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ON THE CURVATURE AND TORSION EFFECTS IN ONE DIMENSIONAL WAVEGUIDES

Guy Bouchitté[1], M. Luísa Mascarenhas[2] and Luís Trabucho[3]

Abstract. We consider the Laplace operator in a thin tube of $\mathbb{R}^3$ with a Dirichlet condition on its boundary. We study asymptotically the spectrum of such an operator as the thickness of the tube's cross section goes to zero. In particular we analyse how the energy levels depend simultaneously on the curvature of the tube's central axis and on the rotation of the cross section with respect to the Frenet frame. The main argument is a Γ-convergence theorem for a suitable sequence of quadratic energies.

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1. Introduction

We are interested in the 3D-1D reduction analysis for the following elementary spectral problem:

$$-\Delta u_\varepsilon = \lambda^\varepsilon u_\varepsilon, \qquad u_\varepsilon \in H^1_0(\Omega_\varepsilon), \tag{1.1}$$

where $\Omega_\varepsilon \subset \mathbb{R}^3$ is a thin and long domain generated by a cross section $\omega_\varepsilon = \varepsilon\,\omega$ ($\omega \subset \mathbb{R}^2$) which rotates along a curve $r(s) \in \mathbb{R}^3$ parametrized by s, the usual arc length variable. Here ε is a small parameter and the rotation angle $\alpha(s)$ of the section with respect to the Frénet frame is given. We are going to show the following behavior of the spectrum $\{\lambda_i^\varepsilon; i \in \mathbb{N}\}$ as $\varepsilon \to 0$:

$$\lambda_i^\varepsilon = \frac{\lambda_0}{\varepsilon^2} + \mu_i^\varepsilon, \quad \mu_i^\varepsilon \to \mu_i, \tag{1.2}$$

where λ_0 is the first eigenvalue of the Laplace operator on ω and the μ_i's are the eigenvalues of a one dimensional problem of the kind

$$-w'' + q(s)w = \mu\, w, \quad w \in H^1_0(0,L).$$

Here $q(s)$ is an effective potential which we are able to characterize in terms of the parameters $k(s), \tau(s)$ (curvature and torsion), of the shape of ω and of the rotation angle $\alpha(s)$.

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[1] Département de Mathématiques, Université du Sud-Toulon-Var, BP 132, 83957 La Garde Cedex, France; bouchitte@univ-tln.fr
[2] Departamento de Matemática da F.C.T.-U.N.L. e C.M.A.-U.N.L., Quinta da Torre, 2829-516 Caparica, Portugal; mlfm@fct.unl.pt
[3] Departamento de Matemática da F.C.-U.L. e C.M.A.F.-U.L., Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal; trabucho@ptmat.fc.ul.pt

A possible physical motivation for this problem is the understanding of the behavior of the probability density associated with the wave function of a particle confined in a thin waveguide. The interpretation of the convergence result above is that, from the particle's point of view, everything happens as if it will propagate in a one dimensional medium governed by the non zero potential $q(s)$. Several results have been published in this direction, for instance in [2, 5], in the case of tube of infinite length with circular cross section, showing the shift of the spectrum on the left due to the curvature and the possible occurence of localized modes. Here we emphasize on the effects of the torsion and of the shape of cross section which, in the opposite direction, tend to shift the spectrum on the right. Moreover we present a new rigourous variational approach through Γ-convergence which is very flexible and can be adapted to other kind of spectral problems with possibly stiff parameters.

Let us emphasize that the geometrical effects described here are very specific to the Dirichlet boundary condition imposed on the lateral part of the tube. Such effects would disappear if the Dirichlet condition would be replaced by a Neumann condition. We refer to [8] (see also the survey [7]) for related questions where networks of tubes with junctions are considered. Let us finally mention a pioneering work by M. Vanninathan [9] where a behavior of the kind (1.2) was established for the eigenvalues of the Dirichlet problem on a periodically perforated domain as the period ε tends to zero.

In Section 2 we describe the geometric properties of the domain. In Section 3, we present the rescaled spectral problem on a varying Hilbert space and show how the asymptotic behavior of the entire spectrum can be recovered by proving the Γ-convergence of a suitable family of quadratic energies. In Section 4, we preliminary study a perturbed problem for the first eigenvalue in the cross section and then establish the main convergence result. Eventually, some elementary examples of limit models are discussed in Section 5.

2. Geometry of the domain

Let $r : s \in [0, L] \to r(s) \in \mathbb{R}^3$ be a simple C^2 curve in $\mathbb{R}^3$ parametrized by the arc length parameter s. Denoting by T its tangent vector and assuming that $T'(s) \neq 0$ for every $s \in [0, L]$, we may define the usual Frenet system (T, N, B) through the following expressions:

$$T = \frac{\mathrm{d}r}{\mathrm{d}s} = r' \quad (\|r'\|_{\mathbb{R}^3} = 1); \quad N = T'/\|T'\|_{\mathbb{R}^3}; \quad B = T \times N.$$

Denote by $k : s \in [0, L] \to k(s) \in \mathbb{R}$ and by $\tau : s \in [0, L] \to \tau(s) \in \mathbb{R}$, the curvature and torsion functions associated with the curve. They are functions in $L^\infty(0, L)$ and they satisfy the Frenet formulas:

$$T' = k\ N; \quad N' = -k\ T + \tau\ B; \quad B' = -\tau\ N. \tag{2.1}$$

Let now $\omega \subset \mathbb{R}^2$ be an open bounded, simply connected subset of $\mathbb{R}^2$ and consider the following subset of $\mathbb{R}^3$, directly associated with the Frenet system defined above:

$$\Omega^F = \{x \in \mathbb{R}^3 : x = r(s) + y_1\ N(s) + y_2\ B(s),\ s \in [0, L],\ y = (y_1, y_2) \in \omega\}.$$

However this choice is too restrictive as we may like that the possibly non circular cross section ω of our waveguide rotates with respect to curve r in a different way as Frenet frame does. In particular another reference system, denominated *Tang's reference system*, could be considered as well, in which the corresponding domain:

$$\Omega^T = \{x \in \mathbb{R}^3 : x = r(s) + y_1\ X(s) + y_2\ Y(s),\ s \in [0, L],\ y = (y_1, y_2) \in \omega\},$$

is such that its cross section is rotation free with respect to the tangent vector T. The orthonormal basis vectors of Tang's reference system are given by:

$$X' = \lambda\ T; \quad Y' = \mu T; \quad T' = -\lambda\ X - \mu\ Y; \tag{2.2}$$

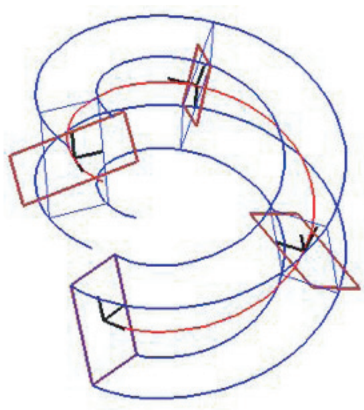

FIGURE 1. Reference domains associated with Frenet's and Tang's systems.

where λ and μ are functions of the arc length parameter s. In Figure 1 we show an illustration of the domains Ω^F and Ω^T (four cross sections only).

For each $s \in [0, L]$ Tang's reference system is such that (X, Y) can be seen as a two dimensional basis, in ω, rotated from (N, B), around T, of an angle $\alpha = \alpha(s)$. In fact if

$$X = \cos\alpha \ N + \sin\alpha \ B, \quad Y = -\sin\alpha \ N + \cos\alpha \ B,$$

using Frenet's formulas (2.1), one obtains:

$$\begin{aligned} X' &= -(\tau + \alpha') \ \sin\alpha \ N + (\tau + \alpha') \ \cos\alpha \ B - k \ \cos\alpha \ T, \\ Y' &= -(\tau + \alpha') \ \cos\alpha \ N - (\tau + \alpha') \ \sin\alpha \ B + k \ \sin\alpha \ T, \end{aligned}$$

which yields (2.2) if, for each $s \in [0, L]$, $\alpha(s)$ satisfies the so called *Tang's relations*:

$$\alpha' = -\tau, \quad \lambda = -k \ \cos\alpha, \quad \mu = k \ \sin\alpha.$$

We notice that, in contrast with Frenet's system, the condition $T'(s) \neq 0$ is not required in order to construct the Tang basis (T, X, Y) (which is uniquely determined as a solution of (2.2)).

We are then faced with many possible choices for the reference set and, in order to model a general twisted tube, we will consider the generic domain Ω^α defined through:

$$\Omega^\alpha = \{x \in \mathbb{R}^3 : x = r(s) + y_1 \ N_\alpha(s) + y_2 \ B_\alpha(s), \ s \in [0, L], \ y = (y_1, y_2) \in \omega\},$$

whose cross section presents an arbitrary rotation of an angle α with respect to Frenet's domain:

$$\begin{aligned} N_\alpha(s) &:= \cos\alpha(s) \ N(s) + \sin\alpha(s) \ B(s), \\ B_\alpha(s) &:= -\sin\alpha(s) \ N(s) + \cos\alpha(s) \ B(s). \end{aligned} \tag{2.3}$$

As is clear from the above notation, if for every $s \in [0, L]$, $\alpha = 0$ then $\Omega^\alpha \equiv \Omega^F$ and if α is such that $\alpha' = -\tau$, then $\Omega^\alpha = \Omega^T$.

We are interested in the (eigenvalue) problem given by (1.1) in a thin domain such that the diameter of the cross section ω is much smaller than its length L. Specifically, we consider a real parameter $\varepsilon > 0$ and a cross section, obtained from the reference one, by an homothety of ratio ε. Our thin waveguide will be then

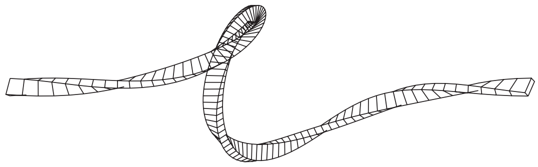

FIGURE 2. A generic domain $\Omega_\varepsilon^\alpha$.

determined as follows (Fig. 2):

$$\Omega_\varepsilon^\alpha := \left\{x \in \mathbb{R}^3 : x = r(s) + \varepsilon\, y_1\, N_\alpha + \varepsilon\, y_2\, B_\alpha,\ s \in [0, L],\ y = (y_1, y_2) \in \omega\right\}.$$

3. Asymptotic spectral problem and γ-convergence approach

We consider, for fixed $\varepsilon > 0$, the thin domain $\Omega_\varepsilon^\alpha$ defined in Section 2 and the following eigenvalue problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda^\varepsilon u_\varepsilon \\ u_\varepsilon \in H_0^1(\Omega_\varepsilon^\alpha). \end{cases} \tag{3.1}$$

As $\Omega_\varepsilon^\alpha$ is bounded, the spectrum σ^ε of this problem is discrete and can be written as $\sigma^\varepsilon := \{\lambda_i^\varepsilon \ :\ i \in \mathbb{N}\}$, where $0 < \lambda_0^\varepsilon \le \lambda_1^\varepsilon \le \cdots \le \lambda_i^\varepsilon \le \lambda_{i+1}^\varepsilon \cdots$ are positive reals, arranged increasingly. As the cross section becomes thiner and thiner, it is clear that all these eigenvalues go to infinity as $\varepsilon \to 0$. More precisely, let λ_0 be the fundamental eigenvalue for the Laplace operator in the cross section ω and let u_0 be the associated normalized eigenvector, that is

$$-\Delta u_0 = \lambda_0 u_0, \quad u_0 \in H_0^1(\omega), \quad u_0 > 0, \quad \int_\omega u_0^2 = 1. \tag{3.2}$$

We are expecting the following asymptotic behavior:

$$\lambda_i^\varepsilon \ = \ \frac{\lambda_0}{\varepsilon^2} \ + \ \mu_i \ + \rho(\varepsilon), \quad \lim_{\varepsilon \to 0} \rho(\varepsilon) = 0, \tag{3.3}$$

where μ_i $(i \in \mathbb{N})$ are suitable real numbers. Our goal is to establish (3.3) and to identify the set $\{\mu_i\}$ as the eigenvalues of a one dimensional spectral problem in $H_0^1(0, L)$ in which the geometric parameters $k(s), \tau(s), \alpha(s)$ are involved.

3.1. **Change of variables**

As usual in the dimension reduction analysis, we first proceed to a rescaling and a change of variables in order to reduce the initial problem to a variational min-max formulation on a fixed domain. Having in mind the asymptotic behavior of the shifted spectrum $\sigma^\varepsilon - \dfrac{\lambda_0}{\varepsilon^2}$, the initial quadratic energy defined in $H_0^1(\Omega_\varepsilon^\alpha)$ reads as:

$$F_\varepsilon(w) := \int_{\Omega_\varepsilon^\alpha} \left(|\nabla w|^2 - \frac{\lambda_0}{\varepsilon^2}|w|^2\right) \mathrm{d}x. \tag{3.4}$$

Recalling (2.3), consider the following transformation, for each $\varepsilon > 0$,

$$\begin{aligned} &\psi_\varepsilon : [0, L] \times \omega \longrightarrow \Omega_\varepsilon^\alpha \\ &(s, (y_1, y_2)) \mapsto x = r(s) + \varepsilon\, y_1 N_\alpha + \varepsilon\, y_2 B_\alpha \end{aligned}$$

and define, for each $w \in H_0^1(\Omega_\varepsilon^\alpha)$, $v(s, (y_1, y_2)) := w(\psi_\varepsilon(s, (y_1, y_2)))$.

We obtain that, in the Frénet frame:

$$\nabla\psi_\varepsilon = \begin{pmatrix} \beta_\varepsilon & 0 & 0 \\ -\varepsilon(\tau+\alpha')(z_\alpha^\perp\cdot y) & \varepsilon\cos\alpha & -\varepsilon\sin\alpha \\ \varepsilon(\tau+\alpha')(z_\alpha\cdot y) & \varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix}, \quad \det\nabla\psi_\varepsilon = \varepsilon^2\beta_\varepsilon,$$

where z_α, $z_\alpha^\perp$ and β_ε are given by

$$\beta_\varepsilon(s,y) := 1-\varepsilon k(s)(z_\alpha\cdot y), \quad z_\alpha := (\cos\alpha, -\sin\alpha), \quad z_\alpha^\perp := (\sin\alpha,\cos\alpha), \tag{3.5}$$

and where, as previously, α' represents the derivative of α with respect to $s\in[0,L]$.

Then

$$\nabla\psi_\varepsilon^{-1} = \begin{pmatrix} \dfrac{1}{\beta_\varepsilon} & 0 & 0 \\ \dfrac{(\tau+\alpha')y_2}{\beta_\varepsilon} & \dfrac{\cos\alpha}{\varepsilon} & \dfrac{\sin\alpha}{\varepsilon} \\ \dfrac{-(\tau+\alpha')y_1}{\beta_\varepsilon} & \dfrac{-\sin\alpha}{\varepsilon} & \dfrac{\cos\alpha}{\varepsilon} \end{pmatrix}.$$

Denote $v(s,y) = w(\psi_\varepsilon(s,y))$, for $w\in H_0^1(\Omega_\varepsilon^\alpha)$, $s\in[0,L]$ and $y\in\omega$. We scale the functional F_ε introduced in (3.4) by dividing it by ε^2. We are led to the quadratic energy G_ε defined by:

$$G_\varepsilon(v) := \frac{1}{\varepsilon^2}F_\varepsilon(v) = \int_0^L\int_\omega \left\{\frac{1}{\beta_\varepsilon}\left|v' + \nabla_y v\cdot R\ y\ (\tau+\alpha')\right|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\nabla_y v|^2 - \lambda_0|v|^2\right)\right\}\,\mathrm{d}y\,\mathrm{d}s, \tag{3.6}$$

where v' stands for the derivative of v in order to s, $\nabla_y v$ for the derivative of v in order to y and R for the rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$.

3.2. **The rescaled spectral problem**

Denote $Q_L = \omega\times(0,L)$ and let H_ε be the Hilbert space $L^2(Q_L,\beta_\varepsilon)$ equipped with the weighted scalar product

$$(u|v)_\varepsilon := \int_{Q_L} u(x)\,v(x)\,\beta_\varepsilon(x)\,\mathrm{d}x.$$

By (3.5) and since the curvature $k(s)$ is assumed to be bounded, $\beta_\varepsilon(s,y)$ converges uniformly to 1 as $\varepsilon\to 0$. Therefore all spaces H_ε are topologically equivalent and the strong convergence in H_ε is equivalent to the convergence in the fixed space $H := L^2(Q_L)$.

Now we define A_ε to be the unique closed self adjoint operator from H_ε to H_ε with dense domain $D(A_\varepsilon) = H^2(Q_L)\cap H_0^1(Q_L)$ such that

$$(A_\varepsilon v|v)_\varepsilon = G_\varepsilon(v), \quad \forall v\in D(A_\varepsilon).$$

In view of (3.3) and (3.6), it turns out that $u_\varepsilon\in H_0^1(\Omega_\varepsilon^\alpha)$ solves the spectral equation $-\Delta u_\varepsilon = \lambda_i^\varepsilon u_\varepsilon$ if and only if the function $v_\varepsilon(s,y) = u_\varepsilon\circ\psi_\varepsilon(s,y)$ satisfies

$$A_\varepsilon v_\varepsilon = \mu_\varepsilon^i v_\varepsilon, \quad v_\varepsilon\in D(A_\varepsilon).$$

It turns out that the sequence $\{A_\varepsilon\}$ is not uniformly bounded and also that the related spectral problem involves a varying scalar product. Therefore it is not possible to use directly an approach through H-convergence and apply the general results to be found in [1]. As our operator becomes stiff as $\varepsilon\to 0$, we adopt a slightly different point of view, reducing the original asymptotic spectral problem to the study of the Γ-convergence of the sequence of functionals $\{G_\varepsilon\}$ on the fixed space $H = L^2(Q_L)$.

3.3. Link with the Γ-convergence theory

First we extend the quadratic functional G_ε given in (3.6) by setting $G_\varepsilon(v) = +\infty$ if $v \in L^2(Q_L) \setminus H_0^1(Q_L)$. We say that the sequence $\{G_\varepsilon\}$ Γ-converges to G in $H = L^2(Q_L)$ if the two following conditions hold:

(i) for any v and $\{v_\varepsilon\}$ such that $v_\varepsilon \to v$ in H, $\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \geq G(v)$;

(ii) for every v, there exists a sequence $\{\tilde{v}_\varepsilon\}$ such that $\tilde{v}_\varepsilon \to v$ in H and $\lim_{\varepsilon\to 0} G_\varepsilon(\tilde{v}_\varepsilon) = G(v)$.

It turns out that such a Γ-limit G always exists, possibly after extracting a subsequence. Also the Γ−convergence of $\{G_\varepsilon\}$ is unchanged if we substitute G_ε by its lower semicontinuous envelope (with respect to the strong topology in H) and the Γ-limit G enjoys the lower semicontinuity property as well. For further features on Γ-convergence theory, we refer to the monograph by Dal Maso [4], in which particular issues concerning the case of quadratic functionals and related linear operators are detailed (see Sect. 12 in this book). We have the following abstract result:

Theorem 3.1. *Let $A_\varepsilon : H_\varepsilon \to H_\varepsilon$ be a sequence of densily defined self-adjoint operators where H_ε coincides algebraically with a fixed Hilbert space H endowed with a scalar product $(\cdot|\cdot)_\varepsilon$ such that*

$$a_\varepsilon \|u\|^2 \ \leq\ (u|u)_\varepsilon \ \leq\ b_\varepsilon \|u\|^2,$$

being $a_\varepsilon, b_\varepsilon$ suitable constants such that $a_\varepsilon, b_\varepsilon \to 1$.

Let $G_\varepsilon : H \to (-\infty, +\infty]$ be defined by $G_\varepsilon(v) := (A_\varepsilon v|v)_\varepsilon$ if $v \in D(A_\varepsilon)$, and $G_\varepsilon(v) := +\infty$ otherwise, and assume that the three following conditions hold:

(i) (Lower bound) $G_\varepsilon(v) \geq\ -c_0 \|v\|^2$ for a suitable constant $c_0 \geq 0$.

(ii) (Compactness) If $\sup_\varepsilon G_\varepsilon(v_\varepsilon) < +\infty$ and $\sup_\varepsilon \|v_\varepsilon\| + < +\infty$, then $\{v_\varepsilon\}$ is strongly relatively compact in H.

(iii) G_ε does Γ−converge to G.

Then the limit functional G determines a unique closed linear operator $A_0 : H \to H$ with compact resolvent (whose domain $D(A_0)$ is a priori non dense in H) such that $G(v) = (A_0 v|v)$ for all $v \in D(A_0)$. Furthermore the spectral problems associated with A_ε converge in the following sense: let $(\mu_i^\varepsilon, v_i^\varepsilon)$ and (μ_i, v_i) be such that

$$\begin{array}{lll} v_i^\varepsilon \in H_\varepsilon, & A_\varepsilon v_i^\varepsilon = \mu_i^\varepsilon v_i^\varepsilon,\ \mu_0^\varepsilon \leq \mu_1^\varepsilon \leq \ldots \leq \mu_i^\varepsilon, \ldots, & (v_i^\varepsilon|v_j^\varepsilon)_\varepsilon = \delta_{i,j} \\ v_i \in H, & A_0 v_i = \mu_i v_i, \quad \mu_0 \leq \mu_1 \leq \ldots \leq \mu_i, \ldots, & (v_i|v_j) = \delta_{i,j}. \end{array}$$

Then, as $\varepsilon \to 0$, $\mu_i^\varepsilon \to \mu_i$ for every $i \in \mathbb{N}$. Moreover, up to a subsequence, $\{v_i^\varepsilon\}$ converges strongly to eigenvectors associated to μ_i. Conversely any eigenvector v_i is the strong limit of a particular sequence of eigenvectors of A_ε associated to μ_i^ε.

Applying this general result to the sequence $\{G_\varepsilon\}$ defined by (3.6), we deduce, in Section 4, the limit of the shifted spectrum $\{\mu_i^\varepsilon\}$, where $\mu_i^\varepsilon = \lambda_i^\varepsilon - \frac{\lambda_0}{\varepsilon^2}$. Let us emphasize that, as already noticed in a similar situation in [1], see Theorem 4.1, the equi-compactness property (ii) is crucial, otherwise, we could only expect the inclusion of the spectrum of A_0 in the set of cluster points of $\{\mu_i^\varepsilon\}$.

Proof. Let $c > c_0$. The condition (i) and (3.7) imply that, for small ε, the operator $A_\varepsilon + c\,\mathrm{I}_{H_\varepsilon}$, where $\mathrm{I}_{H_\varepsilon}$ denotes the identity map on H_ε, is a positive maximal monotone. Let us denote by S_ε its inverse. Since G_ε Γ-converges to G, it is easy to check that G is a quadratic lower semicontinuous functional on H which satisfies condition (i) as well. Therefore, for every $f \in H$, the minimum problem:

$$\inf \left\{ G(v) + c \|v\|_H^2 - 2\,(f|v) \ :\ v \in H \right\}$$

admits a unique minimizer $S_0 f$ and the map $f \mapsto S_0 f$ determines a bounded linear operator. The range of S_0 coincides with the domain $D(A_0)$ of a closed operator $A_0 : H \mapsto H$ such that $S_0 = (A_0 + c\,\mathrm{I}_H)^{-1}$.

We claim that $\{S_\varepsilon\}$ is a uniformly compact family of self adjoint operators on H_ε and that $S_\varepsilon \to S_0$ strongly. Recalling that, due to (3.7), the spaces H_ε share the same topology, this means that

$$\sup \|f_\varepsilon\| < +\infty \Rightarrow \{S_\varepsilon f_\varepsilon\} \text{ strongly relatively compact in } H,\ S_\varepsilon f_\varepsilon \to S_0 f \text{ whenever } f_\varepsilon \to f.$$

The conclusions of Theorem 3.1 will then follow from [6], Theorems 11.4 and 11.5 (see also [3]), after noticing that, for $v \in H$, the following equivalences hold:

$$A_\varepsilon v = \mu_i^\varepsilon v \iff S_\varepsilon v = \frac{1}{\mu_i^\varepsilon + c}\, v, \quad A_0 v = \mu_i v \iff S_0 v = \frac{1}{\mu_i + c}\, v.$$

To prove the claim, it is enough to show that for every weakly convergent sequence f_ε, the following implication holds

$$f_\varepsilon \rightharpoonup f \implies S_\varepsilon f_\varepsilon \to S_0 f \quad (strongly). \tag{3.7}$$

The crucial remark is that v_ε is the unique minimizing point of $\tilde{G}_\varepsilon$ where

$$\tilde{G}_\varepsilon(v) \ := \ G_\varepsilon(v) \ + \ c(v|v)_\varepsilon \ - \ 2\,(f_\varepsilon|v)_\varepsilon.$$

Since $\{f_\varepsilon\}$ is bounded, there exists $M > 0$ such that

$$(c - c_0)\|v_\varepsilon\|^2 - M\|v_\varepsilon\| \le \tilde{G}_\varepsilon(v_\varepsilon) \le 0.$$

It follows that $\{v_\varepsilon\}$ is bounded and that $\sup G_\varepsilon(v_\varepsilon) < +\infty$ Therefore, by the condition (ii), $\{v_\varepsilon\}$ is strongly relatively compact.

On the other hand, $\tilde{G}_\varepsilon$ being a uniformly convergent perturbation of G_ε, it is easy to check that $\tilde{G}_\varepsilon$ does $\Gamma-$converge to the functional $\tilde{G} := G + k\|\cdot\|^2 - 2\,(f|\cdot)$. Therefore, by using the fundamental variational property of the $\Gamma-$convergence, we derive that v_ε converges to a global minimizer of $\tilde{G}$. This minimizer is unique and coincides with $S_0 f$. The claim (3.11) follows and Theorem 3.1 is proved. □

4. Convergence results

In this section we are going to prove that we can apply Theorem 3.1 to the sequence $\{G_\varepsilon\}$ defined by (3.6) and with G defined as follows

$$G(v) := \begin{cases} G_0(w) & \text{if } v(s,y) = w(s)\ u_0(y),\ w \in H_0^1(0,L), \\ +\infty & otherwise, \end{cases} \tag{4.1}$$

where

$$G_0(w) := \int_0^L \left\{ |w'(s)|^2 + \left[(\tau(s) + \alpha'(s))^2\ C(\omega) - \frac{k^2(s)}{4} \right] |w(s)|^2 \right\} \,\mathrm{d}s. \tag{4.2}$$

Here u_0 is the normalized eigenvector (ground state) of the unperturbed problem introduced in (3.2) and the geometric parameter $C(\omega)$ is given by

$$C(\omega) := \int_\omega |\nabla_y u_0 \cdot R\ y|^2 \,\mathrm{d}y. \tag{4.3}$$

Notice that $C(\omega) > 0$, unless u_0 is radial. This parameter which depends only on the shape of the section ω turns out to be very important as it will govern the effect of the torsion.

4.1. **A perturbed spectral problem in the cross section**

The influence of the curvature $k(s)$ goes through the multiplicative coefficient $\beta_\varepsilon(s,y)$ which appears in (3.6) (that is $\int_\omega \beta_\varepsilon(s,y)(|\nabla_y v|^2 - \lambda_0|v|^2)\ \mathrm{d}y$). In order to study this dependence, we consider, for every $\xi \in \mathbb{R}^2$, the following perturbed problem:

$$-\mathrm{div}\,[(1-\xi\cdot y)\nabla_y u] \;=\; \lambda\ (1-\xi\cdot y)u, \quad u \in H_0^1(\omega). \tag{4.4}$$

The parameter ξ will be taken to be $\xi = \varepsilon\, k(s)\, z_\alpha$ so that for small ε the perturbed operator is positive with compact resolvent. Let us denote by $\lambda(\xi) > 0$ its first eigenvalue, that is:

$$\lambda(\xi) = \inf_{\substack{v\in H_0^1(\omega)\\ v\not\equiv 0}} \frac{\int_\omega (1-\xi\cdot y)\ (\nabla_y u)^2\ \mathrm{d}y}{\int_\omega (1-\xi\cdot y)\ (u)^2\ \mathrm{d}y}\cdot$$

Let $v \in H_0^1(Q_L)$; then the following lower bound holds for *a.e.* $s \in (0,L)$:

$$\frac{1}{\varepsilon^2}\int_\omega \beta_\varepsilon(s,y)(|\nabla_y v|^2 - \lambda_0|v|^2)\ \mathrm{d}y \geq \gamma_\varepsilon(s)\ \int_\omega \beta_\varepsilon(s,y)\,|v|^2\,\mathrm{d}y, \tag{4.5}$$

where

$$\gamma_\varepsilon(s) \;:=\; \frac{\lambda\big(\varepsilon\, k(s) z_\alpha(s)\big) - \lambda_0}{\varepsilon^2}\cdot \tag{4.6}$$

The fact that γ_ε remains finite is crucial in order to find a finite $\Gamma-$limit for G_ε. This is also closely related to the validity of the postulated behavior given in (3.3).

Proposition 4.1. *(i) The function $\lambda(\xi)$ is twice differentiable at* 0 *and denoting by I the identity matrix, there holds:*

$$\lambda(0) = \lambda_0, \quad \nabla\lambda(0) = 0, \quad \nabla^2\lambda(0) = -\frac{1}{2}\, I.$$

(ii) Let $\gamma_\varepsilon(s)$ be given by (4.6) *and assume that the curvature $k(s)$ is bounded. Then, as $\varepsilon \to 0$*

$$\gamma_\varepsilon(s) \;\to\; -\ \frac{k^2(s)}{4} \qquad \textit{uniformly on } [0,L].$$

Remark 4.2. It is rather suprising that the Hessian matrix found in the assertion (i) is scalar and independent of the shape of the cross section ω. In fact this situation is very specific to the Laplace operator. If we deal with a more general diffusion operator $-\mathrm{div}\,(a(y)\nabla\cdot)$, being $a(y)$ a non constant positive coefficient, the situation would be quite different. In fact, if in the definition of $\lambda(\xi)$, we replace (4.4) by

$$-\mathrm{div}\,[(1-\xi\cdot y)a(y)\nabla_y u] \;=\; \lambda(1-\xi\cdot y)u, \quad u\in H_0^1(\omega),$$

it turns out that, although the function $\lambda(\cdot)$ is still locally concave in the neighborhood of 0, the symmetric negative matrix $\nabla^2\lambda(0)$ is not, in general, a multiple of the identity matrix. Moreover, and this is the main point, the gradient at 0 does not vanish anymore. This gradient is given by $\nabla\lambda(0) = -2\int_\omega a(y)u_0\nabla_y u_0\,\mathrm{d}y$ where now u_0 is the eigenvector (bound state) of the new unperturbed problem.

In order to prove Proposition 4.1, we introduce, for fixed $\xi \in \mathbb{R}^2$, the solution $u_\xi \in H_0^1(\omega)$ of the following problem:

$$-\Delta\ u_\xi - \lambda_0\ u_\xi = -\xi\cdot\nabla_y\ u_0, \quad u_\xi \perp u_0 \text{in } L^2(\omega). \tag{4.7}$$

The existence of u_ξ falls under Fredholm alternative. Since λ_0 is simple, it is enough to observe that the right hand-side in (4.7) is orthogonal to u_0, which is clear from the fact that $u_0\nabla_y\ u_0 = \frac{1}{2}\nabla_y\ u_0^2$ has zero mean value by the Dirichlet condition on $\partial\omega$. Futhermore, by linearity, denoting by χ_1, χ_2 the solutions of (4.7) for $\xi = e_1$ and $\xi = e_2$, respectively, we have:

$$u_\xi \;=\; \xi_1\,\chi_1 \;+\; \xi_2\,\chi_2. \tag{4.8}$$

Lemma 4.3. *For every $\xi \in \mathbb{R}^2$, we have*

$$\inf_{v\in H_0^1(\omega)} \int_\omega \Big[|\nabla_y v|^2 - \lambda_0|v|^2 + 2\,(\xi\cdot\nabla_y u_0)\,v\Big]\mathrm{d}y = -\frac{|\xi|^2}{4}\cdot \tag{4.9}$$

Furthermore, the above infimum is reached for u_ξ given in (4.7).

Proof. The variational problem in (4.9) is convex and v is a minimizer if and only if it solves the Euler equation: $-\Delta\, v - \lambda_0\, v = -\xi\cdot\nabla_y\, u_0$. Therefore, by (4.7) the minimum is reached at u_ξ. By the equi-repartition of energy, we are then reduced to check the equality

$$\int_\omega (\xi\cdot\nabla_y u_0)\,u_\xi \;=\; -\,\frac{|\xi|^2}{4}\cdot \tag{4.10}$$

We notice that $u_0\nabla_y u_\xi - u_\xi\nabla u_0 = \nabla_y(u_0 u_\xi)$ has zero mean value and that, by (3.2) and (4.7),

$$\mathrm{div}(u_0\nabla_y u_\xi - u_\xi\nabla_y u_0) = u_0\Delta u_\xi - u_0\Delta u_0 = (\xi\cdot\nabla_y u_0)\,u_0.$$

Then (4.10) follows after integrating twice by parts:

$$\begin{aligned}
\int_\omega (\xi\cdot\nabla_y u_0)\,u_\xi \mathrm{d}y &= \frac12\int_\omega [(\xi\cdot\nabla_y u_0)\,u_\xi - (\xi\cdot\nabla_y u_\xi)\,u_0]\,\mathrm{d}y\\
&= \frac12\int_\omega \nabla_y(\xi\cdot y)\;\,(u_\xi\nabla_y u_0\, - u_0\nabla_y u_\xi)\,\mathrm{d}y\\
&= \frac12\int_\omega (\xi\cdot y)\;\mathrm{div}\,[u_0\nabla_y u_\xi\, - u_\xi\nabla_y u_0]\,\mathrm{d}y\\
&= \frac12\int_\omega (\xi\cdot y)\;(\xi\cdot\nabla_y u_0)\,u_0\;\mathrm{d}y\\
&= -\frac14\int_\omega \mathrm{div}[(\xi\cdot y)\xi]\,u_0^2\;\mathrm{d}y\\
&= -\frac14\int_\omega |\xi|^2 u_0^2\;\mathrm{d}y \;\;= -\,\frac14|\xi|^2.
\end{aligned}$$

□

Proof of Proposition 4.1. In view of definition (4.6) assertions (i) and (ii) will be obtained by proving that:

$$\lim_{\xi\to 0}\frac{\lambda(\xi)-\lambda_0}{|\xi|^2} = -\,\frac14\cdot \tag{4.11}$$

In order to obtain (4.11) and recalling the definition of $\lambda(\xi)$, we evaluate the Rayleigh quotient $R_\xi(u) = \dfrac{A(u)}{B(u)}$ where

$$A(u) := \int_\omega (1-\xi\cdot y)|\nabla_y u|^2\,\mathrm{d}y,\quad B(u) := \int_\omega (1-\xi\cdot y)|u|^2\,\mathrm{d}y,$$

and u is written as $u = tu_0 + \varphi$ where $t\in\mathbb{R}$, $\varphi\in H_0^1(\omega)$ and $\varphi\perp u_0$ in $L^2(\omega)$. First we notice that, by integrating by parts $\int_\omega \nabla_y u_0\cdot\nabla_y((\xi\cdot y)w)\,\mathrm{d}y$ and taking (3.2) into account, we have for every $w\in H_0^1(\omega)$:

$$\int_\omega (\xi\cdot y)(\nabla_y u_0\cdot\nabla_y w - \lambda_0\,u_0\,w)\,\mathrm{d}y \;=-\;\int_\omega (\nabla_y u_0\cdot\xi)\,w\,\mathrm{d}y.$$

In particular, using this relation for $w = u_0$ and $w = \varphi$, we easily deduce that

$$A(u) - \lambda_0 B(u) = \int_\omega (1 - \xi \cdot y)(|\nabla_y \varphi|^2 - \lambda_0 |\varphi|^2)\, \mathrm{d}y \;+\; 2t \int_\omega (\nabla_y u_0 \cdot \xi)\, \varphi \,\mathrm{d}y. \tag{4.12}$$

Upper bound. We take $u = u_0 + u_\xi$, *i.e.* $t = 1$ and $\varphi = u_\xi$, where u_ξ is defined by (4.7). In view of (4.8) and (4.12) there is a constant $C > 0$, depending only on ω such that, for ξ small enough,

$$\left| \int_\omega (\xi \cdot y)(|\nabla_y u_\xi|^2 - \lambda_0 |u_\xi|^2)\, \mathrm{d}y \right| \le C|\xi|^3, \quad |B(u_0 + u_\xi) - 1| \le C|\xi|.$$

Also by Lemma 4.3, we have $\int_\omega (|\nabla_y u_\xi|^2 - \lambda_0 |u_\xi|^2)\, \mathrm{d}y + 2 \int_\omega (\nabla_y u_0 \cdot \xi)\, u_\xi \,\mathrm{d}y = -\frac{|\xi|^2}{4}$. Consequently, we deduce the following upper bound:

$$\frac{\lambda(\xi) - \lambda_0}{|\xi|^2} \le \frac{A(u_0 + u_\xi) - \lambda_0 B(u_0 + u_\xi)}{|\xi|^2 B(u_0 + u_\xi)} \le \frac{-\frac{1}{4} + C|\xi|}{1 - C|\xi|}. \tag{4.13}$$

Lower bound. We may choose the constant C large enough and $|\xi|$ small enough so that

$$1 - \xi \cdot y \;\ge\; 1 - C|\xi| \ge 0 \quad \text{in } \omega, \quad B(tu_0 + \varphi) \;\ge\; (1 - C|\xi|)\, t^2, \tag{4.14}$$

for every $t \in \mathbb{R}$, $\varphi \in H_0^1(\omega)$ with $\varphi \perp u_0$ in $L^2(\omega)$. Now we are going to show that, for a suitable positive real s_0, the negative part of $\frac{\lambda(\xi) - \lambda_0}{|\xi|^2}$ satisfies

$$\left(\frac{\lambda(\xi) - \lambda_0}{|\xi|^2} \right)^- \le \frac{1}{4(1 - C|\xi|)^2} + \frac{2C\lambda_0\, s_0^2\, |\xi|}{(1 - C|\xi|)}. \tag{4.15}$$

Indeed, recalling (4.12) and applying (4.9) with ξ substituted by $\frac{t\xi}{(1 - C|\xi|)}$, we have

$$\begin{aligned} (A - \lambda_0 B)(tu_0 + \varphi) &\ge (1 - C|\xi|) \left[\int_\omega (|\nabla_y \varphi|^2 - \lambda_0 |\varphi|^2)\, \mathrm{d}y + \frac{2t}{(1 - C|\xi|)} \int_\omega (\nabla_y u_0 \cdot \xi)\, \varphi \,\mathrm{d}y \right] \\ &\quad - 2C\lambda_0 |\xi| \int_\omega |\varphi|^2 \,\mathrm{d}y \\ &\ge \frac{-t^2 |\xi|^2}{4(1 - C|\xi|)} - 2C\lambda_0 |\xi| \parallel \varphi \parallel_{L^2(\omega)}^2 . \end{aligned} \tag{4.16}$$

Thus by (4.14):

$$\frac{(R_\xi(tu_0 + \varphi) - \lambda_0)^-}{|\xi|^2} \le \frac{1}{4(1 - C|\xi|)^2} + \frac{2C\lambda_0 s_0^2\, |\xi|}{(1 - C|\xi|)},$$

whenever $s := \frac{\|\varphi\|_{L^2(\omega)}}{|t|\,|\xi|}$ satisfies $s \le s_0$. We claim that, for a suitable choice of s_0, the left hand member of the previous inequality vanishes for $s \ge s_0$. The upper bound (4.15) will follow by taking the supremum with respect to $u = tu_0 + \varphi$ and then (4.11) is a consequence of (4.13) and (4.15).

We now prove the claim: let λ_1 denote the second eigenvalue of the Laplace operator in the cross section ω; then $\lambda_1 > \lambda_0$ and since $\varphi \perp u_o$, we have $\int_\omega |\nabla \varphi|^2 \ge \lambda_1 \int_\omega |\varphi|^2$. By (4.17) and Cauchy-Schwartz inequality, it follows that :

$$\begin{aligned} (A - \lambda_0 B)(tu_0 + \varphi) &\ge (\lambda_1 - \lambda_0 - C(\lambda_1 + \lambda_0)|\xi|) \parallel \varphi \parallel_{L^2(\omega)}^2 - 2 \parallel \nabla_y u_0 \parallel_{L^2(\omega)} |t||\xi| \parallel \varphi \parallel_{L^2(\omega)} \\ &\ge |t|^2 |\xi|^2 \left[(\lambda_1 - \lambda_0 - C(\lambda_1 + \lambda_0)|\xi|) s^2 - 2 \parallel \nabla_y u_0 \parallel_{L^2(\omega)} s \right] \end{aligned}$$

yielding clearly that $R_\xi(u_0 + t\varphi) \ge \lambda_0$ for large values of s. The proof of Proposition 4.1 is achieved. □

4.2. **The main result**

Recalling the definitions of G_ε and G (see (3.6) and (4.1)–(4.3)) , we are now in position to state our main result. In what follows we will assume that $k(s), \tau(s)$ belong to $L^\infty(0,L)$ and that the angular parameter $\alpha(s)$ is a Lipschitz function. We introduce the effective potential $q(s) \in L^\infty(0,L)$ given by

$$q(s) := (\tau(s) + \alpha'(s))^2 \; C(\omega) - \frac{k(s)^2}{4} \cdot \tag{4.17}$$

Theorem 4.4.

(i) The sequence $\{G_\varepsilon\}$ *satisfies all conditions (i), (ii), (iii) of Theorem* 3.1.

(ii) The eigenvalues λ_i^ε *of the spectral problem* (3.1) *satisfy* (3.3) *where* μ_i *(*$i \in \mathbb{N}$*) are the eigenvalues of the following Sturm-Liouville problem*

$$-\varphi'' + q(s)\; \varphi = \mu\varphi, \quad \varphi \in H_0^1(0,L), \tag{4.18}$$

(iii) Let u_i^ε *be a normalized eigenvector for problem* (3.1) *associated with* λ_i^ε *(recall* $(u_i^\varepsilon | u_j^\varepsilon) = \delta_{i,j}$*). Then, possibly after extraction of a subsequence,* $v_i^\varepsilon = \psi_\varepsilon(u_i^\varepsilon)$ *converges strongly in* $L^2(Q_L)$ *to* $v_i(s,y) = w_i(s)u_0(y)$ *where* w_i *is a normalized eigenvector of problem* (4.17), (4.18) *associated with* μ_i*. Conversely, any such* v_i *is the limit of a sequence* $\psi_\varepsilon(u_\varepsilon)$ *where* u_ε *is an eigenvector of* (3.1) *associated with* λ_i^ε.

Remark 4.5. In the particular case of a circular cross section of radius R, the ground state u_0 is radial: it is given by $u_0(x) = \frac{\sqrt{2}}{RJ_1(\sqrt{\lambda_0}R)}\; J_0(\sqrt{\lambda_0}\;|x|)$ where $\lambda_0 = \left(\frac{r_0}{R}\right)^2$, $|x|$ is the distance to the axis, J_0, J_1 are the first and second Bessel functions and r_0 denote the first zero of J_0. Therefore the constant $C(\omega)$ defined in (4.3) vanishes and we recover, by variational methods, the curvature dependence obtained in [5] (in [2,5], the length L is infinite and the method used is based on formal operator asymptotic expansions).

In this paper we bring to the fore a new effect due to torsion. This effect appears when the constant $C(\omega)$ is stricly positive; this is the case for example if ω is a rectangle. Then the rotation of the section with respect to the Tang frame $(\tau + \alpha')$ produces a shift of the spectrum on the right. This effect comes into competition with the curvature effect which produces a shift on the left.

Remark 4.6. The tools used in our approach are very flexible and we may as well deal with Neumann conditions on the extremities of the tube (which after the change of variables corresponds to $\{0,L\} \times \omega$) and the limit spectral equation (4.17) would be changed accordingly.

Remark 4.7. The case of a heterogeneous medium decribed by a diffusion coefficient $a(y)$ (see Rem. 4.2), for example $a(y)$ taking two values and jumping across a coaxial cylinder, is beyond the scope of this paper. In that case the behavior of the eignevalues as $\varepsilon \to 0$ include in general an additional term of order ε^{-1} and the asymptotic behavior given in (3.3) has to be replaced by

$$\lambda_i^\varepsilon \;=\; \frac{\lambda_0}{\varepsilon^2} + \frac{\lambda_1}{\varepsilon} + \mu_i \; + \rho(\varepsilon), \quad \lim_{\varepsilon \to 0} \rho(\varepsilon) = 0.$$

The presence of the non zero coeffient λ_1 is a direct consequence of the fact that the gradient of the function $\lambda(\xi)$, introduced in Section 4.1, does not vanish, as emphasized in Remark 4.2.

Remark 4.8. In order to modelize a tube with infinite length, it is natural to send $L \to +\infty$ (after substituting the interval $(0,L)$ by $(-L/2, L/2)$). Doing so, we expect from (4.17), (4.18) that the corresponding spectral set $\{\mu_{i,L}\}$ will converge to the spectrum of the same operator on the whole real line *i.e.* $-w'' + q(s)\, w,\; w \in L^2(\mathbb{R})$. Now it is possible to proceed in different ways namely as in [5] by considering for every ε the spectral problem on the infinite tube and then passing to the limit in ε. It seems to us that both ways lead to the same limit operator. The proof of this fact would require futher analysis and, in particular, a generalization of our results in which we could choose the length $L = L(\varepsilon)$ to be dependant of ε with $L_\varepsilon \to \infty$.

4.3. **Proof of the main theorem**

We proceed in four steps: in Step 1, we prove that $\{G_\varepsilon\}$ satisfies the conditions (i), (ii) of Theorem 3.1. Then to prove the Γ-convergence result (condition (iii)), we establish the lower bound inequality (Step 2) and the upper bound inequality (Step 3). In the last step, we apply Theorem 3.1.

Step 1. Recalling the definition of G_ε in (3.6), we deduce from (4.5) the lower bound

$$G_\varepsilon(v) \geq \int_0^L \int_\omega \left\{ \frac{1}{\beta_\varepsilon} \Big| v' + \nabla_y v \cdot R\ y\ (\tau + \alpha') \Big|^2 + \beta_\varepsilon(y,s)\gamma_\varepsilon(s)\ |v|^2 \right\}\ \mathrm{d}y\ \mathrm{d}s. \tag{4.19}$$

Accordingly, the inequality i) is satisfied for any c_0 so large that $\mathrm{essinf}_{Q_L}\{\gamma_\varepsilon\beta_\varepsilon\} \ \geq -c_0$. Since the curvature $k(s)$ belongs to $L^\infty(0,L)$, $\beta_\varepsilon \to 1$ uniformly on Q_L (see (3.5)) whereas by Proposition 4.1 $\gamma_\varepsilon(s) \to -\frac{1}{4}k^2(s)$ uniformly. The latter lower bound is achieved, for ε small enough, provided $c_0 \ > \ \dfrac{1}{4}\,(\|k\|_\infty)^2$.

Consider now a sequence $\{v_\varepsilon\}$ bounded in $L^2(Q_L)$ such that $G_\varepsilon(v_\varepsilon) \leq M$. Then, as β_ε is unifomly close to 1, we infer from (4.19) that:

$$\limsup_{\varepsilon\to 0} \int_{Q_L} \Big| v'_\varepsilon + \nabla_y v_\varepsilon \cdot R\ y\ (\tau+\alpha') \Big|^2 \ \leq\ C \ < +\infty, \tag{4.20}$$

and from (3.6) that

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{Q_L} |\nabla_y v_\varepsilon|^2 &\leq\ \limsup_{\varepsilon\to 0} \int_{Q_L} \beta_\varepsilon |\nabla_y v_\varepsilon|^2 \\ &\leq\ \limsup_{\varepsilon\to 0} \left\{ \int_{Q_L} \beta_\varepsilon \Big(|\nabla_y v_\varepsilon|^2 - \lambda_0\, |v_\varepsilon|^2 \Big) + \lambda_0 \int_{Q_L} \beta_\varepsilon |v_\varepsilon|^2 \right\} \\ &\leq\ \limsup_{\varepsilon\to 0} \left\{ C\varepsilon^2 + \lambda_0 \limsup_{\varepsilon\to 0} \int_{Q_L} |v_\varepsilon|^2 \right\} < +\infty. \end{aligned} \tag{4.21}$$

From (4.20), (4.21) and the fact that $\tau + \alpha'$ is bounded, we infer that the sequence $\{Dv_\varepsilon\}$, where $Dv_\varepsilon = (v'_\varepsilon, \nabla_y v_\varepsilon)$, is bounded in $L^2(Q_L)$. Thus $\{v_\varepsilon\}$ is bounded in $H^1_0(Q_L)$ and strongly relatively compact in $L^2(Q_L)$ by Rellich-Kondrachov Theorem.

Step 2. Let $\{v_\varepsilon\}$ be a sequence such that $v_\varepsilon \to v$ in $L^2(Q_L)$. Up to a subsequence we may assume that $\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) < +\infty$. Then, as proved in Step 1, the sequence is bounded in $H^1_0(Q_L)$ and inequalities (4.20) and (4.21) apply. Therefore, v belongs to $H^1_0(Q_L)$ and $v'_\varepsilon \rightharpoonup v', \nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ weakly in $L^2(Q_L)$. In particular, as $R\ y\ (\tau+\alpha') \in L^\infty(Q_L)$, we obtain:

$$v'_\varepsilon + \nabla_y v_\varepsilon \cdot R\ y\ (\tau+\alpha') \rightharpoonup v' + \nabla_y v \cdot R\ y\ (\tau+\alpha').$$

Futhermore, from (4.19) and the uniform convergence of $\beta_\varepsilon\gamma_\varepsilon$ to $-\frac{1}{4}k^2(s)$, we deduce that

$$\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \ \geq\ \int_{Q_L} \left\{ \Big| v' + \nabla_y v \cdot R\ y\ (\tau+\alpha') \Big|^2 - \frac{k^2}{4}|v|^2 \right\}\ \mathrm{d}y\ \mathrm{d}s. \tag{4.22}$$

Now, due to (4.21) and the strong convergence of v_ε, we derive that

$$\int_{Q_L} |\nabla_y v|^2 \ \leq\ \liminf_{\varepsilon\to 0} \int_{Q_L} |\nabla_y v|^2 \ \leq\ \lambda_0 \limsup_{\varepsilon\to 0} \int_{Q_L} |v_\varepsilon|^2 \ =\ \lambda_0 \int_{Q_L} |v|^2.$$

In other words, we have $\int_0^L f(s)\,\mathrm{d}s = 0$ where the function $f(s) := \int_\omega (\nabla_y v|^2 - \lambda_0 |v|^2)(s,y)\,\mathrm{d}y$ is nonnegative by the definition of λ_0. Therefore, for *a.e.* $s \in (0,L)$, $f(s)$ vanishes and $v(s,\cdot)$, as an eigenvector associated with λ_0, is proportional to the ground state u_0. We deduce that v can be written in the form $v(s,y) = w(s)\,u_0(y)$ with $w \in H_0^1(0,L)$ (since $v \in H_0^1(Q_L)$). We plug this expression of v into (4.22) and, after straightforward computations where we use (4.3) and the equalities $\int_\omega u_0^2\,\mathrm{d}y = 1$, $\int_\omega u_0 \nabla_y u_0 \cdot Ry\,\mathrm{d}y = 0$, we conclude that $\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \ \geq G(v)$ where $G(v) = G_0(w)$ is given by (4.1) (4.2). This concludes the proof of the lower bound part of the Γ-convergence.

Step 3. Let $v \in L^2(Q_L)$. We have to show the existence of a sequence $\{v_\varepsilon\}$ such that $v_\varepsilon \to v$ and $\limsup_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \leq G(v)$. We may assume that $G(v) < +\infty$ so that, in view of (4.1), we can write $v(s,y) = w(s)u_0(y)$ for a suitable element $w \in H_0^1(0,L)$. We consider v_ε defined by $v_\varepsilon = w(s)[u_0(y) + \varepsilon\varphi(s,y)]$ where $\varphi \in H_0^1(Q_L)$ will be chosen later. Clearly $v_\varepsilon \to v$ strongly in $H_0^1(Q_L)$ and, as β_ε is uniformly close to 1, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{Q_L} \frac{1}{\beta_\varepsilon}\Big|v_\varepsilon' + \nabla_y v_\varepsilon \cdot R\ y\ (\tau+\alpha')\Big|^2 \mathrm{d}s\ \mathrm{d}y &= \int_{Q_L} \Big|v' + \nabla_y v \cdot R\ y\ (\tau+\alpha')\Big|^2 \mathrm{d}s\ \mathrm{d}y \\ &= \int_0^L |w'|^2 + \big[(\tau+\alpha')(s)^2\ C(\omega)\big]\,|w|^2\ \mathrm{d}s. \end{aligned} \tag{4.23}$$

As in the proof of Lemma 4.3, we find that, for *a.e.* s:

$$\int_\omega \frac{\beta_\varepsilon(s,y)}{\varepsilon^2}\Big(|\nabla_y v_\varepsilon|^2 - \lambda_0|v_\varepsilon|^2\Big)\,\mathrm{d}y = \int_\omega \beta_\varepsilon(s,y)\,\big[\big(|\nabla_y\varphi|^2 - \lambda_0|\varphi|^2\big)\ +\ 2k(s)z_\alpha(s)\cdot\nabla_y u_0\ \varphi\big]\ w^2(s)\,\mathrm{d}y.$$

Integrating with repect to s, passing to the limit as $\varepsilon \to 0$ and taking into account (4.23), we are led to

$$\lim_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \quad = \quad \int_0^L |w'|^2 + \big[(\tau+\alpha')(s)^2\ C(\omega)\big]\,|w|^2\ \mathrm{d}s\ +\ F(\varphi), \tag{4.24}$$

where

$$F(\varphi)\ :=\ \int_{Q_L} \big[\big(|\nabla_y\varphi|^2 - \lambda_0|\varphi|^2\big)\ +\ 2k(s)z_\alpha(s)\cdot\nabla_y u_0\ \varphi\big]\ w^2(s)\ \mathrm{d}s\,\mathrm{d}y.$$

We observe that F, as a functional on $H_0^1(Q_L)$, is derivative free with respect to s and then can be extended by continuity to the larger space $L^2(0,L;H_0^1(\omega))$. Recalling (4.7) and (4.8), and in view of Lemma 4.3, the minimizer of this extended functional $\tilde{F}$ is reached for $\varphi = \varphi_0$ where

$$\varphi_0(s,y)\ :=\ u_{(kz_\alpha)(s)}(y)\ =\ k(s)\ \left[\cos\alpha(s)\,\chi_1(y) - \sin\alpha(s)\,\chi_2(y)\right].$$

Therefore

$$\inf\{F(\varphi) : \varphi \in H_0^1(Q_L)\} = \tilde{F}(\varphi_0)\ = -\int_{Q_L} \frac{k^2(s)}{4}\ w^2(s)\,\mathrm{d}s.$$

Now we choose a minimizing sequence $\{\varphi_n\}$ in $H_0^1(Q_L)$ such that $\varphi_n \to \varphi_0$ in $L^2(0,L;H_0^1(\omega))$. Then replacing v_ε by $v_{\varepsilon,n} = w(s)(u_0(y) + \varepsilon\varphi_n(s,y))$ in (4.24), we obtain:

$$\limsup_{n\to\infty}\ \big(\limsup_{\varepsilon\to 0}\ G_\varepsilon(v_{\varepsilon,n})\big)\ \leq\ G_0(w)\ (= G(v)).$$

The conclusion follows by taking a diagonal subsequence.

Step 4. Assume further that $G \geq -c_0\|\cdot\|^2$ for a suitable $c_0 \geq 0$ then, as a functional from $L^2(Q_L)$ into $(-\infty,+\infty]$, G is lower semicontinuous and quadratic (see [5], Th. 11.10). Denote by V the subspace where it is finite. By [4], Theorem 12.13, there exists a bilinear symmetric form $a_0(u,v)$ such that $G(v) = a_0(v,v)$ if

$v \in V$ and such that the associated operator $A_0 : L^2(Q_L) \to L^2(Q_L)$ is self adjoint with dense domain (in fact $A_0 + c_0 \, \mathrm{I_{Q_L}}$ is maximal monotone).

We apply Theorem 3.1 to the sequence $\{G_\varepsilon\}$ which by the previous steps satisfy all the required conditions. The domain of G can be identified with the space $H_0^1(0, L)$ through the map $v(s, y) = w(s)\, u_0(y) \mapsto w(s)$. In this identification, the self-adjoint operator A_0 associated with G becomes

$$A_0 : w \in H_0^1(0, L) \cap H^2(0, L) \;\to\; -w'' \;+\; q(s)\, w \;\in L_2(0, L).$$

5. Physical interpretation and examples

As stated in Theorem 4.4, the result obtained can be put in the form of a Sturm-Liouville problem. Equations (4.17) and (4.18) can be interpreted as a onedimensional problem for the spatial wave equation of a particle confined to move in a onedimensional waveguide with a potential given by $q(s)$. In other words, although we have started from a threedimensional problem without a potential in the interior of the domain under consideration, in the limit, in a onedimensional curved waveguide, the particle sees the curvature, the torsion and the influence of the cross section as a (nonhomogeneous) potential function in an equivalent straight waveguide of the same total length. This potential, induced by the geometry of the waveguide, includes the influence of the curvature and of the rotation of the section through the functions $k(s)$, $\alpha(s)$ and $\tau(s)$; also the influence of the shape of the cross section goes through the constant $C(\omega)$, for it depends both on ω and on the eigenfunction u_0 which changes with ω. Moreover, as pointed out in Remark 4.5, the effects of curvature and rotation compete against each other.

Assume that, from the start, we have a straight waveguide , that is $k \equiv 0$ and $\tau \equiv 0$. If the cross section is circular (or if $\alpha' \equiv 0$) then one obtains the classical (eigenvalue, eigenvector) pairs $\left(\left(\frac{n\pi}{L}\right)^2, \sqrt{\frac{2}{L}} \, \sin \frac{n\pi x}{L} \right)$.

Assume now that the cross section is a square. Then $C(\omega) > 0$ and we can very easily simulate an arbitrary positive potential by suitably choosing the rotation angle $\alpha(s)$. In particular, if we consider a straight waveguide of infinite length twisted so that α is periodic, then our analysis (see Rem. 4.8) allows us to predict the possible presence of *band gaps* in the limit as $L \to \infty$ since, by (4.17), we get an effective potential $q(s) = C(\omega)(\alpha'(s))^2$ on the real line which is periodic.

Coming back to a wave guide of finite length L, let us now illustrate the change of the probability density function through a simple example. Let us consider that q is constant in a certain interval $[a, L] \subset [0, L]$ and zero in $[0, a[$. Then, in this case, solving (4.16), (4.17) leads to search for $\mu \geq q$ such that:

$$\sin(\sqrt{\mu}\, a) \;\cos[\sqrt{\mu - q}(L - a)] + \frac{\sqrt{\mu}}{\sqrt{\mu - q}} \;\cos(\sqrt{\mu}\, a)\; \sin[\sqrt{\mu - q}(L - a)] = 0,$$

(where by convention $\sqrt{\mu} = i\, \sqrt{-\mu}$ if $\mu < 0$).

In Figure 3, we show the dependence of the eigenvalues with respect to a/L for $q = -6$ and $L = 2$.

We remark that for $a/L = 1$ (or $a/L = 0$) , we must obtain the usual eigenvalues

$$\mu_n = \left(\frac{n\,\pi}{L}\right)^2 \quad \text{or} \quad \mu_n = \left(\frac{n\,\pi}{L}\right)^2 + q \qquad (n \in \mathbb{N}).$$

If, for example, one chooses $q = -6$, $a = 1$ and $L = 2$ one gets $\mu \approx -1.363855334$ and the probability density function $P(s) = w^*(s)w(s)$ becomes (Fig. 4).

As is clear from the above example, a local perturbation of the curvature and/or the torsion and/or the shape of the cross section will change not only the energy levels but also the wave function and, consequentely, the probability density function in the waveguide. For example, if one whishes that the probability density function be concentrate near the end $s = L$, then one should strongly bend the waveguide near the end. For the case where one has $q = -80$, $a = 1.8$ and $L = 2$ the result is the following (Fig. 5).

Now, owing to (4.17), (4.18) if the shape contant $C(\omega)$ is stricly positive, we may start from the situation depicted in Figure 5 and apply a rotation $\alpha(s)$ of increasing amplitude between $L/2$ and L in order to compensate

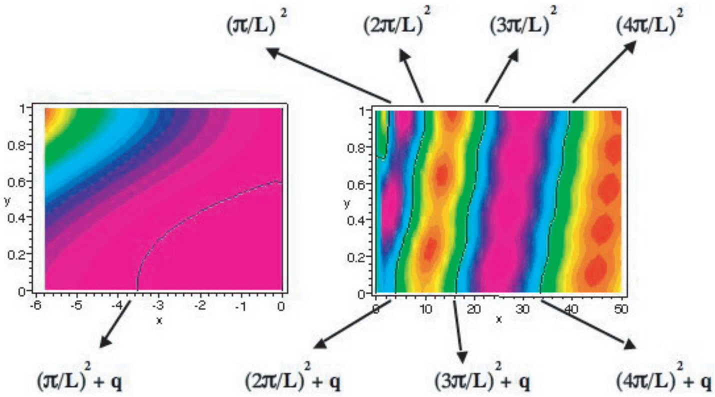

FIGURE 3. $\mu_1, \mu_2, \mu_3, \mu_4$ vs. a/L for $q = -6$ and $L = 2$.

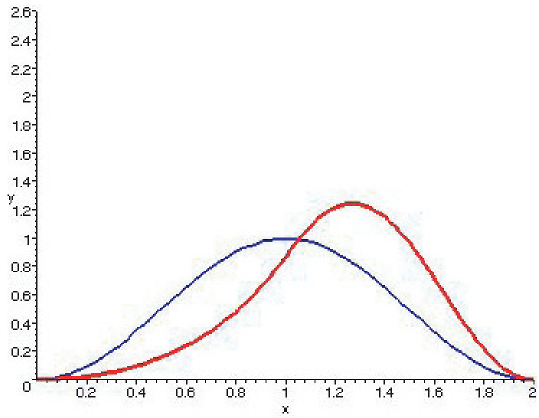

FIGURE 4. Probability density function (thick line) and for the classical case (thin line) ($q = -6$, $a = 1$ and $L = 2$).

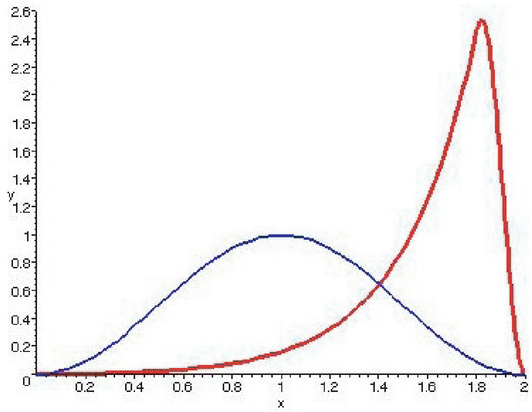

FIGURE 5. Probability density function (thick line) and for the classical case (thin line) ($q = -80$, $a = 1.8$ and $L = 2$).

the curvature and in such a way that we recover the probability density depicted in Figure 4. In other words, by playing with the curvature and the torsion of the waveguide, we can control the energy levels and bound state solutions at will.

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