REGULARITY AND VARIATIONALITY OF SOLUTIONS TO HAMILTON-JACOBI EQUATIONS.
PART I: REGULARITY
(ERRATA)

ANDREA C. G. MENNucci

Abstract. This errata corrects one error in the 2004 version of this paper [Mennucci, ESAIM: COCV 10 (2004) 426–451].

Mathematics Subject Classification. 49L25, 53C22, 53C60.

Received October 24, 2006.

After the publication of [7] in 2004, it became clear that the regularity of the form $\alpha$ in Lemma 4.4 had to be related to the regularity of $K$ and of $u_0$; this influences the minimal regularity of $K, u_0$, as needed in hypotheses in Lemma 4.4, in Theorem 4.1, and in many following relevant discussions. This errata corrects that error; to keep the matter short, all material that is unaffected by the error is omitted; whereas care was taken so that results and discussions that are here corrected retain the original numbering as in [7].

4.1. Regularity of conjugate points

We will prove in this section results regarding the set of focal points; each following result extends to the set $\Gamma$ of conjugate points that is a subset of the focal points.

Theorem 4.1. Assume (CC0,H1,H2). If $u_0, K, H$ are regular enough, then, by Lemma 4.4, there is a (at most) countable number of $n - 1$ dimensional submanifolds of $\mathbb{R} \times O$ that cover all the sets $G^i$; these submanifolds are graphs of functions $\lambda_{i,h} : A_{i,h} \to \mathbb{R}$ (for $h = 1, \ldots$) where $A_{i,h} \subset O$ are open sets. The least regular case is $i = n - 1$, and the regularity of the $\lambda$ functions is related to the regularity of $u_0, K, H$, and to the dimension $\dim(M) = n$ as in the following table:

<table>
<thead>
<tr>
<th>$\dim(M)$</th>
<th>$u_0, K$</th>
<th>$H$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>$C^{(R+2,\theta)}$</td>
<td>$C^{(R+2,\theta)}$</td>
<td>$C^{(R,\theta)}$</td>
</tr>
<tr>
<td>$n \geq 3$</td>
<td>$C^{(R+2,\theta)}$</td>
<td>$C^{(R+n-1,\theta)} \cap C^n$</td>
<td>$C^{(R,\theta)}$</td>
</tr>
</tbody>
</table>

where $R \in \mathbb{N}, \theta \in [0, 1]$.

Keywords and phrases. Hamilton-Jacobi equations, cutlocus, conjugate points.

1 Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy; a.mennucci@sns.it

© EDP Sciences, SMAI 2007
We now infer some explanatory results on the regularity of the focal points \( X(\bigcup_i G^i) \) from the above theorem.

At the lowest regularity, when \( u_0, K \in C^2, H \in C^n \), we know that \( X \in C^1 \) and that the sets \( G^i \) are graphs; we conclude that the set of focal points has measure zero. When \( u_0, K \in C^{(2,\theta)}, H \in C^n \cap C^{(2,\theta)} \), we know that the dimension of the sets \( G^i \) does not exceed \( n - \theta \); so again we conclude that the set of focal points has dimension at most \( n - \theta \). In the case \( \theta = 1 \), we can obtain the set of all focal points is rectifiable; that is, if \( u_0, K \in C^{(2,1)}, H \in C^n \cap C^{(2,1)} \), then the sets \( G^i \) are covered by Lipschitz graphs, so (by known results in [2]) the set of focal points may be covered by \((n-1)\)-dimensional \( C^1 \) regular submanifolds of \( M \), but for a set of Hausdorff \( H^{n-1} \) measure zero.

When we further raise the regularity, we may suppose that \( u_0, K \in C^{s+3}, H \in C^{s+n} \) (with \( s \in \mathbb{N} \))^1; then the sets \( G^i \) are covered by graphs \((\lambda(y), y)\) inside \( \mathbb{R} \times O \) of regularity \( C^{1+s} \); while \( X \in C^{2+s} \) (at least), and we restrict it to those graphs; we can then apply Theorem A.4 to state that the focal points are covered by \( C^{1+s} \) regular submanifolds of \( M \) but for a set of \( H^s \) measure zero, where \( s = n - 2 + 1/(1+s) \).

[... unchanged material deleted ...]

The main tool is this lemma; the complete proof of the lemma is in Section 6.

**Lemma 4.4.** We assume that the hypotheses (CC0,H1,H2) hold.

We set the regularity of the data \( u_0, K, H \) by defining parameters \( R, R' \in \mathbb{N}, \theta, \theta' \in [0,1] \), and assuming that

\[
u_0 \in C^{(R+2,\theta')}, \quad K \in C^{(R+2,\theta')}, \quad H \in C^{(R+2,\theta')}
\]

by Proposition 3.7, the flow \( \Phi = (X, P) \) is \( C^{(R+1,\theta)} \) regular; and \( O \) is a \( C^{(R+1,\theta')} \cup C^{(R+2,\theta')} \) manifold (that is, the least regular of the two).

Let \( i \geq 1 \), \( i \leq n-1 \), and fix a point \((s', y') \in \mathbb{R} \times O\), such that \((s', y') \in G^i \).

Let \( U \) be a neighborhood of \( 0 \) in \( \mathbb{R}^{n-1} \) and let \( \phi : U \to O \) be a local chart to the neighborhood \( \phi(U) \) of \( y' = \phi(0) \). The map \( \phi \) has regularity \( C^{(R+1,\theta')} \cup C^{(R+2,\theta')} \). In the following, \( y \) will be a point in \( \phi(U) \).

To study \( G^i \), we should study the rank of the Jacobian of the map \((t, x) \mapsto (t, \phi(x))\); since the regularity of \( X \) is related only to the regularity of \( H \), it will be useful to decouple this Jacobian in two parts. To this end, we define a \( n \)-form \( \alpha \) on \( \mathbb{R} \times O \), with requirement that \( \alpha(t, y) = \alpha(y) \) (that is, \( \alpha \) does not depend on \( t \)).

Writing \( X(t, y) \) for \((t, y) \), let

\[
X(t, y)^* \alpha
\]

be the push-forward of \( \alpha \) along \( X \); \( X(t, y)^* \alpha \) is then a tangent form defined on \( T_{X(t, y)} M \); it will be precisely defined in equation (6.2). We remark that \( X(t, y)^* \alpha = 0 \) if \((t, y) \in \bigcup_i G^i \). Note that the pushforward \( X(t, y)^* \alpha \) is \( C^{(R,\theta)} \) regular, while the form \( \alpha \) is as regular as \( TO \); that is, \( \alpha \) is \( C^{(R',\theta')} \cup C^{(R+1,\theta')} \).

Note that, since \( X \) solves an O.D.E., then \( X \) and \( \frac{\partial}{\partial t} X \) have the same regularity; note moreover that

\[
\frac{\partial}{\partial t} \left( X(s', y')^* \alpha \right) = \left( \frac{\partial}{\partial t} X \right)(s', y')^* \alpha
\]

since \( \alpha \) does not depend on \( t \). So, by hypotheses and by the definition (6.2) of \( X(t, y)^* \alpha \), the forms \( X(t, y)^* \alpha \) and \( \frac{\partial}{\partial t} X(t, y)^* \alpha \) have regularity \( C^{(R,\theta)} \cap C^{(R',\theta')} \) (see also Eq. (6.3)); the derivatives \( \frac{\partial}{\partial t^j} X(s', y')^* \alpha \) with \( j \geq 1 \) have regularity \( C^{(R-j+1,\theta)} \cup C^{(R',\theta')} \).

Then, when \( R+1 \geq i \), we prove (in Sect. 6) that

\[
X(s', y')^* \alpha = 0, \quad \frac{\partial}{\partial t} X(s', y')^* \alpha = 0, \quad \cdots \quad \frac{\partial^{i-1}}{\partial t^{i-1}} X(s', y')^* \alpha = 0
\]

A similar result may be obtained when \( u_0, K \in C^{(s+3,\theta)}, H \in C^{(s+n,\theta)} \).
whereas
\[ \frac{\partial^i}{\partial t^i} X(s', y')^* \alpha \neq 0. \]

We define eventually the map \( F : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \) given by
\[ F(t, x) = \frac{\partial^{i-1}}{\partial t^{i-1}} X(t, \phi(x))^* \alpha; \]

since
\[ \frac{\partial}{\partial t} F(t, x) = \frac{\partial^i}{\partial t^i} X(t, \phi(x))^* \alpha \neq 0 \]

the above Dini lemma implies that the set \( G^{(i)} \) is locally covered by the graph of a function \( \lambda \), defined on an open subset of \( O \); \( \lambda \) has the same regularity of \( F \), so, if \( i = 1 \) then \( \lambda \) is in \( C^{R, \theta} \cup C^{(R, \theta') \cap C^{(R, \theta')}.} \)

The above directly implies Theorem 4.1.

[... all other results are unchanged ...]

5. APPLICATIONS

5.1. The Cauchy problem

We show now how the above theorems may be used for the Cauchy problem (1.2)
\[
\begin{cases}
\frac{\partial}{\partial t} w(t, x') + H'(t, x', \frac{\partial}{\partial x^2} w(t, x')) = 0 & \text{for } t > 0, x' \in M' \\
w(0, x') = w_0(x') & \forall x' \in M'.
\end{cases}
\]  

[... the preliminary discussion is unchanged ...]

This improves the results of 4.10, 4.12 and 4.17 in [1]; to provide for an easy comparison, we summarize these results

- if \( n' = \dim(M') \), \( n = n' + 1 \), if \( H' \in C^s \) with \( s = n \vee 3 \) and \( w_0 \in C^2 \), then the set \( \Gamma \) has measure zero, so the set \( \Sigma_n = \Sigma \cup \Gamma \) has measure zero;
- if \( H, w_0 \in C^{(2,1)} \), then the set \( \Gamma \) is rectifiable, so the set \( \Sigma_n = \Sigma \cup \Gamma \) is rectifiable;
- and when \( H' \in C^{R+1, \theta} \), \( w_0 \in C^{R+1, \theta} \), \( R \geq 2 \), \( w \) is continuous, we prove that the Hausdorff dimension of \( \Gamma \setminus \Sigma \) is at most \( \beta \), and moreover \( H^3(\Gamma \setminus \Sigma) = 0 \) if \( \theta = 0 \), where \( \beta = n' - 1 + 2/(R + \theta) \).

In the counterexample in Section 4.4 in [1], \( w_0 \) is \( C^{1,1}(M') \) and not \( C^2(M') \); so our results close the gap between the counterexample, where \( w_0 \) is \( C^{1,1}(M') \), and the theorem, where \( w_0 \) is \( C^2(M') \); and actually, studying the counterexample, it is quite clear that, if \( w_0 \) is smoothed to become a \( C^2(M') \) function, then the counterexample would not work.

5.2. Eikonal equation and cut locus

As in Section 3.5, consider a smooth Riemannian manifold \( M \), and a closed set \( K \subset M \) and let \( d_K(x) = d(x, K) \) be the distance to \( K \). We set \( u_0 = 0 \): then \( O \) is the bundle of unit covectors that are normal to \( TK \), and \( d_K(x) \) coincides with the \( \min \) solution \( u(x) \).

We define
\[ \Sigma_{d_K} \overset{\text{def}}{=} \{ x \mid \# \nabla d_K(x) \} \]

If \( K \) is \( C^1 \), then \( \Sigma_{d_K} \) coincides with \( \Sigma \) as defined in (4.1).

Since \( d_K \) is semiconcave in \( M \setminus K \), \( \Sigma_{d_K} \) is always rectifiable.

This primal problem is a good test bed to discuss the differences and synergies of the results in this paper and the results in Itoh and Tanaka [4] and Li and Nirenberg [5].
In the example in Section 3 in [6], there is a curve $K \subset \mathbb{R}^2$, $K \in C^{1,1}$ such that $\Sigma dK$ has positive Lebesgue measure. Note that in this example $\Sigma dK \neq \text{Cut}(K) = \Sigma dK$, so the cutlocus $\text{Cut}(K)$ is rectifiable (but not closed).

We do not know if there is a curve $K \in C^{1,1}$ such that $\Sigma dK$ has positive Lebesgue measure. Note that in this example $\Sigma dK \neq \text{Cut}(K) = \Sigma dK$, so the cutlocus $\text{Cut}(K)$ is rectifiable (but not closed).

We do not know if there is a curve $K \in C^{1,1}$ such that $\text{Cut}(K)$ is not rectifiable. (We recall that, by Prop. 14 in [3], $\text{Cut}(K)$ has always measure zero).

Theorem 4.1 states that if $K$ is $C^2$, then $\Gamma$ has measure zero, so by (1.4) and 4.11.4, we obtain that $\Sigma dK = \text{Cut}(K)$ has measure zero; so Theorem 4.1 closes the gap between the counterexample in Section 3 [6] and the previous available results.

In example in Remark 1.1 in [5], for all $\theta \in (0, 1)$ there is a compact curve $K \in C^{2, \theta}$ such that the distance to the cutlocus is not locally Lipschitz; by Theorem 4.1, the cutlocus has dimension at most $n - \theta$.

We do not know if there exists an example of a compact curve $K \in C^{2, \theta}$ such that $H^{n-1}(\text{Cut}(K)) = \infty$.

By the results in Itoh and Tanaka [4] and Li and Nirenberg [5], when $K \in C^3$, the distance to the cutlocus is locally Lipschitz and the cutlocus is rectifiable, and moreover (by Cor 1.1 in [5]), for any $B$ bounded $H^{n-1}(\text{Cut}(K) \cap B) < \infty$. By Theorem 4.1, the set of (non optimal) focal points is rectifiable as well.

5.2.1. Improvements

[... the discussion is unchanged ...]

**Corollary 5.1.** Consider a 2-dimensional smooth Riemannian manifold $M$; suppose that $K$ is a compact $C^{3+s}$ embedded submanifold.

Then, for any open bounded set $A \subset M$, the set $A \cap \Gamma$ is $C^{s+1} - M^{1/(s+1)}$-rectifiable: that is, it can be covered by at most countably many $C^{s+1}$ curves, but for a set $E$ such that $M^{1/(s+1)}(E) = 0$.

6. Proof of 4.4

[... the two lemma are unchanged ...]

Now we prove Lemma 4.4.

We want to define the $n$ form $\alpha$ so that $\alpha$ does not depend on $t$; and so that $\alpha = e_1 \wedge \cdots \wedge e_n$ where the vectors fields $e_{n-i+1} \ldots e_n$ span the kernel of $\frac{\partial}{\partial t}X$ at the point $(s', y')$ (kernel that we will call $V$) while $\frac{\partial}{\partial t}X$ is full rank on $e_1 \ldots e_{n-1}$ (that generate the space $W$).

One possible way to this is to fix the local chart $\phi : U \subset \mathbb{R}^{n-1} \rightarrow O$, define

$$\hat{e}_1 \equiv \phi \frac{\partial}{\partial x_1}, \ldots \hat{e}_{n-1} \equiv \phi \frac{\partial}{\partial x_{n-1}}, \hat{e}_n \equiv \frac{\partial}{\partial t}$$

and then choose a $n \times n$ constant matrix $A$, so that

$$e_h \equiv \sum_k A_{h,k} \hat{e}_k$$

satisfy the requirements.

[... the rest of the proof is unchanged ...]

**Acknowledgements.** The author thanks Prof. Graziano Crasta for spotting the error that is corrected in this errata.
References


