular monotonicity condition on the nonlinearity is not assumed. In an earlier paper with this assumption, a solution was obtained using a simple application of topological (Brouwer) degree. Here, a more subtle degree theory argument must be used.

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1. Introduction

In this paper we consider an elliptic equation of the form

$$-\Delta u + u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where f is a "superlinear" function of u. For large $|x|$, the equation resembles an autonomous equation

$$-\Delta u + u = f_0(u), \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Under weak assumptions on f and f_0, we prove the existence of a nontrivial solution u of (1.1) with $|u(x)| \to 0$ as $|x| \to \infty$.

Let f satisfy

(f_1) $f \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.
(f_2) $f(x,0) = 0 = f_q(x,0)$ for all $x \in \mathbb{R}^N$, where $f \equiv f(x,q)$.
(f_3) If $N > 2$, there exist $a_1, a_2 > 0$, $s \in (1, (N+2)/(N-2))$ with $|f_q(x,q)| \le a_1 + a_2|q|^{s-1}$ for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$. If $N = 2$, there exist $a_1 > 0$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with $|f_q(x,q)| \le a_1 \exp(\varphi(|q|))$ for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $\varphi(t)/t^2 \to 0$ as $t \to \infty$.

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(f_4) There exists $\mu > 2$ such that

$$0 < \mu F(x,q) \equiv \mu \int_0^q f(x,s)\,\mathrm{d}s \le f(x,q)q \tag{1.3}$$

for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$.

Let $f_0 \in C^2(\mathbb{R},\mathbb{R})$ with satisfy (f_1)-(f_4) (except there is no dependence on x). Let f also satisfy

(f_5) $(f(x,q) - f_0(q))/f_0(q) \to 0$ as $|x| \to \infty$, uniformly in $q \in \mathbb{R}^N \setminus \{0\}$.

In order to state the theorem, we need to outline the variational framework of the problem. Define functionals $I_0, I \in C^2(W^{1,2}(\mathbb{R}^N,\mathbb{R}),\mathbb{R})$ by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F_0(u(x))\,\mathrm{d}x, \tag{1.4}$$

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u(x))\,\mathrm{d}x, \tag{1.5}$$

where $\|u\|$ is the standard norm on $W^{1,2}(\mathbb{R}^N,\mathbb{R})$ given by

$$\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + u(x)^2\,\mathrm{d}x. \tag{1.6}$$

Critical points of I_0 correspond exactly to solutions u of (1.2) with $u(x) \to 0$ as $|x| \to \infty$, and critical points of I correspond exactly to solutions u of (1.1) with $u(x) \to 0$ as $|x| \to \infty$.

By (f_4), F_0 and F are "superquadratic" functions of q, with, for example, $F(x,q)/q^2 \to 0$ as $q \to 0$ and $F(x,q)/q^2 \to \infty$ as $|q| \to \infty$ for all $x \in \mathbb{R}^N$, uniformly in x. Therefore $I(0) = I_0(0) = 0$, and there exists $r_0 > 0$ with $I(u) \ge \|u\|^2/3$ and $I_0(u) \ge \|u\|^2/3$ for all $u \in W^{1,2}(\mathbb{R}^N)$ with $\|u\| \le r_0$, and there also exist $u, u_0 \in W^{1,2}(\mathbb{R}^N,\mathbb{R})$ with $I_0(u_0) < 0$ and $I(u) < 0$. So the sets of "mountain-pass curves" for I_0 and I,

$$\Gamma_0 = \{\gamma \in C([0,1], W^{1,2}(\mathbb{R}^N,\mathbb{R})) \mid \gamma(0) = 0,\ I_0(\gamma(1)) < 0\}, \tag{1.7}$$

$$\Gamma = \{\gamma \in C([0,1], W^{1,2}(\mathbb{R}^N,\mathbb{R})) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\}, \tag{1.8}$$

are nonempty, and the mountain-pass values

$$c_0 = \inf_{\gamma\in\Gamma_0} \max_{\theta\in[0,1]} I_0(\gamma(\theta)) \tag{1.9}$$

$$c = \inf_{\gamma\in\Gamma} \max_{\theta\in[0,1]} I(\gamma(\theta)) \tag{1.10}$$

are positive.

We are now ready to state the theorem.

Theorem 1.1. *If f_0 and f satisfy (f_1)-(f_4) and f satisfies (f_5), and if there exists $\alpha > 0$ such that*

$$I_0 \text{ has no critical values in the interval } [c_0, c_0 + \alpha) \tag{1.11}$$

then there exists $\epsilon_0 = \epsilon_0(f_0) > 0$ with the following property: if f satisfies

$$|f(x,q) - f_0(q)| < \epsilon_0 |f_0(q)| \tag{1.12}$$

for all $x \in \mathbb{R}^N$, $q \in \mathbb{R}$, then (1.2) *has a nontrivial solution $u \not\equiv 0$ with $u(x) \to 0$ as $|x| \to \infty$.*

As shown in [9], (1.12) holds in a wide variety of situations.

The missing monotonicity assumption

One interesting aspect of Theorem 1.1 is a condition that is *not* assumed. We do not assume

$$\begin{aligned}&\text{For all } q\in\mathbb{R} \text{ and } x\in\mathbb{R}^N,\ F_0(q)/q^2 \text{ is}\\&\qquad\text{a nondecreasing function of } q \text{ for } q>0;\\&F_0(q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q<0;\\&F(x,q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q>0;\text{ or}\\&F(x,q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q<0.\end{aligned} \tag{1.13}$$

This condition holds in the power case, $F_0(q)=|q|^\alpha/\alpha,\ \alpha>2$. The condition is due to Nehari.

If (1.13) were case, then for any $u\in W^{1,2}(\mathbb{R}^N,\mathbb{R})\setminus\{0\}$, the mapping $s\mapsto I(su)$ would begin at 0 at $s=0$, increase to a positive maximum, then decrease to $-\infty$ as $s\to\infty$. Defining

$$\mathcal{S}=\{u\in W^{1,2}(\mathbb{R}^N,\mathbb{R})\setminus\{0\}\mid I'(u)u=0\}, \tag{1.14}$$

$\mathcal{S}$ would be a codimension-one submanifold of E, homeomorphic to the unit sphere in $W^{1,2}(\mathbb{R}^N,\mathbb{R})$ *via* radial projection. $\mathcal{S}$ is known as the Nehari manifold in the literature. Any ray of the form $\{su\mid s>0\}$ $(u\neq 0)$ intersects $\mathcal{S}$ exactly once. All nonzero critical points of I are on $\mathcal{S}$. Conversely, under suitable smoothness assumptions on F, any critical point of I constrained to $\mathcal{S}$ would be a critical point of I (in the large) (see [17]). Therefore, one could work with $\mathcal{S}$ instead of $W^{1,2}(\mathbb{R}^N,\mathbb{R})$, and look for, say, a local minimum of I constrained to $\mathcal{S}$ (which may be easier than looking for a saddle point of I). There is another way to use (1.13): for any $u\in\mathcal{S}$, the ray from 0 passing through u can be used (after rescaling in θ) as a mountain-pass curve along which the maximum value of I is $I(u)$. Conversely, any mountain-pass curve $\gamma\in\Gamma$ intersects $\mathcal{S}$ at least once [6]. Therefore, one may work with points on $\mathcal{S}$ instead of paths in Γ. Since assumption (1.13) is so helpful, it is found in many papers, such as [1, 5, 20], and [18].

In the paper [17], a result similar to Theorem 1.1 was proven for the $N=1$ (ODE) case. The proof of Theorem 1.1 is similar except that a simple connectivity argument must be replaced by a degree theory argument [18]. proves a version of Theorem 1.1 under the assumption (1.13). Without 1.13, the manifold $\mathcal{S}$ must be replaced by a set with similar properties.

Define $B_1(0)=\{x\in\mathbb{R}^N\mid |x|<1\}$, and $\overline{\Omega}$ and $\partial\Omega$ to be, respectively, the topological closure and topological boundary of Ω. It is a simple consequence of the Brouwer degree [7] that for any continuous function $h:\overline{B_1(0)}\to\mathbb{R}^N$ with $h(x)=x$ for all $x\in\partial B_1(0)$, there exists $x\in B_1(0)$ with $h(x)=0$. We will need the following generalization:

Lemma 1.2. *Let* $h\in C(\overline{B_1(0)}\times[0,1],\mathbb{R}^N)$ *with, for all* $x\in\overline{B_1(0)}$ *and* $t\in[0,1]$,

(i) $h(x,0)=x=h(x,1)$.
(ii) $x\in\partial B_1(0)\Rightarrow h(x,t)=x$.

Then there exists a connected subset $C_0\subset\overline{B_1(0)}\times[0,1]$ *with* $(0,0),(0,1)\in C_0$ *and* $h(x,t)=0$ *for all* $(x,t)\in C_0$.

Using the Brouwer degree, it is clear that under the hypotheses of Lemma 1.2, for each "horizontal slice" $\overline{B_1(0)}\times\{t\}$ of the cylinder $\overline{B_1(0)}\times[0,1]$, there exists $x\in B_1(0)$ with $h(x,t)=0$. The conclusion of Lemma 1.2 does not follow from this observation. A generalization of Lemma 1.2 is known [16]: however, the reference may be difficult to find, so a proof is given here.

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1. The proof of Lemma 1.2 is deferred until Section 3.

2. Proof of Theorem 1.1

It is fairly easy to show that

$$c \leq c_0, \tag{2.1}$$

where c and c_0 are from (1.9)–(1.10): it is proven in [11] that there exists $\gamma_1 \in \Gamma_0$ with $\max_{\theta\in[0,1]} I_0(\gamma_1(\theta)) = c_0$. Define the translation operator τ as follows: for a function u on $\mathbb{R}^N$ and $a \in \mathbb{R}^N$, define let $\tau_a u$ be u shifted by a, that is, $(\tau_a u)(x) = u(x-a)$. Let $\epsilon > 0$. Let $\mathbf{e}_1 =< 1, 0, 0, \ldots, 0 >\in \mathbb{R}^N$ and define $\tau_{R\mathbf{e}_1}\gamma_1$ by $(\tau_{R\mathbf{e}_1}\gamma_1)(\theta) = \tau_{R\mathbf{e}_1}(\gamma_1(\theta))$. Then for large $R > 0$, by (f_5), $\tau_{R\mathbf{e}_1}\gamma_1 \in \Gamma$ and $\max_{\theta\in[0,1]} I((\tau_{R\mathbf{e}_1}\gamma_1)(\theta)) < c_0 + \epsilon$. Since $\epsilon > 0$ was arbitrary, $c \leq c_0$.

A Palais-Smale sequence for I is a sequence $(u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R})$ with $(I(u_m))$ convergent and $\|I'(u_m)\| \to 0$ as $m \to \infty$. It is well-known that I fails the "Palais-Smale condition". That is, a Palais-Smale sequence need not converge. However, the following proposition states that a Palais-Smale sequence "splits" into the sum of a critical point of I and translates of critical points of I_0:

Proposition 2.1. *If $(u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R})$ with $I'(u_m) \to 0$ and $I(u_m) \to a > 0$, then there exist $k \geq 0$, $v_0, v_1, \ldots, v_k \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$, and sequences $(x_m^i)_{m\geq 1}^{1\leq i\leq k} \subset \mathbb{R}^N$, such that*

(i) $I'(v_0) = 0$;
(ii) $I_0'(v_i) = 0$ *for all* $i = 1, \ldots, k$,

and along a subsequence (also denoted (u_m))

(iii) $\|u_m - (v_0 + \sum_{i=1}^k \tau_{x_m^i} v_i)\| \to 0$ *as* $m \to \infty$;
(iv) $|x_m^i| \to \infty$ *as* $m \to \infty$ *for* $i = 1, \ldots, k$;
(v) $|x_m^i - x_m^j| \to \infty$ *as* $m \to \infty$ *for all* $i \neq j$;
(vi) $I(v_0) + \sum_{i=1}^k I_0(v_i) = a$.

A proof for the case of x-periodic F is found in [6], and essentially the same proof works here. Similar propositions for nonperiodic coefficient functions, for both ODE and PDE, are found in [1, 5], and [19], for example. All are inspired by the "concentration-compactness" theorems of P.-L. Lions [12].

If $c < c_0$, then by standard deformation arguments [15], there exists a Palais-Smale sequence (u_m) with $I(u_m) \to c$. By [11], the smallest nonzero critical value of I_0 is c_0. Applying Proposition 2.1, we obtain $k = 0$, and (u_m) has a convergent subsequence, proving Theorem 1.1. So assume from now on that

$$c = c_0. \tag{2.2}$$

For $u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}$ and $i \in \{1, \ldots, N\}$, define $\mathcal{L}_i$, the ith component of the "location" of u, by

$$\int_{\mathbb{R}^N} u^2 \tan^{-1}(x_i - \mathcal{L}_i(u))\,\mathrm{d}x = 0 \tag{2.3}$$

and the "location" of u by

$$\mathcal{L}(u) = (\mathcal{L}_1(u), \ldots, \mathcal{L}_N(u)) \in \mathbb{R}^N. \tag{2.4}$$

The following lemma establishes the existence and continuity of $\mathcal{L}$.

Lemma 2.2. *$\mathcal{L}$ is well-defined and continuous on $L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}$.*

Proof. It suffices to show that $\mathcal{L}_1$ is well-defined and continuous on $L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}$. Let $u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}$. By Leibniz's Theorem, the mapping $\phi : s \mapsto \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - s)\,\mathrm{d}x$ is continuous, differentiable, and strictly decreasing, with

$$\phi'(s) = -\int_{\mathbb{R}^N} u^2(x)/((x_1 - s)^2 + 1)\,\mathrm{d}x < 0. \tag{2.5}$$

$\phi(s) \to \mp\infty$ as $s \to \pm\infty$. Therefore $\mathcal{L}_1(u)$ is unique and well-defined. Let $\epsilon > 0$ and $u_m \to u$. Now $\int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - (\mathcal{L}_1(u) + \epsilon))\,\mathrm{d}x < 0$. Since $u_m^2 \to u^2$ in $L^1(\mathbb{R}^N, \mathbb{R})$, $\int_{\mathbb{R}^N} u_m^2 \tan^{-1}(x_1 - (\mathcal{L}(u) + \epsilon))\,\mathrm{d}x < 0$ for large m, so for large m, $\mathcal{L}_1(u_m) < \mathcal{L}_1(u) + \epsilon$. Similarly, for large m, $\mathcal{L}_1(u_m) > \mathcal{L}_1(u) - \epsilon$. Since ϵ is arbitrary, $\mathcal{L}_1(u_m) \to \mathcal{L}_1(u)$. □

We are ready to begin the minimax argument. First we construct a mountain-pass curve γ_0 with some special properties:

Lemma 2.3. *There exists $\gamma_0 \in \Gamma_0$ such that for all $\theta \in [0, 1]$,*

(i) $I_0(\gamma_0(\theta)) \le c_0$.
(ii) $\theta > 0 \Rightarrow \gamma_0(\theta) \neq 0$.
(iii) $\theta \le 1/2 \Rightarrow I_0(\gamma(\theta)) \le c_0/2$.
(iv) $\theta > 0 \Rightarrow \mathcal{L}(\gamma(\theta)) = 0$.

Proof. By [10], there exists $\gamma_1 \in \Gamma_0$ with $\max_{\theta\in[0,1]} I_0(\gamma_1(\theta)) = c_0$. Assume without loss of generality that $\gamma_1(\theta) \neq 0$ for $\theta > 0$. By rescaling in θ if necessary, assume that $I_0(\gamma_1(\theta)) \le c_0/2$ for $\theta \le 1/2$. Finally, define γ_0 by $\gamma_0(0) = 0$, $\gamma_0(\theta) = \tau_{-\mathcal{L}(\gamma_1(\theta))}\gamma_1(\theta)$ for $\theta > 0$.

Assume ϵ_0 in (1.12) is small enough so that for all $x \in \mathbb{R}^N$ and $\theta \in [0, 1]$,

$$I(\tau_x(\gamma_0(\theta)) < \min(2c_0, c_0 + \alpha) \text{ and } I(\tau_x(\gamma_0(1))) < 0, \tag{2.6}$$

where α is from (1.11).

A substitute for $\mathcal{S}$

Using the mountain-pass geometry of I and the fact that Palais-Smale sequences of I are bounded in norm [6], we construct a set which has similar properties to $\mathcal{S}$, described in Section 1. Let ∇I denote the gradient of I, that is, $(\nabla I(u), w) = I'(u)w$ for all $u, w \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$. Here, $(\cdot, \cdot)$ is the usual inner product defined by $(u, w) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla w + uw\, dx$. Let $\varphi : W^{1,2}(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ be locally Lipschitz, with $I(u) \ge -1 \Rightarrow \varphi(u) = 1$ and $I(u) \le -2 \Rightarrow \varphi(u) = 0$. Let η be the solution of the initial value problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -\varphi(\eta)\nabla I(u), \quad \eta(0, u) = u. \tag{2.7}$$

In [19] it is proven that η is well-defined on $\mathbb{R}^+ \times W^{1,2}(\mathbb{R}^N)$. Let $\mathcal{B}$ be the basin of attraction of 0 under the flow η, that is,

$$\mathcal{B} = \left\{u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \mid \eta(s, u) \to 0 \text{ as } s \to \infty\right\} \tag{2.8}$$

$\mathcal{B}$ is an open neighborhood of 0 [19]. Let $\partial\mathcal{B}$ be the topological boundary of $\mathcal{B}$ in $W^{1,2}(\mathbb{R}^N, \mathbb{R})$. $\partial\mathcal{B}$ has some properties in common with $\mathcal{S}$. For example, for any $\gamma \in \Gamma$, $\gamma([0, 1])$ intersects $\partial\mathcal{B}$ at least once.

A pseudo-gradient vector field for I' may be used in place of ∇I, in which case $\mathcal{B}$ and $\partial\mathcal{B}$ would be different, but the ensuing arguments would be the same.

Let

$$c^+ = \inf\{I(u) \mid u \in \partial\mathcal{B},\ |\mathcal{L}(u)| \le 1\}. \tag{2.9}$$

The reason for the label "c^+" will become apparent in a moment. From now on, let us assume

$$I \text{ has no critical values in } (0, c_0] = (0, c]. \tag{2.10}$$

This will lead to the conclusion that I has a critical value greater than c_0.

We claim that under assumptions (2.2) and (2.10),

$$c^+ > c_0. \tag{2.11}$$

We use arguments that are sketched here and found in more detail in [19] and [5].

To prove the claim, suppose first that $c^+ < c_0$. Then there exists $u_0 \in \partial\mathcal{B}$ with $I(u_0) < c_0$. By arguments in [19], there exists a large positive constant P with

$$I(u) \leq c_0 \text{ and } \|u\| \geq 2P \Rightarrow I(\eta(s,u)) < 0 \text{ for some } s > 0, \text{ and } \|\eta(s,u)\| > P \tag{2.12}$$

for all $s > 0$. Suppose $a > 0$ and $\|I'(\eta(s_m, U_0))\| \geq a$ for some sequence (s_m) with $s_m \to \infty$. Since $u_0 \in \partial\mathcal{B}$, $\|\eta(u_0)\| < 2P$ for all $s > 0$. I'' is bounded on bounded subsets of $W^{1,2}(\mathbb{R})$, so I' is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R})$. Therefore $I(\eta(s,u_0)) < 0$ for some $s > 0$. This is impossible since $u_0 \in \partial\mathcal{B}$. Therefore $I'(\eta(s,u_0) \to 0$ as $s \to \infty$.

Define $u_n = \eta(n, u_0)$. Since $I'(u_n) \to 0$ and $u_n \in \partial\mathcal{B}$, there exists $b \in (0, c_0)$ with $I(u_n) \to b$. By [11], I_0 has no critical values between 0 and c_0. Therefore, Proposition 2.1, with $k = 0$, implies that (u_n) converges along a subsequence to a critical point w of I with $0 < I(w) < c_0$. This contradicts assumption (2.10).

Next, suppose that $c^+ = c_0$. Then there exists a sequence $(u_n) \subset \partial\mathcal{B}$ with $|\mathcal{L}(u_n)| \leq 1$ for all n and $I(u_n) \to c_0$ as $n \to \infty$. As above, $I'(u_n) \to 0$ as $n \to \infty$; to prove, suppose otherwise. Then there exist $a > 0$ and a subsequence of (u_n) (also called (u_n)) along which $\|I'(u_n)\| > a$. Since $\partial\mathcal{B}$ is forward-η-invariant [19], $\eta(1,u_n) \in \partial\mathcal{B}$ for all n. Since $(\eta(1,u_n))_{n\geq 1}$ is bounded and I' is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R}^N, \mathbb{R})$, for large n, $\eta(1,u_n) \in \partial\mathcal{B}$ with $I(\eta(1,u_n)) < c_0$. By the argument above, this implies that I has a critical value in $(0, c_0)$, contradicting assumption (2.2). Thus $I'(u_n) \to 0$ as $n \to \infty$. Applying Proposition 2.1 and using the fact that $|\mathcal{L}(u_n)| \leq 1$ for all n, (u_n) converges along a subsequence to a critical point of I, contradicting assumption (2.10). (2.11) is proven. □

Let $R > 0$ be big enough so that for all $x \in \partial B_R(0) \subset \mathbb{R}^N$ and $\theta \in [0,1]$,

$$I(\tau_x \gamma_0(\theta)) < c^+. \tag{2.13}$$

This is possible by (1.12), (2.11), and Lemma 2.3(i). Define the minimax class

$$\begin{aligned}\mathcal{H} = \{h \in & C(\overline{B_R(0)} \times [0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \\ & \text{for all } x \in \overline{B_R(0)} \text{ and } t \in [0,1], \\ & \quad t > 0 \Rightarrow h(x,t) \neq 0 \\ & \quad 0 \leq t \leq 1/2 \Rightarrow h(x,t) = \tau_x\gamma_0(t) \\ & \quad x \in \partial B_R(0) \Rightarrow h(x,t) = \tau_x\gamma_0(t) \\ & \quad h(x,1) = \tau_x\gamma_0(1)\}\end{aligned}$$

and the minimax value

$$h_0 = \inf_{h\in\mathcal{H}} \max_{(x,t)\in\overline{B_R(0)}\times[0,1]} I(h(x,t)). \tag{2.14}$$

We claim

$$c_0 < c^+ \leq h_0 < \min(2c_0, c_0 + \alpha). \tag{2.15}$$

Proof of Claim. Define $\bar{h} \in \mathcal{H}$ by $\bar{h}(x,t) = \tau_x(\gamma_0(t))$. Then $\bar{h} \in \mathcal{H}$ and by (2.6), $\max_{(x,t)\in\overline{B_R(0)}\times[0,1]} \bar{h}(x,t) < \min(2c_0, c_0 + \alpha)$. Therefore $h_0 < \min(2c_0, c_0 + \alpha)$.

Next, let $h \in \mathcal{H}$. By Lemma 1.2, and a suitable rescaling of x and t, there exists a connected set $C_2 \subset B_R(0) \times [1/2, 1]$ with $(0, 1/2), (0,1) \in C_2$ and along which for all $(x,t) \in C_2$,

$$\mathcal{L}(h(x,t)) = 0. \tag{2.16}$$

Joining C_2 with the segment $\{0\} \times [0, 1/2]$, we obtain a connected set $C_3 \subset B_R(0) \times [0,1]$ such that $(0,0), (0,1) \in C_3$ and for all $(x,t) \in C_3$, $\mathcal{L}(h(x,t)) = 0$. C_3 is not necessarily path-connected, so let $r > 0$ be small enough so that for all

$$
\begin{aligned}
(x,t) \in & N_r(C_3) \equiv \{(y,s) \in B_R(0) \times [0,1] \mid \\
& \quad \exists (x',t') \in B_R(0) \times [0,1] \text{ with } |y-x'|^2 + (s-t')^2 < r^2\}, \\
& \quad |\mathcal{L}(h(x,t))| < 1.
\end{aligned}
\tag{2.17}
$$

$N_r(C_3)$ is path-connected [21], so there exists a path $g \in C([0,1], N_r(C_3))$ with $g(0) = (0,0)$, $g(1) = (0,1)$, and $g(\theta) \in N_r(C_3)$ for all $\theta \in [0,1]$. If we define $\tilde{\gamma} \in \Gamma$ by $\tilde{\gamma}(\theta) = h(g(\theta))$, then $|\mathcal{L}(\tilde{\gamma}(\theta))| < 1$ for all $\theta \in [0,1]$. Since $\tilde{\gamma}(0) = 0$ and $I(\tilde{\gamma}(1)) < 0$, there exists $\theta^* \in [0,1]$ with $\tilde{\gamma}(\theta^*) \in \partial\mathcal{B}$. By the definition of c^+ (2.9), $I(\tilde{\gamma}(\theta^*)) \geq c^+$.

Since h was an arbitrary element of $\mathcal{H}$, $h_0 \geq c^+$.

By standard deformation arguments, such as described in [15], there exists a Palais-Smale sequence $(u_n) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R})$ with $I'(u_n) \to 0$ and $I(u_n) \to h_0$ as $n \to \infty$. $c_0 < h_0 < \min(2c_0, c_0 + \alpha)$. Apply Proposition 2.1 to (u_n). Since I_0 has no positive critical values smaller than c_0 [11], $k \leq 1$. By (2.10), (u_n) converges along a subsequence to a critical point z of I, with $I(z) = h_0$. Theorem 1.1 is proven.

3. A degree-theoretic lemma

Here, we prove Lemma 1.2. Let h be as in the hypotheses of the lemma. For $l > 0$, define $\mathcal{A}_l \subset \overline{B_1(0)} \times [0,1]$ by

$$\mathcal{A}_l = \{(x,t) \in \overline{B_1(0)} \times [0,1] \mid |f(x,t)| < l\}. \tag{3.1}$$

$\mathcal{A}_l$ is an open neighborhood of $(0,0)$. Let C_l be the component of $\mathcal{A}_l$ containing $(0,0)$. We will prove the following claim:

$$\text{For all } \epsilon > 0,\ (0,1) \in C_\epsilon. \tag{3.2}$$

Then we will use the C_ϵ's to construct C_0. For $l > 0$ and $t \in [0,1]$, define

$$C_l^t = \{x \in \overline{B_1(0)} \mid (x,t) \in C_l\}. \tag{3.3}$$

Fix $\epsilon \in (0,1)$. Define $\phi : [0,1] \to \mathbb{Z}$ by

$$\phi(t) = d(h(\cdot,t), C_\epsilon^t, 0), \tag{3.4}$$

where d is the topological Brouwer degree [7]. We will prove $\phi(t) = 1$ for all $t \in [0,1]$, in particular $\phi(1) = 1$, so (3.2) is satisfied.

f is continuous on a compact domain, so f is uniformly continuous. Let $\rho > 0$ be small enough so that for all $x \in \overline{B_1(0)}$ and $t_1, t_2 \in [0,1]$,

$$|t_1 - t_2| < \rho \Rightarrow |h(x,t_1) - h(x,t_2)| < \epsilon/4. \tag{3.5}$$

Clearly

$$\phi(0) = d(id, B_\epsilon(0), 0) = 1. \tag{3.6}$$

Let $0 \leq t_1 < t_2 \leq 1$ with $t_2 - t_1 < \rho$. We will show $\phi(t_1) = \phi(t_2)$, proving that ϕ is constant, which by (3.6), implies (3.2).

Ω is nonempty. For all $x \in \partial C_\epsilon^{t_1}$, $|h(x,t_1)| = \epsilon$, so by (3.5),

$$x \in \partial C_\epsilon^{t_1} \Rightarrow |h(x,t_1)| \geq \frac{3}{4}\epsilon. \tag{3.7}$$

By the additivity property of d [7],

$$
\begin{aligned}
\phi(t_2) &\equiv d(f(\cdot,t_2), C_\epsilon^{t_2}, 0) \\
&= d(f(\cdot,t_2), C_\epsilon^{t_2} \setminus \overline{C_\epsilon^{t_1}}, 0) + d(f(\cdot,t_2), C_\epsilon^{t_1} \cap C_\epsilon^{t_2}, 0).
\end{aligned}
\tag{3.8}
$$

We will show:

$$\text{There does not exist } x \in C_\epsilon^{t_2} \setminus \overline{C_\epsilon^{t_1}} \text{ with } h(x,t_2)=0. \tag{3.9}$$

Suppose such an x exists. Then by (3.5), $|h| < \epsilon/4$ on the segment $\{x\} \times [t_1, t_2]$. $x \in C_\epsilon^{t_2}$, so $(x,t_2) \in C_\epsilon$, and by the definition of C_ϵ, $(x,t_1) \in C_\epsilon$, and $x \in C_\epsilon^{t_1}$, contradicting $x \in C_\epsilon^{t_2} \setminus \overline{C_\epsilon^{t_1}}$. So (3.9) is true. Therefore by (3.8),

$$\phi(t_2) = d(f(\cdot,t_2), C_\epsilon^{t_1} \cap C_\epsilon^{t_2}, 0). \tag{3.10}$$

By the same argument, switching the roles of t_1 and t_2,

$$\phi(t_1) = d(f(\cdot,t_1), C_\epsilon^{t_1} \cap C_\epsilon^{t_2}, 0). \tag{3.11}$$

For all $t \in [t_1, t_2]$ and $x \in \partial C_\epsilon^{t_1} \cup \partial C_\epsilon^{t_2}$, (3.5) gives $|h(x,t_1)| > 3\epsilon/4$ and $|h(x,t) - h(x,t_1)| < \epsilon/4$. Therefore by the homotopy invariance property of the degree [7],

$$\begin{aligned} \phi(t_1) &= d(f(\cdot,t_1), C_\epsilon^{t_1} \cap C_\epsilon^{t_2}, 0) \\ &= d(f(\cdot,t_2), C_\epsilon^{t_1} \cap C_\epsilon^{t_2}, 0) = \phi(t_2). \end{aligned} \tag{3.12}$$

$\phi(0) = 1$ and $\phi(t_1) = \phi(t_2)$ for any $t_1 < t_2$ with $t_1, t_2 \in [0,1]$ and $t_2 - t_1 < \rho$. Therefore ϕ is constant, and $\phi(1) = 1$. Therefore $(0,1) \in C_\epsilon$.

Now let

$$C_0 = \bigcap_{\epsilon > 0} C_\epsilon. \tag{3.13}$$

Each C_ϵ is a connected set containing $(0,0)$ and $(0,1)$, so it is easy to show that C_0 is a connected set containing $(0,0)$ and $(0,1)$, and clearly for all $(x,t) \in C_0$, $h(x,t) = 0$.

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