

ON A VARIATIONAL PROBLEM ARISING IN CRYSTALLOGRAPHY

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Abstract. We study a variational problem which was introduced by Hannon, Marcus and Mizel [ESAIM: COCV **9** (2003) 145–149] to describe step-terraces on surfaces of so-called “unorthodox” crystals. We show that there is no nondegenerate intervals on which the absolute value of a minimizer is $\pi/2$ identically.

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1. INTRODUCTION

For the understanding of crystalline growth processes, the form of step-terraces on the crystalline surface plays an important role [5]. The edges of these steps usually form oscillations in space that become larger when the equilibrium temperature rises. This behavior is called “orthodox” and had been explained by Herring, Mullins and others (see *e.g.* [6]) by thermodynamical effects. The classical model is given by

$$J_1(y) = \int_0^S \beta(\theta) ds$$

where s is arclength and y is a function defined on a fixed interval $[0, L]$ whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L), \theta = \arctan y' \in [-\pi/2, \pi/2],$$

while β is a positive π -periodic function which satisfies certain properties. Minimization of J_1 subject to appropriate boundary data is a parametric variational problem. It is closely related to the variational problem defining the Wulff crystal shape as that shape for a domain of prescribed area such that the boundary integral with respect to arclength involving the integrand in J_1 [referred to as the surface tension] attains its minimum value [1, 2]. Recently crystals have been studied which are “unorthodox” in the sense that lower temperatures lead to larger oscillations and the step profile takes a saw-tooth structure for low temperatures and not a straight line as the classical theory would predict [3]. To describe this situation, Hannon, Marcus and Mizel [4] suggested a refined model which will be stated below.

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Suppose that a function $\beta \in C(R)$ satisfies the following assumption:
(A)

$$\beta(t) > 0 \text{ for all } t \in R, \tag{1.1}$$

$$\beta(t) = \beta(-t) \text{ for all } t \in R, \tag{1.2}$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for all } t \in R, \tag{1.3}$$

$$\beta(t + \pi) = \beta(t) \text{ for all } t \in R. \tag{1.4}$$

Let $L > 0, \rho > 0, \sigma > 0$. We study the following variational problem introduced in [4]:

$$J_{\rho\sigma}^L(\theta, y) := \int_0^S [\rho(\theta'(s))^2 + \beta(\theta(s)) + \sigma y(s)^2] ds \rightarrow \inf, \tag{1.5}$$

$$y(s) = y(0) + \int_0^s \sin(\theta(\tau)) d\tau, \quad s \in [0, S], \tag{1.6}$$

where $S \geq L, \theta \in W^{1,2}(0, S)$ is subject to the following constraints:

$$\theta(\tau) \in [-\pi/2, \pi/2], \quad \tau \in [0, S], \quad \int_0^S \cos(\theta(\tau)) d\tau = L, \tag{1.7}$$

$$\int_0^S \sin(\theta(\tau)) d\tau = 0.$$

Here θ describes the angle of the step profile relative to a straight line profile. The first constraint in (1.7) expresses the condition that the curve

$$(x(s), y(s)) := \left(\int_0^s \cos(\theta(\tau)) d\tau, y(0) + \int_0^s \sin(\theta(\tau)) d\tau \right)$$

does not “reverse”, the second is the condition that the x -interval is $[0, L]$, while the third condition in (1.7) is the condition that $y(0) = y(S)$.

It was shown in [4] that problem (1.5)–(1.7) has a solution. Actually in [4] it was assumed that $\beta \in C^2(R)$ and that $\beta(0) + \beta''(0) < 0$ but the existence result of [4] holds without these two additional assumptions and with the same proof. Hannon, Marcus and Mizel [4] noted that their theorem does not exclude the possibility that a minimizer (S, θ, y) satisfies $|\theta| = \pi/2$ on one or more nondegenerate intervals. If this occurs, then the locus of the curve $s \rightarrow (x(s), y(s)), s \in [0, S]$ is not the graph of a function defined on $[0, L]$. This fact leads to difficulties in calculating a solution.

Our main result stated below establishes that if a parameter σ is small enough, then the locus of the curve $s \rightarrow (x(s), y(s)), s \in [0, S]$ associated with a minimizer (S, θ, y) is necessarily a graph of a function defined on $[0, L]$. It should be mentioned that the smallness of σ is a natural assumption for the model.

Theorem 1.1. *Let $\rho_1, L_1 > 0$. Then there is $\sigma_1 > 0$ such that for each $\rho \geq \rho_1$, each $L \in (0, L_1]$ and each $\sigma \in (0, \sigma_1]$ the following assertion holds:*

Assume that (S, θ, y) is a solution of the problem (1.5)–(1.7). Then there is no interval $[a, b] \subset [0, S]$ such that $a < b$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

The proof of Theorem 1.1 is long and technical. It is based on a number of auxiliary results. Here we explain the main ideas of the proof.

In the proof of Theorem 1.1 we use two procedures applied to triples (S, θ, y) : a reduction of a triplet and a restriction of a triplet.

Let $S \geq L, \theta \in W^{1,2}(0, S)$ and $y : [0, S] \rightarrow R$ satisfy (1.6) and (1.7).

Assume that $t_0 \in [0, S]$ and $\delta > 0$. An extension of the triplet (S, θ, y) is a triplet $(\tilde{S}, \tilde{\theta}, \tilde{y})$ defined by

$$\tilde{S} = S + \delta, \quad \tilde{\theta}(t) = \theta(t), \quad t \in [0, t_0], \quad \tilde{\theta}(t) = \theta(t_0), \quad t \in (t_0, t_0 + \delta],$$

$$\tilde{\theta}(t) = \theta(t - \delta), \quad t \in (t_0 + \delta, \tilde{S}], \quad \tilde{y}(\tau) = y(0) + \int_0^\tau \sin(\tilde{\theta}(t)) dt, \quad \tau \in [0, \tilde{S}].$$

Let us now describe a reduction of the triplet (S, θ, y) . Assume that $\Delta = [a, b] \subset [0, S]$ and that one of the following cases holds:

(1) $a = 0$; (2) $b = S$; (3) $0 < a, b < S$ and $\theta(a) = \theta(b)$.

Put $\tilde{S} = S - b + a$. In the case (1) set $\tilde{\theta}(t) = \theta(t + b - a)$, $t \in [0, \tilde{S}]$, in the case (2) set $\tilde{\theta}(t) = \theta(t)$, $t \in [0, \tilde{S}]$ and in the case (3) set $\tilde{\theta}(t) = \theta(t)$, $t \in [0, a]$, $\tilde{\theta}(t) = \theta(t + b - a)$, $t \in (a, \tilde{S}]$. Finally we define

$$\tilde{y}(\tau) = y(0) + \int_0^\tau \sin(\tilde{\theta}(t)) dt, \quad \tau \in [0, \tilde{S}].$$

The triplet $(\tilde{S}, \tilde{\theta}, \tilde{y})$ is called a reduction of the triplet (S, θ, y) .

We prove Theorem 1.1 by negation. We assume that a triplet (S_0, θ_0, y_0) is a solution of the variational problem and that $|\theta_0(t)| = \pi/2$ for all t belonging to a subinterval of $[0, S_0]$ with a positive length. Using the extension of triples and the reduction of triples we will construct a new admissible triplet (S_1, θ_1, y_1) such that $J_{\rho\sigma}^L(\theta_1, y_1) < J_{\rho\sigma}^L(\theta_0, y_0)$. In order to meet this goal we will choose a small positive constant r_0 and consider separately two cases:

$$\inf\{\theta_0(t) : t \in [0, S_0]\} \leq -\pi/2 + r_0 \quad \text{and} \quad \inf\{\theta_0(t) : t \in [0, S_0]\} > -\pi/2 + r_0.$$

2. AUXILIARY RESULTS

For each function $f : X \rightarrow R$, where X is nonempty, set $\inf(f) = \inf\{f(x) : x \in X\}$. Denote by $\text{meas}(E)$ the Lebesgue measure of a Lebesgue measurable set $E \subset R$.

Lemma 2.1. *Let $L, \rho, \sigma > 0$. Then $\inf(J_{\rho\sigma}^L) \leq L\beta(0)$.*

Proof. Set $S = L$, $\theta(t) = 0$, $t \in [0, S]$ and $y(t) = 0$, $t \in [0, S]$. Clearly (S, θ, y) satisfies (1.6) and (1.7). Then $\inf(J_{\rho\sigma}^L) \leq J_{\rho\sigma}^L(\theta, y) = L\beta(0)$. Lemma 2.1 is proved. \square

Lemma 2.2. *Let $L, \rho, \sigma > 0$ and let $S \geq L$, $\theta \in W^{1,2}(0, S)$, $y : [0, S] \rightarrow R$ satisfy (1.6) and (1.7). Then*

$$S \leq [L\beta(0) + J_{\rho\sigma}^L(\theta, y) - \inf(J_{\rho\sigma}^L)](\beta(\pi/2))^{-1}. \quad (2.1)$$

Proof. It follows from (1.5), (1.3) and Lemma 2.1 that

$$\begin{aligned} S\beta(\pi/2) &\leq J_{\rho\sigma}^L(\theta, y) = \inf(J_{\rho\sigma}^L) + [J_{\rho\sigma}^L(\theta, y) - \inf(J_{\rho\sigma}^L)] \\ &\leq L\beta(0) + [J_{\rho\sigma}^L(\theta, y) - \inf(J_{\rho\sigma}^L)]. \end{aligned}$$

This inequality implies (2.1). \square

Corollary 2.1. *Let $L, \rho, \sigma > 0$ and let (S, θ, y) be a solution of the problem (1.5)–(1.7). Then $S \leq L\beta(0)(\beta(\pi/2))^{-1}$.*

Corollary 2.2. *Let $L, \rho, \sigma > 0$ and let $S \geq L$, $\theta \in W^{1,2}(0, S)$, $y : [0, S]$ satisfy (1.6) and (1.7). Assume that $J_{\rho\sigma}^L(\theta, y) \leq \inf(J_{\rho\sigma}^L) + L\beta(0)$. Then*

$$S \leq 2L\beta(0)(\beta(\pi/2))^{-1}.$$

Lemma 2.3. *Let $L, \rho, \sigma > 0$ and let $S \geq L$, $\theta \in W^{1,2}(0, S)$, $y : [0, S] \rightarrow R$ satisfy (1.6) and (1.7). Assume that*

$$J_{\rho\sigma}^L(\theta, y) \leq \inf(J_{\rho\sigma}^L) + \min\{L\beta(0), \sigma L^3\}. \quad (2.2)$$

Then

$$|y(t)| \leq 8L\beta(0)(\beta(\pi/2))^{-1} \text{ for all } t \in [0, S]. \quad (2.3)$$

Proof. By (2.2) and Corollary 2.2

$$S \leq 2L\beta(0)(\beta(\pi/2))^{-1}. \quad (2.4)$$

Relations (1.6) and (2.4) imply that for each $t \in [0, S]$

$$|y(t) - y(0)| \leq t \leq S \leq 2L\beta(0)(\beta(\pi/2))^{-1}. \quad (2.5)$$

Set

$$S_1 = S, \theta_1 = \theta, y_1(\tau) = \int_0^\tau \sin(\theta(t))dt, \tau \in [0, S]. \quad (2.6)$$

Clearly

$$|y_1(t)| \leq t \text{ for all } t \in [0, S] \quad (2.7)$$

and (1.6), (1.7) hold with $(S, \theta, y) = (S_1, \theta_1, y_1)$. It follows from (2.2), (2.6) and (1.5) that

$$J_{\rho\sigma}^L(\theta, y) \leq J_{\rho\sigma}^L(\theta_1, y_1) + \sigma L^3 = J_{\rho\sigma}^L(\theta, y) + \sigma \left[\int_0^S (y_1(t))^2 dt - \int_0^S (y(t))^2 dt \right] + \sigma L^3$$

and

$$\int_0^S (y(t))^2 dt \leq \int_0^S (y_1(t))^2 dt + L^3.$$

Combined with (2.7) this implies that

$$\int_0^S (y(t))^2 dt \leq L^3 + S^3 \leq 2S^3. \quad (2.8)$$

We show that $|y(0)| \leq 3S$. Let us assume the converse. Then $|y(0)| > 3S$ and by (2.5) $|y(t)| > 2S$ for all $t \in [0, S]$. This inequality implies that $\int_0^S (y(t))^2 dt \geq 4S^3$. This inequality contradicts (2.8). The contradiction we have reached proves that $|y(0)| \leq 3S$. Combined with (2.5) and (2.4) this inequality implies that for all $t \in [0, S]$

$$|y(t)| \leq |y(0)| + S \leq 4S \leq 8L\beta(0)(\beta(\pi/2))^{-1}.$$

Lemma 2.3 is proved. □

Lemma 2.4. *Let $L, \rho, \sigma > 0$, $S \geq L$, $\theta \in W^{1,2}(0, S)$, $y : [0, S] \rightarrow R$ satisfy (1.6) and (1.7). Suppose that (S, θ, y) is a solution of the problem (1.5)–(1.7). Then*

$$\int_0^S (\theta'(t))^2 dt \leq \rho^{-1} L\beta(0) \quad (2.9)$$

and for each $t_1, t_2 \in [0, S]$ satisfying $t_1 < t_2$ the following inequality holds:

$$|\theta(t_2) - \theta(t_1)| \leq (\rho^{-1} L\beta(0)(t_2 - t_1))^{1/2}. \quad (2.10)$$

Proof. By Corollary 2.1 $L \leq S \leq L\beta(0)\beta(\pi/2)^{-1}$. It follows from (1.5) and Lemma 2.1 that

$$\int_0^S (\theta'(t))^2 dt \leq \rho^{-1} J_{\rho\sigma}^L(\theta, y) = \rho^{-1} \inf(J_{\rho\sigma}^L) \leq \rho^{-1} L\beta(0).$$

Inequality (2.10) follows from (2.9) by the Cauchy-Schwarz inequality. \square

Lemma 2.5. *Let $L_1 > 0$, $\rho > 0$ and let a positive number γ satisfy*

$$\gamma \leq \arcsin(2^{-1}(\beta(0))^{-1}\beta(\pi/2) \min\{1, (\pi^2/16)\rho L_1^{-2}(\beta(0))^{-1}\}). \quad (2.11)$$

Suppose that $\sigma > 0$, $L \in (0, L_1]$ and that $S \geq L$, $\theta \in W^{1,2}(0, S)$ and $y : [0, S] \rightarrow R$ are a solution of the problem (1.5)–(1.7) such that

$$\max\{\theta(t) : t \in [0, S]\} = \pi/2. \quad (2.12)$$

Then

$$\min\{\theta(t) : t \in [0, S]\} \leq -\gamma. \quad (2.13)$$

Proof. By (2.12) there is $t_0 \in [0, S]$ such that

$$\theta(t_0) = \pi/2. \quad (2.14)$$

Set

$$E = [0, S] \cap [t_0 - (\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, t_0 + (\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}]. \quad (2.15)$$

Assume that $t \in E$. By (2.15) and Lemma 2.4

$$|\theta(t) - \theta(t_0)| \leq (\rho^{-1} L\beta(0)|t - t_0|)^{1/2} \leq \pi/4.$$

Combined with (2.14) this inequality implies that $\theta(t) \geq \pi/4$. Thus we have shown that

$$\theta(t) \geq \pi/4 \text{ for all } t \in E. \quad (2.16)$$

Clearly

$$\text{meas}(E) \geq \min\{(\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, S\}. \quad (2.17)$$

Relations (2.15), (2.16), (2.17) and (1.7) imply that

$$\int_E \sin(\theta(t)) dt \geq \sin(\pi/4) \text{meas}(E) \geq 2^{-1} \min\{(\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, S\}. \quad (2.18)$$

By (2.18) and (1.7)

$$\int_{[0, S] \setminus E} \sin(\theta(t)) dt = - \int_E \sin(\theta(t)) dt \leq -2^{-1} \min\{(\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, S\}. \quad (2.19)$$

Since $\inf(\theta) \leq 0$ (see (1.7)) the relation (2.19) implies that

$$\begin{aligned} -2^{-1} \min\{(\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, S\} &\geq \int_{[0, S] \setminus E} \sin(\theta(t)) dt \\ &\geq \sin(\inf(\theta)) \text{meas}([0, S] \setminus E) \geq \sin(\inf(\theta)) S. \end{aligned} \quad (2.20)$$

It follows from (2.20), Corollary 2.1 and the inequality $L \leq L_1$ that

$$\begin{aligned} \sin(\inf(\theta)) &\leq -2^{-1} \min\{(\pi/4)^2 \rho L^{-1}(\beta(0))^{-1}, L\} L^{-1}(\beta(0))^{-1} \beta(\pi/2) \\ &\leq -2^{-1} (\beta(0))^{-1} \beta(\pi/2) \min\{1, (\pi/4)^2 \rho L_1^{-2}(\beta(0))^{-1}\} \end{aligned}$$

and in view of (2.11)

$$\inf(\theta) \leq -\arcsin(2^{-1}(\beta(0))^{-1} \beta(\pi/2) \min\{1, (\pi/4)^2 \rho L_1^{-2}(\beta(0))^{-1}\}) \leq -\gamma.$$

This completes the proof of Lemma 2.5. \square

It is easy to see that the following lemma holds.

Lemma 2.6. *Let $S > 0$, $\theta : [0, S] \rightarrow R$ be a continuous function, $M > 0$, $\delta \in (0, S)$ and let $y : [0, S] \rightarrow R$ satisfy*

$$y(\tau) = y(0) + \int_0^\tau \sin(\theta(t)) dt, \quad \tau \in [0, S] \quad (2.21)$$

and

$$|y(\tau)| \leq M \text{ for all } \tau \in [0, S]. \quad (2.22)$$

Suppose that

$$\tilde{S} = S - \delta, \quad \tilde{\theta}(t) = \theta(t + \delta), \quad t \in [0, \tilde{S}], \quad (2.23)$$

$$\tilde{y}(\tau) = y(0) + \int_0^\tau \sin(\tilde{\theta}(t)) dt, \quad \tau \in [0, \tilde{S}]. \quad (2.24)$$

Then

$$|\tilde{y}(t)| \leq M + \delta \text{ for all } \tau \in [0, \tilde{S}] \quad (2.25)$$

and

$$\left| \int_0^S (y(t))^2 dt - \int_0^{\tilde{S}} (\tilde{y}(t))^2 dt \right| \leq M^2 \delta + \delta S (2M + \delta). \quad (2.26)$$

Lemma 2.7. *Let $S > 0$, $\theta : [0, S] \rightarrow R$ be a continuous function, $M > 0$, $0 < a < b < S$ and let $y : [0, S] \rightarrow R$ satisfy (2.21) and (2.22). Suppose that*

$$\theta(a) = \theta(b), \quad (2.27)$$

$$\tilde{S} = S - b + a, \quad (2.28)$$

$$\tilde{\theta}(t) = \theta(t), \quad t \in [0, a], \quad \tilde{\theta}(t) = \theta(t + b - a), \quad t \in (a, \tilde{S}], \quad (2.29)$$

$$\tilde{y}(\tau) = y(0) + \int_0^\tau \sin(\tilde{\theta}(t)) dt, \quad \tau \in [0, \tilde{S}]. \quad (2.30)$$

Then

$$|\tilde{y}(\tau)| \leq M + b - a \text{ for all } \tau \in [0, \tilde{S}] \quad (2.31)$$

and

$$\left| \int_0^S (y(t))^2 dt - \int_0^{\tilde{S}} (\tilde{y}(t))^2 dt \right| \leq (b - a)M^2 + (b - a)(2M + b - a)S. \quad (2.32)$$

Proof. Relations (2.21), (2.29), (2.30) imply that

$$\tilde{y}(t) = y(t), \quad t \in [0, a]. \quad (2.33)$$

Assume that $\tau \in (a, \tilde{S})$. It follows from (2.30), (2.29) and (2.21) that

$$\begin{aligned} \tilde{y}(\tau) &= y(0) + \int_0^a \sin(\theta(t))dt + \int_a^\tau \sin(\theta(t+b-a))dt \\ &= y(0) + \int_0^{\tau+b-a} \sin(\theta(t))dt - \int_a^b \sin(\theta(t))dt \\ &= y(\tau+b-a) - \int_a^b \sin(\theta(t))dt. \end{aligned}$$

This equality implies that

$$|\tilde{y}(\tau) - y(\tau+b-a)| \leq b-a \quad \text{for all } \tau \in (a, \tilde{S}). \quad (2.34)$$

Combined with (2.22) the inequality (2.34) implies that

$$|\tilde{y}(\tau)| \leq M + b - a \quad \text{for all } \tau \in (a, \tilde{S}). \quad (2.35)$$

Relations (2.22), (2.33), (2.35) imply (2.31). It follows from (2.33), (2.22), (2.28), (2.31) that

$$\begin{aligned} &\left| \int_0^S (y(t))^2 dt - \int_0^{\tilde{S}} (\tilde{y}(t))^2 dt \right| \\ &\leq \int_a^b (y(t))^2 dt + \left| \int_a^{\tilde{S}} (y(t+b-a))^2 dt - \int_a^{\tilde{S}} (\tilde{y}(t))^2 dt \right| \\ &\leq (b-a)M^2 + \int_a^{\tilde{S}} |y(t+b-a) - \tilde{y}(t)| (|y(t+b-a)| + |\tilde{y}(t)|) dt \\ &\leq (b-a)M^2 + (b-a)S(2M+b-a). \end{aligned}$$

Thus (2.32) is true and Lemma 2.7 is proved. \square

The following auxiliary result is proved analogously to Lemma 2.7.

Lemma 2.8. *Let $S > 0$, $\theta : [0, S] \rightarrow R$ be a continuous function, $M > 0$, $0 \leq a \leq S$, $\delta > 0$ and let $y : [0, S] \rightarrow R$ satisfy (2.21) and (2.22). Suppose that*

$$\begin{aligned} \tilde{S} &= S + \delta, \\ \tilde{\theta}(t) &= \theta(t), \quad t \in [0, a], \quad \tilde{\theta}(t) = \theta(a), \quad t \in (a, a + \delta], \\ \tilde{\theta}(t) &= \theta(t - \delta), \quad t \in (a + \delta, \tilde{S}], \\ \tilde{y}(\tau) &= y(0) + \int_0^\tau \sin(\tilde{\theta}(t))dt, \quad \tau \in [0, \tilde{S}]. \end{aligned}$$

Then

$$|\tilde{y}(\tau)| \leq M + \delta \quad \text{for all } \tau \in [0, \tilde{S}]$$

and

$$\left| \int_0^S (y(t))^2 dt - \int_0^{\tilde{S}} (\tilde{y}(t))^2 dt \right| \leq \delta(M + \delta)^2 + \delta S(2M + \delta).$$

3. A WEAKENED VERSION OF THEOREM 1.1

In this section we establish the following result.

Theorem 3.1. *There exists $r_0 \in (0, \pi/8)$ such that for each $L_0 > 0$ there is $\sigma_0 > 0$ for which the following assertion holds:*

Suppose that $L \in (0, L_0]$, $\rho > 0$, $\sigma \in (0, \sigma_0]$, (S, θ, y) is a solution of the problem (1.5)–(1.7) and

$$[-\pi/2 + r_0, \pi/2 - r_0] \subset \theta([0, S]). \quad (3.1)$$

Then there is no interval $[a, b] \subset [0, S]$ such that $a < b$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

Proof. Choose a positive number r_1 such that

$$r_1 \in (0, \pi/16), \quad \beta(0) \cos(\pi/2 - r_1) \leq \beta(\pi/2)/16 \quad (3.2)$$

and choose

$$r_0 \in (0, r_1/2). \quad (3.3)$$

Let $L_0 > 0$. Put

$$\Delta_0 = 16L_0\beta(0)(\beta(\pi/2))^{-1} \quad (3.4)$$

and choose a positive number σ_0 such that

$$\sigma_0\Delta_0^2 < \beta(\pi/2)10^{-2}9^{-1}. \quad (3.5)$$

Let

$$L \in (0, L_0], \quad \rho > 0, \quad \sigma \in (0, \sigma_0]. \quad (3.6)$$

Suppose that $S \geq L$, $\theta \in W^{1,2}(0, S)$ satisfies (3.1) and (1.7), $y : [0, S] \rightarrow R$ satisfies (1.6) and

$$\int_0^S [\rho(\theta'(t))^2 + \beta(\theta(t)) + \sigma(y(t))^2] dt = J_{\rho\sigma}^L(\theta, y) = \inf(J_{\rho\sigma}^L). \quad (3.7)$$

In order to prove Theorem 3.1 it is sufficient to show that there is no interval $[a, b] \subset [0, S]$ such that $a < b$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

Let us assume the converse. Then there is an interval $[a, b] \subset [0, S]$ such that $0 < a < b < S$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$. We may assume without loss of generality that

$$\theta(t) = \pi/2 \text{ for all } t \in [a, b]. \quad (3.8)$$

There is $\tau_0 \in [0, S]$ such that

$$\theta(\tau_0) = \inf\{\theta(t) : t \in [0, S]\}. \quad (3.9)$$

By (3.9) and (3.1)

$$\theta(\tau_0) \leq -\pi/2 + r_0. \quad (3.10)$$

Corollary 2.1 implies that

$$L \leq S \leq L\beta(0)(\beta(\pi/2))^{-1}. \quad (3.11)$$

By Lemma 2.3, (3.6) and (3.4)

$$|y(t)| \leq \Delta_0 \text{ for all } t \in [0, S]. \quad (3.12)$$

By continuity it follows from (3.10) and (3.3) that there is a positive number δ_1 such that

$$\delta_1 < (b - a)/16, \quad (3.13)$$

$$\theta(t) < -\pi/2 + r_1 \text{ for all } t \in [0, S] \cap [\tau_0 - 2\delta_1, \tau_0 + 2\delta_1]. \quad (3.14)$$

It follows from (3.13) that

$$\delta_1 \leq \text{meas}([0, S] \cap [\tau_0 - \delta_1, \tau_0 + \delta_1]) \leq 2\delta_1. \quad (3.15)$$

There are three cases: (1) $\tau_0 \leq \delta_1$; (2) $\tau_0 \geq S - \delta_1 > 4\delta_1$ (see (3.13)); (3) $\delta_1 < \tau_0 < S - \delta_1$.

In the case (1) set

$$\tilde{a} = 0, \tilde{b} = \tau_0 + \delta_1, S_1 = S - \tilde{b} + \tilde{a}, \quad (3.16)$$

$$\theta_1(t) = \theta(t + \tilde{b} - \tilde{a}), t \in [0, S_1]. \quad (3.17)$$

In the case (2) put

$$\tilde{b} = S, \tilde{a} = \tau_0 - \delta_1, S_1 = S - \tilde{b} + \tilde{a}, \quad (3.18)$$

$$\theta_1(t) = \theta(t), t \in [0, S_1]. \quad (3.19)$$

Consider the case (3). Since θ is continuous and τ_0 satisfies (3.9), there exists a closed interval $[\tilde{a}, \tilde{b}] \subset [0, S]$ such that

$$\delta_1 \leq \tilde{b} - \tilde{a} \leq 2\delta_1, \tau_0 \in [\tilde{a}, \tilde{b}] \subset [\tau_0 - \delta_1, \tau_0 + \delta_1], \quad (3.20)$$

$$\theta(\tilde{a}) = \theta(\tilde{b}).$$

We set

$$S_1 = S - \tilde{b} + \tilde{a}, \quad (3.21)$$

$$\theta_1(t) = \theta(t), t \in [0, \tilde{a}], \theta_1(t) = \theta(t + \tilde{b} - \tilde{a}), t \in (\tilde{a}, S_1]. \quad (3.22)$$

It is not difficult to see that in the all three cases $\theta_1 \in W^{1,2}(0, S_1)$,

$$\delta_1 \leq \tilde{b} - \tilde{a} \leq 2\delta_1, \quad (3.23)$$

$$\tau_0 \in [\tilde{a}, \tilde{b}]. \quad (3.24)$$

Relations (3.23), (3.24) and (3.14) imply that

$$\theta(t) \leq -\pi/2 + r_1 \text{ for all } t \in [\tilde{a}, \tilde{b}]. \quad (3.25)$$

Clearly one of the following conditions holds:

$$\tilde{a} = 0; \tilde{b} = S; \tilde{a} > 0, \tilde{b} < S \text{ and } \theta(\tilde{a}) = \theta(\tilde{b}). \quad (3.26)$$

Define $y_1 : [0, S_1] \rightarrow R$ by

$$y_1(\tau) = y(0) + \int_0^\tau \sin(\theta_1(t))dt, t \in [0, S_1]. \quad (3.27)$$

It follows from the definition of y_1 (see (3.27)), θ_1 (see (3.17), (3.19), (3.22)), Lemmas 2.6 and 2.7, (3.12), (3.11), (3.23), (3.6) and (3.4) that

$$\begin{aligned} \left| \int_0^S (y(t))^2 dt - \int_0^{S_1} (y_1(t))^2 dt \right| &\leq (\tilde{b} - \tilde{a})\Delta_0^2 + (\tilde{b} - \tilde{a})(2\Delta_0 + \tilde{b} - \tilde{a})S \\ &\leq (\tilde{b} - \tilde{a})[\Delta_0^2 + (2\Delta_0 + \tilde{b} - \tilde{a})L_0\beta(0)(\beta(\pi/2))^{-1}] \\ &\leq 2\delta_1[\Delta_0^2 + (2\Delta_0 + 2\delta_1)\Delta_0/16]. \end{aligned}$$

Combined with (3.13), (3.11), (3.6) and (3.4) this inequality implies that

$$\begin{aligned} \left| \int_0^S (y(t))^2 dt - \int_0^{S_1} (y_1(t))^2 dt \right| &\leq 2\delta_1[\Delta_0^2 + 16^{-1}\Delta_0(2\Delta_0 + S)] \\ &\leq 2\delta_1[\Delta_0^2 + 16^{-1}\Delta_0(2\Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1})] \\ &\leq 2\delta_1[\Delta_0^2 + 16^{-1}\Delta_0(3\Delta_0)] \leq 3\delta_1\Delta_0^2. \end{aligned}$$

Therefore we have shown that

$$\left| \int_0^S (y(t))^2 dt - \int_0^{S_1} (y_1(t))^2 dt \right| \leq 3\delta_1 \Delta_0^2. \quad (3.28)$$

It follows from the definition of θ_1 (see (3.17), (3.19), (3.22)), (3.25) and (3.8) that there are numbers a_1, b_1 such that

$$a_1, b_1 \in [0, S_1], \quad a_1 < b_1 < S_1, \quad b_1 - a_1 = b - a, \quad (3.29)$$

$$\theta_1(t) = \pi/2 \text{ for all } t \in [a_1, b_1]. \quad (3.30)$$

Set

$$\delta_2 = - \int_{\tilde{a}}^{\tilde{b}} \sin(\theta(t)) dt. \quad (3.31)$$

Relations (3.31) and (3.23) imply that

$$\delta_2 \leq \tilde{b} - \tilde{a} \leq 2\delta_1. \quad (3.32)$$

By (3.25) and (3.2) for each $t \in [\tilde{a}, \tilde{b}]$

$$-\sin \theta(t) \geq \sin(\pi/2 - r_1) \geq \sin(\pi/2 - \pi/16) \geq 1/2. \quad (3.33)$$

In view of (3.33), (3.31) and (3.23)

$$\delta_2 \geq 2^{-1}(\tilde{b} - \tilde{a}) \geq \delta_1/2.$$

Together with (3.32) this implies that

$$\delta_1/2 \leq \delta_2 \leq 2\delta_1. \quad (3.34)$$

Set

$$S_2 = S_1 - \delta_2. \quad (3.35)$$

Relations (3.35), (3.34), (3.16), (3.18), (3.21), (3.23) and (3.13) imply that

$$S_2 \geq S_1 - 2\delta_1 = S - \tilde{b} + \tilde{a} - 2\delta_1 \geq S - 4\delta_1 \geq S/2. \quad (3.36)$$

Define $\theta_2 \in W^{1,2}(0, S_2)$ by

$$\theta_2(t) = \theta_1(t), \quad t \in [0, b_1 - \delta_2], \quad \theta_2(t) = \theta_1(t + \delta_2), \quad t \in (b_1 - \delta_2, S_2] \quad (3.37)$$

(see (3.30), (3.29), (3.13) and (3.34)).

Define $y_2 : [0, S_2] \rightarrow R$ as follows:

$$y_2(\tau) = y_1(0) + \int_0^\tau \sin(\theta_2(t)) dt, \quad \tau \in [0, S_2]. \quad (3.38)$$

Relations (3.38) and (3.27) imply that

$$y_2(0) = y_1(0) = y(0). \quad (3.39)$$

Combined with (3.12) this equality implies that

$$|y_2(0)| = |y_1(0)| = |y(0)| \leq \Delta_0. \quad (3.40)$$

By (3.40), (3.27), (3.11), (3.6) and (3.4) for each $t \in [0, S_1]$

$$|y_1(t)| \leq \Delta_0 + t \leq \Delta_0 + S \leq \Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1} \leq 2\Delta_0. \quad (3.41)$$

It follows from (3.38), (3.40), (3.35), (3.11), (3.6) and (3.4) that for each $t \in [0, S_2]$

$$|y_2(t)| \leq \Delta_0 + S \leq \Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1} \leq 2\Delta_0. \quad (3.42)$$

In view of (3.29), (3.13), (3.34) and (3.30)

$$0 \leq a_1 < b_1 - \delta_2 < b_1 < S_1, \theta_1(b_1 - \delta_2) = \theta_1(b_1) = \pi/2. \quad (3.43)$$

It follows from the definition of y_2 (see (3.38)), θ_2 (see (3.37)), (3.27), (3.41), (3.43), (3.35) and Lemma 2.7 (with $a = b_1 - \delta_2$, $b = b_1$, $\theta = \theta_1$) that

$$\left| \int_0^{S_1} (y_1(t))^2 dt - \int_0^{S_2} (y_2(t))^2 dt \right| \leq \delta_2(2\Delta_0)^2 + \delta_2 S_1(4\Delta_0 + \delta_2).$$

Combined with (3.34), (3.13), (3.11), (3.6) and (3.4) this inequality implies that

$$\begin{aligned} \left| \int_0^{S_1} (y_1(t))^2 dt - \int_0^{S_2} (y(t))^2 dt \right| &\leq 2\delta_1[4\Delta_0^2 + S(4\Delta_0 + 2\delta_1)] \\ &\leq 2\delta_1[4\Delta_0^2 + L_0\beta(0)(\beta(\pi/2))^{-1}(4\Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1})] \\ &\leq 2\delta_1[4\Delta_0^2 + (5/16)\Delta_0^2] \leq 10\delta_1\Delta_0^2. \end{aligned} \quad (3.44)$$

Relations (3.44) and (3.28) imply that

$$\left| \int_0^S (y(t))^2 dt - \int_0^{S_2} (y_2(t))^2 dt \right| \leq 13\delta_1\Delta_0^2. \quad (3.45)$$

By (3.37), (3.35), (3.43), (3.30) and (3.31),

$$\begin{aligned} \int_0^{S_2} \sin(\theta_2(t)) dt &= \int_0^{b_1 - \delta_2} \sin(\theta_1(t)) dt + \int_{b_1}^{S_1} \sin(\theta_1(t)) dt = \int_0^{S_1} \sin(\theta_1(t)) dt \\ &- \int_{b_1 - \delta_2}^{b_1} \sin(\theta_1(t)) dt = \int_0^{S_1} \sin(\theta_1(t)) dt - \delta_2 = \int_0^{S_1} \sin(\theta_1(t)) dt + \int_{\tilde{a}}^{\tilde{b}} \sin(\theta(t)) dt. \end{aligned}$$

It follows from this equality, the definition of θ_1 (see (3.17), (3.19), (3.22)) and (1.7) that

$$\int_0^{S_2} \sin(\theta_2(t)) dt = \int_0^S \sin(\theta(t)) dt = 0. \quad (3.46)$$

By (3.29), (3.30), (3.43), (3.34), (3.13) and (3.37)

$$\sup\{\theta_2(t) : t \in [0, S_2]\} = \pi/2. \quad (3.47)$$

Combined with (3.46) and the mean-value theorem this equality implies that there is $\tau_* \in [0, S_2]$ such that

$$\theta_2(\tau_*) = 0. \quad (3.48)$$

Set

$$\delta_3 = \int_{\tilde{a}}^{\tilde{b}} \cos(\theta(t)) dt. \quad (3.49)$$

By (3.49) and (3.23)

$$\delta_3 \leq \tilde{b} - \tilde{a} \leq 2\delta_1. \quad (3.50)$$

Set

$$S_3 = S_2 + \delta_3 \quad (3.51)$$

and define $\theta_3 \in W^{1,2}(0, S_3)$ by

$$\theta_3(t) = \theta_2(t), \quad t \in [0, \tau_*], \quad \theta_3(t) = 0, \quad t \in (\tau_*, \tau_* + \delta_3], \quad (3.52)$$

$$\theta_3(t) = \theta_2(t - \delta_3), \quad t \in (\tau_* + \delta_3, S_3]$$

(see (3.48)). Define

$$y_3(\tau) = y_2(0) + \int_0^\tau \sin(\theta_3(t)) dt, \quad \tau \in [0, S_3]. \quad (3.53)$$

In view of (3.52), (3.51) and (3.46)

$$\int_0^{S_3} \sin(\theta_3(t)) dt = \int_0^{\tau_*} \sin(\theta_2(t)) dt + \int_{\tau_*}^{S_2} \sin(\theta_2(t)) dt = 0. \quad (3.54)$$

It follows from (3.52) and (3.51) that

$$\int_0^{S_3} \cos(\theta_3(t)) dt = \delta_3 + \int_0^{S_2} \cos(\theta_2(t)) dt. \quad (3.55)$$

By (3.37), (3.35), (3.43), (3.30) and the definition of θ_1 (see (3.17), (3.19), (3.22))

$$\begin{aligned} \int_0^{S_2} \cos(\theta_2(t)) dt &= \int_0^{b_1 - \delta_2} \cos(\theta_1(t)) dt + \int_{b_1}^{S_1} \cos(\theta_1(t)) dt = \int_0^{S_1} \cos(\theta_1(t)) dt \\ &- \int_{b_1 - \delta_2}^{b_1} \cos(\theta_1(t)) dt = \int_0^{S_1} \cos(\theta_1(t)) dt = \int_0^S \cos(\theta(t)) dt - \int_{\bar{a}}^{\bar{b}} \cos(\theta(t)) dt. \end{aligned}$$

Combined with (3.55), (3.49) and (1.7) this equality implies that

$$\int_0^{S_3} \cos(\theta_3(t)) dt = L. \quad (3.56)$$

It follows from Lemma 2.8 (with $\theta = \theta_2$, $\tilde{\theta} = \theta_3$), (3.42), (3.50)–(3.53), (3.13), (3.11) and (3.6) that

$$\begin{aligned} \left| \int_0^{S_3} (y_3(t))^2 dt - \int_0^{S_2} (y_2(t))^2 dt \right| &\leq \delta_3(2\Delta_0 + \delta_3)^2 + \delta_3 S_2(4\Delta_0 + \delta_3) \\ &\leq 2\delta_1[(2\Delta_0 + 2\delta_1)^2 + S(4\Delta_0 + 2\delta_1)] \leq 2\delta_1[(2\Delta_0 + S)^2 + S(4\Delta_0 + S)] \\ &\leq 2\delta_1[(2\Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1})^2 + L_0\beta(0)(\beta(\pi/2))^{-1}(4\Delta_0 + L_0\beta(0)(\beta(\pi/2))^{-1})]. \end{aligned}$$

Combined with (3.4) this inequality implies that

$$\left| \int_0^{S_3} (y_3(t))^2 dt - \int_0^{S_2} (y_2(t))^2 dt \right| \leq 2\delta_1[(2\Delta_0 + \Delta_0/16)^2 + 16^{-1}5\Delta_0^2] \leq 2\delta_1(9\Delta_0^2) \leq 18\delta_1\Delta_0^2.$$

Together with (3.45) this implies that

$$\left| \int_0^{S_3} (y_3(t))^2 dt - \int_0^S (y(t))^2 dt \right| \leq 13\delta_1\Delta_0^2 + 18\delta_1\Delta_0^2 = 31\delta_1\Delta_0^2. \quad (3.57)$$

We will estimate $J_{\rho\sigma}^L(\theta, y) - J_{\rho\sigma}^L(\theta_3, y_3)$. It follows from the definition of $\theta_1, \theta_2, \theta_3$ (see (3.17), (3.19), (3.22), (3.37) and (3.52)) that

$$\int_0^{S_3} (\theta_3'(t))^2 dt = \int_0^{S_2} (\theta_2'(t))^2 dt \leq \int_0^{S_1} (\theta_1'(t))^2 dt \leq \int_0^S (\theta'(t))^2 dt. \quad (3.58)$$

In view of (3.52) and (3.51)

$$\int_0^{S_3} \beta(\theta_3(t)) dt = \int_0^{\tau_*} \beta(\theta_2(t)) dt + \delta_3 \beta(0) + \int_{\tau_*}^{S_2} \beta(\theta_2(t)) dt = \delta_3 \beta(0) + \int_0^{S_2} \beta(\theta_2(t)) dt. \quad (3.59)$$

By (3.37), (3.35), (3.43) and (3.30)

$$\begin{aligned} \int_0^{S_2} \beta(\theta_2(t)) dt &= \int_0^{b_1 - \delta_2} \beta(\theta_1(t)) dt + \int_{b_1}^{S_1} \beta(\theta_1(t)) dt \\ &= \int_0^{S_1} \beta(\theta_1(t)) dt - \int_{b_1 - \delta_2}^{b_1} \beta(\theta_1(t)) dt = \int_0^{S_1} \beta(\theta_1(t)) dt - \delta_2 \beta(\pi/2). \end{aligned} \quad (3.60)$$

It follows from the definition of θ_1 (see (3.17), (3.19) and (3.22)) that

$$\int_0^{S_1} \beta(\theta_1(t)) dt = \int_0^S \beta(\theta(t)) dt - \int_{\tilde{a}}^{\tilde{b}} \beta(\theta(t)) dt. \quad (3.61)$$

Equalities (3.59)–(3.61) imply that

$$\int_0^{S_3} \beta(\theta_3(t)) dt = \delta_3 \beta(0) - \delta_2 \beta(\pi/2) + \int_0^S \beta(\theta(t)) dt - \int_{\tilde{a}}^{\tilde{b}} \beta(\theta(t)) dt. \quad (3.62)$$

Relations (3.49), (3.25), (3.22) and (3.2) imply that

$$\delta_3 \leq (\tilde{b} - \tilde{a}) \cos(\pi/2 - r_1) \leq 2\delta_1 \cos(\pi/2 - r_1). \quad (3.63)$$

By (3.62), (3.63), (3.34) and (3.2)

$$\int_0^{S_3} \beta(\theta_3(t)) dt - \int_0^S \beta(\theta(t)) dt \leq \delta_3 \beta(0) - \delta_2 \beta(\pi/2) \leq 2\delta_1 \cos(\pi/2 - r_1) \beta(0) - 2^{-1} \delta_1 \beta(\pi/2) \leq -4^{-1} \delta_1 \beta(\pi/2). \quad (3.64)$$

In view of (1.5), (3.64), (3.58), (3.57), (3.6) and (3.5)

$$\begin{aligned} J_{\rho\sigma}^L(\theta_3, y_3) - J_{\rho\sigma}^L(\theta, y) &= \int_0^{S_3} \beta(\theta_3(t)) dt - \int_0^S \beta(\theta(t)) dt \\ &\quad + \rho \left[\int_0^{S_3} (\theta_3'(t))^2 dt - \int_0^S (\theta'(t))^2 dt \right] + \sigma \left[\int_0^{S_3} (y_3(t))^2 dt - \int_0^S (y(t))^2 dt \right] \\ &\leq -4^{-1} \delta_1 \beta(\pi/2) + 31\sigma\delta_1\Delta_0^2 \leq -4^{-1} \delta_1 \beta(\pi/2) + 31\sigma_0\delta_1\Delta_0^2 < 0, \end{aligned}$$

a contradiction. The contradiction we have reached proves Theorem 3.1. \square

4. PROOF OF THEOREM 1.1

By Theorem 3.1 there are

$$r_0 \in (0, \pi/8), \sigma_0 > 0 \quad (4.1)$$

such that the following assertion holds:

(A1) If $L \in (0, L_1]$, $\rho > 0$, $\sigma \in (0, \sigma_0]$ and if (S, θ, y) is a solution of the problem (1.5)–(1.7) satisfying

$$[-\pi/2 + r_0, \pi/2 - r_0] \subset \theta([0, S]) \quad (4.2)$$

then there is no interval $[a, b] \subset [0, S]$ such that $a < b$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

Choose a positive number γ such that

$$\gamma \leq \arcsin(2^{-1}(\beta(0))^{-1}\beta(\pi/2) \min\{1, 16^{-1}\pi^2\rho_1 L_1^{-2}(\beta(0))^{-1}\}) \quad (4.3)$$

and choose a number σ_1 such that

$$0 < \sigma_1 < \sigma_0, \quad (4.4)$$

$$\sigma_1(1 + (\cos(\pi/2 - r_0))^{-1})(16L_1\beta(0)(\beta(\pi/2))^{-1} + 4)^3 < \sin(\gamma)\beta(\pi/2)/3200. \quad (4.5)$$

Let

$$L \in (0, L_1], \rho \geq \rho_1, \sigma \in (0, \sigma_1]. \quad (4.6)$$

Suppose that $S \geq L$, $\theta \in W^{1,2}(0, S)$ satisfies (1.7) and $y : [0, S] \rightarrow R$ satisfies (1.6) and

$$\int_0^S [\rho(\theta'(t))^2 + \beta(\theta(t)) + \sigma(y(t))^2] dt = J_{\rho\sigma}^L(\theta, y) = \inf(J_{\rho\sigma}^L). \quad (4.7)$$

In order to prove Theorem 1.1 it is sufficient to show that there is no interval $[a, b] \subset [0, S]$ such that $a < b$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

Let us assume the converse. Then there is an interval $[a, b] \subset [0, S]$ such that $0 < a < b < S$ and $|\theta(t)| = \pi/2$ for all $t \in [a, b]$.

We may assume without loss of generality that

$$\theta(t) = \pi/2 \text{ for all } t \in [a, b]. \quad (4.8)$$

It follows from Lemma 2.5, (4.3), (4.6) and (4.8) that

$$\min\{\theta(t) : t \in [0, S]\} \leq -\gamma. \quad (4.9)$$

Corollary 2.1 implies that

$$L \leq S \leq L\beta(0)(\beta(\pi/2))^{-1}. \quad (4.10)$$

In view of (4.8), assertion (A1), (4.4) and (4.6) the inclusion (4.2) does not hold. Together with (4.8) this implies that

$$\min\{\theta(t) : t \in [0, S]\} \geq -\pi/2 + r_0. \quad (4.11)$$

Choose a positive number ϵ such that

$$\epsilon < \gamma/4 \text{ and } \epsilon(\beta(0) \cos(\pi/2 - r_0)^{-1} + 1) < \beta(\pi/2) \sin(\gamma)/4. \quad (4.12)$$

There is a positive number δ such that

$$\delta < \min\{1/8, L/16, 32^{-1}(b-a) \cos(\pi/2 - r_0)\}, \quad (4.13)$$

$$\delta(1 + \cos(\pi/2 - r_0)^{-1}) < 1/8 \quad (4.14)$$

and

$$\begin{aligned} &\text{if } t_1, t_2 \in [0, S] \text{ satisfies } |t_1 - t_2| \leq 2\delta, \text{ then} \\ &|\theta(t_1) - \theta(t_2)| \leq \epsilon, \quad |\beta(\theta(t_1)) - \beta(\theta(t_2))| \leq \epsilon. \end{aligned} \quad (4.15)$$

There is $t_0 \in [0, S]$ such that

$$\theta(t_0) = \inf\{\theta(t) : t \in [0, S]\}. \quad (4.16)$$

There are three cases: (1) $t_0 \leq \delta$; (2) $t_0 \geq S - \delta$; (3) $\delta < t_0 < S - \delta$. In the case (1) set

$$c = 0, \quad d = t_0 + \delta, \quad S_1 = S - d, \quad (4.17)$$

$$\theta_1(t) = \theta(t + d), \quad t \in [0, S_1]. \quad (4.18)$$

In the case (2) put

$$d = S, \quad c = t_0 - \delta, \quad S_1 = t_0 - \delta, \quad (4.19)$$

$$\theta_1(t) = \theta(t), \quad t \in [0, S_1]. \quad (4.20)$$

Consider the case (3). Since θ is continuous and t_0 satisfies (4.16), there exists a closed interval $[c, d] \subset [0, S]$ such that

$$\delta \leq d - c \leq 2\delta, \quad (4.21)$$

$$t_0 \in [c, d] \subset [t_0 - \delta, t_0 + \delta], \quad (4.22)$$

$$\theta(c) = \theta(d). \quad (4.23)$$

We set

$$S_1 = S - d + c, \quad (4.24)$$

$$\theta_1(t) = \theta(t), \quad t \in [0, c], \quad \theta_1(t) = \theta(t + d - c), \quad t \in (c, S_1]. \quad (4.25)$$

It is not difficult to see that in all three cases $\theta_1 \in W^{1,2}(0, S_1)$, (4.21), (4.22) are true and one of the following conditions holds:

$$c = 0; \quad d = S; \quad c > 0, \quad d < S \text{ and } \theta(c) = \theta(d). \quad (4.26)$$

In view of (4.22) and the choice of δ (see (4.15)) for each $t \in [c, d]$

$$|\theta(t) - \theta(t_0)| \leq \epsilon, \quad |\beta(\theta(t)) - \beta(\theta(t_0))| \leq \epsilon. \quad (4.27)$$

By (4.16), (4.27), (4.9) and (4.12) for each $t \in [c, d]$

$$\theta(t_0) \leq \theta(t) \leq \theta(t_0) + \epsilon \leq -\gamma + \epsilon \leq -(3/4)\gamma. \quad (4.28)$$

Inequality (4.28) implies that

$$\int_c^d \cos(\theta(t)) dt \geq (d - c) \cos(\theta(t_0)). \quad (4.29)$$

In view of (4.16), (4.11) and (4.9) $\cos(\theta(t_0)) \neq 0$. Set

$$\Delta_0 = \left(\int_c^d \cos(\theta(t)) dt \right) (\cos(\theta(t_0)))^{-1}. \quad (4.30)$$

By (4.30), (4.29), (4.9), (4.11) and (4.16)

$$d - c \leq \Delta_0 \leq (d - c)(\cos(\theta(t_0)))^{-1} \leq (d - c)(\cos(\pi/2 - r_0))^{-1}. \quad (4.31)$$

It follows from (4.8), (4.28), and the construction of θ_1 (see (4.18), (4.20), (4.25)) that there is an interval $[a_1, b_1] \subset [0, S_1]$ such that

$$b_1 < S_1, \quad b_1 - a_1 = b - a, \quad \theta_1(t) = \pi/2, \quad t \in [a_1, b_1]. \quad (4.32)$$

In view of (4.28) and the definition of θ_1 (see (4.18), (4.20), (4.25))

$$\inf\{\theta_1(t) : t \in [0, S_1]\} \leq -(3/4)\gamma. \quad (4.33)$$

By (4.32), (4.33) and (4.28) there is $t_1 \in [0, S_1]$ such that

$$\theta_1(t_1) = -\theta(t_0). \quad (4.34)$$

Set

$$S_2 = S_1 + \Delta_0 \quad (4.35)$$

and define

$$\begin{aligned} \theta_2(t) &= \theta_1(t), \quad t \in [0, t_1], \quad \theta_2(t) = \theta_1(t_1), \quad t \in (t_1, t_1 + \Delta_0], \\ \theta_2(t) &= \theta_1(t - \Delta_0), \quad t \in (t_1 + \Delta_0, S_2]. \end{aligned} \quad (4.36)$$

Clearly $\theta_2 \in W^{1,2}(0, S_2)$.

It follows from (4.36), (4.35), (4.34), (1.7), (4.30) and the definition of θ_1 (see (4.18), (4.20), (4.25)) that

$$\begin{aligned} \int_0^{S_2} \cos(\theta_2(t)) dt &= \int_0^{S_1} \cos(\theta_1(t)) dt + \Delta_0 \cos(\theta(t_0)) \\ &= \int_0^{S_1} \cos(\theta_1(t)) dt + \int_c^d \cos(\theta(t)) dt = \int_0^S \cos(\theta(t)) dt = L. \end{aligned} \quad (4.37)$$

By (4.36), (4.34), (4.35), the definition of θ_1 (see (4.18), (4.20), (4.25)) and (1.7)

$$\int_0^{S_2} \sin(\theta_2(t)) dt = \int_0^{S_1} \sin(\theta_1(t)) dt - \Delta_0 \sin(\theta(t_0)) = - \int_c^d \sin(\theta(t)) dt - \Delta_0 \sin(\theta(t_0)). \quad (4.38)$$

Set

$$\Delta_1 = - \int_c^d \sin(\theta(t)) dt - \Delta_0 \sin(\theta(t_0)). \quad (4.39)$$

In view of (4.39), (4.28) and (4.30)

$$0 < \Delta_1 \leq d - c + \Delta_0. \quad (4.40)$$

It follows from (4.32) and the construction of θ_2 (see (4.35), (4.36)) that there is an interval $[a_2, b_2] \subset [0, S_2]$ such that

$$b_2 < S_2, \quad b_2 - a_2 = b_1 - a_1 = b - a, \quad \theta_2(t) = \pi/2, \quad t \in [a_2, b_2]. \quad (4.41)$$

Relations (4.39), (4.31), (4.21) and (4.13) imply that

$$\Delta_1 \leq d - c - \Delta_0 \sin(\theta(t_0)) \leq d - c + \Delta_0 \leq 2(d - c)(\cos(\pi/2 - r_0))^{-1} \leq 4\delta(\cos(\pi/2 - r_0))^{-1} \leq 8^{-1}(b - a). \quad (4.42)$$

Set

$$S_3 = S_2 - \Delta_1. \quad (4.43)$$

Combined with (4.43), (4.35), (4.24), (4.21) and (4.13) the inequality (4.42) implies that

$$S_3 \geq S_1 - \Delta_1 = S - \Delta_1 - d + c \geq S - \Delta_1 - 2\delta \geq b - a - 8^{-1}(b - a) - 16^{-1}(b - a) \geq (3/4)(b - a). \quad (4.44)$$

Define

$$\theta_3(t) = \theta_2(t), \quad t \in [0, b_2 - \Delta_1], \quad \theta_3(t) = \theta_2(t + \Delta_1), \quad t \in [b_2 - \Delta_1, S_3] \quad (4.45)$$

(see (4.41)). Clearly $\theta_3 \in W^{1,2}(0, S_3)$. In view of (4.45), (4.43), (4.41), (4.42) and (4.37)

$$\begin{aligned} \int_0^{S_3} \cos(\theta_3(t))dt &= \int_0^{b_2-\Delta_1} \cos(\theta_2(t))dt + \int_{b_2}^{S_2} \cos(\theta_2(t))dt \\ &= \int_0^{S_2} \cos(\theta_2(t))dt - \int_{b_2-\Delta_1}^{b_2} \cos(\theta_2(t))dt = \int_0^{S_2} \cos(\theta_2(t))dt = L. \end{aligned} \quad (4.46)$$

By (4.45), (4.43), (4.41), (4.42), (4.38) and (4.39)

$$\begin{aligned} \int_0^{S_3} \sin(\theta_3(t))dt &= \int_0^{b_2-\Delta_1} \sin(\theta_2(t))dt + \int_{b_2}^{S_2} \sin(\theta_2(t))dt \\ &= \int_0^{S_2} \sin(\theta_2(t))dt - \Delta_1 = 0. \end{aligned} \quad (4.47)$$

It follows from the definition of θ_1 , θ_2 , θ_3 (see (4.18), (4.20), (4.25), (4.36), (4.45)) that

$$\int_0^{S_3} (\theta'_3(t))^2 dt \leq \int_0^{S_2} (\theta'_2(t))^2 dt = \int_0^{S_1} (\theta'_1(t))^2 dt \leq \int_0^S (\theta'(t))^2 dt. \quad (4.48)$$

We estimate

$$\int_0^{S_3} \beta(\theta_3(t))dt - \int_0^S \beta(\theta(t))dt.$$

By (4.45) and (4.43)

$$\begin{aligned} \int_0^{S_3} \beta(\theta_3(t))dt &= \int_0^{b_2-\Delta_1} \beta(\theta_2(t))dt + \int_{b_2}^{S_2} \beta(\theta_2(t))dt \\ &= \int_0^{S_2} \beta(\theta_2(t))dt - \int_{b_2-\Delta_1}^{b_2} \beta(\theta_2(t))dt. \end{aligned}$$

Combined with (4.41), (4.42), (4.36), (4.34), (4.35) and (1.2) this equality implies that

$$\begin{aligned} \int_0^{S_3} \beta(\theta_3(t))dt &= \int_0^{S_2} \beta(\theta_2(t))dt - \Delta_1 \beta(\pi/2) \\ &= -\Delta_1 \beta(\pi/2) + \Delta_0 \beta(\theta(t_0)) + \int_0^{t_1} \beta(\theta_1(t))dt + \int_{t_1}^{S_1} \beta(\theta_1(t))dt \\ &= \Delta_1 \beta(\pi/2) + \Delta_0 \beta(\theta(t_0)) + \int_0^{S_1} \beta(\theta_1(t))dt. \end{aligned}$$

Together with the definition of θ_1 (see (4.18), (4.20), (4.25)) this equality implies that

$$\int_0^{S_3} \beta(\theta_3(t))dt = -\Delta_1 \beta(\pi/2) + \Delta_0 \beta(\theta(t_0)) + \int_0^S \beta(\theta(t))dt - \int_c^d \beta(\theta(t))dt.$$

Thus

$$\int_0^{S_3} \beta(\theta_3(t))dt - \int_0^S \beta(\theta(t))dt = -\Delta_1 \beta(\pi/2) + \Delta_0 \beta(\theta(t_0)) - \int_c^d \beta(\theta(t))dt. \quad (4.49)$$

In view of (1.3) and (4.27)

$$\begin{aligned} \Delta_0 \beta(\theta(t_0)) - \int_c^d \beta(\theta(t)) dt &= \beta(\theta(t_0))[\Delta_0 - (d - c)] \\ &+ \int_c^d [\beta(\theta(t_0)) - \beta(\theta(t))] dt \leq \beta(0)[\Delta_0 - d + c] + \epsilon(d - c). \end{aligned} \quad (4.50)$$

By (4.30), (4.27), (4.16), (4.9) and (4.11)

$$\begin{aligned} \Delta_0 - d + c &= (\cos(\theta(t_0)))^{-1} \int_c^d [\cos(\theta(t)) - \cos(\theta(t_0))] dt \\ &\leq (\cos(\theta(t_0)))^{-1} \int_c^d |\theta(t) - \theta(t_0)| dt \\ &\leq (\cos(\theta(t_0)))^{-1} \epsilon(d - c) \leq \epsilon(d - c)(\cos(\pi/2 - r_0))^{-1}. \end{aligned} \quad (4.51)$$

Combined with (4.50) the relation (4.51) implies that

$$\Delta_0 \beta(\theta(t_0)) - \int_c^d \beta(\theta(t)) dt \leq \beta(0)(d - c)(\cos(\pi/2 - r_0))^{-1} \epsilon + \epsilon(d - c) = (d - c)[\beta(0)\epsilon(\cos(\pi/2 - r_0))^{-1} + \epsilon]. \quad (4.52)$$

Relations (4.39), (4.28), (4.31), (4.16) and (4.9) imply that

$$\Delta_1 \geq -\Delta_0 \sin(\theta(t_0)) \geq -\sin(\theta(t_0))(d - c) \geq (d - c) \sin(\gamma).$$

Together with (4.49), (4.52) and (4.12) this inequality implies that

$$\begin{aligned} \int_0^{S_3} \beta(\theta_3(t)) dt - \int_0^S \beta(\theta(t)) dt \\ \leq (d - c)[- \beta(\pi/2) \sin(\gamma) + \epsilon(\beta(0)(\cos(\pi/2 - r_0))^{-1} + 1)] \\ \leq -(d - c) \sin(\gamma) \beta(\pi/2)/2. \end{aligned} \quad (4.53)$$

For $i = 1, 2, 3$ set

$$y_i(\tau) = y(0) + \int_0^\tau \sin(\theta_i(t)) dt, \quad \tau \in [0, S_i]. \quad (4.54)$$

We estimate $\int_0^S (y(t))^2 dt - \int_0^{S_3} (y_3(t))^2 dt$. Lemma 2.3 implies that

$$|y(t)| \leq 8L\beta(0)(\beta(\pi/2))^{-1}, \quad t \in [0, S]. \quad (4.55)$$

It follows from (4.55), (4.54), the definition of θ_1 (see (4.18), (4.20), (4.25)), Lemmas 2.6 and 2.7 and (4.10) that

$$\begin{aligned} \left| \int_0^{S_1} (y_1(t))^2 dt - \int_0^S (y(t))^2 dt \right| &\leq (d - c)(8L\beta(0)(\beta(\pi/2))^{-1})^2 \\ &+ (d - c)S(16L\beta(0)(\beta(\pi/2))^{-1} + d - c) \\ &\leq (d - c)[64(L\beta(0)(\beta(\pi/2))^{-1})^2 + 17(L\beta(0)(\beta(\pi/2))^{-1})^2] \\ &= (d - c)81(L\beta(0)(\beta(\pi/2))^{-1})^2. \end{aligned} \quad (4.56)$$

Relations (4.54), (4.55), (4.24), (4.10) and (4.13) imply that for all $t \in [0, S_1]$

$$|y_1(t)| \leq |y(0)| + t \leq |y(0)| + S_1 \leq 8L\beta(0)(\beta(\pi/2))^{-1} + S \leq 9L\beta(0)(\beta(\pi/2))^{-1}.$$

Combined with (4.54), (4.36), Lemma 2.8, (4.31), (4.22), (4.10) and (4.13) this inequality implies that

$$\begin{aligned} \left| \int_0^{S_2} (y_2(t))^2 dt - \int_0^{S_1} (y_1(t))^2 dt \right| &\leq \Delta_0(9L\beta(0)(\beta(\pi/2))^{-1} + \Delta_0)^2 \\ &\quad + \Delta_0 S_1(\Delta_0 + 18L\beta(0)(\beta(\pi/2))^{-1}) \\ &\leq \Delta_0[(10L\beta(0)(\beta(\pi/2))^{-1})^2 + 19(L\beta(0)(\beta(\pi/2))^{-1})^2] \\ &\leq 29(d-c)(\cos(\pi/2 - r_0))^{-1}(L\beta(0)(\beta(\pi/2))^{-1})^2. \end{aligned} \quad (4.57)$$

In view of (4.35), (4.31), (4.21) and (4.13)

$$\begin{aligned} S_2 = S_1 + \Delta_0 &\leq S + (d-c)(\cos(\pi/2 - r_0))^{-1} \\ &\leq S + 2\delta(\cos(\pi/2 - r_0))^{-1} \leq S + b - a \leq 2S. \end{aligned} \quad (4.58)$$

By (4.54), (4.55), (4.58) and (4.10) for each $t \in [0, S_2]$

$$|y_2(t)| \leq |y(0)| + t \leq |y(0)| + S_2 \leq 8L\beta(0)(\beta(\pi/2))^{-1} + 2S \leq 10L\beta(0)(\beta(\pi/2))^{-1}. \quad (4.59)$$

It follows from (4.54), (4.58), (4.59), (4.45), (4.40)–(4.42), Lemma 2.7 and (4.10) that

$$\begin{aligned} \left| \int_0^{S_2} (y_2(t))^2 dt - \int_0^{S_3} (y_3(t))^2 dt \right| &\leq \Delta_1(10L\beta(0)(\beta(\pi/2))^{-1})^2 \\ &\quad + \Delta_1 S_2(20L\beta(0)(\beta(\pi/2))^{-1} + \Delta_1) \\ &\leq \Delta_1[10^2(L\beta(0)(\beta(\pi/2))^{-1})^2 + 2S(20L\beta(0)(\beta(\pi/2))^{-1} + S)] \\ &\leq 142\Delta_1[L\beta(0)(\beta(\pi/2))^{-1}]^2. \end{aligned}$$

Together with (4.42) this implies that

$$\left| \int_0^{S_2} (y_2(t))^2 dt - \int_0^{S_3} (y_3(t))^2 dt \right| \leq 142(L\beta(0)(\beta(\pi/2))^{-1})^2 2(d-c)(\cos(\pi/2 - r_0))^{-1}.$$

Combined with (4.57) and (4.56) this inequality implies that

$$\begin{aligned} \left| \int_0^S (y(t))^2 dt - \int_0^{S_3} (y_3(t))^2 dt \right| &\leq 81(d-c)(L\beta(0)(\beta(\pi/2))^{-1})^2 \\ &\quad + 29(d-c)(\cos(\pi/2 - r_0))^{-1}(L\beta(0)(\beta(\pi/2))^{-1})^2 \\ &\quad + 284(d-c)(\cos(\pi/2 - r_0))^{-1}(L\beta(0)(\beta(\pi/2))^{-1})^2 \\ &\leq 400(d-c)(L\beta(0)(\beta(\pi/2))^{-1})(\cos(\pi/2 - r_0))^{-1}. \end{aligned} \quad (4.60)$$

By (1.5), (4.60), (4.6), (4.48), (4.53) and (4.5)

$$\begin{aligned}
J_{\rho\sigma}^L(\theta_3, y_3) - J_{\rho\sigma}^L(\theta, y) &= \sigma \left[\int_0^{S_3} (y_3(t))^2 dt - \int_0^S (y(t))^2 dt \right] \\
&\quad + \rho \left[\int_0^{S_3} (\theta_3'(t))^2 dt - \int_0^S (\theta'(t))^2 dt \right] + \int_0^{S_3} \beta(\theta_3(t)) dt - \int_0^S \beta(\theta(t)) dt \\
&\leq 400\sigma_1((d-c)(L\beta(0)(\beta(\pi/2))^{-1})^2(\cos(\pi/2 - r_0))^{-1} + \int_0^{S_3} \beta(\theta_3(t)) dt - \int_0^S \beta(\theta(t)) dt) \\
&\leq 400(d-c)\sigma_1(L_1\beta(0)(\beta(\pi/2))^{-1})^2(\cos(\pi/2 - r_0))^{-1} - (d-c)\sin(\gamma)\beta(\pi/2)/2 < 0,
\end{aligned}$$

a contradiction. The contradiction we have reached proves Theorem 1.1.

5. THE PARAMETER σ_1 AS A FUNCTION OF β , L_1 AND ρ_1

Let $\rho_1, L_1 > 0$. We proved the existence of a positive number σ_1 which depends on β, L_1, ρ_1 such that the assertion of Theorem 1.1 holds. In this section we obtain an explicit expression for σ_1 which is a function of β, L_1, ρ_1 . We assume that

$$L_1 > 1 \text{ and } 16^{-1}\pi^2\rho_1L_1^{-2}(\beta(0))^{-1} < 1, \quad (5.1)$$

set

$$r_1 = \beta(\pi/2)(\beta(0))^{-1}16^{-1} \quad (5.2)$$

and observe that r_1 satisfies (3.2). Clearly

$$0 < r_1 < \pi/16, \quad (5.3)$$

so that

$$r_1/2 \leq \cos(\pi/2 - r_1) \leq r_1. \quad (5.4)$$

Relations (5.4) and (5.2) imply that

$$\beta(0)\cos(\pi/2 - r_1) = r_1\beta(0) \leq \beta(\pi/2)16^{-1}.$$

Together with (5.3) this implies that (3.2) holds. Put

$$r_0 = r_1/3. \quad (5.5)$$

Then (3.3) is valid. Set

$$\sigma_0 = 10^{-3}\beta(\pi/2)[16L_1\beta(0)(\beta(\pi/2))^{-1}]^{-2}. \quad (5.6)$$

Clearly (3.5) is true with $L_0 = L_1$ and $\Delta_0 = 16r_1\beta(0)\beta(\pi/2)^{-1}$. We showed that the assertion of Theorem 3.1 holds with $L_0 = L_1$. Thus (A1) holds (see Sect. 4). Now we need to choose positive numbers γ, σ_1 . Set

$$\gamma = \arcsin(2^{-1}(\beta(0))^{-1}\beta(\pi/2)) \min\{1, 16^{-1}\pi^2\rho_1L_1^{-2}(\beta(0))^{-1}\}. \quad (5.7)$$

Clearly (4.3) holds.

Finally we need to choose $\sigma_1 > 0$ which satisfies (4.4) and (4.5). Set

$$\sigma_1 = (\beta(0))^{-6}\beta(\pi/2)^6L_1^{-5}\rho_1(17 \cdot 400 \cdot 16^2 \cdot 20^3)^{-1}. \quad (5.8)$$

It follows from (5.7), (5.5), (5.3), (5.2), (5.1) and (5.8) that

$$\begin{aligned}
& 8^{-1} \sin(\gamma) \beta(\pi/2) [1 + (\cos(\pi/2 - r_0))^{-1}]^{-1} (4 + 16L_1\beta(0)(\beta(\pi/2))^{-1})^{-3} \\
&= 2^{-1} (\beta(0))^{-1} \beta(\pi/2) \min\{1, 16^{-1} \pi^2 \rho_1 L_1^{-2} (\beta(0))^{-1}\} 8^{-1} \beta(\pi/2) [1 \\
&\quad + (\cos(\pi/2 - r_0))^{-1}]^{-1} (4 + 16L_1\beta(0)(\beta(\pi/2))^{-1})^{-3} \\
&\geq 2^{-1} (\beta(0))^{-1} \beta(\pi/2) \min\{1, \pi^2 16^{-1} \rho_1 L_1^{-2} (\beta(0))^{-1}\} \\
&\quad \times 8^{-1} \beta(\pi/2) [1 + 2r_0^{-1}]^{-1} (4 + 16L_1\beta(0)(\beta(\pi/2))^{-1})^{-3} \\
&\geq 2^{-1} (\beta(0))^{-1} \beta(\pi/2) \min\{1, 16^{-1} \pi^2 \rho_1 L_1^{-2} (\beta(0))^{-1}\} 8^{-1} \beta(\pi/2) [1 \\
&\quad + 6 \cdot 16\beta(0)(\beta(\pi/2))^{-1}]^{-1} (4 + 16L_1\beta(0)(\beta(\pi/2))^{-1})^{-3} \\
&\geq 2^{-1} (\beta(0))^{-1} \beta(\pi/2) \min\{1, (\pi^2/16) \rho_1 L_1^{-2} (\beta(0))^{-1}\} \\
&\quad \times 8^{-1} \beta(\pi/2) (17 \cdot 6)^{-1} (\beta(0))^{-1} \beta(\pi/2) (20L_1)^{-3} (\beta(0))^{-3} \beta(\pi/2)^3 \\
&= (\beta(0))^{-5} \beta(\pi/2)^6 (20L_1)^{-3} \min\{1, (\pi^2/16) \rho_1 L_1^{-2} (\beta(0))^{-1}\} (17 \cdot 6)^{-1} 16^{-1} \\
&\quad = (\beta(0))^{-6} (\beta(\pi/2))^6 L_1^{-5} \rho_1 (20^3 \cdot 17 \cdot 16^2 \cdot 6)^{-1} \pi^2 \\
&\quad > (\beta(0))^{-6} (\beta(\pi/2))^6 L_1^{-5} \rho_1 (20^3 \cdot 16^2)^{-1} = 400\sigma_1.
\end{aligned}$$

Thus (4.5) holds.

By (5.8), (5.6) and (5.1)

$$\begin{aligned}
\sigma_1 &= 10^{-3} \beta(\pi/2) [16L_1\beta(0)(\beta(\pi/2))^{-1}]^{-2} (8 \cdot 17)^{-1} (\beta(0))^{-4} (\beta(\pi/2))^3 L_1^{-3} \rho_1 \\
&= \sigma_0 (8 \cdot 17)^{-1} (\beta(0))^{-4} (\beta(\pi/2))^3 L_1^{-3} \rho_1 \\
&< \sigma_0 (8 \cdot 17)^{-1} (\beta(0))^{-4} (\beta(\pi/2))^3 L_1^{-1} 2\beta(0) \leq \sigma_0 (4 \cdot 17)^{-1}.
\end{aligned}$$

Thus (4.4) is true and the assertion of Theorem 1.1 holds with σ_1 defined by (5.8).

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REFERENCES

- [1] B. Dacorogna and C.E. Pfister, Wulff theorem and best constant in Sobolev inequality. *J. Math. Pures Appl.* **71** (1992) 97–118.
- [2] I. Fonseca, The Wulff theorem revisited. *Proc. R. Soc. Lond. A* **432** (1991) 125–145.
- [3] J. Hannon, N.C. Bartelt, B.S. Swartzentruber, J.C. Hamilton and G.L. Kellogg, Step faceting at the (001) surface of boron doped silicon. *Phys. Rev. Lett.* **79** (1997) 4226–4229.
- [4] J. Hannon, M. Marcus and V.J. Mizel, A variational problem modelling behavior of unorthodox silicon crystals. *ESAIM: COCV* **9** (2003) 145–149.
- [5] H.C. Jeng and E.D. Williams, Steps on surfaces: experiment and theory. *Surface Science Reports* **34** (1999) 175–294.
- [6] W.W. Mullins, Theory of thermal grooving. *J. Appl. Physics* **28** (1957) 333–339.